Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **37** (2011), 67–78

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

CONTACT CR-SUBMANIFOLDS OF KENMOTSU MANIFOLDS

Mehmet Atçeken

Communicated by O. Mushkarov

ABSTRACT. In this paper, we research some fundamental properties of contact CR-Submanifolds of a Kenmotsu manifold. We show that the anti-invariant distribution is always integrable and give a necessary and sufficient condition for the invariant distribution to be integrable. After then, properties of the induced structures on submanifold by almost contact metric structure on the ambient manifold are categorized. Finally, we give some results for contact CR-product and totally umbilical contact CR-submanifold in a Kenmotsu manifold and Kenmotsu space form.

1. Introduction. The geometry of semi-invariant submanifolds of a Kenmotsu manifold has been defined and ivestigated by K. Kenmotsu and M. Kobayashi [4, 6]. Furthermore, many geometers contributed to study of several classes of different manifolds with endowed Riemannian metric tensor[see references]. In present paper deal with the geometry of leaves of contact CR-submanifolds of a Kenmotsu manifold.

2010 *Mathematics Subject Classification:* 53C15, 53C42.
Key words: Kenmotsu manifold, contact CR-submanifold and contact CR-product.

In particular, we obtain necessary and sufficient conditions for the contact CR-submanifold to be CR-product, D-geodesic and $D^{\perp}$-geodesic. Furthermore, we get an inequality for the squared norm of the second fundamental form in terms of the dimensional of distributions which are involved definition of contact CR-submanifold in a Kenmotsu manifold. Finally, we discuss contact CR-products and totally umbilical contact CR-submanifolds(CR-products) in a Kenmotsu space form $\bar{M}(c)$.

2. Preliminaries. In this section, we give some notations used throughout this paper. We recall some necessary facts and formulas from the theory of Kenmotsu manifolds and their submanifolds.

A $(2m+1)$-dimensional Riemannian manifold $(\bar{M}, g)$ is said to be an almost contact metric manifold if it admits an endomorphism ϕ of its tangent bundle $T\bar{M}$, a vector field ξ and a 1-form η, satisfying

$$\phi^2 X = -X + \eta(X)\xi, \quad \phi\xi = 0, \quad \eta(\xi) = 1, \quad \eta(\phi X) = 0 \tag{1}$$

and

$$g(\phi X, \phi Y) = g(X, Y) - \eta(X)\eta(Y), \quad \eta(X) = g(X, \xi) \tag{2}$$

for any vector fields X, Y tangent to $\bar{M}$. Furthermore, an almost contact metric manifold is called a Kenmotsu manifold if ϕ and ξ satisfy

$$(\bar{\nabla}_X \phi)Y = g(\phi X, Y)\xi - \eta(Y)\phi X \quad and \quad \bar{\nabla}_X \xi = -\phi^2 X = X - \eta(X)\xi, \tag{3}$$

where $\bar{\nabla}$ denotes the Levi-Civita connection on $\bar{M}$ [2].

Now, let $\bar{M}$ be a $2n+1$-dimensional Kenmotsu manifolds with structure tensors (ϕ, ξ, η, g) and M be an m-dimensional isometrically immersed submanifold in $\bar{M}$. Moreover, we denote the Levi-Civita connections by $\bar{\nabla}$ and ∇, respectively. Then the Gauss and Weingarten formula's for M in $\bar{M}$ are, respectively, given by

$$\bar{\nabla}_X Y = \nabla_X Y + h(X, Y) \tag{4}$$

$$\bar{\nabla}_X V = -A_V X + \nabla^{\perp}_X V \tag{5}$$

for any vector fields X, Y tangent to M and vector V normal to M, where $\nabla^{\perp}$ is the normal connection on $T^{\perp}M$, h and A denote the second fundamental form and shape operator of M in $\bar{M}$, respectively. The A and h are related by

$$g(h(X, Y), V) = g(A_V X, Y). \tag{6}$$

We denote the Riemannian curvature tensor of the induced connection ∇ by R. Then the Gauss and Codazzi equations are, respectively, given by

$$(\bar{R}(X,Y)Z)^{\top} = R(X,Y)Z + A_{h(X,Z)}Y - A_{h(Y,Z)}X \tag{7}$$

and

$$(\bar{R}(X,Y)Z)^{\perp} = (\bar{\nabla}_X h)(Y,Z) - (\bar{\nabla}_Y h)(X,Z) \tag{8}$$

for any vector fields X, Y, Z tangent to M, where the covariant derivative of h is defined by

$$(\bar{\nabla}_X h)(Y,Z) = \nabla^{\perp}_X h(Y,Z) - h(\nabla_X Y, Z) - h(Y, \nabla_X Z) \tag{9}$$

for any vector fields X, Y, Z tangent to M, where $(\bar{R}(X,Y)Z)^{\perp}$ and $(\bar{R}(X,Y)Z)^{\top}$ denote the normal and tangent components of $\bar{R}(X,Y)Z$, respectively.

For any vector field X tangent to M, we set

$$\phi X = fX + \omega X, \tag{10}$$

where fX and ωX are the tangential and normal components of ϕX, respectively. Then f is an endomorphism of the TM and ω is a normal-bundle valued 1-form of TM. For the same reason, any vector field V normal to M, we set

$$\phi V = BV + CV, \tag{11}$$

where BV and CV are the tangential and normal components of ϕV, respectively. Then B is an endomorphism of the normal bundle $T^{\perp}M$ of TM and C is a tangent-bundle valued 1-form of $T^{\perp}M$.

Let $\bar{R}$ be the curvature tensor of the connection $\bar{\nabla}$. The sectional curvature of a ϕ-section is called a ϕ-sectional curvature. A Kenmotsu manifold with constant ϕ-sectional curvature c is said to be a Kenmotsu space form and is denoted by $\bar{M}(c)$. The curvature tensor $\bar{R}$ of a Kenmotsu space form $\bar{M}(c)$ is given

$$\begin{aligned} \bar{R}(X,Y)Z &= \frac{c-3}{4}\{g(Y,Z)X - g(X,Z)Y\} + \frac{c+1}{4}\{\eta(X)\eta(Z)Y - \eta(Y)\eta(Z)X \\ &+ \eta(Y)g(X,Z)\xi - \eta(X)g(Y,Z)\xi + g(X,\phi Z)\phi Y - g(Y,\phi Z)\phi X \\ &+ 2g(X,\phi Y)\phi Z\} \end{aligned} \tag{12}$$

for any vector fields X, Y, Z tangent to $\bar{M}$ [1].

3. Contact CR-submanifolds in Kenmotsu manifolds. In this section, we shall define contact CR-submanifolds in a Kenmotsu manifold and research fundamental properties of their from theory of submanifold.

Let M be an isometrically immersed in an almost contact metric manifold $\bar{M}$, then for every $x \in M$, there exists a maximal invariant subspace denoted by D_x of the tangent space T_xM of M. If the dimension of D_x is the same for all value of $x \in M$, then D_x gives an invariant distribution D on M.

Definition 3.1. *A submanifold M of a Kenmotsu manifold $\bar{M}$ is called contact CR-submanifold if there exists on M a differentiable invariant distribution D whose orthogonal complementary distribution $D^{\perp}$ is anti-invariant, i.e.,*

1.) $TM = D \oplus D^{\perp}, \quad \xi \in D$

2.) $\phi(D_x) = D_x,$

3.) $\phi(D_x^{\perp}) \subseteq (T_x^{\perp}M),$

for any $x \in M$. A contact CR-submanifold is called anti-invariant(or, totally real) if $D_x = \{0\}$ and invariant(or, holomorphic) if $D_x^{\perp} = \{0\}$, respectively, for any $x \in M$. It is called proper contact CR-submanifold if neither $D_x = \{0\}$ nor $D_x^{\perp} = \{0\}$.

Next, let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then from the (1), (10) and (11), we can write as the following way;

$$f^2X + B\omega X = -X + \eta(X)\xi, \quad \omega fX + C\omega X = 0, \tag{13}$$

$$fBV + BCV = 0 \text{ and } \omega BV + C^2V = -V \tag{14}$$

for any vector fields X tangent to M and V normal to M.

Proposition 3.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. For any vector field X tangent to M belong to D (resp. $D^{\perp}$) is necessary and sufficient that $\omega X = 0$ (resp. $fX = 0$).*

Furthermore, taking account of (2) and Proposition 3.1, we have

$$f^2X = -X + \eta(X)\xi \tag{15}$$

for any vector field X in D. Moreover

$$g(fX, fY) = g(X, Y) - \eta(X)\eta(Y) \tag{16}$$

for any vector fields X, Y in D.

Proposition 3.2. *Let M be a contact CR-submanifold of a Kenmotsu manifold M. Then the invariant distribution D has an almost contact metric structure (f, ξ, η, g) and so* $\dim(D)$=*odd.*

We denote by ν the orthogonal distribution $\phi D^{\perp}$ in $T^{\perp}M$. Then we have

$$T^{\perp}M = \phi D^{\perp} \oplus \nu \text{ and } \phi D^{\perp} \perp \nu. \tag{17}$$

From (17), it can easily to see that ν is an invariant distribution with respect to ϕ and it has an almost complex structure C and so ν is even dimensional.

Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. By using (4), (5), (10) and (11) we have

$$(\nabla_X f)Y = A_{\omega Y}X + Bh(X, Y) + g(fX, Y)\xi - \eta(Y)fX \tag{18}$$

and

$$(\nabla_X \omega)Y = Ch(X, Y) - h(X, fY) - \eta(Y)\omega X, \tag{19}$$

where the covariant derivatives of f and ω are defined by

$$(\nabla_X f)Y = \nabla_X fY - f(\nabla_X Y) \text{ and } (\nabla_X \omega)Y = \nabla^{\perp}_X \omega Y - \omega(\nabla_X Y)$$

for any vector fields X, Y tangent to M.

Theorem 3.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then the anti-invariant distribution $D^{\perp}$ is always integrable.*

P r o o f. For any vector fields Z, W tangent to $D^{\perp}$, by using (18), we have

$$f(\nabla_Z W) = -Bh(Z, W) - A_{\phi W}Z$$

which is also equivalent to

$$f[Z, W] = A_{\phi Z}W - A_{\phi W}Z. \tag{20}$$

On the other hand, we obtain

$$\begin{aligned} g(A_{\phi W}Z, U) &= g(h(U, Z), \phi W) = -g(\phi(\bar{\nabla}_U Z), W) = -g(\bar{\nabla}_U \phi Z - (\bar{\nabla}_U \phi)Z, W) \\ &= g(A_{\phi Z}W, U) + g(-\eta(Z)\phi U + g(\phi U, Z)\xi, W) = g(A_{\phi Z}W, U) \end{aligned} \tag{21}$$

for any $U \in \Gamma(TM)$. It implies that

$$A_{\phi Z}W = A_{\phi W}Z, \tag{22}$$

for any vector fields $Z, W \in \Gamma(D^{\perp})$. By combining of (20) and (22), we get $f[Z, W] = 0$, that is, $[Z, W] \in \Gamma(D^{\perp})$ which proves our assertion. □

Theorem 3.2. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then the invariant distribution D is integrable if and only if the second fundamental form of M satisfies*

$$h(X, \phi Y) = h(\phi X, Y) \tag{23}$$

for any $X, Y \in \Gamma(D)$.

Proof. For any vector fields X, Y in D, making use of (3) and (4), we have

$$\begin{aligned} \phi[X, Y] &= \phi(\nabla_X Y - \nabla_Y X) = \phi(\bar{\nabla}_X Y - \bar{\nabla}_Y X) = \bar{\nabla}_X \phi Y - (\bar{\nabla}_X \phi) Y \\ &- \bar{\nabla}_Y \phi X + (\bar{\nabla}_Y \phi) X = \nabla_X \phi Y - \nabla_Y \phi X + h(X, \phi Y) - h(Y, \phi X) \\ &- \eta(X)\phi Y + \eta(Y)\phi X + g(\phi Y, X)\xi - g(\phi X, Y)\xi. \end{aligned} \tag{24}$$

From the normal components of (24), we conclude

$$\omega[X, Y] = h(X, \phi Y) - h(\phi X, Y). \tag{25}$$

Thus D is integrable if and only if (23) is satisfied. □

Proposition 3.3. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. The invariant distribution D is integrable if 1-form ω is parallel.*

Proof. If 1-form ω is parallel, then from (19) we have $Ch(X, Y) = h(X, fY)$ for any $X, Y \in \Gamma(D)$. It implies (23). □

Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. For any vector fields U tangent to M and V normal to M, by using (3), (4), (5) and (11) we have

$$\begin{aligned} (\bar{\nabla}_U \phi)V &= \bar{\nabla}_U \phi V - \phi(\bar{\nabla}_U V) \\ g(\phi U, V)\xi &= (\bar{\nabla}_U B)V + (\bar{\nabla}_U C)V + h(U, BV) + \omega A_V U \\ &+ -A_{CV} U + f A_V U. \end{aligned} \tag{26}$$

The tangential and normal components of (26), respectively, we have

$$(\bar{\nabla}_U B)V = g(\phi U, V)\xi + A_{CV} U - f A_V U \tag{27}$$

and

$$(\bar{\nabla}_U C)V = -h(U, BV) - \omega A_V U, \tag{28}$$

where the covariant derivatives of B and C are, respectively, defined by

$$(\bar{\nabla}_U B)V = \nabla_U BV - B(\nabla^{\perp}_U V) \text{ and } (\bar{\nabla}_U C)V = \nabla^{\perp}_U CV - C(\nabla^{\perp}_U V).$$

Furthermore, for any $X, Y \in \Gamma(D)$, the Equation (18) reduces

$$(\nabla_X f)Y = Bh(X, Y) + g(fX, Y)\xi - \eta(Y)fX. \tag{29}$$

Thus we have the following Proposition.

Proposition 3.4. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then induced structure (f, ξ, η, g) is an almost contact metric structure on D if and only if $Bh(X, Y) = 0$ for any $X, Y \in \Gamma(D)$.*

Lemma 3.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. M is anti-invariant submanifold of Kenmotsu manifold $\bar{M}$ if and only if the endomorphism f is parallel.*

Proof. If f is parallel, then from (29) we have

$$Bh(X, Y) + g(fX, Y)\xi - \eta(Y)fX = 0 \tag{30}$$

for any $X, Y \in \Gamma(D)$. Taking $Y = \xi$ in (30), we get $Bh(X, \xi) + g(fX, \xi) - fX = 0$. Since $h(X, \xi) = 0$ and $f\xi = 0$, we conclude $fX = 0$ which implies M is anti-invariant submanifold.

Conversely, we suppose that M is anti-invariant. Then form (29), we have

$$(\nabla_X f)Y = -f(\nabla_X Y) = Bh(X, Y),$$

for any $X, Y \in \Gamma(TM)$. Because M is anti-invariant, $f(\nabla_X Y) \in \Gamma(TM^{\perp})$ and $Bh(X, Y) \in \Gamma(TM)$, we conclude $-f(\nabla_X Y) = Bh(X, Y) = 0$. Thus we get the desired result. □

Theorem 3.3. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then the anti-invariant distribution $D^{\perp}$ is totally geodesic in M if and only if $h(X, Z) \in \Gamma(\nu)$ for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$.*

Proof. For any $Z, W \in \Gamma(D^{\perp})$ and $X \in \Gamma(D)$, we have

$$\begin{aligned} g(\nabla_Z W, \phi X) &= g(\bar{\nabla}_Z W, \phi X) = -g(\bar{\nabla}_Z \phi X, W) \\ &= -g((\bar{\nabla}_Z \phi)X + \phi \bar{\nabla}_Z X, W) \\ &= -g(-\eta(X)\phi Z + g(\phi Z, X)\xi, W) + g(\bar{\nabla}_Z X, \phi W) \\ &= g(h(X, Z), \phi W) \end{aligned}$$

Thus $\nabla_Z W \in \Gamma(D^{\perp})$ if and only if $h(X, Z) \in \Gamma(\nu)$. $\square$

Theorem 3.4. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then the invariant distribution D is totally geodesic in M if and only if the second fundamental form of M satisfies $h(X, Y) \in \Gamma(\nu)$ for any $X, Y \in \Gamma(D)$.*

Proof.

$$\begin{aligned} g(\nabla_X \phi Y, Z) &= g(\bar{\nabla}_X \phi Y, Z) = g((\bar{\nabla}_X \phi)Y + \phi(\bar{\nabla}_X Y), Z) \\ &= g(g(\phi X, Y)\xi - \eta(Y)\phi X, Z) - (\bar{\nabla}_X Y, \phi Z) \\ &= g(\phi X, Y)\eta(Z) - \eta(Y)g(\phi X, Z) - g(h(X, Y), \phi Z) \\ &= -g(h(X, Y), \phi Z) \end{aligned} \tag{31}$$

for any $X, Y \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Thus $\nabla_X Y \in \Gamma(D)$ if and only if $h(X, Y) \in \Gamma(\nu)$. This completes of the proof. $\square$

4. Contact CR-products in a Kenmotsu manifold. In this section we shall define a contact CR-product in Kenmotsu manifolds, give a necessary and sufficient condition that a contact CR-submanifold to be a contact CR-product and we research contact CR products and totally umbilical contact CR-(submanifolds)products in a Kenmotsu space form $\bar{M}(c)$.

Definition 4.1. *A contact CR-submanifold M of a Kenmotsu manifold $\bar{M}$ is called a contact CR-product if M_T and $M_{\perp}$ are totally geodesic submanifolds of M, where M_T and $M_{\perp}$ denote the integral manifolds of the leaves of D and $D^{\perp}$, respectively.*

Next we shall prove the following Theorem.

Theorem 4.1. *Let M be a contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. M is a contact CR-product if and only if the shape operator of M satisfies*

$$A_{\phi D^{\perp}} D = 0. \tag{32}$$

P r o o f. We suppose that M be a contact CR-product in a Kenmotsu manifold $\bar{M}$. From Theorem 3.3 and Theorem 3.4, respectively, we have $A_{\phi Z}X \in \Gamma(D)$ and $A_{\phi Z}X \in \Gamma(D^{\perp})$ for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. It imply that (32) is satisfied.

Conversely, (32) is satisfied. Then Theorem 3.3 and Theorem 3.4 tell us that M_T and $M_{\perp}$ are totally geodesic submanifolds in M. Thus M is a contact CR-product. Hence the theorem is proved completely. □

Now, let M be a contact CR-product of a Kemotsu space form $\bar{M}(c)$, we shall calculate bisectional sectional curvature of Kenmotsu manifold $\bar{M}(c)$. By using (8), (9) and considering Theorem 3.3 and Theorem 3.4, we have

$$\begin{aligned}
-H_B(X,Z) &= g(R(X,\phi X)Z,\phi Z) = g((\bar{\nabla}_X h)(\phi X,Z) - (\bar{\nabla}_{\phi X}h)(X,Z),\phi Z) \\
&= g(\nabla^{\perp}_X h(\phi X,Z) - h(\nabla_X \phi X,Z) - h(\nabla_X Z,\phi X),\phi Z) \\
&\quad -g(\nabla^{\perp}_{\phi X} h(X,Z) - h(\nabla_{\phi X}X,Z) - h(\nabla_{\phi X}Z,X),\phi Z) \\
&= Xg(h(\phi X,Z),\phi Z) - g(\bar{\nabla}_X \phi Z, h(\phi X,Z)) - \phi X g(h(X,Z),\phi Z) \\
&\quad +g(\bar{\nabla}_{\phi X}\phi Z, h(X,Z)) \\
&= g((\bar{\nabla}_{\phi X}\phi)Z + \phi(\bar{\nabla}_{\phi X}Z), h(X,Z)) - g((\bar{\nabla}_X\phi)Z \\
&\quad +\phi(\bar{\nabla}_X Z), h(\phi X,Z)) \\
&= g(-\eta(Z)\phi^2 X + g(\phi^2 X,Z)\xi + \phi(\bar{\nabla}_{\phi X}Z), h(X,Z)) \\
&\quad -g(-\eta(Z)\phi X + g(\phi X,Z)\xi + \phi(\bar{\nabla}_X Z), h(\phi X,Z)) \\
&= g(\phi h(\phi X,Z),h(X,Z)) - g(\phi h(X,Z),h(\phi X,Z)) \\
&= 2g(\phi h(\phi X,Z),h(X,Z)) \\
&= -2g(\bar{\nabla}_Z\phi X,\phi h(X,Z)) = -2g((\bar{\nabla}_Z\phi)X + \phi(\bar{\nabla}_X Z),\phi h(X,Z)) \\
&= -2g(-\eta(X)\phi Z + g(\phi Z,X)\xi + \phi h(X,Z),\phi h(X,Z)) \\
&= -2g(h(X,Z),h(X,Z)) = -2\|h(X,Z)\|^2,
\end{aligned}$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. So we get

$$H_B(X,Z) = 2\|h(X,Z)\|^2. \tag{33}$$

Thus we have following the Theorem.

Theorem 4.2. *Let M be a contact CR-product of a Kenmotsu space form $\bar{M}(c)$ with constant ϕ-holomorphic sectional curvature c. Then there do not exist contact CR-products in a Kenmotsu space form $\bar{M}(c)$ such that $c < -1$.*

Proof. We suppose that M is a contact CR-product in Kenmotsu space form $\bar{M}(c)$. Then from (3) and (4), we know $h(Z,\xi) = 0$. By using (12) and (33), we have

$$\begin{aligned} g(R(X,\phi X)\phi Z, Z) &= \left(\frac{c+1}{2}\right)\{g(X,X) - \eta^2(X)\}g(Z,Z) \\ &= \left(\frac{c+1}{2}\right) g(\phi X, \phi X)g(Z,Z) = 2\|h(X,Z)\|^2, \end{aligned}$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. So we have

$$\|h(X,Z)\|^2 = \left(\frac{c+1}{4}\right) g(Z,Z)g(\phi X,\phi X), \tag{34}$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^\perp)$. This equality is impossible for $c < -1$. This proves our assertion. □

Theorem 4.3. *Let M be a contact CR-product in Kenmotsu space form $\bar{M}(c)$. Then we have*

$$\| h \|^2 \geq \left\{\frac{c+1}{2}\right\} pq, \tag{35}$$

where $\dim D = 2p+1$ *and* $\dim D^\perp = q$.

Proof. Let $\{e_1, e_2, \ldots, e_{2p}, \xi, e^1, e^2, \ldots, e^q\}$ be an orthonormal basis of $\Gamma(TM)$ such that $e_1, e_2, \ldots, e_{2p}, \xi$ are tangent to $\Gamma(D)$ and $e^1, e^2, \ldots, e^q$ are tangent to $D^\perp$. Then norm of the second fundamental form $\| h \|^2$ is defined by

$$\begin{aligned} \| h \|^2 &= \sum_{i,j=1}^{p} g(h(e_i,e_j), h(e_i,e_j)) + \sum_{r,s=1}^{q} g(h(e^r,e^s), h(e^r,e^s)) \\ &+ 2\sum_{i=1}^{p}\sum_{r=1}^{q} g(h(e_i,e^r), h(e_i,e^r)) \end{aligned}$$

Taking $X = e_1, e_2, \ldots, e_{2p}, \xi$ and $Z = e^1, e^2, \ldots, e^q$ in (34), then we obtain

$$\| h \|^2 > \left(\frac{c+1}{2}\right) pq$$

□

Contact CR-submanifold M is called a totally umbilical contact CR-submanifold if its second fundamental form h satisfies $h(X,Y) = g(X,Y)H$, for any $X, Y \in \Gamma(TM)$, where H denote the mean curvature vector of M.

Theorem 4.4. *Let M be a contact CR-submanifold of a Kenmotsu space form $\bar{M}(c)$. There exist no totally umbilical contact CR-submanifolds in a Kenmotsu space form $\bar{M}(c)$ such that $c \neq -1$.*

P r o o f. We suppose that M is a totally umbilical contact CR-submanifold in Kenmotsu space form $\bar{M}(c)$. Then by using (8), we obtain

$$\begin{aligned} g(\bar{R}(\phi X, X)Z, \phi Z) &= g(X,Z)g(\nabla^{\perp}_{\phi X} H, \phi Z) - g(\phi X, Z)g(\nabla^{\perp}_{X} H, \phi Z) \\ &= 0, \end{aligned} \tag{36}$$

for any $X \in \Gamma(D)$ and $Z \in \Gamma(D^{\perp})$. Since the ambient space $\bar{M}$ is a Kenmotsu space form, from (12) we infer

$$g(\bar{R}(\phi X, X)Z, \phi Z) = \frac{c+1}{2}\{g(\phi X, \phi X)g(Z,Z)\}. \tag{37}$$

Thus from (36) and (37), we obtain the desired result. □

Theorem 4.5. *Let M be a totally umbilical contact CR-submanifold of a Kenmotsu manifold $\bar{M}$. Then at least one of the following is true;*

i.) *M is totally geodesic*

or

ii.) $\dim(D^{\perp}) > 1$

P r o o f. By direct calculations, we have

$$A_{\phi V} X = -A_V \phi X \tag{38}$$

for any $X \in \Gamma(D)$ and $V \in \Gamma(\nu)$. Since M is a totally umbilical contact CR-submanifold and by using (38), we have

$$\begin{aligned} g(A_{\phi CH} X, X) &= -g(A_{CH} \phi X, X) \\ g(X,X)g(H, \phi CH) &= -g(X, \phi X)g(CH, H) = 0 \end{aligned}$$

which is equivalent to $CH = 0$. On the other hand, by using (22) we have

$$\begin{aligned} g(A_{\phi BH} Z, W) &= g(A_{\phi Z} BH, W) \\ g(Z,W)g(H, \phi BH) &= g(BH, W)g(H, \phi Z) \\ g(Z,W)g(BH, BH) &= g(BH, W)g(BH, Z) \end{aligned}$$

for any $Z, W \in \Gamma(D^{\perp})$. This is implies that BH is either identical zero, or BH and Z are linearly dependent. If $BH = 0$, then totally umbilical contact CR-submanifold is totally geodesic otherwise, the anti-invariant distribution $D^{\perp}$ is one dimensional. This completes of the proof of the theorem. □

REFERENCES

[1] B. B. Sinha, A. K. Srivastava. Semi-Invariant Submanifolds of A Kenmotsu Manifold with Constant ϕ-Holomorphic Sectional Curvature. *Indian J. Pure Appl. Math.* **23**, *11* (1992), 783–789.

[2] K. Kenmotsu. A class of almost contact Riemannian Manifold. *Tohoku Math. J.* **24** (1972), 93–103.

[3] K. Matsumoto. On Contact CR-Submanifolds of Sasakian Manifolds. *Internat. J. Math. Math. Sci.* **6**, *2* (1983), 313–326.

[4] M. Kobayahsi. Semi-Invariant Submanifolds of a certain class of almost contact manifolds. *Tensor (N.S)*, **43** (1986), 28–36.

[5] M. Shoeb. CR-Submanifolds of A Normal Almost Paracontact Manifold. *Invertis Journal of Science and Technology* **1**, *3* (2008), 190–194.

[6] N. Papaghuic. Semi-invariant submanifolds in a Kenmotsu manifold. *Rend. Mat. Appl. (7)* **3** (1983), 607–622.

[7] I. Hasegawa, I. Mihai. Contact CR-Warped Product Submanifolds in Sasakian Manifolds. *Geom. Dedicata* **102** (2003), 143–150.

[8] W. Yong. Totally Geodesic Invariant Submanifolds of Contact Metric Manifold. *Chinese Quart. J. Math.* **13**, *4* (1998).

[9] V. A. Khan, K. A. Khan, Sirajuddin. Contact CR-warped product submanifolds of Kenmotsu manifolds. *Thai J. Math.* **6**, *1* (2008), 139–145.

Gaziosmanpasa University
Faculty of Arts and Sciences
Department of Mathematics
60250 Tokat, Turkey
e-mail: `matceken@gop.edu.tr`

Received March 18, 2011
Revised June 7, 2011