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MORE ℓ_r SATURATED $\mathcal{L}^\infty$ SPACES

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Abstract. Given $r \in (1, \infty)$, we construct a new $\mathcal{L}^\infty$ separable Banach space which is ℓ_r saturated.

1. Introduction. The Bourgain-Delbaen spaces [7] are examples of separable $\mathcal{L}^\infty$ spaces containing no isomorphic copy of c_0. They have played a key role in the solution of the scalar-plus-compact problem by Argyros and Haydon [3], where a Hereditarily Indecomposable $\mathcal{L}^\infty$ space is presented with the property that every operator on the space is a compact perturbation of a scalar multiple of the identity.

There has recently been an interest in the study $\mathcal{L}^\infty$ spaces of the Bourgain-Delbaen type. Freeman, Odell and Schlumprecht [8] showed that every Banach space with separable dual is isomorphic to a subspace of a $\mathcal{L}^\infty$ space having a separable dual. The aim of this paper is to present a method of constructing, for every $1 < r < \infty$, a new $\mathcal{L}^\infty$ space which is ℓ_r saturated. Our approach shares common features with the Argyros-Haydon work. More precisely we combine,

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as in [3], the Bourgain-Delbaen method [7] yielding exotic $\mathcal{L}^\infty$ spaces, with the Tsirelson type norms that are equivalent to some ℓ_r norm (see [2], [4], [5], [6], [11]). Recall that in [9], the original Bourgain-Delbaen spaces $\mathfrak{X}_{a,b}$ with $a < 1$, $b < \dfrac{1}{2}$ and $a + b > 1$ where shown to be ℓ_p saturated for p determined by the formulas $\dfrac{1}{p} + \dfrac{1}{q} = 1$ and $a^q + b^q = 1$.

This paper is organized as follows. In the second section, for a given $r \in (1,\infty)$, we construct a Banach space $\mathfrak{X}_r$. To do this, we first choose $n \in \mathbb{N}$, $n > 1$, and a finite sequence $\overline{b} = (b_1, b_2, \dots, b_n)$ of positive real numbers with $b_1 < 1$, $b_2, b_3, \dots, b_n < \dfrac{1}{2}$ such that $\sum\limits_{i=1}^{n} b_i^{r'} = 1$ and $\dfrac{1}{r} + \dfrac{1}{r'} = 1$. The definition of $\mathfrak{X}_r$ combines the Bourgain-Delbaen method with the Tsirelson type space $\mathcal{T}(\mathcal{A}_n, \overline{b})$ which will be later proved to be isomorphic to ℓ_r. In particular, if $b_1 = b_2 = \ldots = b_n = \theta$, $\mathcal{T}(\mathcal{A}_n, \overline{b})$ coincides with $\mathcal{T}(\mathcal{A}_n, \theta)$ and the latter is known to be isomorphic to ℓ_p for some $p \in (1,\infty)$ (see [4]). It is worth noticing that for $n = 2$ the spaces $\mathfrak{X}_r$ essentially coincide with the original Bourgain-Delbaen spaces $\mathfrak{X}_{a,b}$. Thus, our construction of $\mathcal{L}^\infty$ spaces which are ℓ_r saturated spaces, can be considered as a generalization of the Bourgain-Delbaen method. We must point out here that when $n = 2$, our proof of the fact that $\mathfrak{X}_r$ is ℓ_r saturated, differs from Haydon's (see [9]) corresponding one for $\mathfrak{X}_{a,b}$. To be more specific, $\mathfrak{X}_r$ has a natural FDD (M_k). Given a normalized skipped block basis (u_k) of (M_k) with the supports of the u_k's lying far enough apart, then it is not hard to check that (u_k) dominates (e_k), the natural basis of $\mathcal{T}(\mathcal{A}_n, \overline{b})$. The same holds for every normalized block basis of (u_k). To obtain a normalized block basis of (u_k) equivalent to (e_k), we select a sequence $I_1 < I_2 < \ldots$ of successive finite subsets of $\mathbb{N}$ such that $\lim\limits_k \left\| \sum\limits_{i \in I_k} u_i \right\| = \infty$. Such a choice is possible by the domination of (e_k) by (u_k). We set $v_k = \left\| \sum\limits_{i \in I_k} u_i \right\|^{-1} \sum\limits_{i \in I_k} u_i$ and show that some subsequence of (v_k) is dominated by (e_k). To accomplish this we adapt the method of the analysis of the members of a finite block basis of (e_k) with respect to a functional in the natural norming set of $\mathcal{T}(\mathcal{A}_n, \overline{b})$ (see [6]), to the context of the present construction. This approach yields an alternative proof for the saturation of Bourgain-Delbaen type spaces with copies of ℓ_r, which is closer in spirit to the methods of estimating norms in Tsirelson and mixed Tsirelson type spaces.

The rest of the paper is devoted to the proof of the main property, namely that $\mathfrak{X}_r$ is ℓ_r saturated. In Section 3, we define the tree analysis of the functionals

$\{e_\gamma^* : \gamma \in \Gamma\}$ which is a 1-norming subset of the unit ball of $\mathfrak{X}_r^*$. The tree analysis is similar to the corresponding one used in the Tsirelson and mixed Tsirelson spaces [4]. In the following two sections we establish the lower and upper norm estimates for certain block sequences in the space $\mathfrak{X}_r$.

In the final section we show that every block basis of (M_k) admits a further normalized block basis (x_k) such that every normalized block basis of (x_k) is equivalent to the natural basis of the space $\mathcal{T}(\mathcal{A}_n, \overline{b})$. Zippin's theorem [13] yields the desired result.

2. Preliminaries. In this section we define the space $\mathfrak{X}_r$ combining the Bourgain-Delbaen construction [7] and the Tsirelson type constructions [2], [4].

Before proceeding, we recall some notation and terminology from [3]. Let $n \in \mathbb{N}$ and $0 < b_1, b_2, \ldots, b_n < 1$ with $\sum_{i=1}^n b_i > 1$ and there exists $r' \in (1, \infty)$ such that $\sum_{i=1}^n {b_i}^{r'} = 1$. We may also assume without loss of generality that $b_1 > b_2 > \ldots > b_n$. We define $W[(\mathcal{A}_n, \overline{b})]$ to be the smallest subset W of $c_{00}(\mathbb{N})$ with the following properties:

1. $\pm e_k^* \in W$ for all $k \in \mathbb{N}$,
2. whenever $f_i \in W$ and $\max \operatorname{supp} f_i < \min \operatorname{supp} f_{i+1}$ for all i, we have $\sum_{i \le a} b_i f_i \in W$, provided that $a \le n$,

We say that an element f of $W[(\mathcal{A}_n, \overline{b})]$ is of Type 0 if $f = \pm e_k^*$ for some k and of Type I otherwise; an element of Type I is said to have weight b_a for some $a \le n$ if $f = \sum_{i=1}^a f_i$ for a suitable sequence (f_i) of successive elements of $W[\mathcal{A}_n, \overline{b}]$.

The *Tsirelson space* $\mathcal{T}(\mathcal{A}_n, \overline{b})$ is defined to be the completion of c_{00} with respect to the norm

$$\|x\| = \sup\{\langle f, x\rangle : f \in W[\mathcal{A}_n, \overline{b}]\}.$$

We may also characterize the norm of this space implicitly as being the smallest function $x \mapsto \|x\|$ satisfying

$$\|x\| = \max\left\{\|x\|_\infty, \sup \sum_{i=1}^n b_i \|E_i x\|\right\},$$

where the supremum is taken over all sequences of finite subsets $E_1 < E_2 < \cdots < E_n$.

We shall now present the fundamental aspects related to the Bourgain-Delbaen construction.

For the interested readers we mention that the following method can be characterized as the "dual" construction of the construction presented in [3]. This characterization is based on the fact that in [3] a particular kind of basis is given to $\ell_1(\Gamma)$ and the Bourgain-Delbaen type space X is seen as the predual of its dual, which is $\ell_1(\Gamma)$.

Let $(\Gamma_q)_{q\in\mathbb{N}}$ be a strictly increasing sequence of finite sets and denote their union by Γ; $\Gamma = \bigcup_{q\in\mathbb{N}} \Gamma_q$.
We set $\Delta_0 = \Gamma_0$ and $\Delta_q = \Gamma_q \setminus \Gamma_{q-1}$ for $q = 1, 2, \ldots$
Assume furthermore that to each $\gamma \in \Delta_q$, $q \geq 1$, we have assigned a linear functional $c_\gamma^* : \ell^\infty(\Gamma_{q-1}) \to \mathbb{R}$. Next, for $n < m$ in $\mathbb{N}$, we define by induction, a linear operator $i_{n,m} : \ell^\infty(\Gamma_n) \to \ell^\infty(\Gamma_m)$ as follows:
For $m = n+1$, we define $i_{n,n+1} : \ell^\infty(\Gamma_n) \to \ell^\infty(\Gamma_{n+1})$ by the rule

$$(i_{n,n+1}(x))(\gamma) = \begin{cases} x(\gamma), & \text{if } \gamma \in \Gamma_n \\ c_\gamma^*(x), & \text{if } \gamma \in \Delta_{n+1} \end{cases}$$

for every $x \in \ell^\infty(\Gamma_n)$.
Then assuming that $i_{n,m}$ has been defined, we set $i_{n,m+1} = i_{m,m+1} \circ i_{n,m}$. A direct consequence of the above definition is that for $n < l < m$ it holds that $i_{n,m} = i_{l,m} \circ i_{n,l}$. Finally we denote by $i_n : \ell^\infty(\Gamma_n) \to \mathbb{R}^\Gamma$ the direct limit $i_n = \lim_{m\to\infty} i_{n,m}$.

We assume that there exists a $C > 0$ such that for every $n < m$ we have $\|i_{n,m}\| \leq C$. This implies that $\|i_n\| \leq C$ and therefore $i_n : \ell^\infty(\Gamma_n) \to \ell^\infty(\Gamma)$ is a bounded linear map. In particular, setting $X_n = i_n[\ell^\infty(\Gamma_n)]$, we have that $X_n \overset{C}{\approx} \ell^\infty(\Gamma_n)$ and furthermore $(X_n)_{n\in\mathbb{N}}$ is an increasing sequence of subspaces of $\ell^\infty(\Gamma)$. We also set $\mathfrak{X}_{BD} = \overline{\bigcup_{n\in\mathbb{N}} X_n} \hookrightarrow \ell^\infty(\Gamma)$ equipped with the supremum norm. Evidently, $\mathfrak{X}_{BD}$ is an $\mathcal{L}^\infty$ space.

Let us denote by $r_n : \ell^\infty(\Gamma) \to \ell^\infty(\Gamma_n)$ the natural restriction map, i.e. $r_n(x) = x|_{\Gamma_n}$. We will also abuse notation and denote by $r_n : \ell^\infty(\Gamma_m) \to \ell^\infty(\Gamma_n)$ the restriction function from $\ell^\infty(\Gamma_m)$ to $\ell^\infty(\Gamma_n)$ for $n < m$.

Notation 2.1.

(i) We denote by e_γ^* the restriction of the unit vector $e_\gamma \in \ell^1(\Gamma)$ on the space $\mathfrak{X}_{BD}$.

(ii) We also extend the functional $c_\gamma^* : \ell^\infty(\Gamma_n) \to \mathbb{R}$ to a functional $c_\gamma^* : \mathfrak{X}_{BD} \to \mathbb{R}$ by the rule $c_\gamma^*(x) = (c_\gamma^* \circ r_{q-1})(x)$ when $\gamma \in \Delta_q$.

As it is well known from [3] and [7], instead of the Schauder basis of $\mathfrak{X}_{BD}$, it is more convenient to work with a FDD naturally defined as follows:

For each $q \in \mathbb{N}$ we set $M_q = i_q[\ell^\infty(\Delta_q)]$.

We briefly establish this fact in the following proposition and then continue with the details of the construction of $\mathfrak{X}_r$.

Proposition 2.2. *The sequence $(M_q)_{q\in\mathbb{N}}$ is a FDD for $\mathfrak{X}_{BD}$.*

P r o o f. For $q \geq 0$ we define the maps $P_{\{q\}} : \mathfrak{X}_{BD} \to M_q$ with

$$P_{\{q\}}(x) = i_q(r_q(x)) - i_{q-1}(r_{q-1}(x))$$

It is easy to check that each $P_{\{q\}}$ is a projection onto M_q and that for $q_1 \neq q_2$ and $x \in M_{q_2}$ we have $P_{\{q_1\}}(x) = 0$. Also we have that $\|P_q\| \leq 2C$. We point out that in a similar manner one can define projections on intervals of the form $I = (p, q]$ so that $P_I(x) = \sum\limits_{i=p+1}^{q} P_{\{i\}}(x)$ for which we can readily verify the formula

$$P_I(x) = i_q(r_q(x)) - i_p(r_p(x))$$

Note that $\|P_I\| \leq 2C$. This shows that indeed $(M_q)_q$ is a FDD generating $\mathfrak{X}_{BD}$. $\square$

For $x \in \mathfrak{X}_{BD}$ we denote by $\operatorname{supp} x$ the set $\operatorname{supp} x = \{q : P_{\{q\}}(x) \neq 0\}$ and by $\operatorname{ran} x$ the minimal interval of $\mathbb{N}$ containing $\operatorname{supp} x$.

Definition 2.3. *A block sequence $(x_i)_{i=1}^\infty$ in $\mathfrak{X}_{BD}$ is called* skipped *(with respect to $(M_q)_{q\in\mathbb{N}}$), if there is a subsequence $(q_i)_{i=1}^\infty$ of $\mathbb{N}$ so that for all $i \in \mathbb{N}$,* $\operatorname{maxsupp} x_i < q_i < \operatorname{minsupp} x_{i+1}$.

In the sequel, when we refer to a skipped block sequence, we consider it to be with respect to the FDD $(M_q)_{q\in\mathbb{N}}$.

Let $q \geq 0$. For all $\gamma \in \Delta_q$ we set $d_\gamma^* = e_\gamma \circ P_{\{q\}}$. Then the family $(d_\gamma^*)_{\gamma\in\Gamma}$ consists of the biorthogonal functionals of the FDD $(M_q)_{q\geq 0}$. Notice that for $\gamma \in \Delta_q$,

$$\begin{aligned} d_\gamma^*(x) &= P_q(x)(\gamma) = i_q(r_q(x))(\gamma) - i_{q-1}(r_{q-1}(x))(\gamma) = \\ &= r_q(x)(\gamma) - c_\gamma^*(r_{q-1}(x)) = x(\gamma) - c_\gamma^*(x) = \\ &= e_\gamma^*(x) - c_\gamma^*(x). \end{aligned}$$

The sequences $(\Delta_q)_{q\in\mathbb{N}}$ and $(c_\gamma^*)_{\gamma\in\Gamma}$ are determined as in [3], section 4 and Theorem 3.5.

We give some useful notation. For fixed $n \in \mathbb{N}$ and $\overline{b} = (b_1, b_2, \ldots, b_n)$ with $0 < b_1, b_2, \ldots, b_n < 1$, for each $\gamma \in \Delta_q$ we assign

(a) $\operatorname{rank} \gamma = q$

(b) age of γ denoted by $a(\gamma) = a$ such that $2 \leq a \leq n$

(c) weight of γ denoted by $w(\gamma) = b_a$

In order to proceed to the construction, we first need to fix a positive integer n and a descending sequence of positive real numbers $b_1, \ldots, b_n$ such that $b_1 < 1$, $b_i < \frac{1}{2}$, for every $i = 2, \ldots, n$ and $\sum_{i=1}^n b_i > 1$. Let $r \in (1, \infty)$ be such that $\sum_{i=1}^{n} b_i^{r'} = 1$ and $\frac{1}{r} + \frac{1}{r'} = 1$. Now we shall define the space $\mathfrak{X}_r$ by using the Bourgain-Delbaen construction that was presented in the preceding paragraphs.

We set $\Delta_0 = \emptyset$, $\Delta_1 = \{0\}$ and recursively define for each $q > 1$ the set Δ_q. Assume that Δ_p have been defined for all $p \leq q$. We set

$$\begin{aligned} \Delta_{q+1} \;\; = \;\; & \big\{(q+1, a, p, \eta, \varepsilon e_\xi^*) : \; 2 \leq a \leq n, p \leq q, \; \varepsilon = \pm 1, \;\; \xi \in \Gamma_q \setminus \Gamma_p, \\ & \eta \in \Gamma_p, \; b_{a-1} = w(\eta)\big\} \end{aligned}$$

For $\gamma \in \Delta_{q+1}$ it is clear that the first coordinate is the rank of γ, while the second is the age $a(\gamma)$ of γ. The functionals $(c_\gamma^*)_{\gamma \in \Delta_{q+1}}$ are defined in a way that depends on $\gamma \in \Delta_{q+1}$. Namely, let $x \in \ell^\infty(\Gamma_q)$.

(i) For $\gamma = (q+1, 2, p, \eta, \varepsilon e_\xi^*)$ we set

$$c_\gamma^*(x) = b_1 x(\eta) + b_2 \varepsilon e_\xi^* \big(x - i_{p,q}(r_p(x))\big).$$

(ii) For $\gamma = (q+1, a, p, \eta, \varepsilon e_\xi^*)$ with $a > 2$ we set

$$c_\gamma^*(x) = x(\eta) + b_a \varepsilon e_\xi^* \big(x - i_{p,q}(r_p(x))\big).$$

We may now define sequences (i_q), (Γ_q), (X_q) in a similar manner as before and set $\mathfrak{X}_r = \overline{\bigcup_{q \in \mathbb{N}} X_q}$. Assuming that (i_q) is uniformly bounded by a constant C, we conclude that the space $\mathfrak{X}_r$ is a subspace of $\ell_\infty(\Gamma)$. The constant C is determined as in [3] Theorem 3.4, by taking $C = \frac{1}{1 - 2b_2}$. Thus, for every $m \in \mathbb{N}$, $\|i_m\| \leq C$. This implies that $\|P_I\| \leq 2C$ for every I interval.

Remark 2.4. In the case of $n = 2$, i.e. $\overline{b} = (b_1, b_2)$, the space $\mathfrak{X}_{\mathfrak{r}}$ essentially coincides with the Bourgain-Delbaen space $\mathfrak{X}_{b_1,b_2}$, since every $\gamma \in \Gamma$ is of age 2.

Remark 2.5. As it is shown in Proposition 6.2, the choice of r, based on the fixed n and $\overline{b}$, yields that $\mathcal{T}(\mathcal{A}_n, \overline{b}) \cong \ell_r$. Moreover, the ingredients of the "Tsirelson type spaces" theory that are used throughout this paper are essentially the same with the corresponding ones in [3]. The basic difference in our approach is that we use only one family $\mathcal{T}(\mathcal{A}_n, \overline{b})$ for some appropriate n and $\overline{b}$.

3. The Tree Analysis of e^*_γ for $\gamma \in \Gamma$. We begin by recalling the analysis of e^*_γ in [3] section 4. The only difference is that in our case all the functionals e^*_γ have weight depending on their age which is greater or equal to 2.

3.1. The evaluation Analysis of e^*_γ for $\gamma \in \Gamma$. First we point out that for $q \in \mathbb{N}$ every $\gamma \in \Delta_{q+1}$ admits a unique analysis as follows:
Let $a(\gamma) = a \leq n$. Then using backwards induction we determine a sequence of sets $\{p_i, q_i, \varepsilon_i e^*_{\xi_i}\}_{i=1}^a \cup \{\eta_i\}_{i=2}^a$ with the following properties.

(i) $p_1 < q_1 < \cdots < p_a < q_a = q$.

(ii) $\varepsilon_i = \pm 1$, $\operatorname{rank} \xi_i \in (p_i, q_i]$ for $1 \leq i \leq a$ and $\operatorname{rank} \eta_i = q_i + 1$ for $2 \leq i \leq a$.

(iii) $\eta_a = \gamma$, $\eta_i = (\operatorname{rank} \eta_i, i, p_i, \eta_{i-1}, \varepsilon_i e^*_{\xi_i})$ for every $i > 2$
$\eta_2 = (\operatorname{rank} \eta_2, 2, p_2, \varepsilon_1 \xi_1, \varepsilon_2 e^*_{\xi_2})$ and $(p_1, q_1] = (1, \operatorname{rank} \xi_1]$.

Definition 3.1. *Let $q \in \mathbb{N}$ and $\gamma \in \Gamma_q$. Then the sequence $\{p_i, q_i, \varepsilon_i e^*_{\xi_i}\}_{i=1}^a$ $\cup \{\eta_i\}_{i=2}^a$ satisfying all the above properties will be called the analysis of γ.*

Moreover, following similar arguments as in [3] Proposition 4.6 it holds that,

$$e^*_\gamma = \sum_{i=2}^a d^*_{\eta_i} + \sum_{i=1}^a b_i \varepsilon_i e^*_{\xi_i} \circ P_{(p_i,q_i]} = \sum_{i=2}^a e^*_{\eta_i} \circ P_{\{q_i+1\}} + \sum_{i=1}^a b_i \varepsilon_i e^*_{\xi_i} \circ P_{(p_i,q_i]}.$$

We set $g_\gamma = \sum\limits_{i=2}^a d^*_{\eta_i}$ and $f_\gamma = \sum\limits_{i=1}^a b_i \varepsilon_i e^*_{\xi_i} \circ P_{(p_i,q_i]}$.

3.2. The r-Analysis of the functional e^*_γ. Let $r \in \mathbb{N}$ and $\gamma \in \Delta_{q+1}$. Let $a(\gamma) = a \leq n$ and $\{p_i, q_i, \varepsilon_i e^*_{\xi_i}\}_{i=1}^a \bigcup \{\eta_i\}_{i=2}^a$ the evaluation analysis of γ. We define the r-analysis of e^*_γ as follows:

(a) If $r \leq p_1$, then the r-analysis of e^*_γ coincides with the evaluation analysis of e^*_γ.

(b) If $r \geq q_a$, then we assign no r-analysis to e^*_γ and we say that e^*_γ is r-indecomposable.

(c) If $p_1 < r < q_a$, we define $i_r = \min\{i : r < q_i\}$. Note that this is well-defined. The r-analysis of e^*_γ is the following triplet

$$\{(p_i, q_i]\}_{i \geq i_r}, \{\varepsilon_i \xi_i\}_{i \geq i_r}, \{\eta_i\}_{i \geq \max\{2, i_r\}}.$$

where p_{i_r} is either the same or r in the case that $r > p_{i_r}$.

Next we introduce the tree analysis of e^*_γ which is similar to the tree analysis of a functional in a Mixed Tsirelson space (see [4] Chapter II.1). Notice that the evaluation analysis and the r-analysis of e^*_γ form the first level of the tree analysis that we are about to present.

We start with some notation. We denote by $(\mathcal{T}, " \preceq ")$ a finite partially ordered set which is a tree. Its elements are finite sequences of natural numbers ordered by the initial segment partial order. For every $t \in \mathcal{T}$,we denote by S_t the immediate successors of t

Assume now that $(p_t, q_t]_{t \in \mathcal{T}}$ is a tree of intervals of $\mathbb{N}$ such that $t \preceq s$ iff $(p_t, q_t] \supset (p_s, q_s]$ and t, s are incomparable iff $(p_t, q_t] \cap (p_s, q_s] = \emptyset$. For such a family $(p_t, q_t]_{t \in \mathcal{T}}$ and t, s incomparable we shall denote by $t < s$ iff $(p_t, q_t] < (p_s, q_s]$ (i.e. $q_t < p_s$).

3.3. The Tree Analysis of the functional e^*_γ. Let $\gamma \in \Delta_{q+1}$ with $a(\gamma) = a \leq n$. A family of the form $\mathcal{F}_\gamma = \{\xi_t, (p_t, q_t]\}_{t \in \mathcal{T}}$ is called the tree analysis of e^*_γ if the following are satisfied:

(1) $\mathcal{T}$ is a finite tree with a unique root denoted as $\emptyset$.

(2) We set $\xi_\emptyset = \gamma$,$(p_\emptyset, q_\emptyset] = (1, q]$ and let $\{p_i, q_i, \varepsilon_i e^*_{\xi_i}\}_{i=1}^a \bigcup \{\eta_i\}_{i=2}^a$ the evaluation analysis of $\xi_\emptyset$. Set $S_\emptyset = \{(1), (2), \ldots, (a)\}$ and for every $s = (i) \in S_\emptyset$, $\{\xi_s, (p_s, q_s]\} = \{\xi_i, (p_i, q_i]\}$.

(3) Assume that for a $t \in \mathcal{T}$ $\{\xi_t, (p_t, q_t]\}$ has been defined. There are two cases:

   (a) If $e^*_{\xi_t}$ is p_t-decomposable, let

$$\{(p_i, q_i]\}_{i \geq i_{p_t}}, \{\varepsilon_i \xi_i\}_{i \geq i_{p_t}}, \{\eta_i\}_{i \geq \max\{2, i_{p_t}\}}$$

the p_t analysis of $e^*_{\xi_t}$. We set $S_t = \{(t^\frown i) : i \geq i_{p_t}\}$ and

$$S_t^{p_t} = \begin{cases} S_t, & \text{if } \eta_{i_{p_t}} \text{ exists} \\ S_t \setminus \{(t^\frown i_{p_t})\}, & \text{otherwise} \end{cases}$$

Then, for every $s = (t^\frown i) \in S_t$, we set $\{\xi_s, (p_s, q_s]\} = \{\xi_i, (p_i, q_i]\}$ where $\{\varepsilon_i \xi_i, (p_i, q_i]\}$ is a member of the p_t analysis of $e^*_{\xi_t}$.

(b) $e^*_{\xi_t}$ is p_t-indecomposable, then ξ_t consists a maximal node of $\mathcal{F}_\gamma$.

Notation 3.2. *For later use we need the following:*
For every $t \in \mathcal{T}$ $e^*_{\xi_t} = f_t + g_t$, where $f_t = \sum_{s \in S_t} b_s \varepsilon_s e^*_{\xi_s} \circ P_{(p_s, q_s]}$ and

$$g_t = \sum_{s \in S_t^{p_t}} d^*_{\eta_s} \text{ and for } s = (t^\frown i) \in S_t^{p_t},$$

$$\eta_{(t^\frown i)} = (\operatorname{rank} \eta_{(t^\frown i)}, i, p_{(t^\frown i)}, \eta_{(t^\frown i-1)}, \varepsilon_{(t^\frown i)} e^*_{\xi_{(t^\frown i)}}).$$

In the rest of the paper, we set $f_t = f_{\xi_t}$ and $g_t = g_t$.

Lemma 3.3. *Let $x \in \mathfrak{X}_r$ and $\gamma \in \Gamma$. Then,*

$$e^*_\gamma(x) = \prod_{\emptyset \preceq s \preceq t_x} (\varepsilon_s b_s)(f_{t_x} + g_{t_x})(x),$$

where $t_x = \max\{t : \operatorname{ran} x \subseteq (p_t, q_t]\}$.

Proof. Let $\mathcal{F}_\gamma = \{\xi_t, (p_t, q_t]\}_{t \in \mathcal{T}}$ a tree analysis of γ.
If $\{t : \operatorname{ran} x \subseteq (p_t, q_t]\} = \emptyset$, then $e^*_\gamma(x) = f_\emptyset(x) + g_\emptyset(x)$ and the equality holds.
If $\{t : \operatorname{ran} x \subseteq (p_t, q_t]\} \neq \emptyset$, we can find $\{t_1 \prec t_2 \prec \ldots \prec t_m\} \in \mathcal{T}$ such that $t_1 \in S_\emptyset$ and $t_m = t_x$.
For every $t \in \mathcal{T}$ with $t \prec t_x$, $g_t(x) = 0$. Indeed, for every $s \in S_t^{p_t}$, $d^*_{\eta_s}(x) = e^*_{\eta_s} \circ P_{\{q_s+1\}}(x) = 0$ because $\operatorname{ran} x \subseteq (p_{t_x}, q_{t_x}] \subseteq (p_s, q_s]$.
So, we have that

$$\begin{aligned} e^*_\gamma(x) &= f_\emptyset(x) = \sum_{s \in S_\emptyset} b_s \varepsilon_s e^*_{\xi_s} \circ P_{(p_s, q_s]}(x) = b_{t_1} \varepsilon_{t_1} e^*_{\xi_{t_1}}(x) \\ &= b_{t_1} \varepsilon_{t_1} f_{t_1}(x) = b_{t_1} \varepsilon_{t_1} b_{t_2} \varepsilon_{t_2} e^*_{\xi_{t_2}} \circ P_{(p_{t_2}, q_{t_2}]}(x) = b_{t_1} b_{t_2} \varepsilon_{t_1} \varepsilon_{t_2} e^*_{\xi_{t_2}}(x) \\ &= b_{t_1} b_{t_2} \varepsilon_{t_1} \varepsilon_{t_2} f_{t_2}(x) = \ldots = \prod_{\emptyset \preceq s \preceq t_x} (\varepsilon_s b_s)(f_{t_x} + g_{t_x})(x) \end{aligned}$$

setting $\varepsilon_\emptyset = b_\emptyset = 1$. $\square$

Corollary 3.4. *If $\{t : \operatorname{ran} x \subseteq (p_t, q_t]\} \neq \emptyset$ and $(f_{t_x}, (p_{t_x}, q_{t_x}])$ is a maximal node, then $e^*_\gamma(x) = 0$.*

Proof. Let $(f_{t_x}, (p_{t_x}, q_{t_x}])$ be a maximal node. Then $f_{t_x}(x) = 0$ and $g_{t_x}(x) = 0$ and from Lemma 3.3 we deduce that $e^*_\gamma(x) = 0$. $\square$

4. The lower estimate.

Definition 4.1. *An $\phi \in W(\mathcal{A}_n, \overline{b})$ is said to be a proper functional if it admits a tree analysis $(\phi_t)_{t\in\mathcal{T}}$ such that for every non-maximal node $t \in \mathcal{T}$ the set $\{\phi_s : s \in S_t\}$ has at least two non-zero elements.*

We denote by $W_{pr}(\mathcal{A}_n, \overline{b})$ to be the subset of $W(\mathcal{A}_n, \overline{b})$ consisting of all proper functionals. For every $t \in \mathcal{T}$ it holds that $\phi_t = \sum\limits_{s\in S_t} b_s\phi_s$ with $\{b_s\}_{s\in S_t} \subseteq \{b_1, b_2, \dots, b_n\}$ and $b_\emptyset = 1$.

Lemma 4.2. *The set $W_{pr}(\mathcal{A}_n, \overline{b})$ 1-norms the space $\mathcal{T}(\mathcal{A}_n, \overline{b})$.*

Proof. We shall show that for every $\phi \in W(\mathcal{A}_n, \overline{b})$ there exists $g \in W_{pr}(\mathcal{A}_n, \overline{b})$ such that $|\phi(m)| \le g(m)$ $\forall m \in \mathbb{N}$. Since the basis is 1-unconditional the previous statement yields the result.

To this end, let $\phi \in W(\mathcal{A}_n, \overline{b})$. Then using a tree analysis $\{\phi_t\}_{t\in\mathcal{T}}$ of ϕ we easily see that for every $m \in \operatorname{supp} f$, there exists a maximal node $t_m \in \mathcal{T}$ with $\phi_{t_m} = \varepsilon_m e^*_m$ and $\phi(m) = \varepsilon_m \prod\limits_{t<t_m} b_t$.

For every $m \in \operatorname{supp}\phi$ we set $K_m = \{t \in \mathcal{T} : t < t_m \text{ and } \#S_t > 1\}$. Then it is easy to see that the functional $g = \sum\limits_{m\in\operatorname{supp}\phi} \left(\prod\limits_{t\in K_m} b_t\right) e^*_m$ is a functional belonging to $W_{pr}(\mathcal{A}_n, \overline{b})$. Moreover, since $b_t < 1$ for every $t \in \mathcal{T}$ we get that $|\phi(m)| \le g(m)$ $\forall m \in \mathbb{N}$. $\square$

Lemma 4.3. *Let $\phi \in W_{pr}(\mathcal{A}_n, \overline{b})$ and $l \in \mathbb{N}$. If* $\operatorname{maxsupp}\phi = l$, *then $h(\mathcal{T}_\phi) \le l$.*

Proof. Let θ_n be the amount of nodes at the n level of $\mathcal{T}_\phi$. Since ϕ is proper, it holds that $\theta_{n+1} > \theta_n$ for every $n \in \mathbb{N}$. Assume to the contrary that $h(\mathcal{T}_\phi) > l$, i.e. $h(\mathcal{T}_\phi) = l + k$ for some $k \in \mathbb{N}$. Then,

$$\theta_1 = 1,\ \theta_2 \ge 2,\ \dots\ ,\ \theta_{l+k} \ge l + k$$

Since, the $l + k$ level of $\mathcal{T}_\phi$ consists of functionals of the form e^*_i, we deduce that $\operatorname{maxsupp}\phi \ge l + k > l$, which leads to a contradiction. $\square$

Proposition 4.4. *Let $(x_k)_{k\in\mathbb{N}}$ be a normalized skipped block sequence in $\mathfrak{X}_r$ and $(q_k)_{k\in\mathbb{N}}$ a strictly increasing sequence of integers such that* $\operatorname{supp} x_k \subset$

(q_k+k, q_{k+1}). *Then, for every sequence of positive scalars* $(a_k)_{k\in\mathbb{N}}$ *and for every* $l\in\mathbb{N}$, *it holds that*

$$\left\|\sum_{k=1}^{l} a_k e_k\right\|_{\mathcal{T}(\mathcal{A}_n,\overline{b})} \le C\left\|\sum_{k=1}^{l} a_k x_k\right\|_\infty \tag{1}$$

where $(e_k)_{k\in\mathbb{N}}\subseteq \mathcal{T}(\mathcal{A}_n,\overline{b})$ *and* C *is an upper bound for the norms of the operators* i_m *in* $\mathfrak{X}_r$.

Proof. Let $\phi\in W(\mathcal{A}_n,\overline{b})$. From Lemma 4.2 we may assume that ϕ is proper. We will use induction on the height of the tree $\mathcal{T}_\phi$.

If $h(\mathcal{T}_\phi)=0$ (i.e. f is maximal), then ϕ is of the form $\phi=\varepsilon_k e_k^*$ with $\varepsilon_k=\pm1$. We observe that, $\left|\phi\left(\sum\limits_{k=1}^{l} a_k e_k\right)\right| = |a_k| = a_k$. From [3] Proposition 4.8, we can choose $\gamma\in\Gamma_{q_{k+1}-1}\backslash\Gamma_{q_k+k}$ such that $|x_k(\gamma)|\ge \dfrac{1}{C}\|x_k\| = \dfrac{1}{C}$. Then, $\left|\phi\left(\sum\limits_{k=1}^{l} a_k e_k\right)\right| = a_k \le C|a_k||x_k(\gamma)| = C|e_\gamma^*(a_k x_k)| \le C\left|e_\gamma^*\left(\sum\limits_{k=1}^{l} a_k x_k\right)\right|$.

We assume that for every $\phi\in W(\mathcal{A}_n,\overline{b})$ with $h(\mathcal{T}_\phi)=h>0$ and $\operatorname{maxsupp}\phi = l_0$, there exists $\gamma\in\Gamma$, such that:

(1) $\gamma\in\Gamma_{q_{l_0+1}+h}\backslash\Gamma_{q_{l_0+1}}$

(2) $h(\mathcal{T}_\phi)=h(\mathcal{F}_\gamma)\le l_0$

(3) $\left|\phi\left(\sum\limits_{k=1}^{l} a_k e_k\right)\right| \le C\left|\sum\limits_{k=1}^{l} a_k x_k(\gamma)\right|$ for every $l\ge l_0$

Observe that assumption (1) yields $x_{l_0} < \operatorname{rank}\gamma < x_{l_0+1}$, while assumption (2) gives us that $\operatorname{minsupp} x_{l_0+1} - \operatorname{maxsupp} x_{l_0} > h(\mathcal{T}_\phi)$. Indeed,

$$x_{l_0} < q_{l_0+1} < \operatorname{rank}\gamma \le q_{l_0+1}+h \le q_{l_0+1}+l_0 < q_{l_0+1}+(l_0+1) < x_{l_0+1}$$

$$\text{and } \operatorname{minsupp} x_{l_0+1} - \operatorname{maxsupp} x_{l_0} > l_0+1 > l_0 \ge h(\mathcal{F}_\gamma).$$

Let $\phi\in W(\mathcal{A}_n,\overline{b})$ with $h(\mathcal{T}_\phi)=h+1$, $l_0=\operatorname{maxsupp}\phi$ and let $(\phi_t)_{t\in\mathcal{T}}$ the tree analysis of ϕ. Then, ϕ is of the form $\phi=\sum\limits_{s\in S_\emptyset} b_s\phi_s$, $\#S_\emptyset\le n$. We observe that for every $s\in S_\emptyset$, $h(\mathcal{T}_{\phi_s})=h$. We set $p_1=1$, for every $s\in S_\emptyset\backslash\{1\}$ $p_s=\min\{q_k+k : k\in\operatorname{supp}\phi_s\}$ and for every $s\in S_\emptyset$, $r_s=q_{l_s+1}+h$ where $l_s=\operatorname{maxsupp}\phi_s$.

We next apply the inductive hypothesis to obtain $\xi_s \in \Gamma_{r_s} \setminus \Gamma_{q_{l_s}+1}$ with $h(\mathcal{T}_{\phi_s}) = h(\mathcal{F}_{\xi_s})$ such that

$$\begin{aligned} \left|\phi_s\left(\sum_{k=1}^{l} a_k e_k\right)\right| &= \left|\phi_s\left(\sum_{k\in \operatorname{supp}\phi_s} a_k e_k\right)\right| \leq C\varepsilon_s \sum_{k\in \operatorname{supp}\phi_s} a_k x_k(\xi_s) \\ &= C\varepsilon_s e^*_{\xi_s}\left(\sum_{k\in \operatorname{supp}\phi_s} a_k x_k\right) = C\varepsilon_s e^*_{\xi_s} \circ P_{(p_s,r_s]}\left(\sum_{k=1}^{l} a_k x_k\right), \end{aligned}$$

with ε_s such that $\varepsilon_s e^*_{\xi_s}\left(\sum\limits_{k\in \operatorname{supp}\phi_s} a_k x_k\right) = \left|\sum\limits_{k\in \operatorname{supp}\phi_s} a_k x_k(\xi_s)\right|$.

Let $\gamma \in \Gamma$ have analysis $\{p_s, r_s, \varepsilon_s e^*_{\xi_s}\}_{s\in S_\emptyset} \bigcup \{\eta_s\}_{s\in S_\emptyset\setminus\{1\}}$ where $\eta_s \in \Delta_{r_s+1}$. Observe that $\operatorname{rank}\xi_s \in (q_{l_s+1}, r_s] \subset (p_s, r_s]$. It is clear that for every $s \in S_\emptyset \setminus \{1\}$, $d^*_{\eta_s}\left(\sum\limits_{k=1}^{l} a_k x_k\right) = 0$. Indeed,

$$\operatorname{supp} x_{l_s} < q_{l_s+1} < q_{l_s+1} + (h+1) = r_s + 1 \leq q_{l_s+1} + (l_s+1) < \operatorname{supp} x_{l_s+1}.$$

Therefore,

$$\begin{aligned} \left|\phi\left(\sum_{k=1}^{l} a_k e_k\right)\right| &\leq \sum_{s\in S_\emptyset}\left|b_s\phi_s\left(\sum_{k\in \operatorname{supp}\phi_s} a_k e_k\right)\right| \\ &\leq C\sum_{s\in S_\emptyset} b_s\varepsilon_s e^*_{\xi_s} \circ P_{(p_s,r_s]}\left(\sum_{k=1}^{l} a_k x_k\right) \leq C\left|\sum_{k=1}^{l} a_k x_k(\gamma)\right| \end{aligned}$$

It is clear that $h(\mathcal{T}_\phi) = h(\mathcal{F}_\gamma) \leq l_0$ and $x_{l_0} < \operatorname{rank}\gamma < x_{l_0+1}$. $\square$

Corollary 4.5. *For every block sequence in $\mathfrak{X}_r$ there exists a further block sequence satisfying inequality* (1).

5. The upper estimate. Let $(y_l)_{l\in\mathbb{N}}$ be a normalized skipped block sequence in $\mathfrak{X}_r$. From Corollary 4.5, we can find a further block sequence of $(y_l)_l$,

still denoted by $(y_l)_l$, satisfying inequality (1).
Therefore, we have that

$$\left\|\sum_{l=1}^{m} y_l\right\|_\infty \geq \frac{1}{C}\left\|\sum_{l=1}^{m} e_l\right\|_{\mathcal{T}(\mathcal{A}_n,\overline{b})}$$

For every $j \in \mathbb{N}$, set $M_j = \{1,2,\ldots,n\}^j$. It is easily checked, after identifying M_j with $\{1,\ldots,n^j\}$ for every j, that the functional $f_j = \sum\limits_{s\in M_j}\left(\prod\limits_{i=1}^{j} b_{s_i}\right) e_s^*$ belongs to $W(\mathcal{A}_n,\overline{b})$ where s_i is the i-th coordinate of s, for each $i = 1,2,\ldots,n$ and $\sum\limits_{s\in M_j}\prod\limits_{i=1}^{j} b_{s_i} = \left(\sum\limits_{i=1}^{n} b_i\right)^j$. Using the fact that $\#M_j = n^j$, we obtain that

$$\left\|\sum_{l=1}^{n^j} e_l\right\|_{\mathcal{T}(\mathcal{A}_n,\overline{b})} = \left\|\sum_{s\in M_j} e_s\right\|_{\mathcal{T}(\mathcal{A}_n,\overline{b})} \geq f_j\left(\sum_{l=1}^{n^j} e_l\right) = \left(\sum_{i=1}^{n} b_i\right)^j.$$

Also, for every $m \in \mathbb{N}$ large enough we may find $j \in \mathbb{N}$ such that $n^{j+1} > m \geq n^j$. From the above and the unconditionality of the basis of the space $\mathcal{T}(\mathcal{A}_n,\overline{b})$, it follows that

$$\left\|\sum_{l=1}^{m} y_l\right\|_\infty \geq \frac{1}{C}\left\|\sum_{l=1}^{m} e_l\right\|_{\mathcal{T}(\mathcal{A}_n,\overline{b})} \geq \frac{1}{C}\left\|\sum_{l=1}^{n^j} e_l\right\|_{\mathcal{T}(\mathcal{A}_n,\overline{b})} = \left(\sum_{i=1}^{n} b_i\right)^j$$

We conclude that $\left\|\sum\limits_{l=1}^{m} y_l\right\|_\infty \overset{m\to\infty}{\longrightarrow} \infty$ as $\sum\limits_{i=1}^{n} b_i > 1$.

We next choose a further block sequence $(x_k)_{k\in\mathbb{N}}$ of $(y_l)_{l\in\mathbb{N}}$ with some additional properties. Let $\varepsilon > 0$ and choose a descending sequence $(\varepsilon_k)_k$ of positive reals such that $\left(\sum\limits_{k=1}^{\infty}\varepsilon_k\right) < \varepsilon$. We can also find an increasing sequence $(n_k)_k$ of positive integers and a sequence $(F_k)_k$ of successive subsets of $\mathbb{N}$ such that the following are satisfied:

(1) For every $k \in \mathbb{N}$, $\dfrac{1}{n_k} < \varepsilon_k$.

(2) For every $k \in \mathbb{N}$, $\left\|\sum\limits_{l\in F_k} y_l\right\| > n_k$. This is possible, due to the above notation.

We have thus constructed a normalized skipped block sequence $(x_k)_{k\in\mathbb{N}}$ of the form $x_k = \sum_{l\in F_k} \lambda_l y_l$, where $\lambda_l = \dfrac{1}{\left\|\sum_{l\in F_k} y_l\right\|}$. Notice that $|\lambda_l| < \varepsilon_k$ for every $l \in F_k$.

Let $\gamma \in \Gamma$ with tree analysis $\mathcal{F}_\gamma = \{\xi_t, (p_t, q_t]\}_{t\in\mathcal{T}}$.

For every $k \in \mathbb{N}$, we set $t_k = \max\{t : \operatorname{ran} x_k \subset (p_t, q_t]\}$. Notice that if for a given x_k, t_k is non-maximal, then there exist at least two immediate successors of t_k, say s_1, s_2 such that the corresponding intervals $(p_{s_1}, q_{s_1}]$, $(p_{s_2}, q_{s_2}]$ intersect $\operatorname{ran} x_k$. For later use we shall denote by $(p_{s_0}, q_{s_0}]$ the first interval in the natural order of disjoint segments of the natural numbers that intersects x_k. Notice that s_0 is not necessarily the first element of S_t.
For the pair γ, $(x_k)_{k\in\mathbb{N}}$ and for every $t \in \mathcal{T}$ we define the following sets: $D_t = \bigcup_{s\succeq t}\{k : s = t_k\}$, $K_t = D_t \setminus \bigcup_{s\in S_t} D_s \;= \{k : t = t_k\}$ and $E_t = \{s \in S_t : D_s \neq \emptyset\}$.

We now set $x_k = x'_k + x''_k + x'''_k$ where,

$$x'_k = x_k \,|_{(p_{s_0}, q_{s_0}]}, \quad x''_k = x_k \,|_{\bigcup_{s\in S_{t_k}, s\neq s_0}(p_s,q_s]} \text{ and } x'''_k = x_k - x'_k - x''_k.$$

Remark 5.1.

(1) The sets D_t,K_t,E_t are determined by the chosen pair $\gamma, (x_k)_k$. For a different pair, these sets may differ as well. For example, let $k \in K_t$, for the pair $\gamma, (x_k)_k$. Then $t = t_k$ for x_k. By the construction of x'_k, there exists $s_k \in S_t$ such that $x'_k = x_k \,|_{(p_{s_k}, q_{s_k}]}$. Thus, taking the pair $\gamma, (x'_k)_k$ the same k belongs to K_{s_k}.

(2) For every $k \in \mathbb{N}$, $|g_{t_k}(x_k)| \leq 2Cn\varepsilon_k$.
Indeed, from the definition of $(x_k)_{k\in\mathbb{N}}$ we have that

$$\begin{aligned}
|g_{t_k}(x_k)| &\leq \sum_{s\in S_{t_k}^{p_{t_k}}} |d^*_{\eta_s}(x_k)| \leq \sum_{s\in S_{t_k}^{p_{t_k}}} |e^*_{\eta_s} \circ P_{\{q_s+1\}}\left(\sum_{l\in F_k} \lambda_l y_l\right)| \leq \\
&\leq \sum_{s\in S_{t_k}^{p_{t_k}}} \|e^*_{\eta_s}\|\|P_{\{q_s+1\}}\||\lambda^s_l|\|y^s_l\| \leq \sum_{s\in S_{t_k}^{p_{t_k}}} 2C\varepsilon_k \leq \\
&\leq 2C\varepsilon_k(\sharp S_{t_k}) \leq 2Cn\varepsilon_k.
\end{aligned}$$

(3) It is obvious that $g_{t_k}(x_k) = g_{t_k}(x'''_k)$, $f_{t_k}(x'''_k) = 0$ and for every $t \prec t_k$, $g_t(x'''_k) = 0$.

Lemma 5.2. *For the pairs $\gamma, (x'_k)_{k\in\mathbb{N}}$ and $\gamma, (x''_k)_{k\in\mathbb{N}}$ it holds that $\#K_t + \#E_t \leq n$.*

Proof. Let $t \in \mathcal{T}$ and let $k \in K_t$.

We set $s_k = \max\{s \in S_t : (p_s, q_s] \cap \operatorname{ran} x'_k \neq \emptyset\}$. From the definition of t_k, notice that $\#S_t \geq 2$. It holds that $s_k \notin E_t$.

Indeed, from the definition of t_k, s_k we have that $(p_{t_k}, q_{t_k}] \cap \operatorname{ran} x'_k = \operatorname{ran} x'_k$ and $(p_{s_k}, q_{s_k}] \cap \operatorname{ran} x'_k = (p_{s_k}, q_{s_k}]$. Since $s_k \in S_{t_k}$, $(p_{s_k}, q_{s_k}] \subseteq (p_{t_k}, q_{t_k}]$. It follows that $(p_{s_k}, q_{s_k}] \subseteq \operatorname{ran} x'_k$.

Therefore, we can define a one-to-one map $G : K_t \to S_t \setminus E_t$, hence $\#K_t + \#E_t \leq \#S_t \leq n$.

The proof for the pair $\gamma, (x''_k)_{k\in\mathbb{N}}$ is similar. $\square$

Proposition 5.3. *Let $(x_k)_{k\in\mathbb{N}}$ be as above.Then for every $\gamma \in \Gamma$ there exist $\phi_1, \phi_2 \in W(\mathcal{A}_n, \overline{b})$ such that for every sequence $(a_k)_{k\in\mathbb{N}}$ of positive scalars, for every $l \in \mathbb{N}$ it holds that,*

$$\left|\sum_{k=1}^{l} a_k x_k(\gamma)\right| \leq \frac{1}{b_n}(\phi_1 + \phi_2)\left(\sum_{k=1}^{l} a_k e_k\right) + 2Cn\varepsilon \left(\sum_{k=1}^{l} a_k^r\right)^{\frac{1}{r}} \tag{2}$$

Proof. Let $\gamma \in \Delta_{q+1} with a(\gamma) = a \leq n$. Let $\mathcal{F}_\gamma = \{\xi_t, (p_t, q_t]\}_{t\in\mathcal{T}}$, where $\xi_\emptyset = \gamma$, be the tree analysis of γ. We may assume that $\bigcup_{k=1}^{l} \operatorname{ran} x_k \subset (p_\emptyset, q_\emptyset]$.

Claim. *For the pairs $\gamma, (x'_k)_{k\in\mathbb{N}}$ and $\gamma, (x''_k)_{k\in\mathbb{N}}$ there exist $\phi_1, \phi_2 \in W(\mathcal{A}_n, \overline{b})$ such that for every sequence of positive scalars $(a_k)_{k\in\mathbb{N}}$ and for every $l \in \mathbb{N}$, it holds that*

$$\left|f_\emptyset\left(\sum_{k=1}^{l} a_k x'_k\right)\right| \leq \frac{2C}{b_n}\phi_1\left(\sum_{k=1}^{l} a_k e_k\right) \tag{3}$$

$$\left|f_\emptyset\left(\sum_{k=1}^{l} a_k x''_k\right)\right| \leq \frac{2C}{b_n}\phi_2\left(\sum_{k=1}^{l} a_k e_k\right) \tag{4}$$

Proof of the Claim. We only prove inequality (3). The proof of inequality (4) requires the same arguments. We recall that

$f_t = \sum_{s\in S_t} b_s\varepsilon_s(f_s+g_s)\circ P_{(p_s,q_s]}$ for every $t\in\mathcal{T}$ non maximal. From the definition of $(x'_k)_{k\in\mathbb{N}}$, we have that $g_s\circ P_{(p_s,q_s]}(x'_k)=0$ for every $s\in S_t$.Therefore, $f_t\left(\sum_{k\in D_t} a_k x'_k\right) = \left(\sum_{s\in S_t} b_s\varepsilon_s f_s\circ P_{(p_s,q_s]}\right)\left(\sum_{k\in D_t} a_k x'_k\right)$. We will use backwards induction on the levels of the tree $\mathcal{T}$, i.e we shall show that for every $t\in\mathcal{T}$ there exists $\phi_1^t\in W(\mathcal{A}_n,\overline{b})$ with $\operatorname{supp}\phi_1^t\subseteq D_t$ such that

$$\left|f_t\left(\sum_{k\in D_t} a_k x'_k\right)\right| \le \frac{2C}{b_n}\phi_1^t\left(\sum_{k\in D_t} a_k e_k\right).$$

The first inductive step is similar to the general one and therefore we omit it. Let $0<h\le\max\{|t|:t\in\mathcal{T}\}$ and assume that the proposition has been proved for all t with $|t|=h$.

Let $t\in\mathcal{T}$ with $|t|=h-1$.Then we have the following cases:

(1) If f_t is a maximal node, $f_t\left(\sum_{k\in D_t} a_k x'_k\right)=0$, so there is nothing to prove. Indeed, $K_t=D_t$, therefore for every $k\in D_t$, from Corollary 3.4 $f_t(x'_k)=0$ since $t=t_k$.

(2) If f_t is a non-maximal node, then

$$f_t\left(\sum_{k\in D_t} a_k x'_k\right) = \left(\sum_{s\in S_t} b_s\varepsilon_s f_s\circ P_{(p_s,q_s]}\right)\left(\sum_{k\in D_t} a_k x'_k\right) =$$

$$= \sum_{s\in S_t} b_s\varepsilon_s f_s\left(\sum_{k\in D_s} a_k x'_k\right) + \sum_{k\in K_t}\left(\sum_{s\in S_t} b_s\varepsilon_s f_s\right)(a_k x'_k).$$

From the fact that, for every $k\in K_t$, $g_t(x'_k)=0$ we get that

$$|f_t(x'_k)| = |x'_k(\xi_t)| \le \|x'_k\| \le 2C = 2Ce_k^*(e_k).$$

Moreover, for $s\in E_t$ it holds that $|s|=h-1$. For every $k\in D_s$, from the inductive hypothesis we obtain

$$\left|\sum_{s\in S_t} b_s f_s(x'_k)\right| = |b_s f_s(x'_k)| \le b_s\frac{2C}{b_n}\phi_1^s(e_k).$$

with $\phi_1^s \in W(\mathcal{A}_n, \bar{b})$ and $\operatorname{supp}\phi_1^s \subseteq D_s$. We set $\phi_1^t = \left(\sum_{s\in E_t} b_s\phi_1^s + \sum_{k\in K_t} b_k e_k^*\right)$. From Lemma 5.2, it is easily checked that $\phi_1^t \in W(\mathcal{A}_n, \bar{b})$ and it holds that,

$$\left| f_t\left(\sum_{k\in D_t} a_k x_k'\right)\right| \leq \frac{2C}{b_n}\phi_1^t\left(\sum_{k\in D_t} a_k e_k\right). \qquad \square$$

Recall that

$$e_\gamma^*\left(\sum_{k=1}^{l} a_k x_k\right) = g_\emptyset\left(\sum_{k=1}^{l} a_k x_k\right) + f_\emptyset\left(\sum_{k=1}^{l} a_k x_k\right).$$

The fact that

$$\begin{aligned} g_\emptyset\left(\sum_{k=1}^{l} a_k x_k'\right) &= g_\emptyset\left(\sum_{k=1}^{l} a_k x_k''\right) \\ &= g_\emptyset\left(\sum_{k\in\{m:t_m\neq\emptyset\}} a_k x_k'''\right) = f_\emptyset\left(\sum_{k\in\{m:t_m=\emptyset\}} a_k x_k'''\right) = 0 \end{aligned}$$

implies the following:

$$\begin{aligned} \left| e_\gamma^*\left(\sum_{k=1}^{l} a_k x_k\right)\right| &\leq \left| g_\emptyset\left(\sum_{k\in\{m:t_m=\emptyset\}} a_k x_k'''\right)\right| + \left| f_\emptyset\left(\sum_{k=1}^{l} a_k x_k'\right)\right| \\ &+ \left| f_\emptyset\left(\sum_{k=1}^{l} a_k x_k''\right)\right| + \left| f_\emptyset\left(\sum_{k\in\{m:t_m\neq\emptyset\}} a_k x_k'''\right)\right| \end{aligned}$$

From Remark 5.1 we get that,

$$\left| g_\emptyset\left(\sum_{k\in\{m:t_m=\emptyset\}} a_k x_k'''\right)\right| \leq \sum_{k\in\{m:t_m=\emptyset\}} a_k |g_\emptyset(x_k''')| \leq 2Cn \sum_{k\in\{m:t_m=\emptyset\}} a_k\varepsilon_k.$$

From Lemma 3.3 and Remark 5.1 we have that,

$$\begin{aligned} \left| f_\emptyset\left(\sum_{k\in\{m:t_m\neq\emptyset\}} a_k x_k'''\right)\right| &\leq \sum_{k\in\{m:t_m\neq\emptyset\}} a_k (\prod_{t<t_k} b_t)|g_{t_k}(x_k''')| \leq \\ &\leq 2C\frac{1}{2}n \sum_{k\in\{m:t_m\neq\emptyset\}} a_k\varepsilon_k \leq 2Cn \sum_{k\in\{m:t_m\neq\emptyset\}} a_k\varepsilon_k. \end{aligned}$$

Finally, we conclude that

$$\begin{aligned}\left|\sum_{k=1}^{l} a_k x_k(\gamma)\right| &\leq 2Cn \sum_{k\in\{m:t_m=\emptyset\}} a_k\varepsilon_k + \frac{2C}{b_n}\phi_1\left(\sum_{k=1}^{l} a_k e_k\right) \\ &+ \frac{2C}{b_n}\phi_2\left(\sum_{k=1}^{l} a_k e_k\right) + 2Cn \sum_{k\in\{m:t_m\neq\emptyset\}} a_k\varepsilon_k \\ &\leq \frac{2C}{b_n}(\phi_1+\phi_2)\left(\sum_{k=1}^{l} a_k e_k\right) + 2Cn\sum_{k=1}^{l} a_k\varepsilon_k \\ &\leq \frac{2C}{b_n}(\phi_1+\phi_2)\left(\sum_{k=1}^{l} a_k e_k\right) + 2Cn\max\{a_k : k\in\mathbb{N}\}\left(\sum_{k=1}^{l}\varepsilon_k\right) \\ &\leq \frac{2C}{b_n}(\phi_1+\phi_2)\left(\sum_{k=1}^{l} a_k e_k\right) + 2Cn\varepsilon\left(\sum_{k=1}^{l} a_k^r\right)^{\frac{1}{r}}.\end{aligned}$$

where in the last inequality we used the fact that the ℓ_r norm dominates the c_0 norm. $\square$

Remark 5.4. From [4] Theorem I.4, we know that $\|\sum a_k e_k\|_{\mathcal{T}(\mathcal{A}_n,\bar{b})} \geq M\left(\sum a_k^r\right)^{\frac{1}{r}}$. This result and the previous Proposition, yield that

$$\left|\sum_{k=1}^{l} a_k x_k(\gamma)\right| \leq \frac{2C}{b_n}(\phi_1+\phi_2)\left(\sum_{k=1}^{l} a_k e_k\right) + \frac{2Cn\varepsilon}{M}\left\|\sum_{k=1}^{l} a_k e_k\right\|_{\mathcal{T}(\mathcal{A}_n,\bar{b})}.$$

For $\varepsilon = \dfrac{M}{nb_n}$,

$$\left|\sum_{k=1}^{l} a_k x_k(\gamma)\right| \leq \frac{6C}{b_n}\left\|\sum_{k=1}^{l} a_k e_k\right\|_{\mathcal{T}(\mathcal{A}_n,\bar{b})}.$$

Therefore,

$$\left\|\sum_{k=1}^{l} a_k x_k\right\|_{\infty} \leq \frac{6C}{b_n}\left\|\sum_{k=1}^{l} a_k e_k\right\|_{\mathcal{T}(\mathcal{A}_n,\bar{b})}. \tag{5}$$

Corollary 5.5. *For every block sequence in $\mathfrak{X}_r$ there exists a further block sequence satisfying inequality* (5).

6. The main result.

Proposition 6.1. *Let $(x_k)_{k\in\mathbb{N}}$ be a skipped block sequence in $\mathfrak{X}_r$ satisfying* $\operatorname{minsupp} x_{k+1} > \operatorname{maxsupp} x_k + k$ *and the conditions of Proposition* 5.3. *Then $(x_k)_{k\in\mathbb{N}}$ is equivalent to the basis of the Tsirelson space $\mathcal{T}(\mathcal{A}_n, \overline{b})$ for n and $\overline{b}$ determined as before.*

P r o o f. It is an immediate consequence of Propositions 4.4, 5.3 and Remark 5.4. □

Proposition 6.2. *The space $\mathcal{T}(\mathcal{A}_n, \overline{b})$ is isomorphic to ℓ_p for some $p \in (1, \infty)$.*

P r o o f. In a similar manner as in [4] Theorem I.4, one can see that for every normalized block sequence $(x_k)_k$ of the basis $(e_j)_j$ and for every scalar sequence (a_k) it holds that, $\|\sum a_k x_k\| \leq \dfrac{2}{b_n}\|\sum a_k e_k\|$. Zippin's Theorem [13] yields that $\mathcal{T}(\mathcal{A}_n, \overline{b})$ is isomorphic to some ℓ_p for some $p \in (1, \infty)$. □

Remark 6.3. An alternative proof could also be derived using the Results in Sections 4 and 5. Indeed, let $(y_l)_{l\in\mathbb{N}}$ be a skipped block sequence in $\mathfrak{X}_r$. Then, there exists a further block sequence $(x_k)_{k\in\mathbb{N}}$ satisfying simultaneously the assumptions of Corollaries 4.5 and 5.5. Therefore, $(x_k)_{k\in\mathbb{N}}$ satisfies the assumptions of Proposition 6.1.

Let's observe that every further block sequence $(z_k)_k$ of $(x_k)_k$ is also skipped block and satisfies Proposition 6.1, thus it is equivalent to the basis of the space $\mathcal{T}(\mathcal{A}_n, \overline{b})$. Hence, every block sequence $(z_n)_n$ of $(x_k)_k$ is equivalent to $(x_k)_k$. Zippin's theorem [13] yields that the space $< (x_k)_k >$ is isomorphic to some ℓ_p. Therefore, $\mathcal{T}(\mathcal{A}_n, \overline{b}) \cong \ell_p$ for some $p \in (1, \infty)$.

In order to determine the exact value of p, we need the following Proposition.

Proposition 6.4. *The space $\mathcal{T}(\mathcal{A}_n, \overline{b})$ is isomorphic to ℓ_r with $\dfrac{1}{r} + \dfrac{1}{r'} = 1$ and $\sum_{i=1}^{n} b_i^{r'} = 1$.*

P r o o f. Let us observe that for every $x \in c_{00}$, $\|x\| \leq \|x\|_r$. To see this, use induction on the cardinality of $\operatorname{supp} x$. If $|\operatorname{supp} x| = 1$, it is trivial. Assume that it holds for every $y \in c_{00}$ with $|\operatorname{supp} y| \leq n$ and let $x \in c_{00}$ with $|\operatorname{supp} x| = n+1$. Then either $\|x\| = \|x\|_\infty$ or $\|x\| = \sum_{i=1}^{n} b_i \|E_i x\|$ for some appropriate subsets

$E_1 < E_2 < \ldots < E_n$. In the first case, there is nothing to prove as for every $p \in [r, \infty)$ $\|x\|_\infty \leq \|x\|_p$. Therefore we only need to deal with the second case.

It suffices to observe that for every $i = 1, 2, \ldots, n$, the cardinality of $\operatorname{supp} E_i x$ is less than $\operatorname{supp} x$ and thus, using the inductive hypothesis along with $H\ddot{o}lder's$ inequality, we get that

$$\|x\| \leq \sum_{i=1}^{n} b_i \|E_i x\|_r \leq \left(\sum_{i=1}^{n} b_i^{r'}\right)^{\frac{1}{r'}} \left(\sum_{i=1}^{n} \|E_i x\|_r^r\right)^{\frac{1}{r}} = \|x\|_r.$$

By combining the preceding argument with Proposition 6.2, we conclude that $\mathcal{T}(\mathcal{A}_n, \overline{b})$ is isomorphic to ℓ_p for some $p \in [r, \infty)$.

For every $l \in \mathbb{N}$ set $M_l = \{1, 2, \ldots, n\}^l$. We have already mentioned that for every $l \in \mathbb{N}$ the functional $f_l = \sum\limits_{s \in M_l} \left(\prod\limits_{i=1}^{l} b_{s_i}\right) e_s^*$ belongs to $W(\mathcal{A}_n, \overline{b})$ where s_i is the i-th coordinate of s, for each $i = 1, 2, \ldots, n$ and $\sum\limits_{s \in M_l} \prod\limits_{i=1}^{l} b_{s_i} = (\sum_{i=1}^{n} b_i)^l$. We set $a_s = \prod\limits_{i=1}^{l} b_{s_i}$ and $x_l = \sum\limits_{s \in M_l} a_s^{\frac{r'}{r}} e_s$. It is easily seen that for every $l \in \mathbb{N}$, $\|x_l\| = 1$. Indeed,

$$\|x_l\| \leq \|x_l\|_r = \left(\sum_{s \in M_l} a_s^{r'}\right)^{\frac{1}{r}} = \left(\sum_{i=1}^{n} b_i^{r'}\right)^{\frac{l}{r}} = 1 = f_l(x_l) \leq \|x_l\|.$$

We claim that for $p' > r$ and every $\varepsilon > 0$ there exists $l \in \mathbb{N}$ such that $\|x_l\|_{p'} < \varepsilon$. If the claim holds we are done as p coincides with r.

Proof of the Claim. Notice that for $p' > r$, $\sum\limits_{i=1}^{n} b_i^{\frac{r'}{r} p'} = \sum\limits_{i=1}^{n} b_i^{r'(1+\delta)}$ for some $0 < \delta < 1$. But for every $i = 1, 2, \ldots, n$ $b_i < 1$, and therefore

$$\sum_{i=1}^{n} b_i^{r'(1+\delta)} < \sum_{i=1}^{n} b_i^{r'} = 1.$$

Thus, there exists $l \in \mathbb{N}$ such that $\left(\sum\limits_{i=1}^{n} b_i^{r'(1+\delta)}\right)^l < \varepsilon^{p'}$. Then for this l,

$$\|x_l\|_{p'} = \left(\sum_{s \in M_l} a_s^{\frac{r'}{r} p'}\right)^{\frac{1}{p'}} = \left(\sum_{s \in M_l} a_s^{r'(1+\delta)}\right)^{\frac{1}{p'}} = \left(\sum_{i=1}^{n} b_i^{r'(1+\delta)}\right)^{\frac{l}{p'}} < \varepsilon. \qquad \square$$

Theorem 6.5. *For every $r \in (1, \infty)$ the space $\mathfrak{X}_r$ is ℓ_r saturated.*

Proof. As it was mentioned in the above Remark, for every skipped block sequence in $\mathfrak{X}_r$ we can find a further block sequence $(x_k)_k$ such that the space $\overline{\langle (x_k)_k \rangle}$ is isomorphic to ℓ_r. □

Remark 6.6. From the previous Theorem, we deduce that the space $\mathfrak{X}_r$ is a separable $\mathcal{L}^\infty$ space which does not contain ℓ_1. Therefore, the results of D.Lewis-C.Stegall [10] and A. Pelczyński [12] yields that $\mathfrak{X}_r^*$ is isomorphic to ℓ_1. Alternatively, one can use the corresponding argument of D. Alspach [1] and show directly that (M_q) is a shrinking FDD for $\mathfrak{X}_r$. It then follows that $(e_\gamma^*)_{\gamma \in \Gamma}$ is a basis for $\mathfrak{X}_r^*$, equivalent to the usual ℓ_1-basis.

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