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LOW VOLATILITY OPTIONS AND NUMERICAL DIFFUSION OF FINITE DIFFERENCE SCHEMES

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Abstract. In this paper we explore the numerical diffusion introduced by two nonstandard finite difference schemes applied to the Black-Scholes partial differential equation for pricing discontinuous payoff and low volatility options. Discontinuities in the initial conditions require applying nonstandard non-oscillating finite difference schemes such as the exponentially fitted finite difference schemes suggested by D. Duffy and the Crank-Nicolson variant scheme of Milev-Tagliani. We present a short survey of these two schemes, investigate the origin of the respective artificial numerical diffusion and demonstrate how it could be diminished.

1. Introduction. In this paper we discuss the finite difference method for solving a partial differential one factor model problem. In details, nonstandard option pricing models characterized by discontinuities in terminal/boundary conditions and low volatility will be considered. Options characterized by

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discontinuities and low volatility represent a challenge both to finite centred-difference schemes, due to spurious oscillations arising close to discontinuities and to up-wind schemes due to artificial diffusion. Then the most popular Crank-Nicolson one is useless, as it is documented in Duffy (Wilmott Magazine, [1]) and other competitive methods have been introduced in the financial literature such as exponentially fitted schemes (Duffy, [2]) and Crank-Nicolson variant (Milev-Tagliani, [4]).

The aim of the paper is to demonstrate that undesired spurious oscillations could be avoided by applying finite difference schemes that are not standard but very often no attention is given that these schemes introduce artificial numerical diffusion. This requires an additional research of the finite difference scheme that includes exploration not only of the stability and convergence but also of the origin of the numerical diffusion and how it could be diminished.

In order to give an idea of the numerical problems related to discontinuities we consider options satisfying the Black-Scholes equation. If t is the time to expiry T of the contract, $0 \leq t \leq T$, the price $V\,(S,t)$ of the option satisfies

$$-\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{2}\,\sigma^2 S^2\,\frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{1}$$

endowed by its initial and boundary conditions. Here, risk free r and volatility σ satisfy $r = r(t,S)$ and $\sigma = \sigma(t,S)$. The options to be priced will be barrier options with continuous or discrete monitoring. In the latter case the discontinuities can be renewed at each monitoring date. In presence of an inaccurate finite difference scheme the discontinuity arises spurious oscillations close to barriers with solutions that have no financial meaning, for instance, a negative price. These oscillations, which remain well localized, don't reflect instability, but rather that the discontinuities produced by the barriers. As a result, the oscillations cannot decay fast enough. The mathematical reason rests on the spectrum of the matrix originating from the used finite difference scheme. In the spectrum we find complex or negative eigenvalues close to -1. Several remedies to spurious oscillations have been suggested in the financial literature (see Pooley *et al.* (2003), [6]). We focus on the ones based on applying special finite difference schemes. In particular, in Section 2 and 3 we consider

1. The implicit exponentially fitted schemes (see Duffy, [2], 2006);
2. The Crank-Nicolson variant scheme (see Milev-Tagliani, [4]), particularly devoted to the Black-Scholes equation;

Both schemes assure positive prices using M-matrix theory. Nevertheless, *in presence of low volatility, both schemes arise artificial diffusion, so that* *the solution is deteriorated.* In particular, exponentially fitted schemes introduce artificial diffusion given by $\frac{1}{2} rS\Delta S \frac{\partial^2 V}{\partial S^2}$, whilst the Crank-Nicolson variant $\frac{1}{8}\left(\frac{r\Delta S}{\sigma}\right)^2 \frac{\partial^2 V}{\partial S^2}$. Here, r denotes the risk free. In both cases the artificial diffusion is significant for high r values. This is demonstrated in Section 4 with numerical examples including truncated call and discrete double barrier knock-out call options. An accurate solution demands for a very small spatial step ΔS. As a consequence, the exponentially fitted schemes, although uniformly convergent, guarantee accurate results only under a severe spatial step restriction, loosing their peculiarity.

In conclusion, we give some final remarks for successful application and advantages of nonstandard finite difference schemes.

2. Exponentially Fitted Difference Schemes. Exponentially fitted schemes are stable, have good convergence properties and do not produce spurious oscillations. These schemes are implicit finite difference schemes that are characterized by a *fitting factor*.

Let write the Black-Scholes equation (1) in a more general form:

$$-\frac{\partial V}{\partial t} + \mu(S,t)\frac{\partial V}{\partial S} + \sigma(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\,V = 0 \tag{2}$$

When we set the coefficients in front of the derivatives by $\sigma(S,t) = \frac{1}{2}\sigma^2 S^2$, $\mu(S,t) = rS$, $b(S,t) = -r$, we obtain again the original Black-Scholes equation.

Now, let consider the operator L defined by:

$$L\,V \equiv -\frac{\partial V}{\partial t} + \mu(S,t)\frac{\partial V}{\partial S} + \sigma(S,t)\frac{\partial^2 V}{\partial S^2} + b(S,t)\,V$$

We replace the derivatives in $L\,V$ and we define the fitted operator L_k^h by

$$L_k^h\,U_j^n \equiv -\frac{U_j^{n+1} - U_j^n}{k} + \mu_j^{n+1}\frac{U_{j+1}^{n+1} - U_{j-1}^{n+1}}{2h} + \rho_j^{n+1}\frac{\delta_x^2\,U_j^{n+1}}{h^2} + b_j^{n+1}\,U_j^{n+1}$$

where h and k are fixed space and time step.

The fitting factor ρ defined by Duffy is:

$$\rho_j^{n+1} \equiv \frac{\mu_j^{n+1}\,h}{2}\coth\frac{\mu_j^{n+1}\,h}{2\,\sigma_j^{n+1}} \tag{3}$$

This factor is identically equal to 1 in the centred difference scheme. The corresponding finite difference equation is

$$A\,U^{n+1} = U^n \tag{4}$$

where A is the following iteration matrix:

$$A = [a_{i,j}] = \text{tridiag}\left\{\left(-\frac{\rho_j^n}{h^2} + \frac{\mu_j^n}{2h}\right)k;\ \left(\frac{2\rho_j^n}{h^2} - b_j^n + \frac{1}{k}\right)k;\ -\left(\frac{\rho_j^n}{h^2} + \frac{\mu_j^n}{2h}\right)k\right\}$$

with $a_{i,i+1} < 0$, $a_{i+1,i} < 0$ and $a_{i,i} > 0$. Then the iteration matrix A is an irreducible diagonally dominant tridiagonal *M-matrix* (see Ortega, 6.2.3, p. 104 and 6.2.17, p. 110, [5]). Then it follows $A^{-1} > 0$. From A being diagonally dominant it follows (Windisch, [9])

$$\|A^{-1}\|_\infty \le \frac{1}{1-kb} = \frac{1}{1+kr} < 1 \tag{5}$$

The positivity of the numerical solution follows from the induction

$$U^n = A^{-1}\,U^{n-1} = (A^{-1})\,(A^{-1}\,U^{n-2}) = \cdots = (A^{-1})^{\,n}\,U^0 > 0$$

Using the property (5) for the norm $||A^{-1}||_\infty$ we verify that the scheme satisfies the discrete maximum principle

$$\|U^{n+1}\|_\infty = \|A^{-1}\,U^n\|_\infty = \|A^{-1}\|_\infty \|U^n\|_\infty \le 1.\|U^n\|_\infty \le \|U^n\|_\infty$$

Moreover, if $V(S,t)$ and U_j^n are the analytical and discrete solution of equation (1) it is true the following result

$$|V(S_j, t_n) - U_j^n| \le c(h+k) \tag{6}$$

where c is a constant that is independent of h, k and σ. This result shows that convergence is assured regardless of the size of $\sigma(S,t)$.

Thus, advantages of the fitted scheme with *fitting factor* (3) are:

- It is uniformly stable for all values of h, k and σ.
- It is oscillation-free.

Another advantage the scheme is that it is a powerful scheme for the Black-Scholes equation (1) in the case of pure convection/drift, i.e. when $\sigma(S,t) \to 0$

When $\sigma \to 0$ we use the following formula:

$$\lim_{\sigma\to 0} \rho = \lim_{\sigma\to 0} \frac{\mu\, h}{2} \coth \frac{\mu\, h}{2\,\sigma} = \begin{cases} \frac{\mu\, h}{2} & \text{if } \mu > 0 \\ -\frac{\mu\, h}{2} & \text{if } \mu < 0 \end{cases}$$

Inserting these results into the scheme, gives the well known first-order implicit upwind schemes that are stable and convergent.

$$-\frac{U_j^{n+1} - U_j^n}{k} + \mu_j^{n+1} \frac{U_{j+1}^{n+1} - U_j^{n+1}}{2h} + b_j^{n+1}\, U_j^{n+1} = 0\,, \quad \text{for } \mu > 0$$

$$-\frac{U_j^{n+1} - U_j^n}{k} + \mu_j^{n+1} \frac{U_j^{n+1} - U_{j-1}^{n+1}}{2h} + b_j^{n+1}\, U_j^{n+1} = 0\,, \quad \text{for } \mu < 0$$

The latter, through a standard analysis of consistency, introduces numerical diffusion $\frac{1}{2}\mu(S,t)h\frac{\partial^2 V}{\partial S^2}$, so that the above up-wind scheme solves, rather that the hyperbolic equation $-\frac{\partial V}{\partial t} + \mu(S,t)\,\frac{\partial V}{\partial S} - b(S,t)V = 0$, the parabolic one

$$-\frac{\partial V}{\partial t} + \mu(S,t)\,\frac{\partial V}{\partial S} + \frac{1}{2}\mu(S,t)h\,\frac{\partial^2 V}{\partial S^2} - b(S,t)V = 0 \tag{7}$$

The numerical diffusion is significant whenever r is large or h is not small enough. Accurate solution can be given by implicit scheme provided h is chosen small. As a consequence, such a scheme loses its peculiarity consisting in the uniformly convergence above quoted.

3. The Crank-Nicolson variant. Milev-Tagliani, [4] introduced the so called Crank-Nicolson variant as a scheme spurious oscillations free. The scheme is the classical Crank-Nicolson except for the discretization of the reaction term $-rV$ in the Black-Scholes equation. Such a term is discretized by six adjacent nodes, so that the reaction term $V(t + \frac{\Delta t}{2})$ is replaced with

$$\begin{aligned} V(t + \frac{\Delta t}{2}) = \omega_1(U_{j-1}^n + U_{j+1}^n) + \left(\frac{1}{2} - 2\omega_1\right) U_j^n \\ + \omega_2(U_{j-1}^{n+1} + U_{j+1}^{n+1}) + \left(\frac{1}{2} - 2\omega_2\right) U_j^{n+1} \end{aligned} \tag{8}$$

with ω_1 and ω_2 parameters to be determined. The finite difference approximation provides the difference equation

$$PU^{n+1} = NU^n$$

with P and N the following tridiagonal matrices:

$$P = \text{tridiag}\Big\{ + r\omega_2 + \frac{r}{4}\frac{S_j}{\Delta S} - \Big(\frac{\sigma}{2}\frac{S_j}{\Delta S}\Big)^2 \;;\; \frac{1}{\Delta t} + \frac{1}{2}\Big(\frac{\sigma S_j}{\Delta S}\Big)^2 + r\Big(\frac{1}{2} - 2\omega_2\Big); \\ + r\omega_2 - \frac{r}{4}\frac{S_j}{\Delta S} - \Big(\frac{\sigma}{2}\frac{S_j}{\Delta S}\Big)^2 \Big\}$$

$$N = \text{tridiag}\Big\{ - r\omega_1 - \frac{r}{4}\frac{S_j}{\Delta S} + \Big(\frac{\sigma}{2}\frac{S_j}{\Delta S}\Big)^2 \;;\; \frac{1}{\Delta t} - \frac{1}{2}\Big(\frac{\sigma S_j}{\Delta S}\Big)^2 - r\Big(\frac{1}{2} - 2\omega_1\Big); \\ - r\omega_1 + \frac{r}{4}\frac{S_j}{\Delta S} + \Big(\frac{\sigma}{2}\frac{S_j}{\Delta S}\Big)^2 \Big\}$$

The parameters ω_1 and ω_2 are chosen according to the following criteria:

- P is irreducibly diagonally dominant and thus P is an M-matrix, so that $P^{-1} > 0$;
- N has positive entries.

The above requirements are fulfilled if

$$\omega_1 = \omega_2 = -\frac{r}{16\sigma^2} \qquad \text{and} \qquad \Delta t < \frac{1}{r(\frac{1}{2} - 2\omega_1) + \frac{1}{2}(\sigma M)^2} \tag{9}$$

where M denotes the number of nodes in S-direction.

By combining $N \geq 0$ and $P^{-1} > 0$ then the numerical solution $U^{n+1} = P^{-1}NU^n = (P^{-1}N)^n U^0$ is positive, since $U^0 \geq 0$. Then, under (9) the scheme is *positivity-preserving.* Under condition (9) the scheme satisfies the discrete maximum principle. Indeed by combining the norms $\|N\|_\infty = \dfrac{1}{\Delta t} - \dfrac{r}{2}$ and $\|P^{-1}\|_\infty \leq \left(\dfrac{1}{\Delta t} + \dfrac{r}{2}\right)^{-1}$, (see Windisch, 1989, [9]), then we have

$$\|U^{n+1}\|_\infty = \|(P^{-1}N)U^n\|_\infty \\ = \|P^{-1}\|_\infty \|N\|_\infty \|U^n\|_\infty \leq \frac{\frac{1}{\Delta t} - \frac{r}{2}}{\frac{1}{\Delta t} + \frac{r}{2}} \|U^n\|_\infty \leq \|U^n\|_\infty$$

The scheme has a discretization error $O(\Delta S^2, \Delta t^2)$. When σ is small the artificial diffusion is due mainly to the term coming from the discretization of the term

$-rV$. Indeed, from a standard analysis of consistency, the artificial diffusion amounts to $\frac{1}{8}\left(\frac{r}{\sigma}\Delta S\right)^2 \frac{\partial^2 V}{\partial S^2}$, so that we are led to solve the diffusion equation

$$-\frac{\partial V}{\partial t} + rS\,\frac{\partial V}{\partial S} + \frac{1}{8}\Big(\frac{r}{\sigma}\Delta S\Big)^2 \frac{\partial^2 V}{\partial S^2} - rV = 0 \tag{10}$$

When $\sigma S \to 0$ the scheme, even if second order accurate, requires a very small ΔS, equivalently the term $\frac{1}{8}\left(\frac{r}{\sigma}\Delta S\right)^2$ has to become insignificant, and then from (9) small $\Delta t \sim 8\left(\frac{\sigma}{r}\right)^2$. Then the comparison between the two above schemes is led to compare the two terms $\frac{1}{2} rS\Delta S \frac{\partial^2 V}{\partial S^2}$ and $\frac{1}{8}\left(\frac{r}{\sigma}\Delta S\right)^2 \frac{\partial^2 V}{\partial S^2}$ respectively.

4. Numerical results. In this section classical and nonstandard finite difference schemes are applied and compared through numerical experiments involving options having both *discontinuous payoff and low volatility.* We make the following test with the Crank-Nicolson scheme, standard fully implicit scheme, the implicit exponentially fitted scheme of Duffy and the Crank-Nicolson variant scheme:

Test: Truncated call and discrete barrier options are explored when $\sigma^2 \ll r$:

1. The Crank-Nicolson scheme produces undesired spurious oscillations in the numerical solution for every choice of steps ΔS and Δt, see Fig. 1. The same is observed for the standard fully implicit scheme, see Fig. 2.

2. Exponentially fitted schemes of Duffy arise numerical diffusion $\frac{1}{2} rS\Delta S$ which depends on r, S. Thus, we make test for:

$$r = \begin{cases} 0.01 & \text{i.e. } r \text{ is small} \\ 0.5 & \text{i.e. } r \text{ is large} \end{cases}$$

3. Crank-Nicolson variant scheme arises numerical diffusion $\frac{1}{8}(\frac{r}{\sigma}\Delta S)^2$ which depends on r, σ and S. Thus, we make test for:

$$r = \begin{cases} 0.01 & \text{i.e. } r \text{ is small} \\ 0.5 & \text{i.e. } r \text{ is large} \end{cases}$$

4. We will demonstrate that the numerical diffusion introduced by the nonstandard exponential scheme of Duffy and the Crank Nicolson variant one can be diminished by an appropriate choice of the grid steps, see Fig. 5.

Formally, we have divided option with a *discontinuous payoff* in two classes:

1. Options with *continuous monitoring*, i.e. at any instant, but having *themselves discontinuous payoff* such as *digital options*, *truncated options*, ect.

2. Options with *discrete monitoring*, due to the fact that one trading year is considered to consist of 250 working days and a week of 5 days.[1]

Thus, we explore one example of the two classes, i.e. a *truncated call option* and a *discrete double knock-out call option.* Here are the two definitions:

Definition 4.1. *A truncated payoff call option has payoff that is obtained by truncating the payoff of a vanilla call as follows:*

$$f[S(T)] = \begin{cases} S(T) - K & \textit{if } S(T) \in [K, U], \textit{ and} \\ 0 & \textit{otherwise} \end{cases}$$

The stock price U acts as a barrier, canceling the option if $S(T) > U$. In this case, the option is canceled only if the barrier is crossed at maturity. Nothing happens if the barrier is crossed before the maturity.

Definition 4.2. *A discrete double barrier knock-out call option is an option with a payoff condition equal to* $\max(S - K, 0)$ *which expires worthless if before the maturity the asset price has fallen outside the barrier corridor* $[L, U]$ *at the prefixed monitoring dates: at these dates the option becomes zero if the asset falls out of the corridor. If one of the barriers is touched by the asset price at the prefixed dates then the option is canceled, i.e. it becomes zero, but the holder may be compensated by a rebate payment.*

Example 4.1. Let price a truncated call option defined in Definition 4.1 with parameters $r = 0.05$, $\sigma = 0.001$, $T = 5/12$, $U = 70$ and $K = 50$.

In this example we will demonstrate how the *financial provisions* of the contract can *affect strongly the reliability of the numerical solution* by reflecting the *terminal conditions* of the respective Black-Scholes differential equation (1) for the particular derivative. We have chosen to valuate a truncated payoff call option because the payoff function is a slight variation of the payoff function of the standard call option, i.e. the truncated call option is canceled only if the barrier is crossed at maturity. Then probably the standard finite differences schemes that are most frequently used in Finance such as the Crank-Nicolson scheme or

[1]Thus, for one year $T = 1$ the application of barriers occurs with a time increment of 0.004 daily and 0.02 weekly and we have respectively 250 and 50 monitoring dates.

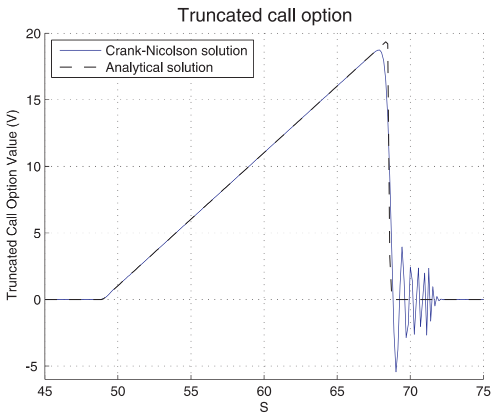

Fig. 1. Numerical oscillations in the solution of the standard Crank-Nicolson scheme when a truncated call option is priced, $\sigma^2 \ll r$. Parameters: $r = 0.05$, $\sigma = 0.001$, $T = 5/12$, $U = 70$, $K = 50$, $S_{\max} = 140$, $\Delta S = 0.05$, $\Delta t = 0.01$

the fully implicit one should work successfully. However, as it is illustrated on Fig. 1, the numerical solution oscillates in neighborhood of the barrier $U = 70$ where there is a discontinuity in the payoff function.

Mathematically, the undesired spurious oscillations stem from the combined effect of the *discontinuous payoff and low volatility.* In this example the volatility takes an extremely low value, i.e. $\sigma = 0.001$ and we have that $\sigma^2 \ll r$. Thus, we confirm that the specific nature of some problems makes the *direct application* of the *Crank-Nicolson method inefficient*, see also Milev-Tagliani, [4]. Giles and Carter (2006) has demonstrated that in case of non-smooth initial data such as the Black-Scholes equation for plain vanilla options there is convergence of the numerical solution in the L^2 norm, but not in *the supremum norm* that is most relevant in Finance, [3].

Example 4.2. Let price a ***discrete double barrier knock-out call option*** having a discontinuous payoff defined by conditions (11)–(13) and for which the strike price is 100, the volatility is 0.1% per annum, the option has twelve months remaining to maturity, the risk-free rate is 5% per annum (compounded continuously), the lower barrier is placed at 95, and the upper barrier is at 110.

Here are the initial and boundary conditions of the Black-Scholes partial differential equation (1) in case of a discrete double barrier knock-out call options:

$$V(S,0) = (S-K)^{+} 1_{[L,U]}(S) \tag{11}$$

$$V(S,t) \to 0 \ \text{ as } \ S \to 0 \ \text{ or } \ S \to \infty \tag{12}$$

with updating of the initial condition at the monitoring dates $t_i, i = 1, \ldots, F$:

$$V(S,t_i) = V(S,t_i^-)1_{[L,U]}(S), \qquad 0 = t_0 < t_1 < \cdots < t_F = T \tag{13}$$

where $\mathbf{1}_{[L,U]}(x)$ is the indicator function, i.e., $\mathbf{1}_{[L,U]} = 1$ if $S \in [L,U]$ and $\mathbf{1}_{[L,u]} = 0$ if $S \notin [L,U]$. Here, we do not explore options that have a rebate payment. We have seen in the previous example that the standard Crank-Nicolson scheme could not manage with options with discontinuous payoff and low volatility values. Then, very often a less accurate finite difference schemes are preferred in computational finance, i.e. $O(\Delta S, \Delta t)$, which usually in practice prevent from undesired spurious oscillations and guarantee a positive solution.

For example, the let consider the standard fully implicit scheme, leading to a difference equation (here $\dfrac{\partial V}{\partial S}$ is discretized through a centered difference):

$$AV^{n+1} = V^n, \text{ where}$$

$$A = \text{tridiag}\Big\{-\frac{\Delta t}{2}\Big[\Big(\frac{\sigma S_j}{\Delta S}\Big)^2 - r\frac{S_j}{\Delta S}\Big]; 1+\Delta t\Big[\Big(\frac{\sigma S_j}{\Delta S}\Big)^2 + r\Big]; -\frac{\Delta t}{2}\Big[\Big(\frac{\sigma S_j}{\Delta S}\Big)^2 + r\frac{S_j}{\Delta S}\Big]\Big\}$$

If the condition $\sigma^2 > r$ is violated then positivity of the solution is not guaranteed, while some $\lambda_i(A^{-1})$ may become complex. As a consequence, spurious oscillations and negative values of V can occur, as it is illustrated in Figure 2.

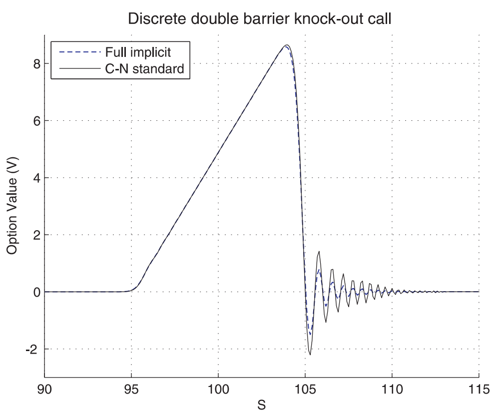

Fig. 2. Option pricing just before first monitoring date $t_1 = T$. The solution is obtained using a fully implicit scheme with $\Delta S = 0.025$ and $\Delta t = 0.001$. Parameters: $L = 90$, $K = 100$, $U = 110$, $r = 0.05$, $\sigma = 0.001$, $T = 1$

Then, when $\sigma^2 < r$ (as occurs in certain regions of the grid for stochastic volatility models) the standard fully implicit scheme may fail.

Under the condition $\sigma^2 \ll r$ more suitable schemes are the discussed schemes in this article, i.e. the exponentially fitted schemes of Duffy or the Crank-Nicolson variant scheme proposed by Milev-Tagliani. Here, it is important to remember that these schemes should be applied having in mind the *numerical diffusion* of the solution which depends on the financial parameters r and σ as well as the applied space step ΔS. This will be demonstrated in the following examples.

Example 4.3. Let apply the implicit exponentially fitted finite difference scheme of Duffy that is presented in Section 2 to price a truncated payoff call option defined in Definition 4.1 for different values of the interest rate r and with fixed parameters $\sigma = 0.001$, $T = 5/12$, $U = 70$ and $K = 50$.

In this example, we demonstrate that the exponentially fitted schemes of Duffy arise numerical diffusion $\dfrac{1}{2}rS\Delta S$ which depends on r, S. Having in mind the term $\dfrac{1}{2}rS\Delta S$, it is obvious that the higher values the interest rate parameter r takes the more the numerical diffusion should be. This is confirmed by the numerical solutions displayed on Fig. 3, where apply the scheme for a small value of r, i.e. $r = 0.01$, and a high value of r, i.e. $r = 0.5$, respectively on the upper and lower graphic. Similar analysis and conclusions are done in the following example when the Crank-Nicolson variant scheme is used instead.

Example 4.4. Let apply the Crank-Nicolson variant scheme of Milev-Tagliani that is presented in Section 3 to price a truncated payoff call option

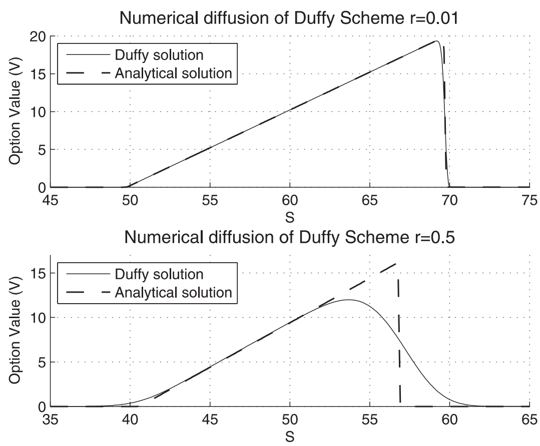

Fig. 3. Numerical diffusion of the implicit exponentially fitted scheme of Duffy for different values of r when a truncated call option is priced. Parameters: $\sigma = 0.001$, $T = 5/12$, $U = 70$, $K = 50$, $S_{\max} = 140$, $\Delta S = 0.05$, $\Delta t = 0.01$

defined in Definition 4.1 for different values of the interest rate r and with fixed parameters $\sigma = 0.001$, $T = 5/12$, $U = 70$ and $K = 50$.

In this example we demonstrate that the Crank-Nicolson variant scheme arises numerical diffusion $\frac{1}{8}\left(\frac{r}{\sigma}\Delta S\right)^2$ which depends on r, σ and S. Obviously, as in the previous case when we have applied the scheme of Duffy, the higher interest rate r is the more the numerical solution should be deteriorated. Here, the numerical diffusion $\frac{1}{8}\left(\frac{r}{\sigma}\Delta S\right)^2$ depends directly also on the volatility parameter σ and we have chosen a very low volatility value, i.e. $\sigma = 0.001$. Thus, comparing the numerical solutions obtained by applying the Crank-Nicolson variant scheme for a small value of r, i.e. $r = 0.01$ and a high value of r, i.e. $r = 0.5$, respectively the upper and lower graphic of Fig. 4, we conclude that:

1. For low volatility values the numerical diffusion is more influent when the interest rate parameter r is increased from $r = 0.01$ to $r = 0.5$.

2. The proposed variant of the Crank-Nicolson scheme works successfully in the case $\sigma^2 \ll r$ when the standard Crank-Nicolson solution suffers of undesired oscillations as we have seen in example and .

Indeed, the numerical solution in both cases $r = 0.01$ and $r = 0.5$ is positive and free of oscillations. It remains to be showed that the numerical diffusion of the Crank-Nicolson variant scheme could be diminished also for high values of the interest rate r, i.e. the second case $r = 0.5$. This could be done by choosing an appropriate choice of the grid steps as in the following example.

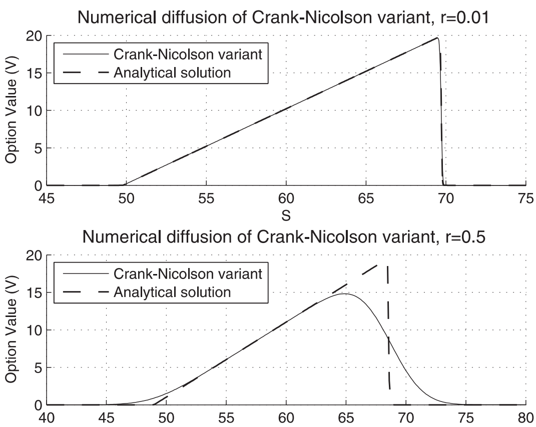

Fig. 4. Numerical diffusion of the Crank-Nicolson variant scheme for different values of r when a truncated call option is priced. Parameters: $\sigma = 0.001$, $T = 5/12$, $U = 70$, $K = 50$, $S_{\max} = 140$, $\Delta S = 0.05$, $\Delta t = 0.01$

Example 4.5. Let price a truncated call option defined in Definition 4.1 with parameters $r = 0.5$, $\sigma = 0.001$, $T = 5/12$, $U = 70$ and $K = 50$.

As in Example 4.4, we apply the Crank-Nicolson variant scheme but using a smaller space step $\Delta S = 0.025$, i.e. two time smaller than $\Delta S = 0.05$. By comparing the numerical solution on Fig. 5 with those obtained for $\Delta S = 0.05$ and the same time step $\Delta S = 0.01$ on the lower graphic on Fig. 4, the numerical diffusion is diminished taking into account the exact analytical solution of truncated payoff call option. If again we take a smaller space and time step, i.e. for instance $\Delta S = 0.01$, $\Delta t = 0.001$, the numerical solutions of the exponential scheme of Duffy and the Crank-Nicolson variant scheme are practically indistinguishable and much more accurate. Evidently an accurate numerical solution requires a restriction on the grid steps. The same conclusions are obtained for discrete double barrier knock-out call options so that the comparison is omitted.

An optimal finite difference scheme does not exist because the numerical diffusion depends on different parameters for the respective numerical scheme. In practice usually the so called characteristic diffusion time $\tau_d = \dfrac{\Delta S^2}{(\sigma S)^2}$ is used, so that whenever $\Delta t \gg \tau_d$ is used, then an oscillating behavior may arise, [8].

5. Conclusions. In the paper is discussed how in option pricing undesired spurious oscillations could be avoided by applying finite difference schemes that are not standard in computational finance such as the traditional implicit and Crank-Nicolson scheme. The advantage of the proposed schemes is that, under severe restrictions, they give highly accurate results, guarantee smooth

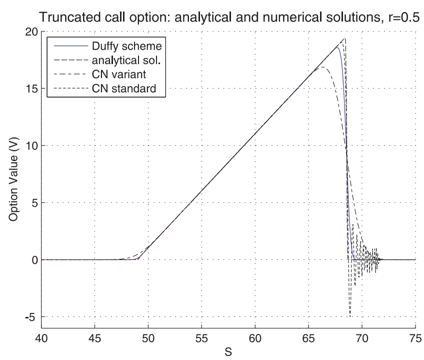

Fig. 5. Numerical diffusion of the Duffy and Crank-Nicolson variant scheme for a value of $r = 0.5$ when a truncated call option is priced. Parameters: $\sigma = 0.001$, $T = 5/12$, $U = 70$, $K = 50$, $S_{\max} = 140$, $\Delta S = 0.025$, $\Delta t = 0.01$

numerical solution, free of undesired spurious oscillations, and manage with low volatility values. Very often these nonstandard schemes introduce a numerical diffusion that could be diminished by an appropriate choice of grid steps. Finally, an optimal method does not exist in extreme cases such as valuation of options with a discontinuous payoff and low volatility values. The choice among the presented schemes depends only on the defined aims, such as accuracy and computational speed.

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