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On a Generalization of a Theorem due to Larcombe on the Sum of a ${}_3F_2$ Series

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In 2007, P. J. Larcombe established the following result for the ${}_3F_2$ hypergeometric series, for $m \geq 2$:

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}(1-m), & \frac{1}{2}(1+m), & 1 \\ \frac{3}{2}, & \frac{3}{2} & \end{matrix} ; 1\right) = \frac{\pi^2}{8}\frac{1}{\Gamma^2\left(1-\frac{1}{4}m\right)\Gamma^2\left(1+\frac{1}{4}m\right)}. $$

The aim of this paper is to provide a generalization of this result. The results are derived with the help of a generalization of the Whipple theorem on the sum of a ${}_3F_2$, obtained earlier by Lavoie et al. A few interesting special cases are also given.

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1. Introduction and statement of results

The generalized hypergeometric functions with p numerator and q denominator parameters [1–3] is defined by

$$ {}_pF_q\left(\begin{matrix} \alpha_1, & \ldots, & \alpha_p \\ \beta_1, & \ldots, & \beta_q \end{matrix} ; z\right) = {}_pF_q\left[\alpha_1,\ldots,\alpha_p;\beta_1,\ldots,\beta_q;z\right] $$
$$ = \sum_{n=0}^{\infty}\frac{(\alpha_1)_n\ldots(\alpha_p)_n}{(\beta_1)_n\ldots(\beta_q)_n}\frac{z^n}{n!}, \quad (1) $$

where $(\alpha)_n$ denotes the Pochhammer symbol (or the shifted factorial, since $(1)_n = n!$) defined by

$$ (\alpha)_n = \begin{cases} \alpha(\alpha+1)\ldots(\alpha+n-1), \, n \in \mathbb{N} \\ 1, \, n = 0. \end{cases} \quad (2) $$

Using the fundamental property $\Gamma(\alpha+1)=\alpha\Gamma(\alpha)$, $(\alpha)_n$ can be written in the form

$$(\alpha)_n=\frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \tag{3}$$

where Γ is the well known gamma function.

It is not out of place to mention here that whenever a hypergeometric function reduce to gamma functions, the results are very important from the applicative point of view. Only a few summation theorems for the series ${}_2F_1$ and ${}_3F_2$ are available in the literature. The classical summation theorems such as of Gauss, Gauss's second, Kummer and Bailey for the series ${}_2F_1$ and Watson, Dixon and Whipple for the series ${}_3F_2$ play an important role in the theory of hypergeometric and generalized hypergeometric series.

Bailey, in his well known and very interesting paper [5] applied the above mentioned classical summation theorems and obtained a large number of known and unknown results involving products of generalized hypergeometric series.

Also, Berndt [6] has pointed out that the interesting summations due to Ramanujan can be obtained quite simply by employing the above mentioned classical summation theorems. For more details about the recent generalizations of such summations theorem and their applications, see [9, 10]

It is well known that the classical summation theorems for ${}_3F_2$ play an important rule in the theory of hypergeometric series. Let us recall some of them:

Watson theorem ([2]):

$$ {}_3F_2\left(\begin{matrix} a, & b, & c \\ & & & ;\ 1 \\ \frac{1}{2}(a+b+1), & 2c & \end{matrix}\right) \tag{4}$$
$$=\frac{\Gamma(\frac{1}{2})\Gamma(c+\frac{1}{2})\Gamma(\frac{1}{2}a+\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2})}{\Gamma(\frac{1}{2}a+\frac{1}{2})\Gamma(\frac{1}{2}b+\frac{1}{2})\Gamma(c-\frac{1}{2}a+\frac{1}{2})\Gamma(c-\frac{1}{2}b+\frac{1}{2})},$$

provided $Re(2c-a-b)>-1$.

Dixon theorem ([2]):

$$ {}_3F_2\left[\begin{matrix} a, & b, & c \\ & & & ;\ 1 \\ 1+a-b, & 1+a-c & \end{matrix}\right] \tag{5}$$
$$=\frac{\Gamma(1+\frac{1}{2}a)\Gamma(1+a-b)\Gamma(1+a-c)\Gamma(1+\frac{1}{2}a-b-c)}{\Gamma(1+a)\Gamma(1+\frac{1}{2}a-b)\Gamma(1+\frac{1}{2}a-c)\Gamma(1+a-b-c)}$$

provided $Re(a-2b-2c)>-2$.

Whipple theorem ([2]):

$$ {}_3F_2\left(\begin{matrix} a, & b, & c \\ & & & ; & 1 \\ & e, & f \end{matrix}\right) \tag{6}$$

$$= \frac{\pi\Gamma(e)\Gamma(f)}{2^{2c-1}\Gamma(\frac{1}{2}a+\frac{1}{2}e)\Gamma(\frac{1}{2}a+\frac{1}{2}f)\Gamma(\frac{1}{2}b+\frac{1}{2}e)\Gamma(\frac{1}{2}b+\frac{1}{2}f)},$$

provided $Re(c) > 0$ *and* $Re(e+f-a-b-c) > 0$ *with* $a+b = 1$ *and* $e+f = 2c+1$.

In [10], explicit expressions of

$$ {}_3F_2\left(\begin{matrix} a, & b, & c \\ & & & ; & 1 \\ \frac{1}{2}(a+b+i+1), & 2c+j \end{matrix}\right) \tag{7}$$

for $i, j = 0, \pm1, \pm2$, and

$$ {}_3F_2\left(\begin{matrix} a, & b, & c \\ & & & ; & 1 \\ 1+a-b+i, & 1+a-c+i+j \end{matrix}\right) \tag{8}$$

for $i = -3, -2, -1, 0, 1, 2$; $j = 0, 1, 2, 3$, as well as

$$ {}_3F_2\left(\begin{matrix} a, & b, & c \\ & & & ; & 1 \\ e, & f \end{matrix}\right) \tag{9}$$

with $a+b = 1+i$ and $e+f = 1+2c+j$ each for $i, j = 0, \pm1, \pm2, \pm3$, are given.

For $i = j = 0$, (7), (8) and (9) reduce to the Watson, Dixon and Whipple summation theorems, (4), (5) and (6) respectively.

By employing (8), very recently in [8] a new hypergeometric transformation and a large number of new and interesting identities are deduced.

In 2007, Larcombe [11, Theorem 3, Eq.II2], established the following result on the sum of a ${}_3F_2$ hypergeometric series

$$ {}_3F_2\left(\frac{1}{2}(1-m), \frac{1}{2}(1+m), 1; \frac{3}{2}, \frac{3}{2}; 1\right) = \frac{\pi^2}{8}\frac{1}{\Gamma^2(1-\frac{m}{4})\Gamma^2(1+\frac{m}{4})}, \text{ for } m \geq 2. \tag{10}$$

This result is derived in two different ways; by employing the formula [2, (3.8.2), p. 21]:

$$ {}_3F_2(a, b, e+f-a-b-1; e, f; 1) = \frac{\Gamma(e)\Gamma(f)\Gamma(e-a-b)\Gamma(f-a-b)}{\Gamma(e-a)\Gamma(e-b)\Gamma(f-a)\Gamma(f-b)}$$

$$+ \frac{1}{(a+b-e)}.\frac{\Gamma(e)\Gamma(f)}{\Gamma(a)\Gamma(b)\Gamma(e+f-a-b)}$$

$$\times {}_3F_2(e-a, e-b, 1; e-a-b+1, e+f-a-b; 1),$$

and by the classical Whipple's theorem (6).

The main aim of this research paper is to obtain the explicit expression for the value of

$$ {}_3F_2\left(\begin{matrix} \frac{1}{2}(1-m), & \frac{1}{2}(1+m)+i+j, & 1 & \\ & & & ;\ 1 \\ \frac{3}{2}, & \frac{3}{2}+i & & \end{matrix}\right) $$

for $i=-3,-2,-1,0,1,2,3$; $j=0,-1,-2,-3$. A few interesting special cases have also been given.

The results are derived with the help of a generalized form of Whipple's theorem given by Lavoie et al. [7, Eq. 4, P. 294], viz.:

$$ {}_3F_2\left[\begin{matrix} a, & b, & c & \\ & & & ;\ 1 \\ & e, & f & \end{matrix}\right] $$

$$ = \frac{\Gamma(e)\Gamma(f)\Gamma\left(c-\frac{1}{2}(j+|j|)\right)\Gamma\left(e-c-\frac{1}{2}(i+|i|)\right)\Gamma\left(a-\frac{1}{2}(i+j+|i+j|)\right)}{2^{2a-i-j}\Gamma(e-a)\Gamma(f-a)\Gamma(e-c)\Gamma(a)\Gamma(c)} $$

$$ \times\left\{A_{i,j}\frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{1}{4}(1-(-1)^i)\right)\Gamma\left(\frac{f}{2}-\frac{a}{2}\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}-\frac{i}{2}+\left[-\frac{j}{2}\right]\right)\Gamma\left(\frac{f}{2}+\frac{a}{2}-\frac{i}{2}+\left(\frac{(-1)^j}{4}\right)((-1)^i-1)+\left[-\frac{j}{2}\right]\right)}\right. $$

$$ +B_{i,j}\frac{\Gamma\left(\frac{e}{2}-\frac{a}{2}+\frac{1}{4}(1+(-1)^i)\right)}{\Gamma\left(\frac{e}{2}+\frac{a}{2}-\frac{1}{2}-\frac{i}{2}+\left[-\frac{j}{2}+\frac{1}{2}\right]\right)} $$

$$ \left.\times\frac{\Gamma\left(\frac{f}{2}-\frac{a}{2}+\frac{1}{2}\right)}{\Gamma\left(\frac{f}{2}+\frac{a}{2}-\frac{1}{2}-\frac{i}{2}+\left(\frac{(-1)^j}{4}\right)(1-(-1)^i)+\left[-\frac{j}{2}+\frac{1}{2}\right]\right)}\right\}, $$

where $a+b=1+i+j, e+f=2c+1+i$, $[x]$ is the greatest integer less than or equal to x and its modulus is denoted by $|x|$. Also, i,j takes values in a subset of $0,\pm1,\pm2,\pm3$ and the coefficients $A_{i,j}, B_{i,j}$ are given in the tables for $A_{i,j}, B_{i,j}$ at the end of this paper, [7, Table 1 and Table 2, p. 295-296].

2. Main result

The result to be derived is the following:

Theorem 1. *With $i = -3, -2, -1, 0, 1, 2, 3$ and $j = 0, -1, -2, -3$, we have*

$$
{}_3F_2\left(\begin{matrix} \frac{1}{2}(1-m), & \frac{1}{2}(1+m)+i+j, & 1 \\ \frac{3}{2}, & \frac{3}{2}+i & \end{matrix} ; 1\right)
$$

$$
= \frac{\Gamma\left(\frac{3}{2}+i\right)\Gamma\left(1-\frac{1}{2}(j+[j])\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}(i+[i])\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}m-\frac{1}{2}(i+j+[i+j])\right)}{2^{2-m-i-j}\,\Gamma\left(1+\frac{1}{2}m\right)\Gamma\left(1+\frac{1}{2}m+i\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}m\right)}
$$

$$
\times\left[A_{ij}\,\frac{\Gamma\left(\frac{1}{2}+\frac{1}{4}m+\frac{1}{4}(1-(-1)^i)\right)\Gamma\left(\frac{1}{2}+\frac{1}{2}i+\frac{1}{4}m\right)}{\Gamma\left(1-\frac{1}{4}m-\frac{1}{2}i+[-\frac{1}{2}j]\right)\Gamma\left(1-\frac{1}{4}m+\frac{(-1)^j}{4}((-1)^i-1)+[-\frac{1}{2}j]\right)}+\right.
$$

$$
\left.B_{ij}\,\frac{\Gamma\left(\frac{1}{2}+\frac{1}{4}m+\frac{1}{4}(1+(-1)^i)\right)\Gamma\left(1+\frac{1}{2}i+\frac{1}{4}m\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{4}m+\frac{(-1)^j}{4}(1-(-1)^i)+[-\frac{1}{2}j+\frac{1}{2}]\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}i-\frac{1}{4}m+[-\frac{1}{2}j+\frac{1}{2}]\right)}\right].
$$

Here as usual $[x]$ denotes the greatest integer less than or equal to x and its modulus is denoted by $|x|$. The coefficients of A_{ij} and B_{ij} can be obtained from the tables of A_{ij} and B_{ij} by setting $a = \frac{1}{2}(1-m), b = \frac{1}{2}(1+m)+i+j, c = 1, e = \frac{3}{2}$ and $f = \frac{3}{2}+i$.

2.1. Derivation

The derivation of this theorem is quite straight forward. It follows by setting $a = \frac{1}{2}(1-m), b = \frac{1}{2}(1+m)+i+j, c = 1, e = \frac{3}{2}$ and $f = \frac{3}{2}+i$ in the generalized Whipple's theorem.

2.2. Special cases

Certain interesting special cases of our main results are given here.

1. For $i = 0, j = -1$

$$
{}_3F_2\left(\begin{matrix} \frac{1}{2}(1-m), & \frac{1}{2}(-1+m), & 1 \\ \frac{3}{2}, & \frac{3}{2} & \end{matrix} ; 1\right)
= \frac{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\frac{3}{2}\right)}{2^{2-m}\,\Gamma\left(\frac{1}{2}m\right)\Gamma\left(1+\frac{1}{2}m\right)}\left[\frac{\Gamma^2\left(\frac{1}{2}+\frac{1}{4}m\right)}{\Gamma^2\left(1-\frac{1}{4}m\right)}+\frac{\Gamma^2\left(1+\frac{1}{4}m\right)}{\Gamma^2\left(\frac{3}{2}-\frac{1}{4}m\right)}\right].
$$

2. For $i=1, j=-1$

$$
{}_3F_2\left(\begin{matrix}\frac{1}{2}(1-m), & \frac{1}{2}(1+m), & 1 \\ & & & ;\ 1 \\ \frac{3}{2}, & \frac{5}{2} & \end{matrix}\right)
$$

$$
= \frac{\Gamma\left(-\frac{1}{2}\right)\Gamma\left(\frac{5}{2}\right)}{2^{2-m}\,\Gamma\left(1+\frac{1}{2}m\right)\Gamma\left(2+\frac{1}{2}m\right)}
\times\left[\frac{\Gamma^2\left(1+\frac{1}{4}m\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{4}m\right)\Gamma\left(\frac{3}{2}-\frac{1}{4}m\right)}-\frac{\Gamma\left(\frac{1}{2}+\frac{1}{4}m\right)\Gamma\left(\frac{3}{2}+\frac{1}{4}m\right)}{\Gamma^2\left(1-\frac{1}{4}m\right)}\right].
$$

3. For $i=1, j=0$

$$
{}_3F_2\left(\begin{matrix}\frac{1}{2}(1-m), & \frac{1}{2}(3+m), & 1 \\ & & & ;\ 1 \\ \frac{3}{2}, & \frac{5}{2} & \end{matrix}\right)
$$

$$
= \frac{-3\pi\,\Gamma\left(\frac{1}{2}-\frac{m}{2}\right)}{2^{2-m}\,\Gamma\left(1+\frac{1}{2}m\right)\Gamma\left(2+\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}-\frac{1}{2}m\right)}
\times\left[\frac{\Gamma^2\left(1+\frac{1}{4}m\right)}{\Gamma^2\left(\frac{1}{2}-\frac{1}{4}m\right)}-\frac{\Gamma\left(\frac{3}{2}+\frac{1}{4}m\right)\Gamma\left(\frac{1}{2}+\frac{1}{4}m\right)}{\Gamma\left(1-\frac{1}{4}m\right)\Gamma^2\left(-\frac{1}{4}m\right)}\right].
$$

4. For $i=-1, j=0$

$$
{}_3F_2\left(\begin{matrix}\frac{1}{2}(1-m), & \frac{1}{2}(-1+m), & 1 \\ & & & ;\ 1 \\ \frac{3}{2}, & \frac{1}{2} & \end{matrix}\right)
$$

$$
= \frac{\pi}{2^{3-m}\,\Gamma\left(1+\frac{1}{2}m\right)\Gamma\left(\frac{1}{2}m\right)}\left[\frac{\Gamma\left(\frac{1}{4}m\right)\Gamma\left(1+\frac{1}{4}m\right)}{\Gamma\left(\frac{1}{2}-\frac{1}{4}m\right)\Gamma\left(\frac{3}{2}-\frac{1}{4}m\right)}+\frac{\Gamma^2\left(\frac{1}{2}+\frac{1}{4}m\right)}{\Gamma^2\left(1-\frac{1}{4}m\right)}\right].
$$

5. For $i=2, j=-1$

$$
{}_3F_2\left(\begin{matrix}\frac{1}{2}(1-m), & \frac{1}{2}(3+m), & 1 \\ & & & ;\ 1 \\ \frac{3}{2}, & \frac{7}{2} & \end{matrix}\right)
$$

$$
= \left[\left(\frac{3}{2}+\frac{1}{2}m\right)\frac{\Gamma\left(\frac{1}{2}+\frac{1}{4}m\right)\Gamma\left(\frac{3}{2}+\frac{1}{4}m\right)}{\Gamma\left(-\frac{1}{4}m\right)\Gamma\left(1-\frac{1}{4}m\right)}
-\left(\frac{1}{2}-\frac{1}{2}m\right)\frac{\Gamma\left(1+\frac{1}{4}m\right)\Gamma\left(2+\frac{1}{4}m\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{4}m\right)\Gamma\left(\frac{1}{2}-\frac{1}{4}m\right)}\right].
$$

6. For $i=-1, j=-1$

$$
{}_3F_2\left(\begin{matrix} \frac{1}{2}(1-m), & \frac{1}{2}(-3+m), & 1 \\ \frac{3}{2}, & \frac{1}{2} & \end{matrix} ; 1\right)
= \frac{\pi}{2^{4-m}\,\Gamma\left(\frac{1}{2}m\right)\Gamma\left(1+\frac{1}{2}m\right)}\left[\frac{1}{2}\frac{\Gamma\left(\frac{1}{4}m\right)\Gamma\left(1+\frac{1}{4}m\right)}{\Gamma^2\left(\frac{3}{2}-\frac{1}{4}m\right)} + \frac{\Gamma^2\left(\frac{1}{2}+\frac{1}{4}m\right)}{\Gamma\left(1-\frac{1}{4}m\right)\Gamma\left(2-\frac{1}{4}m\right)}\right].
$$

7. For $i=-2, j=-1$

$$
{}_3F_2\left(\begin{matrix} \frac{1}{2}(1-m), & \frac{1}{2}(-5+m), & 1 \\ \frac{3}{2}, & -\frac{1}{2} & \end{matrix} ; 1\right)
= \frac{-\pi}{2^{4-m}\,\Gamma\left(1+\frac{1}{2}m\right)\Gamma\left(-1+\frac{1}{2}m\right)}\left[\left(\frac{1}{2}m-\frac{5}{4}\right)\frac{\Gamma\left(\frac{1}{2}+\frac{1}{4}m\right)\Gamma\left(-\frac{1}{2}+\frac{1}{4}m\right)}{\Gamma\left(2-\frac{1}{4}m\right)\Gamma\left(1-\frac{1}{4}m\right)} + \left(\frac{5}{4}-\frac{1}{2}m\right)\frac{\Gamma\left(\frac{1}{4}m\right)\Gamma\left(1+\frac{1}{4}m\right)}{\Gamma\left(\frac{3}{2}-\frac{1}{4}m\right)\Gamma\left(\frac{5}{2}-\frac{1}{4}m\right)}\right].
$$

8. For $i=j=0$

$$
{}_3F_2\left(\begin{matrix} \frac{1}{2}(1-m), & \frac{1}{2}(1+m), & 1 \\ \frac{3}{2}, & \frac{3}{2} & \end{matrix} ; 1\right) = \frac{\pi^2}{8}\frac{1}{\Gamma^2\left(1-\frac{1}{4}m\right)\Gamma^2\left(1+\frac{1}{4}m\right)}.
$$

The result (8) is the Larcombe's result [11, Theorem 3, Eq. II2], and the results (1)-(7) are closely related to it.

3. Tables for A_{ij} and B_{ij}

i/j	-3	-2	-1
3	$-(a+2)(a-3)$ $+3c(c+3)$ $-e(3c-e+5)$	$-(a-1)(a+e-3)$ $+c(a+c)$	$-(a-1)(a-2)$ $+c(2c-e+2)$
2	$-(a+1)(a-2)$ $+c(a+c+3)$ $-e(2c-e+3)$	$-\frac{1}{2}(a+1)(a-2)$ $+c(c+2)$ $-\frac{e}{3}(2c-e+3)$	$c-a+1$
1	$(a-1)(a+2)$ $-2c(c+2)$ $+e(3c-e+1)$	$a-c+e-1$	1
0	$e(2c-e+1)$ $+a(a-c+1)$	$a(a+1)$ $+e(2c-e+1)$	1
-1	-	$e(2c-e+1)$ $+(a+2)(c-e)$	$2c-e$
-2	-	$\frac{1}{2}p_2(a,c,e)$	$e(2c-e-1)-ac$
-3	-	-	$-$

Table 1: Table for the coefficients $A_{i,j} : i=-3,-2,-1$

$$p_2(a,c,e) = a(a+3)[2c(e-c)-e(e+1)] + e(e+2)(2c-e-1)(2c-e+1)$$

i/i	0	1	2	3
3	$e(2c-e) -(a-6) (a-c+e) -c-11$	–	–	–
2	$-\frac{1}{2}(a-1) (a-2) +\frac{1}{2}(e-2) (2c-e+1)$	$e(2c-e+3) -(a+3) (c-1) -6$	$\frac{1}{2}p_1(a,c,e)$	–
1	1	$2c-e$	$e(2c-e+4) +a(c-e+1) -7c-1$	–
0	1	1	$(a-1) (a-2) +(e-2) (2c-e-1)$	$(a+3) (1+a-c) +e(2c-e+1) -6a$
−1	1	1	$a-c+e-1$	$(a-1) (a-2) +(e-c) (2c-e-2)$
−2	$-\frac{a}{2}(a+1) +\frac{e}{2}(2c-e-1)$	$c-a-1$	$-\frac{1}{2}(a+1) (a-2) +c(c-2) -\frac{e}{2}(2c-e-1)$	$-(a-1) (a+1) +c(a+c-2) -e(2c-e-1)$
−3	$e(2c-e-a-3) -(a+2) (a-c+1)$	$-(a-1) (a+2) +(c-1) (2c-e) -2c$	$-(a-1) (a+2) +a(c-e) +c(c-3)$	$-(a+2) (a-3) +3c(c-3) -e(3c-e-4)$

Table 2: Table for the coefficients $A_{i,j}$: $i = 0, 1, 2, 3$

$$\begin{aligned} p_1(a,c,e) = &-a(a-5)[2c(c-e)+(e-1)(e-2)+2] \\ &+(e-1)(e-4)(2c-e)^2-(e-2)[3(e-4)+(2c-e)^2] \end{aligned}$$

i/j	−3	−2	−1
3	$(a+1)(a-2)$ $-c(c+3)$ $+e(c-e+3)$	$(a-1)(a-e+1)$ $+c(a-c-2)$	$(a-1)(a-2)$ $-c(e-2)$
2	$-(a-1)(a+2)$ $+c(a-c+3)$ $-e(2c-e+3)$	-2	$-(a+c-1)$
1	$-a(a+1)$ $-e(c-e+1)$	$-(a+c-e+1)$	-1
0	$e(2c-e+1)$ $+(a+2)(a+c+1)$	2	1
−1	-	$e(2c-e+1)$ $-a(c-e)$	e
−2	-	$2(e+1)(2c-e)$	$c(a+2)$ $+e(2c-e-1)$
−3	-	-	$-$

Table 3: Table for the coefficients $B_{i,j}$: $j=-3,-2,-1$

i/j	**0**	**1**	**2**	**3**
3	$-e(2c-e+a+2)$ $+(a+3)(a+c+1)$ $-6a$	–	–	–
2	-2	$-(a-1)(c-1)$ $-(e-3)(2c-e)$	$-2(e-3)(2c-e)$	–
1	-1	$-(e-2)$	$(a+3)(c+1)$ $-e(2c-e+a)-6$	–
0	0	-1	-2	$-(a-7)(a+c-2)$ $+e(2c-e+1)$ $-3(a-1)$
−1	1	1	$a+c-e+1$	$(a-1)(a-2)$ $+(e-2)(c-e)$
−2	2	$a+c-1$	2	$(a-1)(a-3)$ $-c(c-a)$ $+e(2c-e-1)$
−3	$e(2c-e+a-1)$ $-a(a+c+1)$	$-a(a+1)$ $+e(c-1)$	$-(a+1)(a-2)$ $-a(c-e)$ $+c(c-3)$	$-(a+1)(a-2)$ $+c(c-3)$ $+e(e-c)$

Table 4: Table for the coefficients $B_{i,j}:\ j=0,1,2,3$

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