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WEITZENBÖCK DERIVATIONS AND CLASSICAL INVARIANT THEORY, II: THE SYMBOLIC METHOD

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Abstract. An analogue of the symbolic method of classical invariant theory for a representation and manipulation of the elements of the kernel of Weitzenböck derivations is developed.

1. Introduction. Let $\mathbb{K}$ be a field of characteristic 0 and let $\mathbb{K}[X]$ be a polynomial algebra in a set of variables X. A linear locally nilpotent derivation of the polynomial algebra $\mathbb{K}[X]$ is called a Weitzenböck derivation. Denote by $\mathcal{D}_{\boldsymbol{d}}$, $\boldsymbol{d} := (d_1, d_2, \ldots, d_s)$ the Weitzenböck derivation of the algebra $\mathbb{K}[X]$ if its matrix consists of s Jordan blocks of size d_1+1, d_2+1, $\ldots$, d_s+1, respectively. The only derivation which corresponds to a single Jordan block of size $d+1$ is called the basic Weitzenböck derivation and is denoted by $\mathcal{D}_d$. The algebra

$$\ker \mathcal{D}_{\boldsymbol{d}} = \{f \in \mathbb{K}[X] \mid \mathcal{D}_{\boldsymbol{d}}(f) = 0\},$$

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is called *the kernel* of the derivation $\mathcal{D}_{\boldsymbol{d}}$. It is well known that the kernel $\ker \mathcal{D}_{\boldsymbol{d}}$ is a finitely generated algebra (see [19], [16], [17]). However, it remained an open problem to find a minimal system of homogeneous generators of the algebra $\ker \mathrm{D}_{\boldsymbol{d}}$ even for small tuples $\boldsymbol{d}$.

The aim of this paper is to develop an effective method for the representation and manipulation of the kernel elements of Weitzenböck derivations.

In a previous work [5] we showed that the kernel of the derivations $\mathcal{D}_{\boldsymbol{d}}$, $\boldsymbol{d} := (d_1, d_2, \ldots, d_s)$ is isomorphic to the algebra of joint covariants for s binary forms of orders $d_1, d_2, \ldots, d_s$. Algebras of joint covariants of binary forms were an object of research in invariant theory in the 19th century. To describe the kernel of linear locally nilpotent derivations we should involve computational tools of classical invariant theory, including the famous symbolic method. The symbolic method was developed by Aronhold, Clebsch, and Gordan. It is the most powerful tool of classical invariant theory. A classic presentation of the symbolic method can be found in [9], [11], [10]. Recently, a rigorous foundation for the symbolic method has been given by [15] and by [14].

In this paper we develop an analogue of the classical symbolic method for computing of the kernel of Weitzenböck derivations. We explain the essence of the method by examples. Let $\mathcal{D}_4$ be the basic Weitzenböck derivation of the polynomial algebra $\mathbb{K}[X_4] := \mathbb{K}[x_0, x_1, x_2, x_3, x_4]$ i.e., $\mathcal{D}_4(x_i) = x_{i-1}$, $\mathcal{D}_4(x_0) = 0$, $i = 1, \ldots, 4$. Also, consider the Weitzenböck derivation $\mathcal{D}_{(4,4)}$ of the algebra $\mathbb{K}[X_4, Y_4] : \mathcal{D}_{(4,4)}(x_i) = \mathcal{D}_4(x_i)$, $\mathcal{D}_{(4,4)}(y_i) = y_{i-1}$, $\mathcal{D}_{(4,4)}(y_0) = 0$. The following differential operator $\mathcal{P}^{(n)}_{y,x} : \mathbb{K}[X_n] \to \mathbb{K}[X_n, Y_n]$ defined by

$$\mathcal{P}^{(n)}_{y,x} = y_0 \frac{\partial}{\partial x_0} + y_1 \frac{\partial}{\partial x_1} + \cdots + y_n \frac{\partial}{\partial x_n},$$

is called the *polarization operator.* The operator $\mathcal{R}^{(n)}_{y,x} : \mathbb{K}[X_n, Y_n] \to \mathbb{K}[X_n]$, defined by

$$\mathcal{R}^{(n)}_{y,x}(F) = F\Big|_{y_i = x_i}$$

is called the *restitution operator.* If F is a homogeneous polynomial then Euler's homogeneous function theorem implies that $\mathcal{R}^{(n)}_{y,x}\left(\mathcal{P}^{y}_{x}(F)\right) = \deg(F)\, F$. The polarization operator commutes with the Weitzenböck derivations:

$$\mathcal{P}^{(4)}_{y,x}\left(\mathcal{D}_4(F)\right) = \mathcal{D}_{(4,4)}\left(\mathcal{P}^{(4)}_{y,x}(F)\right).$$

It is easy to verify that the polynomial $F = x_2^2 + 2\,x_0x_4 - 2\,x_1x_3$ belongs to $\ker \mathcal{D}_4$ and its polarization

$$\mathcal{P}^{(4)}_{y,x}(F) = 2\,y_0x_4 - 2\,y_1x_3 + 2\,y_2x_2 - 2\,y_3x_1 + 2\,y_4x_0,$$

belongs to $\ker \mathcal{D}_{(4,4)}$. Let us change the variables by

(1) $$x_i = \frac{1}{i!}\alpha_0^{4-i}\alpha_1^i, y_i = \frac{1}{i!}\beta_0^{4-i}\beta_1^i, i = 1, \ldots, 4.$$

Then we get that

$$\mathcal{P}_{y,x}^{(4)}(F) = \frac{1}{12}\,(\alpha_0\beta_1 - \beta_0\alpha_1) := \frac{1}{12}[\alpha, \beta]^4,$$

where $[\alpha, \beta] := \alpha_0\beta_1 - \beta_0\alpha_1$. The polynomial $\Psi = \dfrac{1}{12}[\alpha, \beta]^4$ is called the *symbolic representation* of the polynomial F. The letters α, β are called the *symbol letters.* Observe that Ψ belongs to the kernel of the derivation $\mathcal{D}_{(1,1)}$ which acts on $\mathbb{K}[\alpha_0, \alpha_1, \beta_0, \beta_1]$. Moreover, the polynomial Ψ has much simpler form than the polynomial F.

On the other hand, let us consider the polynomial $\Phi = \alpha_0^2\beta_0^2[\alpha, \beta]^2 \in \ker \mathcal{D}_{(1,1)}$. Then by (1) we get

$$\Phi = {\alpha_0}^4{\beta_0}^2{\beta_1}^2 - 2\,{\alpha_0}^3{\beta_0}^3\beta_1\alpha_1 + {\alpha_0}^2{\beta_0}^4{\alpha_1}^2 = 2(x_0y_2 - x_1y_1 + x_2y_0),$$

and $\mathcal{R}_{y,x}^{(4)}(\Phi) = 2\,(2x_0x_2 - x_1^2) \in \ker \mathcal{D}_4$. Thus $\dfrac{1}{2}\alpha_0^2\beta_0^2[\alpha, \beta]^2$ is a symbolic representation for $2x_0x_2 - x_1^2$. To get elements of degree 3 we should involve one more symbolic letter γ. Similarly, one may show that

$$\frac{1}{4!}[\alpha, \beta]^2[\alpha, \gamma]^2[\beta, \gamma]^2 \in \ker \mathcal{D}_{(1,1,1)}$$

is a symbolic representation for the following element of the kernel of derivation $\mathcal{D}_4$:

$$12\,x_0x_4x_2 + 6\,x_1x_3x_2 - 2\,{x_2}^3 - 9\,x_0{x_3}^2 - 6\,{x_1}^2x_4$$

and for all its polarizations.

For the general case consider the polynomial algebra $\mathbb{K}[\alpha_i \mid \alpha \in \mathcal{J}, i = 0, 1]$ where α runs on a set $\mathcal{J}$ of symbol letters. Elements of the kernel of the Weitzenböck derivation

$$\mathcal{D}_{\mathcal{J}} := \mathcal{D}_{(\underbrace{1, 1, \ldots, 1}_{|\mathcal{J}|\text{ times}})}$$

defined by $\mathcal{D}_{\mathcal{J}}(\alpha_0) = 0$, $\mathcal{D}_{\mathcal{J}}(\alpha_1) = \alpha_0$, for all $\alpha \in \mathcal{J}$, are called symbolic expressions.

The following statement is the main point of the symbolic method:

The Symbolic Method.

- *Any element of* $\ker \mathcal{D}_{\mathbf{d}}$ *allows a symbolic representation;*
- *Any symbolic expression is a symbolic representation for an element of* $\ker \mathcal{D}_{\mathbf{d}}$ *for some* $\mathbf{d}$.

Thus we arrive at the remarkable fact that the kernel of an arbitrary Weitzenböck derivation $\mathcal{D}_{\boldsymbol{d}}$ is completely defined by the kernel of the special Weitzenböck derivation $\mathcal{D}_{\mathcal{J}}$. The kernel $\ker \mathcal{D}_{\mathcal{J}}$ is well-known and generated by α_0 and the brackets $[\alpha, \beta]$ where α, β run over $\mathcal{J}$.

The paper is organized as follows. In Section 2 we review some of the standard facts on the representation theory of the Lie algebra $\mathfrak{sl}_2$ and its maximal nilpotent subalgebra $\mathfrak{u}_2$.

In Section 3 we develop an analogue of the classical symbolic method for Weitzenböck derivations. In Section 4 we introduce the notions of the convolution and the semi-transvectant which are used for calculation of a generating set of the kernel of the derivations.

2. Basic facts. A representation of the Lie algebra $\mathfrak{g}$ on a finite-dimensional complex vector space V is a homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is the Lie algebra of endomorphisms of V. We say that such a map gives V the structure of a $\mathfrak{g}$-module. The algebra $\mathfrak{g}$ acts on V by linear operators $\rho(g), g \in \mathfrak{g}$. When there is little ambiguity about the map ρ we sometimes call V itself a representation of $\mathfrak{g}$; in this vein we will suppress the symbol ρ and write $g\,v$ for $\rho(g)v$.

If U, V are representations, then the tensor product $U \otimes V$ is also a representation with the action

$$g(u \otimes v) = gu \otimes v + u \otimes gv.$$

For a representation V, the tensor algebra $\mathrm{T}(V)$ is again a representation of $\mathfrak{g}$ by this rule, and the symmetric algebra $\mathrm{Sym}(V)$ is its subrepresentation. Thus, the algebra $\mathfrak{g}$ acts on $\mathrm{Sym}(V)$ by derivations. An element $v \in V$ is called an *invariant* of the $\mathfrak{g}$-module V if $gv = 0$. Denote by $V^{\mathfrak{g}}$ the set of all invariants of the $\mathfrak{g}$-module V.

Let $\mathfrak{sl}_2$ be the Lie algebra of 2×2 traceless matrices and let $\mathfrak{u}_2$ be its maximal nilpotent subalgebra. The canonical basis of $\mathfrak{sl}_2$ is the basis (e, f, h), where

$$e = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We have

$$[h,e] = 2\,e, [h,f] = -2\,f, [e,f] = h. \tag{2}$$

For each nonnegative integer n, the algebra $\mathfrak{sl}_2$ has an irreducible representation V_n of dimension $n+1$, which is unique up to isomorphism. The endomorphisms $\rho(e)$, $\rho(f)$, $\rho(h)$ act on $V_n := \langle v_0, v_1, \dots, v_n\rangle$ by the formulas

$$\rho(e)v_i = v_{i-1}, \quad \rho(e)v_i = (n-i)(i+1)v_{i+1}, \quad \rho(h)v_i = (n-2i)v_i.$$

The action of $\mathfrak{sl}_2$ extends by derivations to the symmetric algebra $\mathrm{Sym}(V_n)$ and to the algebra

$$\mathrm{Sym}(V_{\mathbf{d}}) := \mathrm{Sym}(V_{d_1} \oplus V_{d_2} \oplus \ldots \oplus V_{d_s}), \boldsymbol{d} := (d_1, d_2, \dots, d_s).$$

For convenience, the derivations of $\mathrm{Sym}(V_{\mathbf{d}})$ which correspond to the operators $\rho(e)$, $\rho(f)$, $\rho(h)$ are denoted by $\mathbf{D}$, $\mathbf{D}_*$ and $\mathbf{E}$, respectively. Let us identify the algebra $\mathrm{Sym}(V_n)$ with the polynomial algebra $\mathbb{K}[V_n] := \mathbb{K}[v_0, v_1, \dots, v_n]$ and the algebra $\mathrm{Sym}(V_{\mathbf{d}})$ with the polynomial algebra $\mathbb{K}[V_{d_1}, V_{d_2}, \dots V_{d_s}]$. Under this identification, the kernel of the derivation $\mathbf{D}$ coincides with the algebra of invariants $\mathrm{Sym}(V_{\mathbf{d}})^{\mathfrak{u}_2}$, and the algebra $\ker \mathbf{D} \cap \ker \mathbf{D}_*$ coincides with the algebra of invariants $\mathrm{Sym}(V_{\mathbf{d}})^{\mathfrak{sl}_2}$. Any $\mathfrak{u}_2$-invariant is called a *semi-invariant.* It is clear that the derivation $\mathbf{D}$ is exactly the Weitzenböck derivation $\mathcal{D}_{\boldsymbol{d}}$.

To begin, let us describe the algebras of invariants and semi-invariants of the symmetric algebra $\mathrm{Sym}(V_1 \oplus V_1 \oplus \ldots \oplus V_1)$.

Let $\mathfrak{G} = \{\alpha, \beta, \ldots\}$ be an alphabet consisting of an infinite supply of Greek letters. The letters in $\mathfrak{G}$ are called *symbol letters.* To each symbol letter α we associate two variables α_0, α_1 and the two-dimensional vector space $V_\alpha := \mathbb{K}\alpha_0 \oplus \mathbb{K}\alpha_1$. For a finite subset $\mathcal{J} \subset \mathfrak{G}$ let $V_{\mathcal{J}} := \oplus_{\alpha \in J} V_\alpha$. The algebra $\mathrm{Sym}\,(V_{\mathcal{J}})$ turns into an $\mathfrak{sl}_2$-module by the action

$$\begin{aligned}\mathbf{D}(\alpha_0) = 0, \quad \mathbf{D}(\alpha_1) = \alpha_0, \quad \mathbf{D}_*(\alpha_0) = \alpha_1, \quad \mathbf{D}_*(\alpha_2) = 0,\\ \mathbf{E}(\alpha_i) = (1-2i)\alpha_i, \quad i = 0, 1, \alpha \in \mathcal{J}.\end{aligned} \tag{3}$$

A direct check shows that the conditions (2) hold. We note that the symmetric group $S_{|\mathcal{J}|}$ naturally acts on $\mathrm{Sym}\,(V_{\mathcal{J}})$, where $|\mathcal{J}|$ is the cardinality of the set $\mathcal{J}$.

The First Fundamental Theorem for $\mathfrak{sl}_2$. *The algebra of semi-invariants* $\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}$ *is generated by* α_0 *and by the brackets*

$$[\alpha, \beta] := \begin{vmatrix} \alpha_0 & \alpha_1 \\ \beta_0 & \beta_1 \end{vmatrix}, \quad \alpha, \beta \in \mathcal{J}.$$

The algebra of invariants $\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{sl}_2}$ *is generated by the brackets* $[\alpha,\beta]$.

This is a well-known result of classical invariant theory, see [12]. In the theory of locally nilpotent derivations the first part of this statement is known as the Nowicki conjecture, see, for instance, [3].

Corollary. *As a vector space the algebra* $\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}$ *is generated by the polynomials*

$$P_{l,m} := \prod_{\alpha\neq\beta}[\alpha,\beta]^{l_{\alpha,\beta}}\prod_{\gamma}\gamma_0^{m_\gamma}, \alpha,\beta,\gamma\in\mathcal{J}.$$

The *order* $\mathrm{ord}\,P$ and the *weight* $\mathrm{wt}\,P := (\mathrm{wt}_\alpha P)_{\alpha\in\mathcal{J}}$ of the symbolic expression P are defined by

$$\mathrm{ord}\,P := \sum_\gamma m_\gamma, \mathrm{wt}_\alpha\,P := \sum_\beta (l_{\alpha,\beta}+l_{\beta,\alpha}) + m_\alpha.$$

In particular, $\mathrm{wt}_\alpha P$ is equal to the number of times the symbol α occurs in the symbolic expression P. Observe, that

$$\mathrm{ord}\,P = \min_k\{k\in\mathbb{N}\mid \mathbf{D}_*^{k+1}(P)=0\}, \mathbf{E}\,(P) = \mathrm{wt}\,P\,\cdot P.$$

The symbolic expression P is called *decomposable* if it can be written as a product $P = P_1\,P_2$ in a non-trivial way where P_1 and P_2 are *disjoint,* i.e., no symbol occurs in both. We denote by $\mathrm{supp}\,P$ the support of P, i.e., the set of symbols $\alpha\in\mathcal{J}$ occurring in P. Put $\alpha\sim\beta$ if $\mathrm{wt}_\alpha = \mathrm{wt}_\beta$. Then the relation $\sim$ is an equivalence relation defined on the set $\mathrm{supp}\,P\subseteq\mathcal{J}$. Denote by $\mathcal{J}_1$, $\mathcal{J}_2,\ldots,\mathcal{J}_t$ the equivalence classes and by m_1, m_2, $\ldots m_t$ denote their cardinality, $t = |\,\mathrm{supp}\,P/\sim|$. Denote by n_1, n_2, $\ldots n_t$ the corresponding weights of elements of the classes J_i.

Let $x_{\alpha,\beta}, x_\gamma$ for $\alpha,\beta,\gamma\in\mathcal{J}$, $\alpha\neq\beta$, denote independent variables and define the free polynomial algebra

$$\mathrm{Sym}_{\mathcal{J}} := \mathbb{K}[x_{\alpha,\beta}, x_\gamma\mid\alpha,\beta,\gamma\in\mathcal{J},\alpha\neq\beta].$$

Define the map $\chi:\mathrm{Sym}_{\mathcal{J}}\to\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}$ by

$$\chi(x_{\alpha,\beta}) = [\alpha,\beta], \chi(x_\gamma) = \gamma_0,$$

and extend it in the natural way to monomials and all of $\mathrm{Sym}_{\mathcal{J}}$.

The Second Fundamental Theorem $\mathfrak{sl}_2$. *There is a canonical isomorphism*

$$\mathrm{Sym}_{\mathcal{J}}/\ker\chi\cong\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}$$

where the ideal $\ker \chi$ *is generated by the elements*

$$x_{\alpha,\beta} + x_{\beta,\alpha} = 0, x_\gamma x_{\alpha,\beta} + x_\beta x_{\gamma,\alpha} + x_\alpha x_{\beta,\gamma} = 0,$$
$$x_{\alpha,\beta} x_{\gamma,\delta} + x_{\gamma,\alpha} x_{\beta,\delta} + x_{\beta,\gamma} x_{\alpha,\delta} = 0.$$

The theorem implies that the following three relations (syzygies)

$$[\alpha,\beta] + [\beta,\alpha] = 0,$$
$$\gamma_0[\alpha,\beta] + \beta_0[\gamma,\alpha] + \alpha_0[\beta,\gamma] = 0,$$
$$[\alpha,\beta][\gamma,\delta] + [\gamma,\alpha][\beta,\delta] + [\beta,\gamma][\alpha,\delta] = 0,$$

for distinct $\alpha,\beta,\gamma,\delta \in \mathcal{J}$ generate all the relationship among the semi-invariants $\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}$.

The algebra $\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}$ is graded by weight. Let $\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}_{\mathbf{w}}$ be the vector space of the elements of weight $\mathbf{w}$. Then $\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2} = \oplus_{\mathbf{w}} \mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}_{\mathbf{w}}$. If $\mathbf{w} = (n, n, \ldots, n)$ we will write $\mathbf{w} = (n)^{|\mathcal{J}|}$. If

$$\mathbf{w} = (\underbrace{n_1, n_1, \ldots, n_1}_{m_1 \text{ times}}, \underbrace{n_2, n_2, \ldots, n_2}_{m_2 \text{ times}}, \ldots, \underbrace{n_t, n_t, \ldots, n_t}_{m_t \text{ times}}),$$

we write $\mathbf{w} = (n_1)^{m_1}(n_2)^{m_2} \cdots (n_t)^{m_t}$, or more compact $\mathbf{w} = \mathbf{n}^{\mathbf{m}}$. Here $\mathbf{n} := (n_1, n_2, \ldots n_t)$, $\mathbf{m} := (m_1, m_2, \ldots m_t)$ and $m_1 + m_2 + \cdots + m_t = |\mathcal{J}|$.

3. Symbolic method. We will show that symbolic expressions can be used in a very efficient way to describe and manipulate semi-invariants of the $\mathfrak{u}_2$-module $\mathrm{Sym}(V_{\mathbf{d}})$, $\boldsymbol{d} := (d_1, d_2, \ldots, d_s)$. Recall that $\mathrm{Sym}(V_{\mathbf{d}})^{\mathfrak{u}_2} = \ker \mathcal{D}_{\boldsymbol{d}}$.

Let $\mathfrak{R} = \{x, y, z, \ldots\}$ be an alphabet consisting of an infinite supply of ordered roman letters. Denote by n_x the ordinal number of the letter x in the set $\mathfrak{R}$. To each letter x and to each integer number n associate the $n+1$-dimension vector space

$$V_{x,n} := \mathbb{K}x_0 \oplus \mathbb{K}x_1 \oplus \cdots \oplus \mathbb{K}x_n \cong V_n.$$

For a finite subset $\mathcal{I} \subset \mathfrak{R}$ and for $\boldsymbol{d} := (d_1, d_2, \ldots, d_{|\mathcal{I}|})$ let $V_{\mathcal{I},\boldsymbol{d}} := \oplus_{x \in \mathcal{I}} V_{x,d_i}$. The action

$$\text{(4)} \quad \mathbf{D}(x_i) = x_{i-1}, \quad \mathbf{D}_*(x_i) = (i+1)(d_{n_x} - i)x_{i+1},$$
$$\mathbf{E}(x_i) = (d_{n_x} - 2i)x_i, \quad i = 0, 1, \ldots, d_{n_x}, x \in \mathcal{I}.$$

gives $\mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})$ the structure of an $\mathfrak{sl}_2$-module. The *order* $\mathrm{ord}\, S$ of a homogeneous semi-invariant $S \in \mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}$ is defined by

$$\mathrm{ord}\, S := \min_k \{k \mid \mathbf{D}_*^{k+1}(S) = 0\}.$$

The algebra $\mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}$ is graded by multidegree. Let $\left(\mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}\right)_{\mathbf{m}}$ be the homogeneous component of multidegree $\mathbf{m}$. Then

$$\mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2} = \oplus_{\mathbf{m}} \left(\mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}\right)_{\mathbf{m}}.$$

The following result summarizes what is classically called “symbolic method”:

Theorem 3.1. *There is a surjective $\mathfrak{u}_2$-homomorphism of vector spaces*

$$\Lambda : \left(\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}\right)_{\mathbf{d}^{\mathbf{m}}} \to \left(\mathrm{Sym}\,(V_{\mathcal{I},\mathbf{d}})^{\mathfrak{u}_2}\right)_{\mathbf{m}}, \mathbf{m} := (m_1, m_2, \ldots m_t), |\mathcal{I}| = t,$$

such that the composition $\Lambda \circ \chi : \left(\mathrm{Sym}_{\mathcal{J}}\right)_{\mathbf{d}^{\mathbf{m}}} \to \left(\mathrm{Sym}\,(V_{\mathcal{I},\mathbf{d}})^{\mathfrak{u}_2}\right)_{\mathbf{m}}$ is surjective with kernel

$$\ker \Lambda \circ \chi = (\ker \chi)_{\mathbf{d}^{\mathbf{m}}} + \big\langle P - \sigma P \mid P \in (\mathrm{Sym}_{\mathcal{J}})_{\mathbf{d}^{\mathbf{m}}}, \\ \sigma \in S_{|\mathcal{J}_i|}, \mathcal{J}_i \in \mathrm{Supp}\, P/\sim,\ i = 1, \ldots t\big\rangle.$$

P r o o f. Let $V_\alpha = \langle \alpha_0, \alpha_1 \rangle$ and $V_{x,n} = \langle x_0, x_1, \ldots x_n \rangle$ be two $\mathfrak{sl}_2$-modules as above. It is well-known that the linear map $\alpha_0^{n-i}\alpha_1^i \longmapsto i!x_i$ is an $\mathfrak{sl}_2$-isomorphism of the vector spaces $\mathrm{Sym}^n(V_\alpha)$ and $V_{x,n}$. In fact, we have

$$\mathbf{D}(\alpha_0^{n-i}\alpha_1^i) = i\alpha_0^{n-i+1}\alpha_1^{i-1} \longmapsto i(i-1)!\, x_{i-1} = i!\,\mathbf{D}(x_i),$$
$$\mathbf{D}_*(\alpha_0^{n-i}\alpha_1^i) = (n-i)\alpha_0^{n-i-1}\alpha_1^{i+1} \longmapsto (n-i)(i+1)!\, x_{i+1} = i!\,\mathbf{D}_*(x_i).$$

Let us consider a set $\mathcal{J} \subset \mathfrak{G}$, $|\mathcal{J}| = n$. The linear multiplicative map $\alpha_0^{d-i}\alpha_1^i \longmapsto i!\, x_i$ for all $\alpha \in \mathcal{J}$ determines the $\mathfrak{sl}_2$-homomorphism of the component $(\mathrm{Sym}\,(V_{\mathcal{J}}))_{\mathbf{w}=(d)^n}$ into $\mathrm{Sym}^n(V_d)$.

Let us now consider the component $\left(\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}\right)_{\mathbf{w}}$, where $\mathbf{w} = (n_1)^{m_1}(n_2)^{m_2}\cdots(n_t)^{m_t}$ and $m_1 + \cdots + m_t = |\mathcal{J}|$. Let P be a symbol expression of $\left(\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}\right)_{\mathbf{w}}$. Let φ be a surjective map of the coset $\mathrm{supp} P/\sim$ into a finite set $\mathcal{I} \subset \mathfrak{R}$. Define the map Λ by

$$\alpha_0^{\mathrm{wt}_\alpha - i}\alpha_1^i \mapsto i!\, \varphi(\alpha)_i,$$

$i = 0, 1, \ldots \mathrm{wt}_\alpha$, for all $\alpha \in \operatorname{supp} P$ and extend it in the natural way to monomials. Then $\Lambda(P)$ is a multihomogeneous polynomial of multidegree $\mathbf{m} := (m_1, m_2, \ldots m_t)$ in the set of t letters $\mathcal{I}$. We associate to each roman letter $x \in \mathcal{I}$ the variable set $x_0, x_1, \ldots, x_{\mathrm{wt}_\alpha}$, where $\varphi(\alpha) = x$, and $\mathrm{wt}_\alpha = n_i$ for some i. Therefore we can conclude that $\Lambda(P) \in (\operatorname{Sym}(V_{\mathcal{I},\boldsymbol{d}}))_{\mathbf{m}}$. Since Λ is a $\mathfrak{u}_2$-homomorphism and $\mathbf{D}(P) = 0$, we see that the following inclusion holds:

$$\Lambda\left(\left(\operatorname{Sym}(V_{\mathcal{J}})^{\mathfrak{u}_2}\right)_{\mathbf{w}}\right) \subseteq \left(\operatorname{Sym}(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}\right)_{\mathbf{m}}.$$

Let us prove the surjectivity of the map Λ. Rewrite the $\mathfrak{u}_2$-module $V_{\mathcal{I},\boldsymbol{d}}$ as $V_{\mathcal{I},\boldsymbol{d}} = \oplus_\alpha m_i V_{\varphi(\alpha),n_i}$, where α runs over all coset representatives. Every semi-invariant is a sum of multihomogeneous semi-invariants. Moreover, a multihomogeneous semi-invariant $S \in \left(\operatorname{Sym}(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}\right)_{\mathbf{m}}$ of multidegree $(m_1, m_2, \ldots m_t)$ can be polarized to produce a *multilinear* semi-invariant $\mathcal{P}(S)$ of multidegree $(\underbrace{1, 1, \ldots, 1}_{|\mathcal{J}| \text{ times}})$. Note that

$$\mathcal{P}(S) \in \left(\operatorname{Sym}\left(\widetilde{V}_{\mathcal{I},\boldsymbol{d}}\right)^{\mathfrak{u}_2}\right)_{(1,1,\ldots,1)},$$

where $\widetilde{V}_{\mathcal{I},\boldsymbol{d}} = \oplus_{x \in \widetilde{\mathcal{I}}} V_{x,n_i}$, $\widetilde{\mathcal{I}}$ is union $\mathcal{I}$ with the set of new polarizing variables, $|\widetilde{\mathcal{I}}| = |\mathcal{J}|$. Clearly, $\mathcal{P}(S)$ can be reconstructed from S by restitution. Since $|\widetilde{\mathcal{I}}| = |\mathcal{J}|$ we can associate to each symbol letter $\alpha \in \mathcal{J}$ a vector space V_{x,n^*}, for some natural n^*. Extend the map $x_k \mapsto 1/k!\, \alpha_0^{n^*-k} \alpha_1^k$ multiplicatively to all monomials and denote it by $\widetilde{\Lambda}$. We have the commutative diagram

$$\begin{tikzcd}
\left(\operatorname{Sym}(V_{\mathcal{J}})^{\mathfrak{u}_2}\right)_{\mathbf{d}^{\mathbf{m}}} \arrow[rr, "\Lambda"] & & \left(\operatorname{Sym}(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}\right)_{\mathbf{m}} \arrow[dl, "\mathcal{P}"] \\
& \left(\operatorname{Sym}\left(\widetilde{V}_{\mathcal{I},\boldsymbol{d}}\right)^{\mathfrak{u}_2}\right)_{(\underbrace{1,1,\ldots,1}_{|\mathcal{I}| \text{ times}})} \arrow[ul, "\widetilde{\Lambda}"] &
\end{tikzcd}$$

where all arrows are $\mathfrak{sl}_2$-homomorphisms, and $\Lambda\left(\widetilde{\Lambda}(\mathcal{P}(S))\right) = S$. Thus Λ is a surjective map.

The part of the theorem concerning $\ker \Lambda \circ \chi$ can be proved in the same manner as the proof in the preprint [14], see also [13]. $\square$

Observe that $\left(\mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}\right)_{\mathbf{m}} = (\ker \mathcal{D}_{\boldsymbol{d}})_{\mathbf{m}}$, where $\mathcal{D}_{\boldsymbol{d}}$ is a derivation of the polynomial algebra $\mathbb{K}[\mathcal{I}]$. Therefore we obtain a handy tool for writing elements of the kernel of arbitrary linear locally nilpotent derivations.

It is best to look now at some examples.

Example 3.1. Let us to consider the symbolic expression $P := [\alpha,\beta]^2[\beta,\gamma]\gamma_0^2$. Then $\mathrm{supp}\,P = \{\alpha,\beta,\gamma\}$, $\mathrm{wt}\,P = (2,3,3) = (2)^1(3)^2$, $\mathrm{supp}\,P/\sim = \{\{\alpha\},\{\beta,\gamma\}\}$, $|\mathrm{supp}\,P/\sim| = 2$. Let $\mathcal{I} = \{x,y\}$. Associate α to x and both β,γ associate to y. Since $\boldsymbol{d} = (2,3)$ then the map Λ acts by

$$\begin{aligned}
&\alpha_0^2 \mapsto x_0, \alpha_0\alpha_1 \mapsto x_1, \alpha_1^2 \mapsto 2!\,x_2,\\
&\beta_0^3 \mapsto y_0, \beta_0^2\beta_1 \mapsto y_1, \beta_0\beta_1^2 \mapsto 2!\,y_2, \beta_1^3 \mapsto 3!\,y_3,\\
&\gamma_0^3 \mapsto y_0, \gamma_0^2\gamma_1 \mapsto y_1, \gamma_0\gamma_1^2 \mapsto 2!\,y_2, \gamma_1^3 \mapsto 3!\,y_3.
\end{aligned}$$

We have

$$\begin{aligned}
P := [\alpha,\beta]^2[\beta,\gamma]\gamma_0^2 &= {\alpha_0}^2{\beta_1}^2\beta_0{\gamma_0}^2\gamma_1 - {\alpha_0}^2{\beta_1}^3{\gamma_0}^3 - 2\,\alpha_0\alpha_1\beta_1{\beta_0}^2{\gamma_0}^2\gamma_1 +\\
&+2\,\alpha_0\alpha_1{\beta_1}^2\beta_0{\gamma_0}^3 + {\alpha_1}^2{\beta_0}^3{\gamma_0}^2\gamma_1 - {\alpha_1}^2{\beta_0}^2\beta_1{\gamma_0}^3.
\end{aligned}$$

Thus

$$\begin{aligned}
\Lambda(P) &= 2\,x_0y_2y_1 - 6\,x_0y_3y_0 - 2\,x_1y_1^2 + 4\,x_1y_2y_0 + 2\,x_2y_0y_1 - 2\,x_2y_1y_0 =\\
&= 2\,x_0y_2y_1 - 6\,x_0y_3y_0 - 2\,x_1y_1^2 + 4\,x_1y_2y_0.
\end{aligned}$$

Consider the polynomial algebra $\mathbb{K}[X_2,Y_3] := \mathbb{K}[x_0,x_1,x_2,y_0,y_1,y_2,y_3]$. The polynomial $\Lambda(P)$ belongs to the kernel of the derivation $\mathcal{D}_{(2,3)}$ of the algebra $\mathbb{K}[X_2,Y_3]$ defined by

$$\mathcal{D}_{(2,3)}(x_i) = x_{i-1}, i = 1,2, \mathcal{D}_{(2,3)}(y_i) = y_{i-1}, i = 1,2,3, \mathcal{D}_{(2,3)}(x_0) = \mathcal{D}_{(2,3)}(y_0) = 0.$$

Example 3.2. Let $P = [\alpha,\beta]^n$. Then $\mathrm{supp}\,P = \{\alpha,\beta\}$, $\mathrm{wt}\,P = (n,n) = (n)^2$, $\mathrm{supp}\,P/\sim = \{\{\alpha,\beta\}\}$, $|\mathrm{supp}\,P/\sim| = 1$. Let $\mathcal{I} = \{x\}$ and associate α,β to x. The map Λ acts by

$$\alpha_0^{n-i}\alpha_1^i \mapsto i!\,x_i, \beta_0^{n-i}\beta_1^i \mapsto i!\,x_i, i = 0,1,\ldots,n.$$

We have

$$\Lambda\left([\alpha,\beta]^n\right) = \Lambda\left(\sum_{i=0}^{n}(-1)^i\alpha_0^{n-i}\alpha_1^i\beta_0^i\beta_1^{n-i}\right) = \sum_{i=0}^{n}(-1)^i i!\,(n-i)!\,x_ix_{n-i}.$$

Observe that $[\alpha,\beta]^n = (-1)^n[\beta,\alpha]^n$ but, obviously, $\Lambda\left([\alpha,\beta]^n\right) = \Lambda\left([\beta,\alpha]^n\right)$. It follows that $\Lambda\left([\alpha,\beta]^n\right) = 0$, for odd n. Note, that the polynomial belongs to the kernel of the basic Weitzenböck derivation $\mathcal{D}_n$ of $\mathbb{K}[V_{x,n}]$ defined by $\mathcal{D}_n(x_i) = x_{i-1}$.

Example 3.3. Consider the polynomial $A = 3\,x_1x_2x_0 - 3\,x_3{x_0}^2 - {x_1}^3 \in \ker\mathcal{D}_3$. Find a symbolic expression of the semi-invariant A. To get a multilinear polynomial polarize A two times with respect to the letters y and z:

$$\begin{aligned}
P_y(A) =& 3\,y_0x_1x_2 - 6\,y_0x_3x_0 + 3\,y_1x_0x_2 - 3\,y_1{x_1}^2 + 3\,y_2x_0x_1 - 3\,y_3{x_0}^2,\\
\mathcal{P}(A) =& P_z(P_y(A)) = 3\,z_0y_1x_2 - 6\,z_0y_0x_3 + 3\,z_0y_2x_1 - 6\,z_0y_3x_0 + 3\,z_1y_0x_2-\\
&- 6\,z_1y_1x_1 + 3\,z_1y_2x_0 + 3\,z_2y_0x_1 + 3\,z_2y_1x_0 - 6\,z_3y_0x_0.
\end{aligned}$$

The polynomial $\mathcal{P}(A)$ has multidegree $(1,1,1)$. The map $\widetilde{\Lambda}$ acts by

$$x_i \mapsto 1/i!\,\alpha_0^{3-i}\alpha_1^i,\, y_i \mapsto 1/i!\,\beta_0^{3-i}\beta_1^i,\, z_i \mapsto 1/i!\,\gamma_0^{3-i}\gamma_1^i.$$

We have

$$\begin{aligned}
\widetilde{\Lambda}\left(\mathcal{P}(A)\right) = & -{\gamma_0}^3{\beta_0}^3{\alpha_1}^3 + 3/2\,{\gamma_0}^3{\beta_0}^2\beta_1\alpha_0{\alpha_1}^2 + 3/2\,{\gamma_0}^3\beta_0{\beta_1}^2{\alpha_0}^2\alpha_1 - {\gamma_0}^3{\beta_1}^3{\alpha_0}^3+\\
&+3/2\,{\gamma_0}^2\gamma_1{\beta_0}^3\alpha_0{\alpha_1}^2 - 6\,{\gamma_0}^2\gamma_1{\beta_0}^2\beta_1{\alpha_0}^2\alpha_1 + 3/2\,{\gamma_0}^2\gamma_1\beta_0{\beta_1}^2{\alpha_0}^3+\\
&+3/2\,\gamma_0{\gamma_1}^2{\beta_0}^3{\alpha_0}^2\alpha_1 + 3/2\,\gamma_0{\gamma_1}^2{\beta_0}^2\beta_1{\alpha_0}^3 - {\gamma_1}^3{\beta_0}^3{\alpha_0}^3.
\end{aligned}$$

After simplification we obtain

$$2\,\widetilde{\Lambda}\left(\mathcal{P}(A)\right) = 3\,\beta_0{\gamma_0}^2[\alpha,\beta]^2[\alpha,\gamma] + 3\,{\beta_0}^2\gamma_0[\alpha,\beta][\alpha,\gamma]^2 - 2\,{\beta_0}^3[\alpha,\gamma]^3 - 2\,{\gamma_0}^3[\alpha,\beta]^3.$$

Taking into account $\Lambda({\gamma_0}^3[\alpha,\beta]^3) = \Lambda({\beta_0}^3[\alpha,\gamma]^3) = 0$, and

$$\Lambda(\beta_0{\gamma_0}^2[\alpha,\beta]^2[\alpha,\gamma]) = \Lambda({\beta_0}^2\gamma_0[\alpha,\beta][\alpha,\gamma]^2),$$

we get that $\widetilde{\Lambda}\left(\mathcal{P}(A)\right) = 3\,\beta_0{\gamma_0}^2[\alpha,\beta]^2[\alpha,\gamma]$. Thus $\beta_0{\gamma_0}^2[\alpha,\beta]^2[\alpha,\gamma]$ is a symbolic expression of A and

$$A = \Lambda\left(\beta_0{\gamma_0}^2[\alpha,\beta]^2[\alpha,\gamma]\right).$$

Example 3.4. Let $P = \prod_{\alpha<\beta}[\alpha,\beta]^2$, $|\,\mathrm{supp}P| = 2\,k$. We have, see the proof in [15], that

$$\Lambda\left(\prod_{\alpha<\beta}[\alpha,\beta]^2\right)=$$

$$=(k+1)!\begin{vmatrix} x_0 & x_1 & 2!\,x_2 & \cdots & k!\,x_k \\ x_1 & 2x_2 & 3!x_3 & \cdots & (k+1)!x_{k+1} \\ \dots & \dots & \dots & \dots & \dots \\ (k-1)!x_{k-1} & k!x_k & (k+1)!x_{k+1} & \cdots & (2k-1)!x_{2k-1} \\ k!x_k & (k+1)!\,x_{k+1} & (k+2)!\,x_{k+2} & \cdots & (2k)!\,x_{2k} \end{vmatrix}.$$

This semi-invariant belongs to $\ker \mathcal{D}_{2\,k}$, has degree $k+1$ and is called the *catalecticant.*

4. Convolution and semi-tranvectant. There is a simple and effective way to find semi-invariants of given multidegree $\mathbf{m}$. The following differential operator on $\mathrm{Sym}\,(V_{\mathcal{J}})$

$$\mathrm{Conv}_{\alpha,\beta}:=[\alpha,\beta]\frac{\partial^2}{\partial\alpha_0\partial\beta_0},\alpha,\beta\in\mathcal{J},$$

is called a *convolution* with respect to the symbol letters α and β. Obviously, the convolution operator does not change the weight of a symbolic expression, so $\mathrm{Conv}_{\alpha,\beta}$ is an endomorphism of the vector space $\mathrm{Sym}\left(V_{\mathcal{J}}^{\mathfrak{u}_2}\right)_{\mathbf{d}^{\mathbf{m}}}$.

Example 4.1. Let us find elements of the kernel of the derivation $\mathcal{D}_{1,2,3}$ of multidegree $(1,2,1)$ and weight $(1,2,2,3)=(1)^1(2)^2(3)^1$. Let $\mathcal{J}=\{\alpha,\beta,\gamma,\delta\}$, $\mathcal{I}=\{x,y,z\}$. We associate the roman letter x to the symbol letter α, the letter y to both letters β,γ and the letter z to the symbol letter δ, respectively. The map Λ acts by $\alpha_i\mapsto x_i$, $i=0,1$, $\beta_0^{2-i}\beta_1^i\mapsto i!\,y_i$, $\gamma_0^{2-i}\gamma_1^i\mapsto i!\,y_i$, $i=0,1,2$, and $\delta_0^{3-i}\delta_1^i\mapsto i!\,z_i$, $z=0,1,2,3$. It is clear that the following symbolic expression $\Phi=\alpha_0\beta_0^2\gamma_0^2\delta_0^3$ has weight $(1)^1(2)^2(3)^1$. The polynomial $\Lambda(\Phi)$ is equal to $x_0y_0^2z_0$ and has multidegree $(1,2,1)$. By direct calculations we get

$$\mathrm{Conv}_{\alpha,\beta}(\Phi)=2\,[\alpha,\beta]\beta_0\gamma_0^2\delta_0^3\longmapsto 2\,y_0z_0\begin{vmatrix}x_0 & x_1\\ y_0 & y_1\end{vmatrix},$$

$$\mathrm{Conv}_{\gamma,\delta}\left(\mathrm{Conv}_{\alpha,\beta}(\Phi)\right)=12\,[\alpha,\beta][\gamma,\delta]\beta_0\gamma_0\delta_0^2\mapsto 12\begin{vmatrix}x_0 & x_1\\ y_0 & y_1\end{vmatrix}\begin{vmatrix}y_0 & y_1\\ z_0 & z_1\end{vmatrix},$$

$$\mathrm{Conv}^2_{\gamma,\delta}\left(\mathrm{Conv}_{\alpha,\beta}(\Phi)\right)=24\,[\alpha,\beta][\gamma,\delta]^2\beta_0\delta_0\mapsto 48\,(z_0y_2-y_1z_1+y_0z_2)\begin{vmatrix}x_0 & x_1\\ y_0 & y_1\end{vmatrix},$$

$\mathrm{Conv}_{\beta,\delta}\left(\mathrm{Conv}^2_{\gamma,\delta}\left(\mathrm{Conv}_{\alpha,\beta}(\Phi)\right)\right) = 24\,[\alpha,\beta][\gamma,\delta]^2[\beta\delta] \mapsto 48\,(3\,x_0y_0z_3y_1-$

$-2\,x_0y_2y_0z_2 - 2\,x_0{y_1}^2z_2 + 3\,x_0y_2y_1z_1 - 2\,x_0{y_2}^2z_0 - 3\,{y_0}^2x_1z_3 + 3\,y_1x_1y_0z_2-$

$-{y_1}^2x_1z_1 - y_0x_1z_1y_2 + y_1x_1z_0y_2)$.

For a symbolic expression P denote by $\mathrm{Conv}(P)$ the set of all its possible convolutions. For a subalgebra $\Delta \in \mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}$ denote by $\mathrm{Conv}(\Delta)$ the subalgebra generated by all possible convolutions $\mathrm{Conv}(P)$, $P \in \Delta$. The following statement holds:

Lemma 4.1.

$$\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2} = \mathrm{Conv}(\mathrm{Sym}\,(\oplus_{\alpha\in\mathcal{J}}\mathbb{K}\alpha_0)).$$

P r o o f. Let $P \in (\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2})_{\mathbf{w}}$. Then P is obtained by convolutions of the semi-invariants $\prod_{\alpha\in\mathcal{J}}\alpha_0^{\mathrm{wt}_\alpha} \in \mathrm{Sym}\,(\oplus_{\alpha\in\mathcal{J}}\mathbb{K}\alpha_0)$. $\square$

Example 4.2. Let $\mathcal{J} = \{\alpha,\beta,\gamma\}$. The component $\left(\mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}\right)_{(2,1,1)}$ is generated by the following 5 semi-invariants

$$\alpha_0^2\beta_0\gamma_0, \alpha_0\gamma_0[\alpha,\beta], \alpha_0\beta_0[\alpha,\gamma], \alpha_0^2[\beta,\gamma], [\alpha,\gamma][\alpha,\beta].$$

All of them are convolutions of the semi-invariant $\alpha_0^2\beta_0\gamma_0$.

Let $F \in \mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}$ and let $\Phi \in \mathrm{Sym}\,(V_{\mathcal{J}})^{\mathfrak{u}_2}$ be its symbolic representation. Denote by $\mathrm{Conv}_\Lambda(F)$ the set of elements $\Lambda\,(\mathrm{Conv}(\Phi))$. The elements of $\mathrm{Conv}_\Lambda(F)$ are called Λ-*convolutions*. For a subalgebra $\mathcal{T} \subset \mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}$ denote by $\mathrm{Conv}_\Lambda(\mathcal{T})$ the algebra generated by all Λ-convolutions of all elements of the algebra $\mathcal{T}$.

Theorem 4.1. *Let* $\mathcal{N}_{\mathcal{I}} := \mathbb{K}[x_0 \mid x \in \mathcal{I}]$. *Then* $\mathrm{Sym}\,(V_{\mathcal{I},\mathbf{d}})^{\mathfrak{u}_2} = \mathrm{Conv}_\Lambda(\mathcal{N}_{\mathcal{I}})$.

P r o o f. Consider an arbitrary homogeneous semi-invariant $F \in \mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}$. Let Φ be its symbolic expression. By Lemma 4.1, Φ belongs to $\mathrm{Conv}(\mathrm{Sym}\,(\oplus_{\alpha\in\mathcal{J}}\mathbb{K}\alpha_0))$ and $|\mathcal{J}| = \deg F$. It is easy to see that

$$\Lambda(\mathrm{Sym}\,(\oplus_{\alpha\in\mathcal{J}}\mathbb{K}\alpha_0)) = \mathcal{N}_{\mathcal{I}} \text{ and } \Lambda\,(\mathrm{Conv}(\mathrm{Sym}\,(\oplus_{\alpha\in\mathcal{J}}\mathbb{K}\alpha_0))) = \mathrm{Conv}_\Lambda(\mathcal{N}_{\mathcal{I}}),$$

thus $\Lambda(\Phi) = F \in \mathrm{Conv}(\mathcal{N}_{\mathcal{I}})$. We get $\mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2} \subseteq \mathrm{Conv}_\Lambda\,(\mathcal{N}_{\mathcal{I}})$. The inclusion $\mathrm{Conv}_\Lambda\,(\mathcal{N}_{\mathcal{I}}) \subseteq \mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}$ is obvious. $\square$

We have obtained a tool to find a generating system of the kernel for Weitzenböck derivations.

Example 4.3. Let $|\mathcal{I}| = n$ and $\mathbf{d} = (1, 1, \ldots, 1)$. Prove that

$$\ker \mathcal{D}_{\boldsymbol{d}} = \mathbb{K}[x_0, x_0y_1 - x_1y_0 \mid x \neq y, x, y \in \mathcal{I}].$$

In fact, let F be a homogeneous polynomial of $\mathcal{N}_{\mathcal{I}}$ of degree m. Then its symbolic representation Φ has the form $\Phi = \prod_{\alpha \in \mathcal{J}} \alpha_0$, $\mathcal{J} \subset \mathfrak{G}$, $|\mathcal{J}| = m$. Since all factors of Φ have degrees 1, we obtain that all possible convolutions of the polynomial Φ have the form $\prod_{\alpha \in \mathcal{J}} \alpha_0 \prod_{\beta,\gamma \in \mathcal{J}} [\beta, \gamma]$, for $\alpha \neq \beta, \gamma$. It follows

$$\Lambda \left(\prod_{\alpha \in \mathcal{J}} \alpha_0 \prod_{\beta,\gamma \in \mathcal{J}} [\beta, \gamma] \right) = \prod_{x \in \mathcal{I}} x_0 \prod_{y,z \in \mathcal{I}} (y_0z_1 - y_1z_0), x \neq y, z.$$

Thus $\mathrm{Conv}_\Lambda(F) \in \mathbb{K}[x_0, x_0y_1 - x_1y_0 \mid x \neq y, x, y \in \mathcal{I}]$ and

$$\mathrm{Conv}_\Lambda(\mathcal{N}_{\mathcal{I}}) = \mathbb{K}[x_0, x_0y_1 - x_1y_0 \mid x \neq y, x, y \in \mathcal{I}].$$

Example 4.4. Let $|\mathcal{I}| = n \geq 3$, $\mathbf{d} = (2, 2, \ldots, 2)$ and let F be a homogeneous polynomial of degree m in $\mathcal{N}_{\mathcal{I}}$. Then its symbolic representation Φ has the form $\Phi = \prod_{\alpha \in \mathcal{J}} \alpha_0^2$, $\mathcal{J} \subset \mathfrak{G}$, $|\mathcal{J}| = m \cdot n$. For $m = 1$ we have $\Phi = \alpha_0^2, \alpha \in \mathcal{J}$. It is obvious that $\mathrm{Conv}(\Phi) = \{\alpha_0^2, \alpha \in \mathcal{J}\}$.

For $m = 2$ we have $\Phi = \alpha_0^2\beta_0^2$, $\alpha \neq \beta$, α, $\beta \in \mathcal{J}$. There are only two convolutions of $\Phi : \alpha_0\beta_0[\alpha, \beta]$ and $[\alpha, \beta]^2$.

For $m = 3$ we have $\Phi = \alpha_0^2\beta_0^2\gamma_0^2$, α, β, $\gamma \in \mathcal{J}$. There exists a unique indecomposable convolution: $[\alpha, \beta][\alpha, \gamma][\beta, \gamma]$.

If $m > 3$ then any symbol expression is either decomposable or belongs to $\ker \Lambda$, see [12, p. 162]. We have

$$\Lambda(\alpha_0^2) = x_0, \Lambda\left(\alpha_0\beta_0[\alpha, \beta]\right) = \begin{vmatrix} x_0 & x_1 \\ y_0 & y_1 \end{vmatrix}, \Lambda\left([\alpha, \beta]^2\right) = 2\left(x_0y_2 - x_1y_1 + x_2y_2\right),$$

$$\begin{gathered} \Lambda\left([\alpha, \beta][\alpha, \gamma][\beta, \gamma]\right) = \Lambda({\alpha_0}^2\beta_1{\gamma_1}^2\beta_0 - {\alpha_0}^2{\beta_1}^2\gamma_1\gamma_0 + \\ +\alpha_0{\beta_1}^2{\gamma_0}^2\alpha_1 - {\beta_0}^2\alpha_1\alpha_0{\gamma_1}^2 + {\beta_0}^2{\alpha_1}^2\gamma_0\gamma_1 - \beta_0{\alpha_1}^2{\gamma_0}^2\beta_1) = \\ = 2\,x_0z_2y_1 - 2\,x_0y_2z_1 + 2\,x_1y_2z_0 - 2\,y_0x_1z_2 + 2\,y_0x_2z_1 - 2\,y_1x_2z_0 = \\ = 2 \begin{vmatrix} x_0 & y_0 & z_0 \\ x_1 & y_1 & z_1 \\ x_2 & y_2 & z_2 \end{vmatrix}. \end{gathered}$$

Thus, the kernel $\ker \mathcal{D}_{(2,2,\dots,2)}$ is generated by the semi-invariants of these four types.

As d_i increases it becomes harder to apply Theorem 4.1. We will offer other similar but more effective approach to find the kernel of Weitzenböck derivations. In the paper [1] we introduced the concept of semi-transvectant, an analogue of the classical transvectant. Recall that the algebra $\mathfrak{u}_2$ acts on $\mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})$ by the locally nilpotent derivation $\mathbf{D} = \mathcal{D}_{\boldsymbol{d}}$. Let $F, G \in \mathrm{Sym}\,(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}$ be two semi-invariants of degrees p and q, respectively. The semi-invariant of the form

$$[F,G]^r := \sum_{i=0}^{r} (-1)^i \binom{r}{i} \frac{\mathbf{D}_*^i(F)}{[p]_i} \frac{\mathbf{D}_*^{r-i}(G)}{[q]_{r-i}}, \tag{5}$$

$0 \le r \le \min(p,q)$, $[m]_i := m(m-1)\dots(m-(i-1)), m \in \mathbb{Z}$, is called the r-*th semi-transvectant* of the semi-invariants F and G.

Example 4.5. The semi-transvectant $[F,G]^1 := [F,G]$ is called *a semi-Jacobian.* If F, G, H are three semi-invariants of orders greater than unity, then the iterated semi-Jacobian $[[F,G],H]$ is reducible [10] and

$$[[F,G],H] = \frac{\mathrm{ord}(F) - \mathrm{ord}(G)}{2\,(\mathrm{ord}(F) + \mathrm{ord}(G) - 2)} + \frac{1}{2}[F,G]^2 H + \frac{1}{2}[F,H]^2 G - \frac{1}{2}[G,H]^2 F.$$

Example 4.6. The semi-invariant $[F,F]^2 := \mathrm{Hes}(F)$ is called *a semi-Hessian.* The square of a semi-Jacobian $[F,G]$ is given by the formula

$$[F,G][F,G] = [F,G]^2 F\,G - \frac{1}{2}\mathrm{Hes}(F)G^2 - \frac{1}{2}\mathrm{Hes}(G)F^2.$$

Example 4.7. We have

$$\begin{aligned}\left(\alpha_0^2, \beta_0\gamma_0^2\right)^2 = \frac{1}{3}\,{\alpha_0}^2\beta_1\gamma_1\gamma_0 + \frac{1}{6}\,{\alpha_0}^2\beta_0{\gamma_1}^2 - \frac{1}{3}\,\alpha_1\alpha_0\beta_1{\gamma_0}^2 - \frac{2}{3}\,\alpha_1\alpha_0\gamma_1\beta_0\gamma_0 + \\ + \frac{1}{2}\,{\alpha_1}^2\beta_0{\gamma_0}^2 = \frac{1}{2}\,[\alpha,\gamma]^2\beta_0 - \frac{1}{3}\,\alpha_0[\alpha,\gamma][\beta,\gamma].\end{aligned}$$

The following statement holds:

Lemma 4.2 ([1]).
(*i*) *For* $0 \le r \le \min(\mathrm{ord}(x_0), \max(\mathrm{ord}(F), \mathrm{ord}(G))$ *the semi-transvectant* $[x_0, F\,G]^r$ *is reducible;*
(*ii*) *If* $\mathrm{ord}(F) = 0$, *then* $[x_0, F\,G]^r = F[x_0, G]^r$;

(iii) $\operatorname{ord}([F,G]^r) = \operatorname{ord}(F) + \operatorname{ord}(G) - 2r$.

There is a close relationship between convolutions and semi-transvectants. A *k-fold contraction* of two disjoint symbolic expressions Φ and Ψ is the symbolic expression

$$\left(\prod_{\alpha,\beta} \operatorname{Conv}_{\alpha,\beta}\right)(\Phi\cdot\Psi),$$

where the product runs over k pairs $(\alpha,\beta) \in \operatorname{Supp}\Phi \times \operatorname{Supp}\Psi$.

Lemma 4.3 ([14]). *Let Φ and Ψ be two disjoint symbolic expressions. The semi-transvectant $[\Phi,\Psi]^k$ is a linear combination of semi-invariants T, where T runs through the k-fold contractions of Φ and Ψ, and each such T occurs with a positive rational coefficient q_T, where $\sum_T q_T = 1$.*

Using semi-transvectants we offer an algorithm for computation of the kernel of Weitzenböck derivations. For $x \in \mathcal{I}$ let $\tau_{x,i}(F) := (x_0, F)^i$, $i \le \min(\operatorname{ord}(x_0), \operatorname{ord}(F))$. For a subalgebra $T \subseteq \operatorname{Sym}(V_{\mathcal{I},\boldsymbol{d}})^{\mathfrak{u}_2}$ denote by $\tau(T)$ the algebra generated by the elements $\tau_{x,i}(F), F \in T$, $i \leqslant \min(\operatorname{ord}(x_0),\operatorname{ord}(F))$. The following theorem is a weak form of Gordan's theorem:

Theorem 4.2 ([4]). *Let $\mathcal{T}$ be a subalgebra of $\ker\mathcal{D}_{\mathbf{d}}$ containing $\mathcal{N}_I$ and $\tau(\mathcal{T}) \subseteq \mathcal{T}$. Then $\ker\mathcal{D}_{\mathbf{d}} = \mathcal{T}$.*

The theorem implies the following algorithm for $\ker\mathcal{D}_{\boldsymbol{d}}$. Define the series of subalgebras

$$\mathcal{T}_1 \subseteq \mathcal{T}_2 \subseteq \mathcal{T}_3 \subseteq \cdots \subset \mathcal{T}_k \subseteq \cdots,$$

where $\mathcal{T}_1 = \mathcal{N}_{\mathcal{I}}$ and $\mathcal{T}_i := \tau(\mathcal{T}_{i-1})$. If for some i we have that $\mathcal{T}_i = \mathcal{T}_{i+1}$, then $\mathcal{T}_i = \ker\mathcal{D}_{\boldsymbol{d}}$.

Example 4.8. Consider the basic Weitzenböck derivation $\mathcal{D}_3$, $\mathcal{I} = \{x\}$. We have $\mathcal{T}_1 = \mathbb{K}[x_0]$. The subalgebra $\mathcal{T}_2$ is generated by the elements $\tau_i(x_0^j)$. By Lemma 4.2, (i) for $j > 1$ all of them are reducible ones. The only irreducible semi-invariant is $dv = \tau_2(x_0)$. Therefore, $\mathcal{T}_2 = \mathbb{K}[x_0, dv]$. The algebra $\mathcal{T}_3$ is generated by $\tau_i(x_0^k dv^l)$, $i,k,l \in \mathbb{N}$. Since $\operatorname{ord}(dv) = 2$, then Lemma 4.2, (i) implies that the algebra $\mathcal{T}_3$ is generated only by the elements $\tau_1(dv), \tau_2(dv)$ and $\tau_3(dv^2)$. By (5) we obtain that $\tau_3(dv^2) = 0$, $\tau_2(dv) = 0$ and $tr := \tau_1(dv) \neq 0$. The direct calculation shows that tr does not belong to T_2, thus $T_3 = \mathbb{K}[t, dv, tr]$. The algebra $\mathcal{T}_4$ is generated by $\tau_i(x_0^k dv^l tr^m)$, $i,k,l,m \in \mathbb{N}$. As above we find the only new element $ch = \tau_3(tr)$ for $\mathcal{T}_4$. We have $\operatorname{ord}(ch) = 0$ but the algebra $\mathcal{T}_3$ does not contain any such invariants. It implies that $\mathcal{T}_4 = \mathbb{K}[t, dv, tr, ch]$. By Lemma 4.2, (ii) we have that the algebra $\mathcal{T}_5$ does not contain any new semi-invariants. Thus $\mathcal{T}_5 = T_4$ and $\ker \mathbf{D} = \mathbb{K}[t, dv, tr, ch]$, where

$$dv = {x_1}^2 - 2\,x_0\,x_2,$$
$$tr = 3\,x_3{x_0}^2 + {x_1}^3 - 3\,x_0x_1x_2,$$
$$ch = 8\,x_0{x_2}^3 + 9\,{x_3}^2{x_0}^2 + 6\,{x_1}^3x_3 - 3\,{x_1}^2{x_2}^2 - 18\,x_0x_1x_2x_3.$$

Up to a constant factor the symbolic representations of these semi-invariants have the form $\alpha_0\beta_0[\alpha,\beta]^2$, $\beta_0{\gamma_0}^2{[\alpha,\beta]}^2[\alpha,\gamma]$ and $[\alpha,\beta]^2[\alpha,\gamma][\beta,\delta][\gamma,\delta]^2$, respectively.

For $d = 4, 5, 6$ the kernel of the basic Weitzenböck derivation was calculated in [4]. The cases $d = 7, 8$ were considered in [2], [6]. For $d > 8$ the problem is still open, however the corresponding algebras of invariants were calculated for $d = 9, 10$ in [8], [7].

Example 4.9. Consider the derivation $\mathcal{D}_{(2,3)}$, $\mathcal{I} = \{x, y\}$, $\mathrm{ord}(x_0) = 2$, $\mathrm{ord}(y_0) = 3$. We associate the letter x to the set $\mathcal{J}_1 := \{\alpha, \beta, \gamma, \delta\}$ and the letter y to the set $\mathcal{J}_2 := \{\varepsilon, \varkappa, \eta, \mu\}$, respectively. The map Λ acts by $\nu_0^{2-i}\nu_1^i \mapsto i!\,x_i$, $i = 0, 1, 2$ for $\nu \in \mathcal{J}_1$ and by $\nu_0^{3-i}\nu_1^i \mapsto i!\,y_i$, $i = 0, 1, 2, 3$ for $\nu \in \mathcal{J}_2$. Let $D := \Lambda\left([\alpha,\beta]^2\right)$, $\Delta := \Lambda\left([\epsilon,\varkappa]^2\epsilon_0\varkappa_0\right)$, $Q := \Lambda\left([\varepsilon,\varkappa]^2[\eta,\varepsilon]\varkappa_0\eta_0^3\right)$, $R := \Lambda\left([\varepsilon,\varkappa]^2[\varepsilon,\eta][\varkappa,\mu][\eta,\mu]^2\right)$.

The minimal generating set for $\ker \mathcal{D}_{(2,3)}$ consists of the following 15 elements:

$$x_0, y_0,$$
$$D, [x_0, y_0]^2, \Delta, [x_0, y_0],$$
$$[x_0, \Delta]^2, [x_0^2, y_0]^3, [x_0, \Delta], Q,$$
$$R, [x_0, Q]^2,$$
$$[x_0^3, y_0^2]^6, [x_0^2, Q]^3,$$
$$[x_0^3, y_0 Q]^6.$$

The proof for the corresponding algebras of covariants is in [11] or in [12], [10].

For instance, let us calculate the explicit form of the semi-invariant $(x_0, Q)^2$. We have

$$\left(\alpha_0^2, [\varepsilon,\varkappa]^2[\eta,\varepsilon]\varkappa_0\eta_0^3\right)^2 = [\varepsilon,\varkappa]^2[\eta,\varepsilon]\left(\alpha_0^2, \varkappa_0\eta_0^3\right)^2 =$$

$$= [\varepsilon,\varkappa]^2[\eta,\varepsilon]\left(\frac{1}{6}\alpha_0^2\varkappa_0\eta_1^2+\frac{1}{3}\alpha_0^2\varkappa_1\eta_1\eta_0-\frac{2}{3}\alpha_1\alpha_0\eta_1\varkappa_0\eta_0- -\frac{1}{3}\alpha_1\alpha_0\varkappa_1\eta_0^2+\frac{1}{2}\,\varkappa_0\eta_0^2\alpha_1^2\right)=$$

$$=\frac{5}{6}\,{\alpha_0}^2\varkappa_0{\eta_1}^2{\varepsilon_0}^2{\varkappa_1}^2\eta_0\varepsilon_1-\frac{2}{3}\,{\alpha_0}^2{\varkappa_0}^2{\eta_1}^2\varepsilon_0\varkappa_1{\varepsilon_1}^2\eta_0-\frac{2}{3}\,{\alpha_0}^2{\varkappa_1}^2\eta_1{\eta_0}^2\varepsilon_0\varkappa_0{\varepsilon_1}^2+$$

$$+\frac{2}{3}\,\alpha_1\alpha_0{\eta_1}^2\varkappa_0\eta_0{\varepsilon_0}^3{\varkappa_1}^2+\frac{2}{3}\,\alpha_1\alpha_0{\eta_1}^2{\varkappa_0}^3\eta_0{\varepsilon_1}^2\varepsilon_0+\frac{2}{3}\,\alpha_1\alpha_0{\varkappa_1}^2{\eta_0}^3\varepsilon_0\varkappa_0{\varepsilon_1}^2+$$

$$+{\varkappa_0}^2{\eta_0}^2{\alpha_1}^2{\varepsilon_0}^2\varkappa_1\varepsilon_1\eta_1+\frac{1}{2}\,{\varkappa_0}^3{\eta_0}^3{\alpha_1}^2{\varepsilon_1}^3-\frac{1}{6}\,{\alpha_0}^2\varkappa_0{\eta_1}^3{\varepsilon_0}^3{\varkappa_1}^2+$$

$$+\frac{1}{6}\,{\alpha_0}^2{\varkappa_0}^3{\eta_1}^2{\varepsilon_1}^3\eta_0-\frac{1}{6}\,{\alpha_0}^2{\varkappa_0}^3{\eta_1}^3{\varepsilon_1}^2\varepsilon_0-\frac{1}{3}\,{\alpha_0}^2{\varkappa_1}^3{\eta_1}^2\eta_0{\varepsilon_0}^3+$$

$$+\frac{1}{3}\,{\alpha_0}^2{\varkappa_1}^3\eta_1{\eta_0}^2{\varepsilon_0}^2\varepsilon_1+\frac{1}{3}\,{\alpha_0}^2\varkappa_1\eta_1{\eta_0}^2{\varkappa_0}^2{\varepsilon_1}^3-\frac{2}{3}\,\alpha_1\alpha_0\eta_1{\varkappa_0}^3{\eta_0}^2{\varepsilon_1}^3+$$

$$+\frac{1}{3}\,\alpha_1\alpha_0{\varkappa_1}^3{\eta_0}^2{\varepsilon_0}^3\eta_1-\frac{1}{3}\,\alpha_1\alpha_0\varkappa_1{\eta_0}^3{\varkappa_0}^2{\varepsilon_1}^3+\frac{1}{2}\,\varkappa_0{\eta_0}^3{\alpha_1}^2{\varepsilon_0}^2{\varkappa_1}^2\varepsilon_1-$$

$$-\frac{1}{2}\,\varkappa_0{\eta_0}^2{\alpha_1}^2{\varepsilon_0}^3{\varkappa_1}^2\eta_1-{\varkappa_0}^2{\eta_0}^3{\alpha_1}^2\varepsilon_0\varkappa_1{\varepsilon_1}^2-\frac{1}{2}\,{\varkappa_0}^3{\eta_0}^2{\alpha_1}^2{\varepsilon_1}^2\varepsilon_0\eta_1-$$

$$-\frac{4}{3}\,\alpha_1\alpha_0\eta_1\varkappa_0{\eta_0}^2{\varepsilon_0}^2{\varkappa_1}^2\varepsilon_1-\frac{4}{3}\,\alpha_1\alpha_0{\eta_1}^2{\varkappa_0}^2\eta_0{\varepsilon_0}^2\varkappa_1\varepsilon_1+\frac{5}{3}\,\alpha_1\alpha_0\eta_1{\varkappa_0}^2{\eta_0}^2\varepsilon_0\varkappa_1{\varepsilon_1}^2+$$

$$+\frac{1}{3}\,{\alpha_0}^2{\varkappa_0}^2{\eta_1}^3{\varepsilon_0}^2\varkappa_1\varepsilon_1-\frac{1}{3}\,\alpha_1\alpha_0{\varkappa_1}^3{\eta_0}^3{\varepsilon_0}^2\varepsilon_1.$$

Then

$$(x_0,Q)^2=\Lambda\left(\left(\alpha_0^2,[\varepsilon,\varkappa]^2[\eta,\varepsilon]\varkappa_0\eta_0^3\right)^2\right)=$$

$$=6\,x_0{y_1}^2y_3-6\,y_2y_0x_2y_1-6\,x_1y_0y_1y_3-2\,x_0y_1{y_2}^2+6\,{y_0}^2x_2y_3-6\,x_0y_2y_3y_0+ +8\,x_1y_0{y_2}^2-2\,x_1y_2{y_1}^2+2\,{y_1}^3x_2.$$

The algebras of joint covariants which are isomorphic to the kernels of the following derivations $\mathcal{D}_{(2,4)}$, $\mathcal{D}_{(3,3)}$ and $\mathcal{D}_{(3,3,3)}$ were calculated in [11], [12], [18].

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