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ON THE ASYMPTOTIC DISTRIBUTION OF CLOSED ORBITS FOR A CLASS OF OPEN BILLIARDS

Julien Giol

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Abstract. We explain why the lengths of the closed orbits in a certain class of open billiards are asymptotically equidistributed with respect to the number of their reflections.

1. Introduction. We consider an obstacle $K = K_1 \cup \cdots \cup K_N$ made of $N \geq 3$ strictly convex and pairwise disjoint compact sets with smooth boundary in the Euclidean space $\mathbb{R}^d$ $(d \geq 2)$. The following condition was introduced by Ikawa [1] and it is commonly denoted by (H): for every triple (K_i, K_j, K_k) of pairwise disjoint components of K, the intersection of K_i with the convex hull of $K_j \cup K_k$ is empty.

Then let us imagine that particles move within $\mathbb{R}^d \setminus K$ with constant velocity 1. Their trajectories follow straight lines between obstacles. Whenever a particle meets an obstacle, the reflection obeys Snell's Law. We refer the reader

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to [8] for a rigorous definition of the broken geodesic flow obtained this way: the so-called *billiard flow*.

We are interested in the set $\mathcal{P}$ of closed orbits of this flow. For every $\gamma \in \mathcal{P}$ we denote by $\ell(\gamma)$ the least period and by $r(\gamma)$ the number of reflections during such a period. Thanks to the work of Morita [2] (case $d = 2$) and Stoyanov [8] ($d \geq 3$), we can use a symbolic model to study the bounded trajectories of such a flow. Like in the case of an Axiom A flow restricted to a basic set, this dynamical system is isomorphic to a certain suspended flow over a subshift of finite type. Therefore it is possible to apply the techniques developed by Parry and Pollicott [3] to obtain an analogue of the Prime Number Theorem for the periodic trajectories of the billiard under consideration.

Theorem 1 (Morita-Stoyanov)**.** *There exists a constant $h > 0$ such that* $\#\{\gamma \in \mathcal{P} \mid \ell(\gamma) \leq t\} \sim \mathrm{li}(e^{ht})$ *as* $t \to +\infty$.

Note that the constant h is the topological entropy of the billiard, and that li stands for the integral logarithm , i.e. $\mathrm{li}(x) = \int_2^x (1/\log u)du \sim x/\log x$.

This work was motivated by the following question: what does this asymptotic become if we simply count the trajectories with an even number of reflections? The answer follows by application of techniques *à la* Chebotarev inspired by [4]: $\mathrm{li}(e^{ht})/2$. More generally, we have the following result.

Theorem 2. *Let $n \geq 2$. Counting the trajectories with number of reflections congruent to a certain integer $j \in \{0, 1, \ldots, n-1\}$ modulo n, we have* $\#\{\gamma \in \mathcal{P} \mid r(\gamma) \equiv j\ [n]\ ,\ \ell(\gamma) \leq t\} \sim \frac{1}{n}\,\mathrm{li}(e^{ht})$ *as* $t \to +\infty$.

The third section is devoted to the proof of this result. In the second section, we recall the definitions and the results needed from the underlying symbolic model. Our strategy differs slightly from that of [4] for cyclic extensions of suspended flows by the fact that we introduce dynamical zeta functions well-adapted to the problem. We generalize this way the functions ζ_+ et ζ_- used by Petkov in [6]. Note that we recall the outline of the proof of Parry and Pollicott's Prime Orbit Theorem in Section 2. Our purpose is to leave to Section 3 what is really specific in our case.

Acknowledgements. This article is an extended version of an unpublished note of 2005 which is referred to in [7]. It builds upon a result which was part of my M.S. Thesis (2000), and which also appears in an article of Xia [9]: the case $n = 2$ of trajectories with an even number of reflections. I would like to thank Frédéric Naud for fruitful conversations. I am grateful to Vesselin Petkov for asking the question that motivated this work. Finally, let me thank the referee

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2. Symbolic dynamics.

Bernoulli Shift. Recall that N denotes the number of obstacles in $\mathbb{R}^d$. Then let $\Sigma = \{x \in \{1, \dots, N\}^{\mathbb{Z}} \ : \ x_j \neq x_{j+1} \, \forall j\}$ be the set of infinite words in the alphabet $\{1, \dots, N\}$ with no equal consecutive letters. We equip $\{1, \dots, N\}$ with the discrete topology and Σ with the product topology. The latter can be metrized for any fixed $0 < \theta < 1$ by the distance

$$d_\theta(x, y) = \theta^{\inf\{n \geq 0 \, : \, x_n \neq y_n \text{ or } x_{-n} \neq y_{-n}\}}.$$

The *Bernoulli shift* is the homeomorphism of the compact space Σ defined by $(\sigma x)_n = x_{n+1}$.

Suspended Flow. Let us fix a continuous function $f : \Sigma \to]0, +\infty[$. The so-called *roof function* allows us to define the compact space

$$\Sigma^f = \{(x, u) \ : \ x \in \Sigma, 0 \leq u \leq f(x)\} / \sim$$

with the identification $(x, f(x)) \sim (\sigma x, 0)$. The *suspended flow* on Σ^f is defined by the formula $\sigma_t^f(x, u) = (x, t + u)$.

Note that an element $x = (\dots, x_0, x_1, \dots)$ in the symbolic model Σ corresponds to a trajectory in the billiard leaving the obstacle K_{x_0}, with first reflection on K_{x_1}, etc... The function $f(x)$ is equal to the distance, in the billiard again, between these two initial reflection points.

Closed Orbits. The periodic orbits of the Bernoulli shift are clearly related to the closed orbits of the suspended flow. We denote

$$Fix_n = \{x \ : \ \sigma^n x = x\}$$

the set of words with period n under the action of σ. Let $\mathcal{P}$ denote the set of periodic orbits. For each $\gamma \in \mathcal{P}$, observe that $\gamma = \{x, \sigma x, \dots, \sigma^{k-1} x\}$ has a minimal period k which we denote $r(\gamma)$. To each such orbit corresponds a closed orbit of the suspended flow and conversely. The length of the latter will be denoted by $\ell(\gamma)$. Note that the iterated function

$$f^k(x) := f(x) + f(\sigma x) + \cdots + f(\sigma^{k-1} x)$$

satisfies $f^k(x) = \ell(\gamma)$. Also note that the length spectrum $\{\ell(\gamma) \ : \ \gamma \in \mathcal{P}\}$ is bounded away from 0. Indeed, by continuity of f on the compact Σ^+, we have

$$\ell(\gamma) \geq \inf_{x \in \Sigma^+} f(x) = \min_{x \in \Sigma^+} f(x) > 0$$

for every $\gamma \in \mathcal{P}$.

Ruelle Transfer Operators. Thanks to Sinai's lemma (see e.g. [5, Proposition 1.2]), we can assume that the roof function f depends only on the *future coordinates* $(x_0, x_1, \ldots)$. So we introduce now the set

$$\Sigma^+ = \{x \in \{1, \ldots, N\}^{\mathbb{N}} \ : \ x_j \neq x_{j+1} \, \forall j\}$$

which can easily be embedded in Σ.

The new roof function has the same length spectrum, so there is no loss of generality for the problem under study. One great advantage of this is that we can use the Ruelle transfer operator.

Let us fix a function ϕ continuous on Σ^+. The Banach space of Lispschitz functions on Σ^+ is denoted by $\mathcal{F}_\theta^+$. It is the space of continuous functions w on Σ^+ such that

$$\|w\|_\theta := \sup_{x \neq y} \frac{|f(x) - f(y)|}{d_\theta(x, y)}$$

Then the formula

$$\mathcal{L}_\phi w(x) = \sum_{\sigma y = x} e^{\phi(y)} w(y)$$

defines an operator from $\mathcal{F}_\theta^+$ to itself. The key technical aspect of the theory is to estimate the norms of the iterates of such operators. This is related to the following notion.

Topological Pressure. For a continuous function ϕ, the topological pressure is defined by

$$P(\phi) = \sup\left\{h(\nu) + \int \phi d\nu : \nu \text{ is a } \sigma - \text{invariant probability measure}\right\}$$

where $h(\nu)$ denotes the entropy of the shift σ with respect to the measure ν. The *Basic Inequality* in [5] says essentially that there exists a constant $C_0 > 0$ such that the iterates of the Ruelle operator obey the norm inequalities:

$$\frac{\|\mathcal{L}_{-sf}^k w\|}{e^{kP(-Re(s)\, f)}} \leq C_0 |Im(s)| \|w\|_\infty + \theta^k \|w\|_\theta$$

for all $k \geq 0$ and all w in $\mathcal{F}_\theta^+$. It follows in particular that the spectral radius of the operator $\mathcal{L}_{-sf}^k w$ is not greater than $e^{P(-Re(s)\, f)}$. Actually, $e^{P(-Re(s)\, f)}$ is known to be a simple positive eigenvalue, and the rest of the spectrum is contained in a disk of radius strictly smaller.

Dynamical Zeta Function. Let $h > 0$ denote the topological entropy of the flow, which is characterized by $P(-hf) = 0$. The formula

$$\zeta(s) = \exp\left(\sum_{n\geq 1} \frac{1}{n} \sum_{x\in Fix_n} e^{-shf^n(x)}\right)$$

defines the *Dynamical Zeta Function.* It is convenient to introduce the notation

$$Z_n(s) = \sum_{x\in Fix_n} e^{-shf^n(x)}$$

and to observe that

$$\sum_{n=1}^{\infty} \frac{Z_n(s)}{n} = \sum_{\gamma\in\mathcal{P}} \sum_{k\geq 1} \frac{N(\gamma)^{-sk}}{k}$$

if we put $N(\gamma) = e^{h\ell(\gamma)}$. It follows that

$$\zeta(s) = \prod_{\gamma\in\mathcal{P}} \frac{1}{1 - N(\gamma)^{-s}}.$$

The convergence of the sum, or the product, defining the Dynamical Zeta Function at s_0 is related to the spectrum of the Ruelle operator

$$L_{s_0} = \mathcal{L}_{-s_0 hf}.$$

By introduction of the topological entropy h and the corresponding normalization of the zeta function, the series $\sum_{n=1}^{+\infty} Z_n(s)/n$ is absolutely convergent on every closed half-plane $\{Re\ s \geq \sigma_0 > 1\}$. Hence the dynamical zeta function is analytic without zeros on the open half-plane $\{Re\ s > 1\}$.

The behavior on the axis is settled by the following alternative. According to [5, Theorem 5.5], two cases can occur depending on the spectral radius of L_{s_0}:

(1) If $\rho(L_{s_0}) < 1$, the series $\sum_{n=1}^{+\infty} Z_n(s)/n$ is absolutely convergent in a certain neighborhood of s_0.

(2) If $\rho(L_{s_0}) = 1$, then we have an analytic function $\lambda(s)$ giving the eigenvalue of maximal modulus of L_s in a certain neighborhood of s_0 where moreover the series $\sum_{k=1}^{+\infty}(Z_k(s) - \lambda(s)^k)/k$ is absolutely convergent.

Since the billiard flow is weakly mixing (cf. [8, Lemma 5.2]), so is the underlying symbolic flow. Consequently the dynamical zeta function is analytic in a neighborhood of the closed half-plane $\{Re\, s \geq 1\}$, except for a simple pole at $s = 1$ (see [3, Proposition 9]). It follows that $\lambda(s_0)$ can never be equal to 1, except for $s_0 = 1$ where $\lambda(1) = 1$ and $\lambda'(1) \neq 0$. Thus the logarithmic derivative $\frac{\zeta'}{\zeta}(s)$ is analytic in a neighborhood of the open half-plane $\{Re\, s > 1\}$, except for a simple pole with residue 1 at $s = 1$.

Prime Orbit Theorem. The celebrated Prime Orbit Theorem of Parry and Pollicott [3] asserts that the counting function

$$\pi(u) = \#\{\gamma \in \mathcal{P} \ : \ N(\gamma) \leq u\}$$

admits the asymptotic equivalent $\mathrm{li}(u)$ as $u \to +\infty$.

By definition of the dynamical zeta function, the logarithmic derivative of $\zeta(s)$ is equal to the derivative of

$$\sum_{n=1}^{\infty} \frac{Z_n(s)}{n} = \sum_{\gamma\in\mathcal{P}} \sum_{k\geq 1} \frac{N(\gamma)^{-sk}}{k},$$

namely

$$-\frac{\zeta'}{\zeta}(s) = \sum_{\gamma\in\mathcal{P}} \sum_{k\geq 1} N(\gamma)^{-sk} \log N(\gamma).$$

The strategy consists in studying the *essential part* of the latter by introducing the function

$$\eta(s) = \sum_{\gamma\in\mathcal{P}} N(\gamma)^{-s} \log N(\gamma)$$

which is related to the zeta function by the formula:

$$-\frac{\zeta'}{\zeta}(s) = \eta(s) + \sum_{\gamma\in\mathcal{P}} \sum_{k\geq 2} N(\gamma)^{-sk} \log N(\gamma).$$

Lemma 2.1. *Let $\sigma_0 > 1/2$. There exists a constant $C > 0$ such that for every orbit $\gamma \in \mathcal{P}$ and for every complex number s in the closed half-plane $\{Re\, s \geq \sigma_0\}$, we have:*

$$\sum_{k\geq 2} |N(\gamma)^{-sk}| \leq C N(\gamma)^{-2\sigma_0}.$$

P r o o f. Recall that the length spectrum is bounded away from 0 so that there exists $\alpha > 0$ such that $\ell(\gamma) \geq \alpha$ for every $\gamma \in \mathcal{P}$. With $C := (1 - e^{-\sigma_0 h\alpha})^{-1}$, we have the following estimates:

$$\begin{aligned}\sum_{k\geq 2} |N(\gamma)^{-sk}| &\leq \sum_{k\geq 2} N(\gamma)^{-\sigma_0 k} \\ &= \frac{N(\gamma)^{-2\sigma_0}}{1 - N(\gamma)^{-\sigma_0}} \\ &\leq CN(\gamma)^{-2\sigma_0}.\end{aligned}$$

□

It follows that

$$\sum_{\gamma\in\mathcal{P}} \sum_{k\geq 2} \left| N(\gamma)^{-sk} \log N(\gamma) \right| \leq C \sum_{\gamma\in\mathcal{P}} N(\gamma)^{-2\sigma_0} \log N(\gamma) = C\eta(2\sigma_0).$$

Recall that $\zeta(s)$ is analytic and nonzero on the open half-plane $\{Re\, s > 1\}$. Note that by definition, the absolute convergence of $-\frac{\zeta'}{\zeta}(s)$ implies that of $\eta(s)$, which is merely a sum of fewer terms. In particular $\eta(2\sigma_0)$ is finite for every $\sigma_0 > 1/2$. So the difference between $\eta(s)$ and $-\frac{\zeta'}{\zeta}(s)$ is analytic in the open half-plane $\{Re\, s > 1/2\}$. Hence there is a function $\phi(s)$ analytic in a certain neighborhood of the open half-plane $\{Re\, s > 1\}$ where

$$\eta(s) = \frac{1}{s-1} + \phi(s).$$

By Stieltjes integration, we have the formula

$$\eta(s) = \int_1^{+\infty} u^{-s} \log u \, d\pi(u).$$

It only remains to apply Wiener-Ikehara Tauberian Theorem to conclude that $\pi(u) \sim \mathrm{li}(u)$ as $u \to +\infty$.

3. Proof of Theorem 2. This proof follows closely the one for the Prime Orbit Theorem. All the difference lies in the change of zeta function, which essentially results in a change of residue when the Tauberian argument comes.

Thanks to the work of Stoyanov [8, Theorem 1.3] (see also Morita [2, Theorem 1]), the length spectrum $\{\ell(\gamma) \mid \gamma \in \mathcal{P}\}$ of the billiard is the same as the one of a certain suspended flow σ^f over a subshift σ of the type introduced above. To each periodic billiard trajectory γ corresponds a closed orbit of σ^f of minimal length $\ell(\gamma)$, whose discrete orbit under σ has $r(\gamma)$ elements. So we start by recasting this way the problem in the symbolic model studied in the previous section.

Let $\mathcal{P}_{n,j} := \{\gamma \in \mathcal{P} \mid r(\gamma) \equiv j \ [n]\}$ denote the subset of orbits $\mathcal{P}$ under consideration in the statement of the theorem. Let also

$$\pi(u) := \#\{\gamma \in \mathcal{P}_{n,j} \mid N(\gamma) \leq u\}$$

denote the corresponding counting function, where $N(\gamma) = e^{h\ell(\gamma)}$.

The asymptotic behavior of π depends on the singularities of the complex function

$$\eta(s) := \int_1^{+\infty} u^{-s} \log u \, \mathrm{d}\pi(u) = \sum_{\gamma \in \mathcal{P}_{n,j}} N(\gamma)^{-s} \log N(\gamma).$$

By Wiener-Ikehara Tauberian Theorem, it suffices to show that η is analytic in a certain neighborhood of the closed half-plane $\{Re\ s \geq 1\}$, except for a simple pole with residue $1/n$ at $s = 1$.

Let us introduce now a zeta function well-adapted to our situation:

$$\zeta_{n,j}(s) := \exp\left(\sum_{k=0}^{+\infty} \frac{Z_{j+kn}(s)}{j+kn}\right).$$

For $Re\ s > 1$, we see that, like in the previous section, $\zeta_{n,j}$ is analytic without zeros and that

$$-(\zeta'_{n,j}/\zeta_{n,j})(s) = \eta(s) + \sum_{k=2}^{+\infty} \sum_{r(\gamma)k \equiv j\ [n]} N(\gamma)^{-sk} \log N(\gamma).$$

In the sum above, the right-hand term defines a Dirichlet series whose absolute value is bounded above by $C \cdot \sum_{\gamma \in \mathcal{P}} N(\gamma)^{-2Re(s)} \log N(\gamma)$ by Lemma 2.1 It follows that η has exactly the same singularities as $-(\zeta'_{n,j}/\zeta_{n,j})$ in a certain neighborhood of the half-plane $\{s \in \mathbb{C} \mid \mathrm{Re}(s) \geq 1\}$.

To study the latter at $s_0 = 1 + it$, we need to consider the Ruelle operator L_{s_0}.

(1) If $\rho(L_{s_0}) < 1$, then the series $\sum_{k=1}^{+\infty} |Z_k(s)|/k$ is absolutely convergent in a certain neighborhood of s_0. *A fortiori*, $\zeta_{n,j}$ is analytic without zeros in the same neighborhood.So η admits an analytic continuation at s_0.

(2) If $\rho(L_{s_0}) = 1$, then we have an analytic function $\lambda(s)$ giving the eigenvalue of maximal modulus of L_s in a certain neighborhood of s_0 where moreover the series $\sum_{k=1}^{+\infty} |Z_k(s) - \lambda(s)^k|/k$ is absolutely convergent. Thus for s close enough to s_0 and for $Re\ s > 1$, the function

$$\sum_{k=0}^{+\infty} \frac{Z_{j+kn}(s)}{j+kn}$$

has the same singularities as

$$\sum_{k=0}^{+\infty} \frac{\lambda(s)^{j+kn}}{j+kn}$$

So the logarithmic derivative $-(\zeta'_{n,j}/\zeta_{n,j})(s)$ has the same singularities as

$$-\sum_{k=0}^{+\infty} \lambda'(s)\lambda(s)^{j+kn-1} = \frac{\lambda'(s)\lambda(s)^{j-1}}{\lambda(s)^n - 1}.$$

Since the flow is weakly mixing (cf. [8, Lemma 5.2]), $\lambda(s_0)$ can never be a root of unity except for $s_0 = 1$, where $\lambda(1) = 1$ and $\lambda'(1) \neq 0$ (see [3, Proposition 7]). Thus η can always be continued analytically at each point of the axis $\{s \in \mathbb{C} \mid \mathrm{Re}(s) = 1\}$, except for the simple pole with residue

$$\lim_{s \to 1} \frac{(s-1)\lambda'(s)\lambda(s)^{j-1}}{\lambda(s)^n - 1} = \frac{\lambda'(1)}{n\lambda'(1)} = \frac{1}{n}$$

at $s = 1$.

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Department of Mathematics
Bucknell University
Lewisburg, PA 17837
e-mail: `julien.giol@bucknell.edu`

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