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$SO(2,1)$-Invariant Double Integral Transforms and Formulas for the Whittaker Functions

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The paper contains some new formulas involving the Whittaker functions and arising as the values of some double integrals, which are invariant with respect to the representation of the group $SO(2,1)$.

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1. Introduction

From the group-theoretical point of view, the special functions are matrix elements of the representations of the corresponding groups. It is clear that this definition can not be applied to all special functions, but it is permissible for quite a wide class of functions arising in problems of mathematical physics. We can say also, that the special functions are matrix elements of the linear operators acting in the representation space. This point of view can be extended in the following direction. We can consider the special functions as the values of the functionals defined on representation spaces.

In this paper, we derive some formulas involving the Whittaker functions

$$M_{\mu,\nu}(z) = \exp\left(-\frac{z}{2}\right) z^{\nu+\frac{1}{2}} {}_1F_1\left(\nu - \mu + \frac{1}{2}\ 2\nu + 1;\, z\right),$$

$$W_{\mu,\nu}(z) = \frac{\Gamma(-2\nu)}{\Gamma\left(\mu - \nu + \frac{1}{2}\right)} M_{\mu,\nu}(z) + \frac{\Gamma(2\nu)}{\Gamma\left(\mu + \nu + \frac{1}{2}\right)} M_{\mu,-\nu}(z).$$

These formulas are related to the 3-dimensional special Lorentz group $SO(2,1)$, which represents the simplest case of the special pseudo-orthogonal group. On the other hand, for derivation of all formulas, we use some bilinear functionals, which are defined on a pair of the representation spaces of the corresponding case of group $SO(2,1)$. These functionals satisfy some good properties. First, these functionals are invariant with respect to all linear operators of the group representation. Let us note that the functionals, satisfying such condition, play an important role in the study of the properties of the representation. For instance, if $D_{(n_1,n_2)}$ be the representation space of the group $SL(2,\mathbb{C})$ and consists of all (n_1-1, n_2-1)-homogeneous and infinitely differentiable functions $f(z_1,z_2)$, than the invariant bilinear functional

$$B(\phi,\psi) = -\frac{1}{4}\int (z_1-z_2)^{-n_1-1}\,\overline{(z_1-z_2)}^{-n_2-1}\,\phi(z_1)\,\psi(z_2)\,\mathrm{d}z_1\,\mathrm{d}z_2\,\mathrm{d}\bar{z}_1,\,\mathrm{d}\bar{z}_2,$$

defined on a pair $(D_{(n_1,n_2)}, D_{(n_1^*,n_2^*)})$, can be applied [1] for the investigation of operator irreducibility of the representation

$$T_{(n_1,n_2)}(g)[f(z_1,z_2)] = f(\alpha z_1+\gamma z_2, \beta z_1+\delta z_2),$$

where

$$g = \begin{pmatrix}\alpha & \beta\\ \gamma & \delta\end{pmatrix}, \quad \det g = 1.$$

The main property of our functionals is that these functionals, defined as double integrals, do not depend on the integration domain. Using this property, we can evaluate the corresponding integrals along two different contours, compare the two results and, finally, obtain new formulas containing the Whittaker functions.

In [7] and [5], we have considered the general case of the group $SO(p,q)$ and the invariant bilinear functionals of another kind. In particular, in [7], we have obtained formulas involving the Legendre functions.

2. Functionals D_i and their properties

Let the space $\mathbb{R}^{p+q}$ is endowed with the quadratic form $\mathsf{q}(x) := \sum\limits_{l=1}^{p} x_l^2 - \sum\limits_{l=1}^{q} x_{p+l}^2$. The group $SO(p,q)$ preserves this form and divides $\mathbb{R}^{p+q}$ into orbits, one of which is the cone $\mathsf{C}: q(x) = 0$. Let σ be an arbitrary complex number. We introduce the linear space $\mathfrak{D}_\sigma$ consisting of all functions f, defined on C and satisfying the following conditions: first, f is smooth, i.e. infinitely differentiable, and, second, f is a σ-homogeneous function, i.e., for any $\alpha\in\mathbb{C}$, the equality $f(\alpha x) = \alpha^\sigma f(x)$ holds. We define the representation T_σ in $\mathfrak{D}_\sigma$ by left shifts according to the formula $T_\sigma(g)[f(x)] := f(g^{-1}x)$. It is known that T_σ is irreducible, if σ is not an integer number [4].

Let us introduce three contours on C. Let $\gamma_1 := \{x : \sum\limits_{l=0}^{p+q} x_i^2 = 2$ (i.e. γ_1 be a circle or a sphere), $\gamma_2 := \{x : x_i + x_j = 1\}$, $i\in\{1,\dots,p\}$, $j\in\{p+1,\dots,p+q\}$

(parabola or paraboloid), and $\gamma_3 := \gamma_{3,+} \bigcup \gamma_{3,-}$ with $\gamma_{3,\pm} := \{x : x_l = \pm 1\}$, $l \in \{1, \dots, p+q\}$ (hyperbola or hyperboloid).

Let H_i mean the subgroup, which acts transitively on γ_i. Let $\mathrm{d}x$ be an H_i-invariant measure on γ_i. According to the definition, $H_1 \simeq SO(p) \times SO(q)$, $H_3 \simeq SO(p-1, q)$ or $H_3 \simeq SO(p, q-1)$, and $H_2 \simeq \exp[\mathbb{R}(e_{12} + e_{13} + e_{31} - e_{21})]$, where $\{e_{ij} : i, j \in \{1, \dots, p+q\}\}$ be the canonical basis of the linear space of all real matrices of size $(p+q) \times (p+q)$. We denote by $\mathrm{d}x$ the H_i-invariant measure on γ_i, i.e.

$$\int\limits_{\gamma_i} f(gx)\,\mathrm{d}x = \int\limits_{\gamma_i} f(x)\,\mathrm{d}x,$$

for any $g \in H_i$.

For each $i \in \{1, 2, 3\}$, we consider the bilinear functionals

$$\mathsf{D}_i : \mathfrak{D}_\sigma^2 \longrightarrow \mathbb{C},\ (u, v) \longmapsto \int\limits_{\gamma_i}\int\limits_{\gamma_i} k(x, \hat{x})\, u(x)\, v(\hat{x})\,\mathrm{d}x\,\mathrm{d}\hat{x}.$$

We say that D_i is invariant with respect to T_σ, if, for any $g \in SO(p, q)$,

$$\mathsf{D}_i(T_\sigma(g)[u], T_\sigma(g)[v]) = \mathsf{D}_i(u, v). \tag{1}$$

Let $\hat{\mathsf{q}}$ be the bilinear form, which is polar for q.

Lemma 1. *For the case of* $SO(2,1)$*, the conditions* (1) *and*

$$k(x, \hat{x})\,|_{\gamma_i \times \gamma_i} = C_i[\hat{\mathsf{q}}(x, \hat{x})]^{-\sigma-1} \tag{2}$$

are equivalent.

P r o o f. Since $SO(2,1)$ is generated by subgroup $H_1 \simeq SO(2)$ and an one-parameter subgroup of hyperbolic rotations in the plane of any pair of axes [4], then it is sufficient to prove the lemma for these subgroups. The proof of this lemma for D_2 is more difficult, than for D_1 and D_3, because of γ_1 and γ_3 are invariant with respect to H_1 and H_3 respectively. Let us prove the lemma only for the case $i = 2$. Let $H := \exp[\mathbb{R}(e_{12} + e_{21})]$.

If $g(\phi) \in H_1$ and $x \in \mathsf{C}$, then, from the equality

$$g^{-1}(\phi)x = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & -\sin\phi & \cos\phi \end{pmatrix} \begin{pmatrix} \frac{1+y^2}{2} \\ \frac{1-y^2}{2} \\ y \end{pmatrix} t =$$

$$\begin{pmatrix} \frac{1+y^2}{2} \\ \frac{1+y^2}{2}\cos\phi + y\sin\phi \\ y\cos\phi + \frac{y^2-1}{2}\sin\phi \end{pmatrix} t = \begin{pmatrix} \frac{1+\tilde{y}^2}{2} \\ \frac{1-\tilde{y}^2}{2} \\ \tilde{y} \end{pmatrix} \tilde{t},$$

we have

$$\mathrm{d}y = \frac{\tilde{t}\,\mathrm{d}\tilde{y}}{t}. \tag{3}$$

If $g(r) \in H$ and $x \in \mathsf{C}$, then from

$$g^{-1}(r)x = \begin{pmatrix} \operatorname{ch} r & -\operatorname{sh} r & 0 \\ -\operatorname{sh} r & \operatorname{ch} r & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \frac{1+y^2}{2} \\ \frac{1-y^2}{2} \\ y \end{pmatrix} t =$$

$$\begin{pmatrix} e^{-r} + e^{r} y^2 \\ e^{-r} - e^{r} y^2 \\ y \end{pmatrix} t = \begin{pmatrix} \frac{1+\tilde{y}^2}{2} \\ \frac{1-\tilde{y}^2}{2} \\ \tilde{y} \end{pmatrix} \tilde{t},$$

we obtain the same formula (3).

So,

$$\mathsf{D}_2(T_\sigma(g)[u], T_{\hat{\sigma}}(g)[v]) =$$

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} k\left(\frac{1+y^2}{2}, \frac{1-y^2}{2}, y, \frac{1+z^2}{2}, \frac{1-z^2}{2}, z\right) \times$$

$$u\left(\frac{\tilde{t}(1+\tilde{y}^2)}{2}, \frac{\tilde{t}(1-\tilde{y}^2)}{2}, \tilde{t}\tilde{y}\right) |_{t=1} \times$$

$$v\left(\frac{\tilde{s}(1+\tilde{z}^2)}{2}, \frac{\tilde{s}(1-\tilde{z}^2)}{2}, \tilde{s}\tilde{z}\right) |_{s=1} \,\mathrm{d}y\,\mathrm{d}z =$$

$$\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} \tilde{t}^{\sigma+1}\,\tilde{s}^{\sigma+1}\,k\left(\frac{1+y^2}{2}, \frac{1-y^2}{2}, y, \frac{1+z^2}{2}, \frac{1-z^2}{2}, z\right) \times$$

$$u\left(\frac{1+\tilde{y}^2}{2}, \frac{1-\tilde{y}^2}{2}, \tilde{y}\right)\, v\left(\frac{1+\tilde{z}^2}{2}, \frac{1-\tilde{z}^2}{2}, \tilde{z}\right) \mathrm{d}\tilde{y}\,\mathrm{d}\tilde{z},$$

because of the homogeneity of u and v. This formula yields

$$k(\tilde{y}, \tilde{z}) = \tilde{t}^{\sigma+1}\,\tilde{s}^{\sigma+1}\,k(y,z). \tag{4}$$

If $g \in H_2$, then we have implication $x \in \gamma_2 \Longrightarrow gx \in \gamma_2$, therefore, $\tilde{s} = \tilde{t} = 1$. It means that the kernel $k(y,z)$, i.e. the restriction $k(x,\hat{x})\,|_{\gamma_2\times\gamma_2}$, depends on the difference $y-z$.

The function $[\hat{\mathsf{q}}(y,z)]^{-\sigma-1} = \left[\frac{1}{2}(y-z)\right]^{-\sigma-1}$ depends on the difference $y-z$ too and satisfies the condition (4). So we can choose $k(y,z)$ as $k(y,z) = C_2\,\left[\frac{1}{2}(y-z)^2\right]^{-\sigma-1}$.

Now consider $\mathsf{D}_i(u,v)$ as a value of the regular generalized function at finite function from the test function space $\mathfrak{D}_{2\sigma}$. From the viewpoint of the homogeneous generalized functions, we can write k as a linear combination of the functions $\left[\frac{1}{2}(y-z)^2\right]_+^{-\sigma-1}$ and $\left[\frac{1}{2}(y-z)^2\right]_-^{-\sigma-1}$, where ([2]

$$x_+^\zeta = \begin{cases} x^\zeta & x>0, \\ 0 & x\le 0, \end{cases} \qquad x_-^\zeta = \begin{cases} |x|^\zeta & x<0, \\ 0 & x\ge 0. \end{cases}$$

Since $\frac{1}{2}(y-z)^2 \ge 0$, then $k(y,z) \in \operatorname{span}\{\frac{1}{2}(y-z)^2\}$.

Thus, we know $k(x,\hat{x})\mid_{\gamma_2\times\gamma_2}$ up to a constant factor. ∎

Lemma 1 describes all restrictions of the kernel k to $\gamma_i \times \gamma_i$:

$$\begin{aligned}
k(x,\hat{x})\mid_{\gamma_1\times\gamma_1} &\equiv k(\alpha,\beta) = C_1\left[1-\cos(\alpha-\beta)\right]^{-\sigma-1}, \\
k(x,\hat{x})\mid_{\gamma_2\times\gamma_2} &\equiv k(y,z) = C_2\left[\frac{1}{2}(y-z)^2\right]^{-\sigma-1}, \\
k(x,\hat{x})\mid_{\gamma_{3,+}\times\gamma_{3,+}} &\equiv k(s,t) = C_{3,+}\left[\cosh(s-t)-1\right]^{-\sigma-1}, \\
k(x,\hat{x})\mid_{\gamma_{3,+}\times\gamma_{3,-}} &\equiv k(s,t) = C_{3,-}\left[\cosh(s-t)+1\right]^{-\sigma-1}, \\
k(x,\hat{x})\mid_{\gamma_{3,-}\times\gamma_{3,+}} &\equiv k(s,t) = C_{3,-}\left[\cosh(s-t)+1\right]^{-\sigma-1}, \\
k(x,\hat{x})\mid_{\gamma_{3,-}\times\gamma_{3,-}} &\equiv k(s,t) = C_{3,+}\left[\cosh(s-t)-1\right]^{-\sigma-1},
\end{aligned}$$

where $C_1, C_2, C_{3,+}, C_{3,-}$ are complex numbers.

Let us now consider another invariant property of our functionals, which is associated with the integration domain.

Lemma 2. *Let the condition* (1) *hold. Then* $\mathsf{D}_1 = \mathsf{D}_2 = \mathsf{D}_3$.

P r o o f. If $x, \hat{x} \in \gamma_i$, then

$$\mathsf{D}_j(u,v) = \int\limits_{\gamma_j}\int\limits_{\gamma_j} k(tx, s\hat{x})\, u(tx)\, v(s\hat{x})\, \mathrm{d}(tx)\, \mathrm{d}(s\hat{x}) =$$

$$\int\limits_{\gamma_j} (ts)^{-\sigma-1}\, k(x,\hat{x})\, (ts)^{\sigma}\, u(x)\, v(\hat{x})\, t\, \mathrm{d}x\, s\, \mathrm{d}\hat{x} =$$

$$\int\limits_{\gamma_i} k(x,\hat{x})\, u(x)\, v(\hat{x})\, \mathrm{d}x\, \mathrm{d}\hat{x} = \mathsf{D}_i(u,v).$$

It completes the proof. ∎

3. Two series converging to the Whittaker function

In this section, we will obtain two formulas related to the rudimentary pseudo-orthogonal group $SO(2,1)$. Let us introduce the bases $\{f_n : n \in \mathbb{Z}\}$, $\{f_\lambda : \lambda \in \mathbb{R}\}$, $\{f_{\rho,\pm} : \rho \in \mathbb{R}\}$ in $\mathfrak{D}_\sigma$ consisting of the functions

$$f_n(x) = x_0^{\sigma-n}\,(x_1 + \mathbf{i}x_2)^n,$$
$$f_\lambda(x) = (x_0 + x_1)^\sigma\,\exp\left(\frac{\mathbf{i}\lambda x_2}{x_0 + x_1}\right),$$
$$f_{\rho,\pm}(x) = (x_1)_\pm^{\sigma-\mathbf{i}\rho}\,(x_0 + x_2).$$

Theorem 1. *Let $n \in \{1, 2, 3, \dots\}$, $\lambda \neq 0$, and $-1 < \operatorname{re}\sigma < -\frac{1}{2}$. Then,*

$$\sum_{l=0}^{\infty} (\mathbf{i}\lambda)^l\,\left[l!\,\mathrm{B}(\sigma-n+1, l+n-\sigma+1)\right]^{-1}\,{}_2F_1(l-2\sigma,\, n-\sigma;\, l+n-\sigma+1;\, -1) =$$
$$(-1)^n\,8\pi^2\,(\mathbf{i}\lambda)^{-2\sigma}\,|\lambda|^{\sigma+\frac{1}{2}}\,\Gamma(-2\sigma-1)\,\Gamma^2(\sigma+1)\,\Gamma^{-2}(-\sigma)\,\Gamma^{-1}(-n\operatorname{sign}\lambda - \sigma)$$
$$\times\,\mathrm{B}(n-\sigma,\, -n-\sigma)\,W_{-n\operatorname{sign}\lambda,\sigma+\frac{1}{2}}(2|\lambda|).$$

P r o o f. Let us make the change of variables $t := y - z$ in the integral

$$\mathsf{D}_2(f_n, f_\lambda) = 2\,C_2 \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} (y-z)^{-2\sigma-2}\,(1+\mathbf{i}y)^{\sigma+n}\,(1-\mathbf{i}y)^{\sigma-n}\,\mathrm{e}^{\mathbf{i}\lambda z}\,\mathrm{d}y\,\mathrm{d}z.$$

We obtain

$$\mathsf{D}_2(f_n, f_\lambda) = 2\,C_3 \int_{-\infty}^{+\infty} t^{-2\sigma-2}\,e^{-\mathbf{i}\lambda t}\,\mathrm{d}t \;\cdot\; \int_{-\infty}^{+\infty} (1+\mathbf{i}y)^{\sigma+n}\,(1-\mathbf{i}y)^{\sigma-n}\,\mathrm{e}^{\mathbf{i}\lambda y}\,\mathrm{d}y.$$

We apply the known formulas

$$\int_0^{+\infty} x^{\alpha-1}\,\mathrm{e}^{-px}\,\mathrm{d}x = \Gamma(\alpha)\,p^{-\alpha},\;\; 0 < \operatorname{re}p < 1,\; \operatorname{re}p = 0,$$

[6, 2.3.3.1], and

$$\int_{-\infty}^{+\infty} (a-\mathbf{i}x)^{-\mu}\,(b+\mathbf{i}x)^{-\nu}\,\mathrm{e}^{\mathbf{i}xy}\,\mathrm{d}y = \qquad (5)$$

$$\begin{cases} 2\pi\,\Gamma^{-1}(\nu)\,(a+b)^{-\frac{\nu+\mu}{2}}\,y^{\frac{\nu+\mu-1}{2}}\,\mathrm{e}^{\frac{(b-a)y}{2}}\,W_{\frac{\nu-\mu}{2},\frac{1-\nu-\mu}{2}}(ay+by), & y>0,\\ 2\pi\,\Gamma^{-1}(\mu)\,(a+b)^{-\frac{\nu+\mu}{2}}\,(-y)^{\frac{\nu+\mu-1}{2}}\,\mathrm{e}^{\frac{(a-b)y}{2}}\,W_{\frac{\mu-\nu}{2},\frac{1-\nu-\mu}{2}}(-ay-by), & y<0,\end{cases}$$

where $\mathrm{re}(\mu+\nu)>0$, [6, 2.3.6.19].

Further,

$$\mathsf{D}_1(f_n,f_\lambda)=C_1\int_0^{2\pi}\int_0^{2\pi}[1-\cos(\alpha-\beta)]^{-\sigma-1}\,(1+\cos\beta)^{\sigma}\times\exp(\mathbf{i}n\alpha)\,\exp\left(\frac{\mathbf{i}\lambda\sin\beta}{1+\cos\beta}\right)\mathrm{d}\alpha\,\mathrm{d}\beta.$$

We make the change $t:=\alpha-\beta$, so

$$\mathsf{D}_1(f_n,f_\lambda)=2^{-\sigma}\,C_1\underbrace{\int_0^{\pi}\sin^{-2\sigma-2}t\,\mathrm{e}^{2\mathbf{i}nt}\,\mathrm{d}t}_{I_1}\times\underbrace{\int_0^{2\pi}(1+\cos\beta)^{\sigma}\,\exp(\mathbf{i}n\beta)\,\exp\left(\frac{\mathbf{i}\lambda\sin\beta}{1+\cos\beta}\right)\mathrm{d}\beta}_{I_2}.$$

Using the Taylor series for $\exp\left(\frac{\mathbf{i}\lambda\sin\beta}{1+\cos\beta}\right)$, we can evaluate I_1 and I_2 according to the formula ([3, 3.892.4]):

$$\int_0^{\pi}\mathrm{e}^{\mathbf{i}2\beta x}\,\sin^{2\mu}x\,\cos^{2\nu}x\,\mathrm{d}x=4^{-(\mu+\nu)}\,\pi\times(2\mu+1)^{-1}\,\mathrm{e}^{\mathbf{i}\pi(\beta-\nu)}\,\mathrm{B}^{-1}(1-\beta+\mu+\nu,1+\beta+\mu-\nu)\times{}_2F_1(-2\nu,b-\mu-\nu;1+\beta+\mu-\nu;-1).$$

The equality $\mathsf{D}_1(f_n,f_\lambda)=\mathsf{D}_2(f_n,f_\lambda)$ contains numbers C_1 and C_2, which don't depend of n and λ. So let us assume $n:=0$ and $\lambda:=1$. Then

$$\frac{C_3}{C_1}=\frac{\Gamma^2(\sigma+1)\,K_{\sigma+\frac12}(1)}{\Gamma^2(-\sigma)\,K_{-\sigma-\frac12}(1)}=\frac{\Gamma^2(\sigma+1)}{\Gamma^2(-\sigma)}.$$

■

The functionals D_i can be applied to the derivation of matrix elements of the representation T_σ and bases transform linear operators. For instance, from the equality

$$\mathsf{D}_i(f_{\rho,+}, f_m) = \sum_{n\in\mathbb{Z}} c_{\rho,+,n}\, \mathsf{D}_1(f_n, f_m),$$

for any i, we have

$$\mathsf{D}_i(f_{\rho,+}, f_m) = \sum_{n\in\mathbb{Z}} c_{\rho,+,n} \int\limits_0^{2\pi} (1-\cos t)^{-\sigma-1}\, \mathrm{e}^{\mathbf{i}tn}\, \mathrm{d}t \cdot \int\limits_0^{2\pi} \mathrm{e}^{\mathbf{i}(m+n)\beta}\, \mathrm{d}\beta,$$

i.e.

$$c_{\rho,+,n} = (-1)^n\, 2^{-\sigma-2}\, \pi^{-2}\, (2n+1)^{-1}\, \mathrm{B}(-\sigma-n, n-\sigma+1)\, \mathsf{D}_i(f_{\rho,+}, f_{-n}).$$

In the same way, we have

$$t_{nm}(g) = (-1)^n\, 2^{-\sigma-2}\, \pi^{-2}\, (2n+1)^{-1}\, \mathrm{B}(-\sigma-n, n-\sigma+1)\, \mathsf{D}_i(T_\sigma(g)[f_n], f_{-m}).$$

Let us derive the matrix elements $t_{\lambda,n}(g)$ of the subrepresentation T_σ to the subgroup H_1H_2 with respect to the "mixed basis". In other words, let us derive the coefficients $t_{\lambda,n}(g)$ of the the decomposition

$$T_\sigma(g)[f_\lambda] = \sum_{n\in\mathbb{Z}} t_{\lambda,n}(g)\, f_n.$$

Since D_i is invariant with respect to T_σ, then, for any $g(\phi) \in H_1$ and $g^*(b) \in H_2$,

$$\mathsf{D}_i(T_\sigma(g(\phi)g^*(b))[f_\lambda], f_n) = \mathsf{D}_i(T_\sigma(g^*(b))[f_\lambda], T_\sigma(g^{-1}(\phi))[f_n]) = \mathsf{D}_i(T_\sigma(g^*(b))[f_\lambda], T_\sigma(g(-\phi))[f_n]).$$

If $i=2$ and $\lambda>0$, then

$$\mathsf{D}_2(T_\sigma(g^*(b))[f_\lambda], T_\sigma(g(-\phi))[f_n]) = 2\,\mathrm{e}^{(\phi n - b\lambda)\mathbf{i}} \times \int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty} (y-z)^{-2\sigma-2}\, (1+\mathbf{i}z)^{\sigma+n}\, (1-\mathbf{i}z)^{\sigma-n}\, \mathrm{e}^{\mathbf{i}\lambda y}\, \mathrm{d}y\, \mathrm{d}z = 2^{-\sigma+1}\, \pi^{\frac{3}{2}}\, \mathrm{e}^{(\phi n - b\lambda)\mathbf{i}}\, \lambda^{\sigma+\frac{1}{2}}\, \Gamma\left(-\sigma-\frac{1}{2}\right) \times \Gamma^{-1}(\sigma+1)\, \Gamma^{-1}(-\sigma-n)\, W_{n\lambda,\sigma+\frac{1}{2}}(2\lambda).$$

If $i=1$, then

$$\mathsf{D}_1(T_\sigma(g^*(b))[f_\lambda], T_\sigma(g(-\phi))[f_n]) = \mathrm{e}^{\mathbf{i}n\phi} \int_0^{2\pi}\int_0^{2\pi} [1-\cos(\alpha-\beta)]^{-\sigma-1} \times$$
$$(\cos\alpha+1)^\sigma \exp\frac{(\sin\alpha - b\cos\alpha - b)\mathbf{i}\lambda}{\cos\alpha+1}\,\mathrm{e}^{\mathbf{i}n\beta}\,\mathrm{d}\alpha\,\mathrm{d}\beta =$$
$$2^3\,(-1)^n\,\mathbf{i}^{n+1}\,\pi^2\,\mathrm{e}^{\mathbf{i}n\phi}\,(-2\sigma-1)^{-1}\,\mathrm{B}(-\sigma, n-\sigma-1)\times$$
$$\sum_{l=0}^{\infty}\sum_{k=0}^{l}\binom{n}{k}(-\mathbf{i}\lambda b)^l\,[l!]^{-1}\,\mathrm{e}^{\left(n-\sigma+\frac{l-k-1}{2}\right)\pi\mathbf{i}}\,(2\sigma+k-l)^{-1}\times$$
$$\mathrm{B}^{-1}(\sigma-n+1, \sigma+n+l-l+1)\,{}_2F_1(l-k-2\sigma,\, n-\sigma-1;\, \sigma+n+l-k;\, -1).$$

Consequently,

$$\sum_{l=0}^{\infty}\sum_{k=0}^{l}\binom{n}{k}(-\mathbf{i}\lambda b)^l\,[l!]^{-1}\,\mathrm{e}^{\left(n-\sigma+\frac{l-k-1}{2}\right)\pi\mathbf{i}}\,(2\sigma+k-l)^{-1}\times$$
$$\mathrm{B}^{-1}(\sigma-n+1, \sigma+n+l-l+1)\,{}_2F_1(l-k-2\sigma,\, n-\sigma-1;\, \sigma+n+l-k;\, -1) =$$
$$2^{-\sigma-2}\,\mathbf{i}^{n-1}\pi^{-\frac{1}{2}}\,\mathrm{e}^{-\mathbf{i}b\lambda}\,\lambda^{\sigma+\frac{1}{2}}\,(-2\sigma-1)\,\Gamma\left(-\sigma-\frac{1}{2}\right)\,\Gamma^{-1}(\sigma+1)\,\Gamma^{-1}(-\sigma-n)\times$$
$$\mathrm{B}^{-1}(-\sigma, n-\sigma-1)\,W_{n\lambda,\sigma+\frac{1}{2}}(2\lambda).$$

4. Some linear combinations of Whittaker functions

Lemma 3. *Let* $-1 < \operatorname{re}\sigma < -\frac{1}{2}$. *Then*

$$\mathsf{D}_2(f_\lambda, f_{\rho,+}) = C_2\cdot 2^{-\sigma-1}\,\pi^{\frac{1}{2}}\,\mathbf{i}^{-\sigma-1}\,\lambda^\sigma\,\Gamma\left(-\sigma-\frac{1}{2}\right)\times$$
$$\Gamma^{-1}(\sigma+1)\,\mathrm{B}(\sigma+1+\mathbf{i}\rho, \sigma+1-\mathbf{i}\rho)\,M_{-\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda).$$

P r o o f. Let us make the change $t := y - z$ in the integral

$$\mathsf{D}_2(f_\lambda, f_{\rho,+}) = 2\,C_2\int_{-\infty}^{+\infty}\int_{-1}^{1}(y-z)^{-2\sigma-2}\,(1-z)^{\sigma-\mathbf{i}\rho}\,(1+z)^{\sigma+\mathbf{i}\rho}\,\mathrm{e}^{\mathbf{i}\lambda z}\,\mathrm{d}y\,\mathrm{d}z.$$

We obtain

$$\mathsf{D}_2(f_\lambda, f_{\rho,+}) = 2\, C_2 \int\limits_0^{+\infty} t^{-2\sigma-2} \cos\lambda t \,\mathrm{d}t \int\limits_{-1}^{1} (1-z)^{\sigma-\mathbf{i}\rho}\,(1+z)^{\sigma+\mathbf{i}\rho}\,\mathrm{e}^{-\mathbf{i}\lambda z}\,\mathrm{d}z.$$

Let us apply the formulas

$$\int\limits_0^{+\infty} x^{\alpha-1}\cos^{2n+1} bx\,\mathrm{d}x = 2^{\alpha-2n-1}\sqrt{\pi}\,(2n+1)!\,\times$$

$$\times\, b^{-\alpha}\,\Gamma\left(\frac{\alpha}{2}\right)\,\Gamma^{-1}\left(\frac{1-\alpha}{2}\right)\sum_{k=0}^{n}\frac{(2n-2k+1)^{-\alpha}}{k!\,(2n-k+1)!}, \quad (6)$$

where $0 < \operatorname{re}\alpha < 1$, [6, 2.5.4.14], and

$$\int\limits_0^{a} x^{\alpha-1}\,(a-x)^{\beta-1}\,\mathrm{e}^{-px}\,\mathrm{d}x = \mathrm{B}(\alpha,\beta)\,a^{\alpha+\beta-1}\,{}_1F_1(\alpha;\,\alpha+\beta;\,-ap), \quad (7)$$

where $\operatorname{re}\alpha, \operatorname{re}\beta > 0$, [6, 2.3.6.1]. ■

Lemma 4. *Let* $-1 < \operatorname{re}\sigma < -\frac{1}{2}$. *Then*

$$\mathsf{D}_3(f_\lambda, f_{\rho,+}) = \Gamma(\sigma+1-\mathbf{i}\rho)\,\Gamma(\sigma+1+\mathbf{i}\rho)\,\Big[C_{3,+}\cdot 2^{-\sigma}\,\times$$
$$\pi^{-\frac{1}{2}}\cosh(\pi\rho)\,\Gamma\left(-\sigma-\frac{1}{2}\right)\,\Gamma^{-1}\left(\sigma+\frac{3}{2}\right)\,\mathrm{B}(-\sigma-\mathbf{i}\rho, -\sigma+\mathbf{i}\rho)\,\times$$
$$M_{-\mathbf{i}\rho,-\sigma-\frac{1}{2}}(2\mathbf{i}\lambda) + C_{3,-}\cdot 2^{\sigma+1}\,(\mathbf{i}\lambda)^{\sigma}\,\Gamma(2\sigma+2)\,\times$$
$$\Big[\Gamma(-\sigma-\mathbf{i}\rho)\,W_{-\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda) + (-1)^{\sigma}\,\Gamma(-\sigma+\mathbf{i}\rho)\,W_{\mathbf{i}\rho,\sigma+\frac{1}{2}}(-2\mathbf{i}\lambda)\Big]\Big].$$

P r o o f. Making the change $t := \frac{p-q}{2}$ in the integral

$$\mathsf{D}_3(f_\lambda, f_{\rho,+}) = C_{3,+}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}[\cosh(p-q)-1]^{-\sigma-1}\,\times$$
$$(\cosh p+1)^{\sigma}\,\exp\left(\frac{\mathbf{i}\lambda\sinh p}{\cosh p+1}\right)\,\mathrm{e}^{\mathbf{i}\rho q}\,\mathrm{d}p\,\mathrm{d}q +$$
$$C_{3,-}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}[\cosh(p-q)+1]^{-\sigma-1}\,(\cosh p-1)^{\sigma}\,\exp\left(\frac{\mathbf{i}\lambda\sinh p}{\cosh p-1}\right)\,\mathrm{e}^{\mathbf{i}\rho q}\,\mathrm{d}p\,\mathrm{d}q,$$

we obtain

$$\mathsf{D}_3(f_\lambda, f_{\rho,+}) = C_{3,+} \cdot 2^{-\sigma+1} \underbrace{\int_0^{+\infty} \sinh^{-2\sigma-2} t \, \cos 2qt \, \mathrm{d}t}_{I_3} \times$$

$$\underbrace{\int_{-\infty}^{+\infty} (\cosh p + 1)^\sigma \, \exp\left(\frac{\mathbf{i}\lambda \sinh p}{\cosh p + 1}\right) \mathrm{e}^{\mathbf{i}\rho p} \, \mathrm{d}p}_{I_4} + C_{3,-} \cdot 2^{-\sigma+1} \times$$

$$\underbrace{\int_0^{+\infty} \cosh^{-2\sigma-2} t \, \cos 2qt \, \mathrm{d}t}_{I_5} \cdot \underbrace{\int_{-\infty}^{+\infty} (\cosh p - 1)^\sigma \, \exp\left(\frac{\mathbf{i}\lambda \sinh p}{\cosh p - 1}\right) \mathrm{e}^{\mathbf{i}\rho p} \, \mathrm{d}p}_{I_6}.$$

For I_3, we apply

$$\int_0^{+\infty} \sinh^{\nu-1} ax \, \cos bx \, \mathrm{d}x = \frac{1}{2a\sqrt{\pi}} \cosh \frac{b\pi}{2a} \times$$
$$\Gamma\left(1 - \frac{\nu}{2}\right) \Gamma\left(\frac{1-\nu}{2} + \frac{\mathbf{i}b}{2a}\right) \Gamma\left(\frac{1-\nu}{2} - \frac{\mathbf{i}b}{2a}\right),$$

where $\operatorname{re}\nu > 0$, $\operatorname{re}[(\nu-1)a] < -|\operatorname{im} b|$, [6, 2.5.47.15].

For I_4, we make the change $u := \tanh \frac{p}{2} + 1$ and apply (7).

For I_5, we apply

$$\int_0^{+\infty} \frac{\cos bx \, \mathrm{d}x}{\cosh^\nu cx} = \frac{2^{\nu-2}}{c\Gamma(\nu)} \Gamma\left(\frac{\nu}{2} - \frac{\mathbf{i}b}{2c}\right) \Gamma\left(\frac{\nu}{2} + \frac{\mathbf{i}b}{2c}\right),$$

where $\operatorname{re}(\nu c) > |\operatorname{im} b|$, [6, 2.5.47.6].

For I_6, we make the change $u := \coth \frac{p}{2}$ and apply

$$\int_0^{+\infty} x^{\alpha-1}(y+x)^{\tau-1} \mathrm{e}^{-sx} \, \mathrm{d}x = y^{\alpha+\tau-1} \, \mathrm{B}(\alpha, 1-\alpha-\tau) \, {}_1F_1(\alpha;\, \alpha+\tau;\, sy) +$$
$$(1-\tau) \, s^{1-\alpha-\tau} \, \Gamma(\alpha+\tau-1) \, {}_1F_1(1-\tau;\, 2-\alpha-\tau;\, sy),$$

where $\operatorname{re}\alpha > 0, |\arg y| < \pi$, $\operatorname{re} s = 0$, $\operatorname{re}(\alpha+\tau) < 2$, [6, 2.3.2.3].

Applying the formulas

$$M_{\mu,\nu}(z) = z^{\mu+\frac{1}{2}}\, \mathrm{e}^{-\frac{z}{2}}\, {}_1F_1\left(\nu-\mu+\frac{1}{2};\, 2\nu+1;\, z\right),$$
$$(-x)^{-\mu-\frac{1}{2}}\, M_{-\lambda,\mu}(-x) = x^{-\mu-\frac{1}{2}}\, M_{\lambda,\mu}(x),$$

[8, 8.3.4.2], and

$$W_{\lambda,\mu}(z) = \frac{\Gamma(-2\mu)}{\Gamma\left(\lambda-\mu+\frac{1}{2}\right)}\, M_{\lambda,\mu}(z) + \frac{\Gamma(2\mu)}{\Gamma\left(\lambda+\mu+\frac{1}{2}\right)}\, M_{\lambda,-\mu}(z)$$

[8, 8.3.4.4], we have

$$\begin{aligned} &\mathrm{B}(-\sigma\pm\mathbf{i}\rho, 2\sigma+1)\, M_{\pm\mathbf{i}\rho,-\sigma-\frac{1}{2}}(\mp 2\mathbf{i}\lambda) + \Gamma(-2\sigma-1)\, M_{\pm\mathbf{i}\rho,\sigma+\frac{1}{2}}(\mp 2\mathbf{i}\rho) = \\ &= \frac{\Gamma(-\sigma\pm\mathbf{i}\rho)\,\Gamma(2\sigma+1)}{\Gamma(\sigma+1\pm\mathbf{i}\rho)} M_{\pm\mathbf{i}\rho,-\sigma-\frac{1}{2}}(\mp 2\mathbf{i}\lambda) + \\ &\frac{\Gamma(-\sigma\pm\mathbf{i}\rho)\,\Gamma(-2\sigma-1)}{\Gamma(-\sigma\pm\mathbf{i}\rho)}\, M_{\pm\mathbf{i}\rho,\sigma+\frac{1}{2}}(\mp 2\mathbf{i}\rho) = \\ &\Gamma(-\sigma\pm\mathbf{i}\rho)\, W_{\pm\mathbf{i}\rho,\sigma+\frac{1}{2}}(\mp 2\mathbf{i}\rho). \end{aligned}$$

This completes the proof. ∎

Lemmas 3 and 4 immediately yield the following:

Theorem 2. *Let* $-1 < \operatorname{re}\sigma < -\frac{1}{2}$. *Then*

$$\begin{aligned} &\Gamma(2\sigma+2)\,\Bigg[C_{2,+}\cdot 2^{-\sigma}\,\pi^{-\frac{1}{2}}\,\cosh(\pi\rho)\,\Gamma\left(-\sigma-\frac{1}{2}\right)\,\Gamma^{-1}\left(\sigma+\frac{3}{2}\right)\times \\ &\mathrm{B}(-\sigma-\mathbf{i}\rho,-\sigma+\mathbf{i}\rho)\, M_{-\mathbf{i}\rho,-\sigma-\frac{1}{2}}(2\mathbf{i}\lambda) + C_{2,-}\cdot 2^{\sigma+1}\,(\mathbf{i}\lambda)^{\sigma}\,\Gamma(2\sigma+2)\times \\ &\Big[\Gamma(-\sigma-\mathbf{i}\rho)\, W_{-\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda) + (-1)^{\sigma}\,\Gamma(-\sigma+\mathbf{i}\rho)\, W_{\mathbf{i}\rho,\sigma+\frac{1}{2}}(-2\mathbf{i}\lambda)\Big]\Bigg] = \\ &C_3\cdot 2^{-\sigma-1}\,\pi^{\frac{1}{2}}\,\mathbf{i}^{-\sigma-1}\,\Gamma\left(-\sigma-\frac{1}{2}\right)\,\Gamma^{-1}(\sigma+1)\, M_{-\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda), \end{aligned}$$

where

$$\frac{C_{3,+}}{C_2} = \frac{A_1}{A_0}, \qquad \frac{C_{3,-}}{C_2} = \frac{A_2}{A_0}$$

and

$$\begin{aligned} A_0 := {} & 2^{-4\sigma}\,\Gamma(-\sigma)\,\Gamma(2\sigma+2)\,\Gamma\left(-\sigma-\frac{1}{2}\right)\,\mathrm{B}(-\sigma,-\sigma)\times \\ &\Big[I_{-\sigma-\frac{1}{2}}(\mathbf{i})\,\Big[K_{\sigma+\frac{1}{2}}(-\mathbf{i}) + (-1)^{\sigma+\frac{1}{2}}\,K_{\sigma+\frac{1}{2}}(\mathbf{i})\Big] - \\ &I_{-\sigma-\frac{1}{2}}(-\mathbf{i})\,\Big[K_{\sigma+\frac{1}{2}}(\mathbf{i}) + (-1)^{\sigma+\frac{1}{2}}\,K_{\sigma+\frac{1}{2}}(-\mathbf{i})\Big]\Big], \end{aligned}$$

$$A_1 := 2\,\pi\,\mathbf{i}^{-\sigma-1}\left(\sigma+\frac{3}{2}\right)\,\sin^{-1}\left[\left(\sigma+\frac{3}{2}\right)\pi\right]\Gamma(-\sigma)\times \left[I_{\sigma+\frac{1}{2}}(\mathbf{i})\left[K_{\sigma+\frac{1}{2}}(-\mathbf{i})+(-1)^{\sigma+\frac{1}{2}}\,K_{\sigma+\frac{1}{2}}(\mathbf{i})\right]- I_{\sigma+\frac{1}{2}}(-\mathbf{i})\left[K_{\sigma+\frac{1}{2}}(\mathbf{i})+(-1)^{\sigma+\frac{1}{2}}\,K_{\sigma+\frac{1}{2}}(-\mathbf{i})\right]\right],$$

$$A_2 := 2^{-\sigma-1}\,\pi^2\,\mathbf{i}^{-\sigma-1}\left(\sigma+\frac{3}{2}\right)\,\sin^{-1}\left[\left(\sigma+\frac{3}{2}\right)\pi\right]\times \Gamma\left(-\sigma-\frac{1}{2}\right)\,\mathrm{B}(-\sigma,-\sigma)\,\Gamma^{-1}(\sigma+1)\,\Gamma^{-1}(2\sigma+2)\times \left[I_{-\sigma-\frac{1}{2}}(\mathbf{i})\,I_{\sigma+\frac{1}{2}}(-\mathbf{i})-I_{\sigma+\frac{1}{2}}(\mathbf{i})\,I_{-\sigma-\frac{1}{2}}(-\mathbf{i})\right].$$

P r o o f. Since $C_2, C_{3,+}, C_{3,-}$ do not depend of λ and ρ, we can use the "initial conditions" $(\lambda_1,\rho_1) := (1,0)$ and $(\lambda_2,\rho_2) := (-1,0)$, i.e. we can solve the corresponding system of linear equations with respect to $\frac{C_{3,+}}{C_2}$ and $\frac{C_{3,-}}{C_2}$. ■

Lemma 5. *Let* $-1 < \operatorname{re}\sigma < -\frac{1}{2}$. *Then*

$$\mathsf{D}_2(f_\lambda, f_{\rho,-}) = C_2\cdot 2^{-\sigma-1}\,\pi^{\frac{1}{2}}\,\mathbf{i}^{-\sigma-1}\,\lambda^\sigma\,\Gamma\left(-\sigma-\frac{1}{2}\right)\,\Gamma^{-1}(\sigma+1)\times \left[2\,\mathrm{B}(\sigma+1+\mathbf{i}\rho,\sigma+1-\mathbf{i}\rho)\;M_{\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda)+(-1)^{\mathbf{i}\rho}\,\left[\Gamma(\sigma+1+\mathbf{i}\rho)\;\mathrm{e}^{-2\mathbf{i}\lambda}\times W_{\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda)+\Gamma(\sigma+1-\mathbf{i}\rho)\,\mathrm{e}^{2\mathbf{i}\rho}\,W_{-\mathbf{i}\rho,\sigma+\frac{1}{2}}(-2\mathbf{i}\lambda)\right]\right].$$

P r o o f. In order to evaluate

$$\mathsf{D}_2(f_\lambda, f_{\rho,-}) = 2\,C_2\left[\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{-1}(y-z)^{-2\sigma-2}\;(1-z)^{\sigma-\mathbf{i}\rho}\,(1+z)^{\sigma+\mathbf{i}\rho}\,\mathrm{e}^{\mathbf{i}\lambda z}\,\mathrm{d}y\,\mathrm{d}z+ \int\limits_{-\infty}^{+\infty}\int\limits_{1}^{+\infty}(y-z)^{-2\sigma-2}\,(1-z)^{\sigma-\mathbf{i}\rho}\,(1+z)^{\sigma+\mathbf{i}\rho}\,\mathrm{e}^{\mathbf{i}\lambda z}\,\mathrm{d}y\,\mathrm{d}z\right],$$

we make the change $t := y - z$. So,

$$\mathsf{F}_2(f_\lambda, f_{\rho,-}) = 2\, C_2 \underbrace{\int\limits_0^{+\infty} t^{-2\sigma-2} \cos\lambda t \,\mathrm{d}t}_{I_7} \times$$

$$\left[\underbrace{\int\limits_1^{+\infty} (1-z)^{\sigma-\mathbf{i}\rho}\,(1+z)^{\sigma+\mathbf{i}\rho}\,\mathrm{e}^{-\mathbf{i}\lambda z}\,\mathrm{d}z}_{I_8} + \underbrace{\int\limits_1^{+\infty} (1+z)^{\sigma-\mathbf{i}\rho}\,(1-z)^{\sigma+\mathbf{i}\rho}\,\mathrm{e}^{\mathbf{i}\lambda z}\,\mathrm{d}z}_{I_9} \right].$$

For I_7, we apply (6). For I_8 and F_9, we make the change $t := 1 - z$ and apply the formula

$$\int\limits_a^{+\infty} x^{\alpha-1}\,(x-a)^{\beta-1}\,\mathrm{e}^{-px}\,\mathrm{d}x = \Gamma(\beta)\,a^{\alpha+\beta-1}\,\mathrm{e}^{-ap}\,\Psi(\beta,\,\alpha+\beta;\,ap),$$

where $\mathrm{re}\,\alpha, \mathrm{re}\,\beta > 0$, [6, 2.3.6.6]. Let us replace the Ψ-function by the linear combination of ${}_1F_1$ confluent hypergeometric functions. Each ${}_1F_1$ function can be replaced by Whittaker function. ■

Lemma 6. *Let* $-1 < \mathrm{re}\,\sigma < -\frac{1}{2}$. *Then*

$$\mathsf{D}_3(f_\lambda, f_{\rho,-}) = \Gamma(\sigma+1-\mathbf{i}\rho)\,\Gamma(\sigma+1+\mathbf{i}\rho)\,\Big[C_{3,+}\cdot 2^{-\sigma}\,\pi^{-\frac{1}{2}}\,\cosh(\pi\rho)\,\Gamma\left(-\sigma-\frac{1}{2}\right)\times$$
$$\Gamma^{-1}\left(\sigma+\frac{3}{2}\right)\,\mathrm{B}(-\sigma-\mathbf{i}\rho, -\sigma+\mathbf{i}\rho)\,[\Gamma(-\sigma-\mathbf{i}\rho)\;W_{-\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda)+$$
$$(-1)^\sigma\,\Gamma(-\sigma+\mathbf{i}\rho)\,W_{\mathbf{i}\rho,\sigma+\frac{1}{2}}(-2\mathbf{i}\lambda)\Big] + C_{3,-}\cdot 2^{\sigma+1}\,(\mathbf{i}\lambda)^\sigma\,\Gamma(2\sigma+2)\,M_{-\mathbf{i}\rho,-\sigma-\frac{1}{2}}(2\mathbf{i}\lambda)\Big].$$

P r o o f. Let us make the change $t := \frac{p-q}{2}$ in

$$\mathsf{D}_3(f_\lambda, f_{\rho,-}) = C_{3,-}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}[\cosh(p-q)+1]^{-\sigma-1}\,(\cosh p+1)^\sigma\times$$
$$\exp\left(\frac{\mathbf{i}\lambda\sinh p}{\cosh p+1}\right)\,\mathrm{e}^{\mathbf{i}\rho q}\,\mathrm{d}p\,\mathrm{d}q+$$
$$C_{3,+}\int\limits_{-\infty}^{+\infty}\int\limits_{-\infty}^{+\infty}[\cosh(p-q)-1]^{-\sigma-1}\,(\cosh p-1)^\sigma\,\exp\left(\frac{\mathbf{i}\lambda\sinh p}{\cosh p-1}\right)\,\mathrm{e}^{\mathbf{i}\rho q}\,\mathrm{d}p\,\mathrm{d}q.$$

We obtain

$$\mathsf{F}_3(f_\lambda, f_{\rho,-}) = C_{3,-}\cdot 2^{-\sigma+1} \underbrace{\int_0^{+\infty} \cosh^{-2\sigma-2} t\, \cos 2qt\, \mathrm{d}t}_{I_{10}} \times$$

$$\underbrace{\int_{-\infty}^{+\infty} (\cosh p+1)^{\sigma} \exp\left(\frac{\mathbf{i}\lambda \sinh p}{\cosh p+1}\right) \mathrm{e}^{\mathbf{i}\rho p}\, \mathrm{d}p}_{I_{11}} + C_{3,+}\cdot 2^{-\sigma+1} \times$$

$$\underbrace{\int_0^{+\infty} \sinh^{-2\sigma-2} t\, \cos 2qt\, \mathrm{d}t}_{I_{12}} \cdot \underbrace{\int_{-\infty}^{+\infty} (\cosh p-1)^{\sigma} \exp\left(\frac{\mathbf{i}\lambda \sinh p}{\cosh p-1}\right) \mathrm{e}^{\mathbf{i}\rho p}\, \mathrm{d}p}_{I_{13}}.$$

The formulas for evaluating of the integrals $I_{10}, I_{11}, I_{12}, I_{13}$ have been described in the proofs of the previous lemmas. ■

Lemmas 5 and 6 yield the following

Theorem 3. *Let* $-1 < \operatorname{re}\sigma < -\frac{1}{2}$. *Then*

$$C_2 \cdot 2^{-\sigma-1}\, \pi\, \mathbf{i}^{-\sigma-1}\, \Gamma\left(-\sigma-\frac{1}{2}\right) \Gamma^{-1}(\sigma+1)\, \Big[2\, \mathrm{B}(\sigma+1+\mathbf{i}\rho, \sigma+1-\mathbf{i}\rho) M_{\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda)$$
$$+ (-1)^{\mathbf{i}\rho} \Big[\Gamma(\sigma+1+\mathbf{i}\rho)\mathrm{e}^{-2\mathbf{i}\lambda} W_{\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda) + \Gamma(\sigma+1-\mathbf{i}\rho)\, \mathrm{e}^{2\mathbf{i}\lambda}\, W_{-\mathbf{i}\rho,\sigma+\frac{1}{2}}(-2\mathbf{i}\lambda)\Big]\Big]$$
$$= \Gamma(\sigma+1-\mathbf{i}\rho)\, \Gamma(\sigma+1+\mathbf{i}\rho)\, \Big[C_{3,+}\cdot 2^{-\sigma}\ \cosh(\pi\rho)\, \Gamma\left(-\sigma-\frac{1}{2}\right) \Gamma^{-1}\left(\sigma+\frac{3}{2}\right)$$
$$\times\ \mathrm{B}(-\sigma-\mathbf{i}\rho, -\sigma+\mathbf{i}\rho)\, \Big[\Gamma(-\sigma-\mathbf{i}\rho)\, W_{-\mathbf{i}\rho,\sigma+\frac{1}{2}}(2\mathbf{i}\lambda)$$
$$+ (-1)^{\sigma}\Gamma(-\sigma+\mathbf{i}\rho)\, W_{\mathbf{i}\rho,\sigma+\frac{1}{2}}(-2\mathbf{i}\lambda)\Big] + C_{3,-}\cdot 2^{\sigma+1}\, (\mathbf{i}\lambda)^{\sigma}\, \Gamma(2\sigma+2)\, M_{-\mathbf{i}\rho,-\sigma-\frac{1}{2}}(2\mathbf{i}\lambda)\Big].$$

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