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NEW COEFFICIENT CONDITIONS FOR FUNCTIONS STARLIKE WITH RESPECT TO SYMMETRIC POINTS

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Abstract. We consider some familiar subclasses of functions starlike with respect to symmetric points and obtain sufficient conditions for these classes in terms of their Taylor coefficient. This leads to obtain several new examples of these subclasses.

1. Introduction and preliminaries. Let $\mathcal{A}$ denote the class of functions

$$f(z) = \sum_{n=1}^{\infty} a_n z^n, \ a_1 = 1, \tag{1.1}$$

which are analytic in the open unit disc $U = \{z : |z| < 1\}$. Let S denote the univalent subclass of $\mathcal{A}$, and S^* denote the subclass of S for which $f(U)$ is starlike with respect to the origin. It is well known that $f \in S^*$ if and only if

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$Re(zf'(z)/f(z)) > 0$ for $z \in U$. A function $f \in \mathcal{A}$ is starlike with respect to symmetric points in U if for every r close 1, $r < 1$ and every z_0 on $|z| = 1$ the angular velocity of $f(z)$ about $f(z_0)$ is positive at $z = z_0$ as z traverses the circle $|z| = r$ in the positive direction. This class was introduced and studied by Sakaguchi [9]. He proved that the condition is equivalent to

$$Re\frac{zf'(z)}{f(z)-f(-z)} > 0.$$

Recall the prominent subclasses studied in the theory of univalent functions, for $-1 \le B < A \le 1, 0 \le \beta < 1:$

$$\begin{aligned}
ST(A,B) &= \left\{ f \in \mathcal{A} : \frac{zf'(z)}{f(z)} \prec \frac{1+Az}{1+Bz}; \quad z \in U \right\} \\
S_s^*(\beta) &= \left\{ f \in \mathcal{A} : Re\frac{2zf'(z)}{f(z)-f(-z)} > \beta; \quad z \in U \right\} \\
S_s^*(A,B) &= \left\{ f \in \mathcal{A} : \frac{2zf'(z)}{f(z)-f(-z)} \prec \frac{1+Az}{1+Bz}; \quad z \in U \right\} \\
S_{s\beta} &= \left\{ f \in \mathcal{A} : \left| \arg\frac{2zf'(z)}{f(z)-f(-z)} \right| < \frac{\beta\pi}{2}; \quad z \in U \right\},
\end{aligned}$$

where '$\prec$' stands for the subordinate of two functions in $\mathcal{A}$. Set $S_s^* \equiv S_s^*(0)$.

Recently Wang et all [11], Elashwa and Thomas [1], Sudharasan et al. [10], Reddy et al. [7], and Parvatham and Premabai [6] have obtained various results concerning functions in $S_s^*(0), S_s^*(A,B), S_{s\beta}$.

Moreover Nezhmetdinov and Ponnusamy [2] has shown that any of the following inequalities

$$2 \le 3a_2 \le 4a_3 \le \cdots \le (n+1)a_n \le \cdots; \quad na_n \le 2 \ \text{ for } \ n \ge 2,$$

or

$$2/3 \ge a_2 \ge 2a_3 \ge 3a_4 \ge \cdots \ge (n-1)a_n \ge \cdots \ge 0; \quad na_n \ge a_2 \ \text{ for } \ n \ge 3$$

implies that $f(z) = z + \sum\limits_{n=2}^{\infty} a_n z^n$ is starlike.

Hence from the above inequalities it is easy to see that the function $f(z) = z + \frac{2}{3}z^2 + \frac{1}{3}z^3$ is starlike, but a simple calculation shows that this function is not starlike with respect to symmetric points. Indeed, for $z = e^{i\theta}$ we have

$$Re\frac{zf'(z)}{f(z)-f(-z)} = \frac{8\cos\theta\left(\cos\theta+\frac{2}{3}\right)}{3\left|1+\frac{1}{3}e^{2i\theta}\right|^2},$$

which is negative for $\theta = \frac{2\pi}{3}$, so that f is not starlike with respect to symmetric points.

Also Ozaki [4], Ponnusamy [5] and Obradovic and Ponnusamy [3] have obtained coefficient conditions for close-to-convex functions and starlike and convex functions.

Our main result is motivated by this problem. Find conditions on the Maclaurin coefficients of f which guarantee the corresponding f belongs to $S_s^*(0)$, $S_s^*(A,B)$, $S_{s\beta}$. We use the duality technique developed by Ruscheweyh [8] to obtain our results.

2. Main results.

Theorem 2.1. *If a function $f \in \mathcal{A}$ defined by* (1.1) *satisfies the condition*

$$\sum_{n=1}^{\infty} |(n+1)a_{2n+1} - na_{2n-1}| + |na_{2n+1} - (n-1)a_{2n-1}| + 2|na_{2n} - (n-1)a_{2(n-1)}| \le 1 \tag{2.1}$$

with $a_0 = 0$, then $f \in S_s^*$.

P r o o f. It is well known that the function $\frac{1+\omega}{1-\omega}$ maps $G = \{\omega : |\omega| = 1\}$ onto the imaginary axis. At $z = 0$, $\frac{2zf'(z)}{f(z)-f(-z)} = 1$, so that $f \in S_s^*$ if and only if

$$\frac{2zf'(z)}{f(z)-f(-z)} \neq \frac{1+x}{1-x} \quad \text{for all} \quad |x| = 1, z \in U,$$

or equivalently if and only if

$$\frac{f(z)}{z} * \left[\frac{1-x}{(1-z)^2} - \frac{1+x}{(1-z^2)}\right] \neq 0 \quad \text{for all} \quad |x| = 1, z \in U,$$

where $*$ stands for the Hadamard product of the two functions. Equivalently, this can be written as follows

$$-2x+\sum_{n=1}^{\infty}a_{2n+1}[2n(1-x)-2x]z^{2n}+(1-x)\sum_{n=1}^{\infty}2na_{2n}z^{2n-1}\neq 0.$$

After dividing the above inequality by $-x$ and multiplying by a non vanishing factor $1-z^2$, it can be easily seen that $f\in S_s^*$ if only

$$2+\sum_{n=1}^{\infty}a_{2n+1}[2n(t+1)+2]z^{2n}+(t+1)\sum_{n=1}^{\infty}2na_{2n}z^{2n-1}$$
$$-\sum_{n=1}^{\infty}a_{2n-1}[2(n-1)(t+1)+2]z^{2n}-(t+1)\sum_{n=1}^{\infty}2(n-1)a_{2(n-1)}z^{2n-1}\neq 0$$

Clearly, for all $|t|=1$, from (2.1) we find that the above inequality holds true and we complete the proof. □

Corollary 2.1. *Let the coefficients of* $f(z)=z+\sum\limits_{n=2}^{\infty}a_nz^n$ *satisfy either of the two conditions*

1) $1\leq 2a_3\leq 3a_5\leq 4a_7\leq 5a_9\leq\cdots\leq(n+1)a_{2n+1}\leq\cdots$ *and*

2) $0\leq a_2\leq 2a_4\leq 3a_6\leq 4a_8\leq\cdots\leq na_{2n}\leq\cdots$ *and*

3) $(2n+1)a_{2n+1}+2na_{2n}\leq 2,\ n\geq 1,$

or

1) $\frac{1}{2}\geq a_3\geq 2a_5\geq 3a_7\geq 4a_9\geq 5a_{11}\geq\cdots\geq(n-1)a_{2n-1}\geq\cdots\geq 0$ *and*

2) $0\leq a_2\leq 2a_4\leq 3a_6\leq 4a_8\leq\cdots\leq na_{2n}\leq\cdots$ *and*

3) $(2n+1)a_{2n+1}\geq 2na_{2n}+2a_3,\ n\geq 1,$

then $f\in S_s^*$.

It is interesting to state a counterpart of Corollary 2.1 for odd functions $f(z)$.

Corollary 2.2. *Let* $f\in\mathcal{A}$ *defined by* (1.1) *is an odd function satisfying either of the following conditions:*

$$1\leq 2a_3\leq 3a_5\leq 4a_7\leq 5a_9\leq\cdots\leq(n+1)a_{2n+1}\leq\cdots;\ (2n+1)a_{2n+1}\leq 2,\ n\geq 1,$$

or

$$\frac{1}{2} \geq a_3 \geq 2a_5 \geq 3a_7 \geq \cdots \geq (n-1)a_{2n-1} \geq \cdots \geq 0;\ (2n+1)a_{2n+1} \geq 2a_3,\ n \geq 1,$$

then $f \in S_s^*$.

Theorem 2.2. *Let* $a_0 = 0, a_1 = 1$ *and* $-1 \leq B < 0 < A \leq 1$. *If a function* $f \in \mathcal{A}$ *defined by* (1.1) *satisfies the condition*

$$\sum_{n=1}^{\infty} |[A - B(n+1)]a_{2n+1} - [A - B(2n-1)]a_{2n-1}| + |2na_{2n+1} - 2(n-1)a_{2n-1}| + 2|2na_{2n} - 2(n-1)a_{2(n-1)}| \leq A - B,$$

then $f \in S_s^*(A, B)$.

Proof. We note that $f \in S_s^*(A, B)$ if and only if

$$\frac{f(z)}{z} * \left[\frac{1+Bx}{(1-z)^2} - \frac{1+Ax}{(1-z^2)}\right] \neq 0 \quad \textit{for all} \quad |x| = 1, z \in U.$$

Now by proceeding the same line of the proof Theorem 2.1 we get our result and we omit the details. □

Corollary 2.3. *Suppose the coefficients of* $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ *satisfies either of the two conditions*

1) $(A-B) \leq (A-3B)a_3 \leq (A-5B)a_5 \leq \cdots \leq [A - B(2n+1)]a_{2n+1} \leq \cdots$ *and*

2) $0 \leq a_2 \leq 2a_4 \leq 3a_6 \leq 4a_8 \leq \cdots \leq na_{2n} \leq \cdots$ *and*

3) $[2n(1-B) + (A-B)]a_{2n+1} + 4na_{2n} \leq 2(A-B),\ n \geq 1,$

or

1) $\dfrac{A-B}{A-3B} \geq a_3 \geq 2a_5 \geq 3a_7 \geq 4a_9 \geq 5a_{11} \geq \cdots \geq na_{2n-1} \geq \cdots \geq 0$ *and*

2) $0 \leq a_2 \leq 2a_4 \leq 3a_6 \leq 4a_8 \leq \cdots \leq na_{2n} \leq \cdots$ *and*

3) $[2n(1-B) + A - B]a_{2n+1} \geq 4na_{2n} + 4a_3,\ n \geq 1,$

then $f \in S_s^*(A, B)$.

P r o o f. Under the first assumption we have

$$a_{2n+1} \geq \frac{A-B(2n-1)}{A-B(2n+1)} a_{2n-1} \geq \frac{n-1}{n} a_{2n-1}, \quad n \geq 1$$

and from the second assumption we have

$$na_{2n} \geq (n-1)a_{2(n-1)}, \quad n \geq 1.$$

Thus for any $N \geq 1$,

$$\sum_{n=1}^{N} |[A-B(2n-1)](a_{2n+1}-a_{2n-1}) - 2Ba_{2n+1}| + |2n(a_{2n+1}-a_{2n-1}) + 2a_{2n+1}|$$

$$+2|2n(a_{2n}-a_{2(n-1)})+2a_{2n}| = [2N(1-B)+(A-B)]a_{2n+1}+4Na_{2n} \leq 2(A-B),$$

and Theorem 2.1 implies that $f \in S_s^*(A,B)$. The second assertion is verified in a similar way. □

The proof of the following corollary is similar to Corollary 2.3 and we omit the details.

Corollary 2.4. *Let* $f \in \mathcal{A}$ *defined by* (1.1) *satisfies either of the two conditions*

1) $(A-B) \leq (A-3B)a_3 \leq (A-5B)a_5 \leq \cdots \leq [A-B(2n+1)]a_{2n+1} \leq \cdots$ *and*

2) $a_2 \geq 2a_4 \geq 3a_6 \geq 4a_8 \geq \cdots \geq na_{2n} \geq \cdots \geq 0$ *and*

3) $[2n(1-B)+(A-B)]a_{2n+1} - 4na_{2n} + 8a_2 \leq 2(A-B),\ n \geq 1,$

or

1) $\dfrac{A-B}{A-3B} \geq a_3 \geq 2a_5 \geq 3a_7 \geq 4a_9 \geq 5a_{11} \geq \cdots \geq na_{2n-1} \geq \cdots \geq 0$ *and*

2) $a_2 \geq 2a_4 \geq 3a_6 \geq 4a_8 \geq \cdots \geq na_{2n} \geq \cdots \geq 0$ *and*

3) $[2n(1-B)+A-B]a_{2n+1} + 4na_{2n} \geq 8a_2 + 4a_3,\ n \geq 1,$

then $f \in S_s^*(A,B)$.

In the following corollary we generalize the results obtained in [2] for odd functions.

Corollary 2.5. *Suppose that* $f \in \mathcal{A}$ *defined by* (1.1) *is an odd function satisfying either of the two conditions*

1) $(A-B) \le (A-3B)a_3 \le (A-5B)a_5 \le \cdots \le [A-B(2n+1)]a_{2n+1} \le \cdots$ *and*

$[2n(1-B)+(A-B)]a_{2n+1} \le 2(A-B),\ n \ge 1$

or

1) $\dfrac{A-B}{A-3B} \ge a_3 \ge 2a_5 \ge 3a_7 \ge 4a_9 \ge 5a_{11} \ge \cdots \ge na_{2n-1} \ge \cdots \ge 0$ *and*

$[2n(1-B)+(A-B)]a_{2n+1} \ge 4a_3,\ n \ge 1$

then $f \in ST(A,B)$.

By putting $B=-1$ and $a_{2n+1} = \dfrac{A+2n+1}{A+2n-1}a_{2n-1}$, $(n=1,2,3,\dots)$ with $a_1 = 1$ in the first part of Corollary 2.5 we obtain

Example 2.1. The function $f(z) = z + \sum_{n=1}^{\infty} \dfrac{1+A}{2n+1+A} z^{2n+1}$ belongs to $ST(A,-1)$.

Also by taking $B=-1$ and $a_3 = \dfrac{A+1}{A+3}$ and $a_{2n+1} = \dfrac{n-1}{n}a_{2n-1}$, $(n = 2,3,\dots)$ in the second part of Corollary 2.5 we obtain

Example 2.2. The function $f(z) = z + \dfrac{A+1}{A+3}\sum_{n=1}^{\infty} \dfrac{1}{n} z^{2n+1}$ belongs to $ST(A,-1)$.

Moreover by choosing $a_{2n+1} = \dfrac{A+1}{n(A+3)}$ and $a_{2n} = \dfrac{(A+1)^2}{4nN(A+3)}$, $(N \ge 1)$ it is easy to see that the conditions of the second part of Corollary 2.4 is satisfied, so we have

Example 2.3. The function

$$f(z) = z + \sum_{n=1}^{N} \frac{A+1}{n(A+3)} z^{2n+1} + \sum_{n=1}^{N} \frac{(A+1)^2}{4nN(A+3)} z^{2n}$$

belongs to $S_s^*(A,-1)$.

Theorem 2.3. *Let* $a_0 = 0, a_1 = 1$ *and* $0 < \alpha \le 1$. *If a function* $f \in \mathcal{A}$

defined by (1.1) *satisfies the condition*

$$\sum_{n=1}^{\infty}|(2n-e^{-i\pi\alpha})(a_{2n+1}-a_{2n-1})+(a_{2n+1}-a_{2n-1})|+|2na_{2n+1}-2(n-1)a_{2n-1}|$$
$$+2|2na_{2n}-2(n-1)a_{2(n-1)}|\le 2\sin\frac{\pi}{2}\alpha,$$

then $f\in S_{s\alpha}$.

Proof. It is well known that $f\in S_{s\alpha}$ if and only if

$$\frac{f(z)}{z}*\frac{1}{1-te^{\pm(i\alpha\pi/2)}}\left[\frac{(1+z)-(1-z)te^{\pm(i\alpha\pi/2)}}{(1-z)^2(1+z)}\right]\neq 0\quad (z\in U,t\ge 0)$$

Equivalently, this can be written as follows

$$1+\sum_{n=1}^{\infty}a_{2n+1}\left[\frac{2n+1-te^{\pm(i\alpha\pi/2)}}{1-te^{\pm(i\alpha\pi/2)}}\right]z^{2n}+\sum_{n=1}^{\infty}a_{2n}\left[\frac{2n}{1-te^{\pm(i\alpha\pi/2)}}\right]z^{2n-1}\neq 0. \tag{2.2}$$

Then after multiplying (2.2) by $1-z^2$, we need to maximize the modulus of

$$H_n(\omega)=\frac{A_n-B_n\omega}{1-\omega}\quad \textit{and}\quad G_n(\omega)=\frac{C_n}{1-\omega}$$

where $A_n=[(2n+1)a_{2n+1}-(2n-1)a_{2n-1}]$, $B_n=(a_{2n+1}-a_{2n-1})$, $C_n=2na_{2n}-2(n-1)a_{2(n-1)}$ and $\omega=te^{\pm(i\alpha\pi/2)}$, $t\ge 0$.

Note that the functions H_n and G_n maps the two rays $\omega=te^{\pm(i\alpha\pi/2)}$ onto two circles with radii

$$R'_n=\frac{1}{2}\csc\left(\frac{\pi\alpha}{2}\right)|A_n-B_n|,\quad R''_n=\frac{1}{2}\csc\left(\frac{\pi\alpha}{2}\right)|C_n|,$$

whereas their centers are at the points

$$P_n^{\pm}=\frac{1}{2}\left[(A_n+B_n)\pm i(A_n-B_n)\cot\left(\frac{\pi\alpha}{2}\right)\right]$$

and

$$T_n^{\pm}=\frac{1}{2}\left[C_n\pm iC_n\cot\left(\frac{\pi\alpha}{2}\right)\right],$$

respectively. Since, by our assumption, A_n,B_n and C_n are real, and so the required maxima are $|P_n^{\pm}|+R'_n, |T_n^{\pm}|+R''_n$, and the rest of the proof readily follows. □

3. Applications.

Theorem 3.1. *Let* $0 < A \leq 1$. *Suppose that* $a > 0$ *and, in addition,*

$$b > \max\left\{1 + a, a\frac{11 + 2A - A^2}{(A+1)^2}\right\}. \tag{3.1}$$

Then the function

$$\Phi(z) = z + \frac{4a}{(a+b)(1+A)}z^3 + \sum_{n=1}^{\infty}\frac{(a,n)}{(a+b,n)}z^{2n} \tag{3.2}$$

belongs to $S_s^*(A, -1)$.

P r o o f. Consider

$$\Phi(z) = z + A_3 z^3 + \sum_{n=2}^{\infty} A_{2n} z^{2n}.$$

Then we find that

$$(n-1)A_{2n-2} - nA_{2n} = \frac{(a, n-1)}{(a+b, n)}[(n-1)(b-1) - a]$$

which is nonnegative for all $n \geq 2$ if $b > 1 + a$.

Now we let $T_1 = \sum\limits_{n=2}^{\infty} |2nA_{2n} - 2(n-1)A_{2(n-1)}|$, and

$$T = |A_3(A+3) - (A+1)| + |2A_3| + |2A_2| + T_1.$$

Next, we evaluate T_1. We obtain

$$\begin{aligned}
T_1 &= 2\sum_{n=2}^{\infty}\frac{(a, n-1)}{(a+b, n)}[(n-1)(b-1) - a] \\
&= 2(b-1)\sum_{n=2}^{\infty}\frac{(a,n)}{(a+b,n)} - 2a(b-1)\sum_{n=2}^{\infty}\frac{(a,n-1)}{(a+b,n)} - 2a\sum_{n=2}^{\infty}\frac{(a,n-1)}{(a+b,n)} \\
&= 2(b-1)\frac{a}{a+b}\left[\frac{\Gamma(a+b+1)\Gamma(b-1)}{\Gamma(b)\Gamma(a+b)} - 1\right] - \frac{2ab}{a+b}\left[\frac{\Gamma(a+b+1)\Gamma(b)}{\Gamma(b+1)\Gamma(a+b)} - 1\right] \\
&= \frac{2a}{a+b}.
\end{aligned}$$

We note that, under the condition (3.2), we have

$$T = (A+1) - A_3(A+3) + 2A_3 + \frac{4a}{a+b} = A+1$$

and by Theorem 2.2 we get our result. □

Theorem 3.2. *Let* $0 < A \leq 1$. *Suppose that* $a, b > 0$ *and, in addition,*

$$\frac{\Gamma(a+b)}{\Gamma(a)\Gamma(b)} < \frac{(A+1)^2}{2(A+3)}. \tag{3.3}$$

Then the function

$$\Psi(z) = z + \frac{2\Gamma(a+b)}{\Gamma(a)\Gamma(b)(1+A)} z^3 + \sum_{n=1}^{\infty} \frac{(a,n)(b,n)}{(a+b,n)(1,n)} z^{2n} \tag{3.4}$$

belongs to $S_s^*(A, -1)$.

Proof. Write the function $\Psi(z)$ as

$$\Psi(z) = z + B_3 z^3 + \sum_{n=1}^{\infty} B_{2n} z^{2n}.$$

First we observe that, if $c = a+b$, then

$$\begin{aligned} nB_{2n} - (n-1)B_{2n-2} &= n\frac{(a,n)(b,n)}{(c,n)(1,n)} - (n-1)\frac{(a,n-1)(b,n-1)}{(c,n-1)(1,n-1)} \\ &= \frac{(a,n-1)(b,n-1)}{(a+b,n)(1,n-1)} ab, \end{aligned}$$

and, therefore we get

$$\begin{aligned} M_1 &:= \sum_{n=2}^{\infty} |2nB_{2n} - 2(n-1)B_{2(n-1)}| \\ &= 2ab \sum_{n=2}^{\infty} \frac{(a,n-1)(b,n-1)}{(a+b,n)(1,n-1)} \\ &= \frac{2\Gamma(a+b)}{\Gamma(a)\Gamma(b)} - \frac{2ab}{a+b}. \end{aligned}$$

Now if we let

$$M = |B_3(A+3) - (A+1)| + |2B_3| + |2B_2| + M_1,$$

then, by (3.4) and the definition of B_3 we find that

$$\begin{aligned} M &= (A+1) - \frac{2\Gamma(a+b)(A+3)}{\Gamma(a)\Gamma(b)(1+A)} + \frac{4\Gamma(a+b)}{\Gamma(a)\Gamma(b)(1+A)} + \frac{2ab}{a+b} + \frac{2\Gamma(a+b)}{\Gamma(a)\Gamma(b)} - \frac{2ab}{a+b} \\ &= 1 + A, \end{aligned}$$

and by Theorem 2.2 our proof is complete. □

By putting $A = 1, a = b = \dfrac{1}{2}$ in the Theorem 3.2 we have

Example 3.1. The function

$$\Psi(z) = z + \frac{1}{\pi}z^3 + \sum_{n=1}^{\infty} \frac{[(2n)!]^2}{2^{4n}(n!)^4} z^{2n}$$

belongs to $S_s^*(1, -1)$.

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