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POTAPOV-GINSBURG TRANSFORMATION AND FUNCTIONAL MODELS OF NON-DISSIPATIVE OPERATORS

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Abstract. A relation between an arbitrary bounded operator A and dissipative operator A_+, built by A in the following way $A_+ = A + i\varphi^* Q_- \varphi$, where $A - A^* = i\varphi^* J\varphi$, ($J = Q_+ - Q_-$ is involution), is studied.

The characteristic functions of the operators A and A_+ are expressed by each other using the known Potapov-Ginsburg linear-fractional transformations. The explicit form of the resolvent $(A - \lambda I)^{-1}$ is expressed by $(A_+ - \lambda I)^{-1}$ and $(A_+^* - \lambda I)^{-1}$ in terms of these transformations. Furthermore, the functional model [10, 12] of non-dissipative operator A in terms of a model for A_+, which evolves the results, was obtained by Naboko, S. N. [7].

The main constructive elements of the present construction are shown to be the elements of the Potapov-Ginsburg transformation for corresponding characteristic functions.

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Introduction. Construction of functional models for a non-dissipative bounded operator A comes across to considerable difficulties. First, a dilation of A is self-adjoint in J-metric (but not in Hilbert metric) and it does not have an appropriate spectral decomposition. Second, the operation of taking an orthogonal complement in the space of dilation, due to the indefiniteness of J, has its own problem in connection with the existence of the isotropic subspace. Naboko, S. N. in his work [7] concerning a construction of functional model for non-dissipative operator used Potapov-Ginsburg's transformation.

This work is a continuation of the subject, and extends the approach of Naboko, S. N. on the other case.

1. Potapov-Ginsburg transformation.

1.1. Consider a colligation $\triangle$ [1, 3, 5],

$$\triangle = (A, H, \varphi, E, J), \tag{1}$$

which has the colligation relationship of the form $A - A^* = i\varphi^* J\varphi$, where H, E are Hilbert spaces $A : H \to H$, $\varphi : H \to E$, $J = E \to E$ and $J = J^* = J^{-1}$ is an involution, i.e. $J = Q_+ - Q_-$, and $Q_\pm = \frac{1}{2}(I \pm J)$ are orthoprojectors in E onto $E_\pm = Q_\pm E$, where $E_+ \perp E_-$ since $Q_+Q_- = 0$.

The open system equations $\mathcal{F}_\triangle = \{R_\triangle, S_\triangle\}$ associated with $\triangle$ (1) have the form [4]

$$R_\triangle \quad : \quad \begin{cases} i\dfrac{d}{dt}h(t) + Ah(t) = \varphi^* Ju(t) = \varphi^*(u_+(t) - u_-(t)); \\ \qquad\qquad h(0) = h_0; \end{cases} \tag{2}$$

$$S_\triangle \quad : \quad v_+(t) + v_-(t) = u_+(t) + u_-(t) - i\varphi h(t), \tag{3}$$

where $u(t), v(t)$ are vector functions from E, $h(t)$ is a vector function from H, $t \in \mathbb{R}_+$ and $v_\pm(t) = Q_\pm v(t)$, $u_\pm(t) = Q_\pm u(t)$.

Since $u_-(t) = v_-(t) + iQ_-\varphi h(t)$, then equation (2) may bewritten as

$$i\frac{d}{dt}h(t) + Ah(t) + i\varphi^* Q_-\varphi h(t) = \varphi^*(u_+(t) - v_-(t)).$$

Note that the operator

$$A_+ = A + i\varphi^* Q_-\varphi \tag{4}$$

is dissipative since $A_+ - A_+^* = i\varphi^* J\varphi + 2i\varphi^* Q_-\varphi = i\varphi^*\varphi$, $\varphi^*\varphi \geq 0$. Therefore, the family

$$\triangle_+ = (A_+, H, \varphi, E, I) \tag{5}$$

is a dissipative colligation [4].

The open system $\mathcal{F}_{\triangle_+} = \left\{R_{\triangle_+}, S_{\triangle_+}\right\}$ associated with $\triangle_+$ has the form [4]

$$R_{\triangle_+} \quad : \quad \begin{cases} i\dfrac{d}{dt}h(t) + A_+h(t) = \varphi^*(u_+(t) - v_-(t)); \\ h(0) = h_0; \end{cases} \tag{6}$$

$$S_{\triangle_+} \quad : \quad v_+(t) - u_-(t) = u_+(t) - v_-(t) - i\varphi h(t), \tag{7}$$

where $u_-(t) = v_-(t) + iQ_-\varphi h(t)$.

Thus, $h(t)$ is a solution of two Cauchy problems simultaneously. On the one hand, $h(t)$ is a solution of (2) when in the right hand side of the equation is $u_+(t) - u_-(t)$. On the other hand, $h(t)$ satisfies equation (6) when in the right hand side is $u_+(t) - v_-(t)$, where $v_-(t) = u_-(t) - iQ_-h(t)$.

Determine the connection between the transfer mappings $S_\triangle$ and $S_{\triangle_+}$. From (3) and (7) it follows that

$$Q_\pm S_\triangle u(t) = v_\pm(t)$$

and

$$S_{\triangle_+}(u_+(t) - v_-(t)) = v_+(t) - u_-(t).$$

Therefore

$$S_{\triangle_+}(Q_+ - Q_-S_\triangle)u(t) = (Q_+S_\triangle - Q_-)u(t).$$

Similarly, since

$$Q_+S_{\triangle_+}(u_+(t) - v_-(t)) = v_+(t)$$

and

$$Q_-S_{\triangle_+}(u_+(t) - v_-(t)) = -u_-(t),$$

it is obvious that

$$S_\triangle(Q_+ - Q_-S_{\triangle_+})(u_+(t) - v_-(t)) = (Q_+S_{\triangle_+} - Q_-)(u_+(t) - v_-(t)).$$

Proposition 1. *The mappings $S_\triangle$ (3) and $S_{\triangle_+}$ (7) corresponding to the colligations $\triangle$ (1) and $\triangle_+$ (5), where A and A_+ satisfy the relations (4), are connected by the equalities*

$$(8)\qquad \begin{aligned} S_{\triangle_+}(Q_+ - Q_-S_\triangle) &= (Q_+S_\triangle - Q_-);\\ S_\triangle(Q_+ - Q_-S_{\triangle_+}) &= (Q_+S_{\triangle_+} - Q_-). \end{aligned}$$

Note that transformation (8) also has been obtained in [1].

In a similar way, from (8) it follows that

$$(9)\qquad \begin{aligned} (Q_+ + S_{\triangle_+}Q_-)S_\triangle &= Q_- + S_{\triangle_+}Q_+;\\ (Q_+ + S_\triangle Q_-)S_{\triangle_+} &= Q_- + S_\triangle Q_+. \end{aligned}$$

Transfer mappings $S_\triangle$ (3) and $S_{\triangle_+}$ (7) correspond to the characteristic functions [4, 10],

$$(10)\qquad S_\triangle(\lambda) = I - i\varphi(A-\lambda I)^{-1}\varphi^* J,\ S_{\triangle_+}(\lambda) = I - i\varphi(A_+ - \lambda I)^{-1}\varphi^*$$

of the colligations $\triangle$ (1) and $\triangle_+$ (5) respectively.

Theorem 1 ([2])**.** *Each of the characteristic functions $S_\triangle(\lambda)$ and $S_{\triangle_+}(\lambda)$ (10) of colligations $\triangle$ (1) and $\triangle_+$ (5) under condition (4) can be expressed by each other using the Potapov-Ginsburg transformation,*

$$(11)\qquad \begin{aligned} &1.\quad S_{\triangle_+}(\lambda) = (Q_+ \cdot S_\triangle(\lambda)\cdot Q_-)(Q_+ - Q_-S_\triangle(\lambda))^{-1};\\ &2.\quad S_\triangle(\lambda) = (Q_+ + S_{\triangle_+}(\lambda)Q_-)^{-1}(Q_- + S_{\triangle_+}(\lambda)Q_+);\\ &3.\quad S_\triangle(\lambda) = (Q_+S_{\triangle_+}(\lambda) - Q_-)(Q_+ - Q_-S_{\triangle_+}(\lambda))^{-1};\\ &4.\quad S_{\triangle_+}(\lambda) = (Q_+ + S_\triangle(\lambda)Q_-)^{-1}(Q_- + S_\triangle(\lambda)Q_+). \end{aligned}$$

One can easily see that the respective inverses in (11) exist and are bounded in appropriate domains. For example, the inversibility of $Q_+ + S_{\triangle_+}(\lambda)Q_-$ in $(11)_2$ follows from the formula

$$(Q_+ + S_{\triangle_+}(\lambda)Q_-)S_\triangle(\lambda) = Q_- + S_{\triangle_+}(\lambda)Q_+$$

by virtue of boundedness and holomorphy of $Q_- + S_{\triangle_+}(\lambda)Q_+$ in $\mathbb{C}_-$ when $|\lambda| \gg 1$. Furthermore,

$$(Q_+ + S_{\triangle_+}(\lambda)Q_-)^{-1} = Q_+ + S_{\triangle}(\lambda)Q_-. \tag{12}$$

$$\begin{aligned}
& (Q_+ + S_{\triangle_+}(\lambda)Q_-)(Q_+ + S_{\triangle}(\lambda)Q_-) \\
= \; & (I - i\varphi(A_+ - \lambda I)^{-1}\varphi^* Q_-)(I + i\varphi(A - \lambda I)^{-1}\varphi^* Q_-) \\
= \; & I - i\varphi(A_+ - \lambda I)^{-1}\varphi^* Q_- + i\varphi(A - \lambda I)^{-1}\varphi^* Q_- \\
& - i\varphi(A_+ - \lambda I)^{-1} i\varphi^* Q_- \varphi(A - \lambda I)^{-1}\varphi^* Q_-
\end{aligned}$$

and using (4) we get

$$I - i\varphi\left\{(A_+ - \lambda I)^{-1} - (A - \lambda I)^{-1} + (A_+ - \lambda I)^{-1}(A_+ - A)(A - \lambda I)^{-1}\varphi^* Q_-\right\} = I.$$

1.2. In this section we derive formulas similar to (11) which connect the linear-fractional transforms $S_{\triangle}(\lambda)$ and $S_{\triangle_+}(\lambda)$. First of all, note that from (4) it follows that

$$A = A_+^* + i\varphi^* Q_+ \varphi, \tag{13}$$

as it is evident that

$$A = A_R + \frac{i}{2}(\varphi^* Q_+ \varphi - \varphi^* Q_- \varphi) = (A_R + \frac{i}{2}\varphi^*\varphi)^* + i\varphi^* Q_+ \varphi.$$

Theorem 2 ([2]). *For characteristic functions $S_{\triangle}(\lambda)$ and $S_{\triangle_+}(\lambda)$* (10) *of colligations $\triangle$* (1) *and $\triangle_+$* (5)*, the formulas of Potapov and Ginsburg* [9],

$$\begin{aligned}
& 1. \quad S^*_{\triangle_+}(\overline{\lambda}) = (Q_+ - Q_- S_{\triangle}(\lambda))(Q_+ S_{\triangle}(\lambda) - Q_-)^{-1}; \\
& 2. \quad S_{\triangle}(\lambda) = (Q_- + S^*_{\triangle_+}(\overline{\lambda})Q_+)^{-1}(Q_+ + S^*_{\triangle_+}(\overline{\lambda})Q_-); \\
& 3. \quad S_{\triangle}(\lambda) = (Q_+ - Q_- S^*_{\triangle_+}(\overline{\lambda}))(Q_+ S^*_{\triangle_+}(\overline{\lambda}) - Q_-)^{-1}; \\
& 4. \quad S^*_{\triangle_+}(\overline{\lambda}) = (Q_- + S_{\triangle}(\lambda)Q_+)^{-1}(Q_+ + S_{\triangle}(\lambda)Q_-).
\end{aligned} \tag{14}$$

are valid when (4) *holds.*

Proof. The formulas (14) may be derived by argumentation similar to that of subsection 1.1 using the transfer mappings $S_{\triangle}$ and $S_{\triangle_+}$. We present, as an example, a direct proof of formula $(14)_2$ (other formulas are proved similarly).

To this end, we calculate

$$\begin{aligned}
& (Q_- + S^*_{\triangle_+}(\overline{\lambda})Q_+)S_\triangle(\lambda) \\
= \; & (I + i\varphi(A^*_+ - \lambda I)^{-1}\varphi Q_+)(I - i\varphi(A-\lambda I)^{-1}\varphi^* J) \\
= \; & I + i\varphi(A^*_+ - \lambda I)^{-1}\varphi Q_+ - i\varphi(A-\lambda I)^{-1}\varphi^* J \\
& -i\varphi(A^*_+ - \lambda I)^{-1} i\varphi Q_+ \varphi(A-\lambda I)^{-1}\varphi^* J,
\end{aligned}$$

and after using (13) we get

$$\begin{aligned}
(Q_- + S^*_{\triangle_+}(\overline{\lambda})Q_+)S_\triangle(\lambda) &= I + i\varphi(A^*_+ - \lambda I)^{-1}\varphi^* Q_- \\
&= Q_+ + S^*_{\triangle_+}(\overline{\lambda})Q_-,
\end{aligned}$$

which was to be proved. Invertibility of the operators is proved similarly to (12). □

Among all the formulas (11) and (14), we mark out the following two important formulas:

$$\begin{aligned}
&1. \quad S_\triangle(\lambda) = (Q_+ + S_{\triangle_+}(\lambda)Q_-)^{-1}(Q_- + S_{\triangle_+}(\lambda)Q_+); \\
&2. \quad S_\triangle(\lambda) = (Q_- + S^*_{\triangle_+}(\overline{\lambda})Q_+)^{-1}(Q_+ + S^*_{\triangle_+}(\overline{\lambda})Q_-).
\end{aligned} \tag{15}$$

Corollary 1. *The nonreal set of singularitiy points of $S_\triangle(\lambda)$ in $\mathbb{C}_-$ and in $\mathbb{C}_+$ belongs to the singularitiy points of $(Q_+ + S_{\triangle_+}(\lambda)Q_-)^{-1}$ in $\mathbb{C}_-$ and of $(Q_- + S^*_{\triangle_+}(\overline{\lambda})Q_+)^{-1}$ in $\mathbb{C}_+$ respectively.*

Indeed, from $(15)_1$ owing to the holomorphy of $Q_- + S_{\triangle_+}(\lambda)Q_+$ in $\mathbb{C}_-$ we conclude that the function $S_\triangle(\lambda)$ may have singularities in the lower half plane $\mathbb{C}_-$ only in zeroes of $Q_+ + S_{\triangle_+}(\lambda)Q_-$. Similar reasoning for $(15)_2$ shows that nonreal singularities of $S_\triangle(\lambda)$ in $\mathbb{C}_+$ are in zeroes of the function $Q_- + S^*_{\triangle_+}(\overline{\lambda})Q_+$.

Thus, the Potapov-Ginsburg formulas (15) factor out the nonreal singularities of the characteristic function $S_\triangle(\lambda)$ and hence decompose nonreal spectra of operator A relatively to $\mathbb{C}_+$ and $\mathbb{C}_-$.

Resuming the results of 1.1, 1.2 we note that the Potapov-Ginsburg triangular linear-fractional transforms (11), (14) ascertain one-to-one correspondence between the class of operator functions $S_{\triangle_+}(\lambda)$ and the class function $S_\triangle(\lambda)$ (in corresponding domains). Then the dissipative operator A_+ changes into the bounded arbitrary operator A and formula (4) holds.

2. Relation between the resolvents of A and A_+.

2.1. Below we derive the explicit form of the resolvent operator of A, expressed by the resolvent of A_+. Let $f \in H$ and $(A-\lambda I)^{-1} f = g$ or $f = Ag - \lambda g$; then using (4) we get

$$f = A_+ g - \lambda g - i\varphi^* Q_- \varphi g.$$

If $\lambda \in \mathbb{C}_-$, it is evident that

$$(A_+ - \lambda I)^{-1} f = g - i(A_+ - \lambda I)^{-1} \varphi^* Q_- \varphi g. \tag{16}$$

Applying φ to the both sides of the equality we obtain

$$\varphi(A_+ - \lambda I)^{-1} f = \left\{ I - i\varphi(A_+ - \lambda I)^{-1} \varphi^* Q_- \right\} \varphi g,$$

and since

$$I - i\varphi(A_+ - \lambda I)^{-1} \varphi^* Q_- = Q_+ + S_{\triangle_+}(\lambda) Q_-,$$

then

$$\varphi g = (Q_+ + S_{\triangle_+}(\lambda) Q_-)^{-1} \varphi (A_+ - \lambda I)^{-1} f.$$

Substituting this expression in (16), we find that

$$\begin{aligned} g \;\; = \;\; & (A - \lambda I)^{-1} f = (A_+ - \lambda I)^{-1} f + i(A_+ - \lambda I)^{-1} \varphi^* Q_- (Q_+ + S_{\triangle_+}(\lambda) Q_-)^{-1} \times \\ & \varphi (A_+ - \lambda I)^{-1} f \end{aligned}$$

where $\lambda \in \mathbb{C}_-$ and $|\lambda| \gg 1$.

Similarly, from (13), if $\lambda \in \mathbb{C}_+$, we get

$$\begin{aligned} g \;\; = \;\; & (A - \lambda I)^{-1} f = (A_+^* - \lambda I)^{-1} f - i(A_+^* - \lambda I)^{-1} \varphi^* Q_+ (Q_- + S_{\triangle_+}^*(\overline{\lambda}) Q_+)^{-1} \times \\ & \varphi (A_+^* - \lambda I)^{-1} f. \end{aligned}$$

Thus, we come to the theorem.

Theorem 3. *If $A_+ = A + i\varphi^* Q_- \varphi$ (4), then the resolvent of A is expressed by the dissipative operator A_+ as follows*

$$\begin{aligned} (17) \quad & (A - \lambda I)^{-1} \\ & = (A_+ - \lambda I)^{-1} + i(A_+ - \lambda I)^{-1} \varphi^* Q_- (Q_+ + S_{\triangle_+}(\lambda) Q_-)^{-1} \varphi (A_+ - \lambda I)^{-1} \end{aligned}$$

when $\lambda \in \mathbb{C}_-$, $|\lambda| \gg 1$,
and

$$(18) \quad (A-\lambda I)^{-1}$$
$$= (A_+^* - \lambda I)^{-1} - i(A_+^* - \lambda I)^{-1}\varphi^* Q_+ (Q_- + S^*_{\triangle_+}(\overline{\lambda})Q_+)^{-1}\varphi(A_+^* - \lambda I)^{-1}$$

when $\lambda \in \mathbb{C}_+$, $|\lambda| \gg 1$.

2.2. It is known that the self-adjoint operator B_+ acting in a Hilbert space G is called [7, 9] a self-adjoint dilation of bounded dissipative operator A_+ if

$$(19) \qquad G \supseteq H, \quad (A_+ - \lambda I) = P_H (B_+ - \lambda I)^{-1} |_H \, , \quad \forall \lambda \in \mathbb{C}_-.$$

We recall [12] that a self-adjoint dilation B_+ of bounded dissipative operator A_+ is acting in the space [6, 8]

$$\mathcal{H} = \left\{ f = (v(\xi); h; u(\xi)); \int\limits_{-\infty}^{0} \|v(\xi)\|^2 \, d\xi + \|h\|^2 + \int\limits_{0}^{\infty} \|u(\xi)\|^2 \, d\xi < \infty \right\}$$

where $v(\xi)$, $u(\xi) \in E$, $\operatorname{Supp} v(\xi) \in \mathbb{R}_-$, $\operatorname{Supp} u(\xi) \in \mathbb{R}_+$, $h \in H$ and is define on the functions $f = (v(\xi); h; u(\xi)) \in \mathcal{H}$ by formula

$$(20) \qquad B_+ f = \left(\frac{1}{i}\frac{d}{d\xi} v(\xi); A_+ h - \varphi^* u(0); \frac{1}{i}\frac{d}{d\xi} u(\xi) \right),$$

where f belongs to the domain of operator B_+,

$$(21) \qquad D(B_+) = \left\{ \begin{array}{c} f \in \mathcal{H} : \dfrac{d}{d\xi} v(\xi) \in L^2_{\mathbb{R}_-}(E), \dfrac{d}{d\xi} u(\xi) \in L^2_{\mathbb{R}_+}(E), \\ v(0) = u(0) - i\varphi h \end{array} \right\},$$

provided that all corresponding derivatives exist in the standard sence. Define the operator J on the Hilbert space $\mathcal{H}$, namely:

$$Jf = (Jv(\xi), h, Ju(\xi))$$

where $f = (v(\xi); h; u(\xi)) \in \mathcal{H}$.

Define an operator B in $\mathcal{H}$ by

$$(22) \qquad B = B_+ J,$$

in other words,

$$Bf = \left(J\frac{1}{i}\frac{d}{d\xi}v(\xi); A_+h - \varphi^* Ju(0); J\frac{1}{i}\frac{d}{d\xi}u(\xi)\right), \tag{23}$$

where f belongs to the domain of B,

$$D(B) = \left\{ \begin{array}{c} f \in \mathcal{H} : \frac{d}{d\xi}v(\xi) \in L^2_{\mathbb{R}_-}(E), \frac{d}{d\xi}u(\xi) \in L^2_{\mathbb{R}_+}(E); \\ v(0) = u(0) - iJ\varphi h \end{array} \right\}. \tag{24}$$

Theorem 4. *The resolvent operator of* B (23), (24) *may be expressed as follows*
if $\lambda \in \mathbb{C}_+ \cap \rho(A^*)$ *then*

$$\begin{aligned} (B-\lambda I)^{-1}f \quad = \quad & \left(i\int\limits_{-\infty}^{\zeta} e^{i\lambda(\xi-s)}Q_+v(s)ds + i\int\limits_{\xi}^{0} e^{-i\lambda(\xi-s)}Q_-v(s)ds \right. \\ & +e^{-i\lambda\zeta}\{\tilde{u}_-(0) + iQ_-\varphi[(A^*-\lambda I)^{-1}h \\ & +(A^*-\lambda I)^{-1}\varphi^*(\tilde{v}_+(0) - \tilde{u}_-(0))]\}; \\ & (A^*-\lambda I)^{-1}h + (A^*-\lambda I)^{-1}\varphi^*(\tilde{v}_+(0) - \tilde{u}_-(0)); \\ & i\int\limits_{\xi}^{\infty} e^{-i\lambda(\xi-s)}Q_-u(s)ds + i\int\limits_{0}^{\zeta} e^{i\lambda(\xi-s)}Q_+u(s)ds \\ & +e^{i\lambda\zeta}\{\tilde{v}_+(0) + iQ_+\varphi[(A^*-\lambda I)^{-1}h \\ & \left. +(A^*-\lambda I)^{-1}\varphi^*(\tilde{v}_+(0) - \tilde{u}_-(0))]\} \right), \end{aligned} \tag{25}$$

where,

$$\tilde{u}_-(0) = i\int\limits_{0}^{\infty} e^{i\lambda s}Q_-u(s)ds, \quad \tilde{v}_+(0) = \int\limits_{-\infty}^{0} e^{-i\lambda s}Q_+v(s)ds,$$

and if $\lambda \in \mathbb{C}_- \cap \rho(A)$ *then*

$$(25') \quad (B-\lambda I)^{-1}f = \begin{pmatrix} -i\int\limits_{-\infty}^{\zeta} e^{i\lambda(\xi-s)}Q_-v(s)ds - i\int\limits_{\xi}^{0} e^{i\lambda(\xi-s)}Q_+v(s)ds \\ +e^{-i\lambda\zeta}\{\tilde{u}_+(0) - iQ_+\varphi[(A-\lambda I)^{-1}h \\ +(A-\lambda I)^{-1}\varphi^*\left(\tilde{u}_+(0)-\tilde{v}_-(0)\right)]\}; \\ (A-\lambda I)^{-1}h + (A-\lambda I)^{-1}\varphi^*\left(\tilde{u}_+(0)-\tilde{v}_-(0)\right); \\ -i\int\limits_{\xi}^{\infty} e^{i\lambda(\xi-s)}Q_+u(s)ds - i\int\limits_{0}^{\zeta} e^{-i\lambda(\xi-s)}Q_-u(s)ds \\ +e^{-i\lambda\zeta}\{\tilde{v}_-(0) - iQ_-\varphi[(A-\lambda I)^{-1}h \\ +(A-\lambda I)^{-1}\varphi^*\left(\tilde{u}_+(0)-\tilde{v}_-(0)\right)]\} \end{pmatrix},$$

where,

$$\tilde{u}_+(0) = -i\int\limits_0^{\infty} e^{-i\lambda s}Q_+u(s)ds, \quad \tilde{v}_-(0) = -i\int\limits_{-\infty}^{0} e^{i\lambda s}Q_-v(s)ds.$$

P r o o f. Let us derive a formula for resolvent when $\lambda \in \mathbb{C}_+$ (for $\mathbb{C}_-$ the proof is similar). Let $(B-\lambda I)^{-1}f = \widetilde{f}$ or $B\widetilde{f} - \lambda\widetilde{f} = f$. This means that

$$(26) \quad \begin{cases} J\widetilde{v}'(\xi) - i\lambda\widetilde{v}(\xi) = iv(\xi) & (\xi \in \mathbb{R}_-); \\ J\widetilde{u}'(\xi) - i\lambda\widetilde{u}(\xi) = iu(\xi) & (\xi \in \mathbb{R}_+); \\ A_+\widetilde{h} - \lambda\widetilde{h} - \varphi^*J\widetilde{u}(0) = h; \\ \widetilde{v}(0) = \widetilde{u}(0) - iJ\varphi\widetilde{h}. \end{cases}$$

From $\widetilde{u}_\pm(\xi) = Q_\pm\widetilde{u}(\xi)$ we have the following equations:

$$\begin{aligned} \widetilde{u}'_+(\xi) &= i\lambda\widetilde{u}_+(\xi) + iu_+(\xi) & (\xi \in \mathbb{R}_+); \\ \widetilde{u}'_-(\xi) &= -i\lambda\widetilde{u}_-(\xi) - iu_-(\xi) & (\xi \in \mathbb{R}_+). \end{aligned}$$

Hence for $\widetilde{u}_{\pm}(\xi)$ we have

(27)
$$\begin{aligned}
\widetilde{u}_{+}(\xi) &= \widetilde{u}_{+}(0)e^{i\lambda\xi} + i\int\limits_{0}^{\xi} e^{i\lambda(\xi-s)}u_{+}(s)ds, \qquad (\xi\in\mathbb{R}_{+}),\\
\widetilde{u}_{-}(\xi) &= i\int\limits_{0}^{\xi} e^{i-\lambda(\xi-s)}u_{-}(s)ds, \qquad (\xi\in\mathbb{R}_{+}).
\end{aligned}$$

Similarly, for $\tilde{v}_{\pm}(\xi) = Q_{\pm}\tilde{v}_{\pm}(\xi)$, the equations

$$\begin{aligned}
\widetilde{v}'_{+}(\xi) &= i\lambda\widetilde{v}_{+}(\xi) + iv_{+}(\xi) \qquad (\xi\in\mathbb{R}_{-});\\
\widetilde{v}'_{-}(\xi) &= -i\lambda\widetilde{v}_{-}(\xi) - iv_{-}(\xi) \quad (\xi\in\mathbb{R}-).
\end{aligned}$$

imply that

(28)
$$\begin{aligned}
\widetilde{v}_{-}(\xi) &= e^{i-\lambda\xi}\widetilde{v}_{-}(0) + i\int\limits_{0}^{\xi} e^{-i\lambda(\xi-s)}v_{-}(s)ds, \qquad (\xi\in\mathbb{R}_{-}),\\
\widetilde{v}_{+}(\xi) &= i\int\limits_{-\infty}^{\xi} e^{i\lambda(\zeta-s)}v_{+}(s)ds, \qquad (\xi\in\mathbb{R}_{-}).
\end{aligned}$$

Since $\widetilde{v}(0) = \widetilde{u}(0) - iJ\varphi\widetilde{h}$ (26), then

(29)
$$\tilde{u}_{+}(0) = \tilde{v}_{+}(0) + iQ_{+}\varphi\tilde{h} \;\;;\;\; \tilde{v}_{-}(0) = \tilde{u}_{-}(0) + iQ_{-}\varphi\tilde{h}.$$

Thus it follows that the formula $A_{+}\widetilde{h} - \lambda\widetilde{h} - \varphi^{*}J\widetilde{u}(0) = h$ (26) by virtue of (29) can be written in the following form

$$A_{+}\widetilde{h} - \lambda\widetilde{h} - \varphi^{*}[\tilde{v}_{+}(0) + iQ_{+}\varphi\tilde{h} - \tilde{u}_{-}(0)] = h.$$

Therefore

$$\tilde{h} = (A^{*} - \lambda I)^{-1}h + (A^{*} - \lambda I)^{-1}\varphi^{*}\left(\tilde{v}_{+}(0) - \tilde{u}_{-}(0)\right),$$

which proves the first formula in (25) by virtue of (27), (28), (29). $\square$

Theorem 5. *If $\lambda\in\mathbb{C}_{-}\cap\rho(A)$, the operator B (23), (24) is a dilation of the operator A,*

$$P_{H}(B - \lambda I)^{-1}|_{H} = (A - \lambda I)^{-1},$$

and if $\lambda \in \mathbb{C}_+ \cap \rho(A^*)$*, the operator* B *is a dilation of operator* A^**,*

$$P_H(B - \lambda I)^{-1}|_H = (A^* - \lambda I)^{-1}.$$

The proof follows from (24).

3. Functional Model of the Operator A. Here we derive an explicit form of the operator A in terms of functional model of dissipative operator A_+ [10]. Let us consider that A_+ is acting as in the the functional model [8], namely, let H coincides with

$$H_P = L^2 \begin{pmatrix} I & S^*_{\triangle_+}(\xi) \\ S_{\triangle_+}(\xi) & I \end{pmatrix} \ominus \begin{pmatrix} H^2_-(E) \\ H^2_+(E) \end{pmatrix}, \tag{30}$$

where operators A_+ and A_+^* act in in the following way, [8],

$$\begin{aligned} (A_+ f)(\xi) &= \begin{pmatrix} \xi f_1(\xi) \\ \xi f_2(\xi) + i g_2(0) \end{pmatrix}; \\ (A_+^* f)(\xi) &= \begin{pmatrix} \xi f_1(\xi) - i g_1(0) \\ \xi f_2(\xi) \end{pmatrix}, \end{aligned} \tag{31}$$

where $S_{\triangle_+}(\xi)$ is the characteristic function (10) of colligation $\triangle_+$ (5) and $H^2_\pm(E)$ are Hardy classes of E-valued functions corresponding to the half-planes $\mathbb{C}_\pm$, then $g_2(0)$ and $g_1(0)$ are the values of Fourier transform of $g_2(x)$ and $g_1(x)$ at zero respectively, and

$$\begin{aligned} g_2(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (f_2(\xi) + S_{\triangle_+}(\xi) f_1(\xi)) e^{i\xi x} d\xi; \\ g_1(x) &= \frac{1}{2\pi} \int_{-\infty}^{\infty} (f_1(\xi) + S^*_{\triangle_+}(\xi) f_2(\xi)) e^{i\xi x} d\xi. \end{aligned} \tag{32}$$

where $g_1(x), g_2(x) \in L^2$.

Using the form φ functional realization [8, 12], we derive that

$$\varphi^* Q_- \varphi f = \begin{pmatrix} Q_- g_1(0) \\ Q_- g_2(0) \end{pmatrix},$$

therefore, in virtue of (4), we have

$$(Af)(\xi) = \begin{pmatrix} \xi f_1(\xi) - iQ_- g_1(0) \\ \xi f_2(\xi) + iQ_+ g_2(0) \end{pmatrix}. \tag{32}$$

Remark. The collegation is simple [6, 12] if $H = \mathrm{span}\{A^n \varphi^* E; n \in \mathbb{Z}_+\}$.

Thus, we have the following result.

Theorem 6. *Let a simple colligation $\triangle$ (1) be defined, where $J = Q_+ - Q_-$ is involution ($Q_\pm$ are orthoprojectors and $Q_+ Q_- = 0$), and $S_{\triangle_+}(\lambda)$ be a function built by the characteristic function $S_\triangle(\lambda) = I - i\varphi(A - \lambda I)^{-1}\varphi^* J$ of the colligation $\triangle$ with the help of the Potapov-Ginsburg triangular linear-fractional transform* (11)*. Then the main operator of colligation $\triangle$ is unitary equivalent to the functional model* (33) *acting in the space H_P* (30)*.*

Obviously, A^* in H_P is presented by

$$(A^* f)(\xi) = \begin{pmatrix} \xi f_1(\xi) - iQ_+ g_1(0) \\ \xi f_2(\xi) + iQ_- g_2(0) \end{pmatrix}. \tag{33}$$

Let us derive the resolvent of operator A (33), let $f = (A - \lambda I)^{-1} u$, where $u(\xi) \in H_P$ (30), then

$$\begin{cases} f_1(\xi) = \dfrac{u_1(\xi) + iQ_- g_1(0)}{\xi - \lambda}; \\ f_2(\xi) = \dfrac{u_2(\xi) - iQ_+ g_2(0)}{\xi - \lambda}. \end{cases} \tag{34}$$

Let $\lambda \in \mathbb{C}_-$. Multiplying the first equation of (35) by $S_{\triangle_+}(\xi)$, adding result to the second equation, and integrating, we get

$$\begin{aligned} -ig_2(0) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}(f_2(\xi)+S_{\triangle_+}(\xi)f_1(\xi))d\xi \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{u_2(\xi)+S_{\triangle_+}(\xi)u_1(\xi)}{\xi-\lambda}d\xi \\ &-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Q_+g_2(0)-S_{\triangle_+}(\xi)Q_-g_1(0)}{\xi-\lambda}d\xi. \end{aligned} \tag{35}$$

Now using $u_2(\xi)+S_{\triangle_+}(\xi)u_1(\xi)\in H^2_-(E)$ and an analogue of the Cauchy theorem for $H^2_-(E)$ [11], we get

$$ig_2(0)=(u_2+S_{\triangle_+}u_1)(\lambda).$$

To calculate $g_1(0)$, we multiply the second equation of (35) by $S^*_{\triangle_+}(\xi)$ and add to the first equation, then after the integration we get

$$\begin{aligned} -ig_1(0) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}(f_1(\xi)+S^*_{\triangle_+}(\xi)f_2(\xi))d\xi \\ &= \frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{u_1(\xi)+S^*_{\triangle_+}(\xi)u_2(\xi)}{\xi-\lambda}d\xi \\ &+\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{Q_-g_1(0)-S^*_{\triangle_+}(\xi)Q_+g_2(0)}{\xi-\lambda}d\xi. \end{aligned} \tag{36}$$

It is not difficult to see that each integral on the right side (37) is equal to zero. To confirm this, use the Cauchy theorem for $H^2_+(E)$ [11] and the fact that $u_1(\xi)+S_{\triangle_+}(\xi)u_2(\xi)\in H^2_+(E)$. Thus, we have

$$\left[(A-\lambda I)^{-1}u\right](\xi)=\frac{1}{\xi-\lambda}\begin{pmatrix} u_1(\xi) \\ u_2(\xi)-Q_+(u_2+S_{\triangle_+}u_1)(\lambda)\end{pmatrix} \tag{37}$$

when $\lambda \in \mathbb{C}_-$.

If we take $\lambda \in \mathbb{C}_+$ then by the similar consideration we get $g_2(0) = 0$ and $g_1(0) = i(u_1 + S^*_{\triangle_+} u_2)(\lambda)$. Consequently,

$$\left[(A - \lambda I)^{-1} u\right](\xi) = \frac{1}{\xi - \lambda} \begin{pmatrix} u_1(\xi) - Q_-(u_1 + S^*_{\triangle_+} u_2)(\lambda) \\ u_2(\xi) \end{pmatrix}. \tag{38}$$

Theorem 7. *In each regular point λ the resolvent $(A - \lambda I)^{-1}$ of the operator A (33) acting in the space H_P (30) has the form (38), when $\lambda \in \mathbb{C}_-$, and (39), when $\lambda \in \mathbb{C}_+$.*

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