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OUTER ENDOMORPHISMS OF FREE METABELIAN LIE ALGEBRAS

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Abstract. Let F_m be the free metabelian Lie algebra of rank m over a field K of characteristic 0. We consider the semigroup $\mathrm{IE}(F_m)$ of the endomorphisms of F_m which are identical modulo the commutator ideal of F_m. We describe the factor semigroup of $\mathrm{IE}(F_m)$ modulo the congruence induced by the group of inner automorphisms.

Introduction. Let L_m be the free Lie algebra of rank $m \geq 2$ over a field K of characteristic 0 and let $F_m = L_m/L_m''$ be the free metabelian Lie algebra. This is the relatively free algebra of rank m in the metabelian (solvable of class 2) variety of Lie algebras $\mathfrak{A}^2$ defined by the identity

$$[[x_1, x_2], [x_3, x_4]] = 0.$$

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The description of the group of automorphisms of free algebras of finite rank is one of the most interesting problems of modern algebra. In 1964 Cohn [7] proved that automorphisms of finitely generated free Lie algebras are tame, i.e., the group of automorphisms is generated by the so called elementary automorphisms. A description of the defining relations between elementary automorphisms was given in 2006 by Umirbaev [19].

Free metabelian algebras form another interesting class of Lie algebras. There are many papers devoted to the study of automorphisms of free metabelian algebras, although many crucial questions are still open. Quite often earlier results on automorphisms of free metabelian groups serve as a model of the investigations in the Lie case. In 1965 Bachmuth [1] proved that automorphisms of free metabelian groups of rank 2 are tame and the IA-automorphisms which induce the identity automorphism in the abelianized group (i.e., modulo the commutator subgroup) are inner. In the 1980's Bachmuth and Mochizuki established that the group of automorphisms of the free metabelian group of rank 3 is not finitely generated and hence contains a lot of wild automorphisms [2]. On the other hand, all automorphisms of free metabelian groups of rank ≥ 4 are tame [3]. There are analogues of these results for metabelian Lie algebras. Shmel'kin [18] proved that all IA-automorphisms (i.e., the automorphisms which induce the identity map modulo the commutator ideal of F_m), are inner when $m = 2$. Since the by the theorem of Cohn [7] the only automorphisms of the free Lie algebra L_2 are the linear ones, this immediately implies that F_2 has wild automorphisms. Bahturin and Nabiyev [5] proved that the inner automorphisms of F_m are wild for all $m \geq 2$. Although wild, inner automorphisms form a very natural class. Roman'kov [17] established that the algebra F_3 has automorphisms which do not belong to the group generated by the tame and inner automorphisms. He announced in [16] that for $m \geq 4$ all automorphisms of F_m are products of tame and inner automorphisms. Let us mention also the recent papers by Daniyarova, Kazatchkov and Remeslennikov [8, 9, 10] on algebraic geometry of free metabelian Lie algebra, by Drensky and the author [11, 12] on automorphisms of free metabelian nilpotent Lie algebras, and Gerritzen [13] and Kurlin [15] on the Baker-Campbell-Hausdorff formula for free metabelian Lie algebras.

The automorphism group $\operatorname{Aut}(F_m)$ is a semidirect product of the normal subgroup $\operatorname{IA}(F_m)$ of IA-automorphisms and the general linear group $\operatorname{GL}_m(K)$. Since the group of inner automorphisms $\operatorname{Inn}(F_m)$ is a subgroup of $\operatorname{IA}(F_m)$, for the description of the factor group of the outer automorphisms $\operatorname{Out}(F_m) = \operatorname{Aut}(F_m)/\operatorname{Inn}(F_m)$ it is sufficient to know only $\operatorname{IA}(F_m)/\operatorname{Inn}(F_m)$. In the present paper we study the semigroup $\mathrm{IE}(F_m)$ of IA-endomorphisms of F_m, i.e., the endomorphisms which induce the identity automorphisms modulo the commutator ideal of F_m. Our main result describes the factor semigroup of $\mathrm{IE}(F_m)$ modulo the congruence induced by the group of inner automorphisms $\mathrm{Inn}(F_m)$ of the Lie algebra F_m. Since $\mathrm{IOut}(F_m) = \mathrm{IA}(F_m)/\mathrm{Inn}(F_m)$ is canonically embedded into $\mathrm{IE}(F_m)/\mathrm{Inn}(F_m)$, it is natural to work in the semigroup of outer IA-endomorphisms and then to recognize the automorphisms in $\mathrm{Out}(F_m)$.

A result of Shmel'kin [18] states that the free metabelian Lie algebra F_m can be embedded into the abelian wreath product $A_m \operatorname{wr} B_m$, where A_m and B_m are m-dimensional abelian Lie algebras with bases $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_m\}$, respectively. The elements of $A_m \operatorname{wr} B_m$ are of the form $\sum\limits_{i=1}^{m} a_i f_i(t_1, \dots, t_m) + \sum\limits_{i=1}^{m} \beta_i b_i$, where f_i are polynomials in $K[t_1, \dots, t_m]$ and $\beta_i \in K$. This allows to introduce partial derivatives in F_m with values in $K[t_1, \dots, t_m]$ and the Jacobian matrix $J(\phi)$ of an endomorphism ϕ of F_m. Restricted on the semigroup $\mathrm{IE}(F_m)$, the map $J : \phi \to J(\phi)$ is a semigroup monomorphism of $\mathrm{IE}(F_m)$ into the multiplicative semigroup of the algebra $M_m(K[t_1, \dots, t_m])$ of $m \times m$ matrices with entries from $K[t_1, \dots, t_m]$. We use the explicit form of the Jacobian matrices of inner automorphisms of F_m and give the coset representatives of the outer IA-endomorphisms in $\mathrm{IE}(F_m)/\mathrm{Inn}(F_m)$.

1. Preliminaries. Let L_m be the free Lie algebra of rank $m \geq 2$ over a field K of characteristic 0 with free generators $y_1, \dots, y_m$ and let $F_m = L_m/L_m''$ be the free metabelian Lie algebra, where $L_m' = [L_m, L_m]$ and $L_m'' = [L_m', L_m']$. It is freely generated by the set $X = \{x_1, \dots, x_m\}$, where $x_i = y_i + L_m''$, $i = 1, \dots, m$. We use the commutator notation for the Lie multiplication. Our commutators are left normed:

$$[u_1, \dots, u_{n-1}, u_n] = [[u_1, \dots, u_{n-1}], u_n], \quad n = 3, 4, \dots.$$

It is well known, see e.g. [4], that

$$[x_{i_1}, x_{i_2}, x_{i_{\sigma(i_3)}}, \dots, x_{i_{\sigma(k)}}] = [x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_k}],$$

where σ is an arbitrary permutation of $3, \dots, k$ and that F_m' has a basis consisting of all

$$[x_{i_1}, x_{i_2}, x_{i_3}, \dots, x_{i_k}], \quad 1 \leq i_j \leq m, \quad i_1 > i_2 \leq i_3 \leq \dots \leq i_k.$$

For each $v \in F'_m$, the linear operator $\operatorname{ad} v : F_m \to F_m$ defined by

$$\operatorname{ad} v(u) = [u, v], \quad u \in F_m,$$

is a derivation of F_m which is nilpotent and $\operatorname{ad}^2 v = 0$ because $F''_m = 0$. Hence the linear operator

$$\exp(\operatorname{ad} v) = 1 + \frac{\operatorname{ad} v}{1!} + \frac{\operatorname{ad}^2 v}{2!} + \cdots = 1 + \operatorname{ad} v$$

is well defined and is an automorphism of F_m. The set of all such automorphisms forms a normal subgroup of the group of all automorphisms $\operatorname{Aut}(F_m)$ of F_m. This group is called the inner automorphism group of F_m and is denoted by $\operatorname{Inn}(F_m)$. It is abelian because

$$\exp(\operatorname{ad} u)\exp(\operatorname{ad} v) = \exp(\operatorname{ad}(u+v)), \quad u, v \in F'_m.$$

The set of all endomorphisms $\operatorname{End}(F_m)$ of F_m forms a semigroup. Let $\operatorname{IE}(F_m)$ be the subsemigroup of all endomorphisms of F_m which are identical modulo the commutator ideal F'_m.

Lemma 1. *The set* $\{\operatorname{Inn}(F_m)\theta \mid \theta \in \operatorname{IE}(F_m)\}$ *is a semigroup with the multiplication*

$$(\operatorname{Inn}(F_m)\theta_1) \cdot (\operatorname{Inn}(F_m)\theta_2) = \operatorname{Inn}(F_m)(\theta_1\theta_2).$$

Proof. Since $\operatorname{Inn}(F_m)$ is a subgroup of the semigroup $\operatorname{IE}(F_m)$, the relation

$$\sigma = \{(\theta_1, \theta_2) \in (\operatorname{IE}(F_m), \operatorname{IE}(F_m)) \mid \operatorname{Inn}(F_m)\theta_1 = \operatorname{Inn}(F_m)\theta_2\}$$

is an equivalence and $(\theta_1, \theta_2) \in \sigma$ if and only if $\theta_1 = \exp(\operatorname{ad} u)\theta_2$ for some $u \in F'_m$.

Now let $\theta, \theta_1, \theta_2 \in \operatorname{IE}(F_m)$ be such that $(\theta_1, \theta_2) \in \sigma$ and $\theta_1 = \exp(\operatorname{ad} u)\theta_2$, $u \in F'_m$. Hence

$$\theta_1\theta = \exp(\operatorname{ad} u)(\theta_2\theta),$$

we have that $(\theta_1\theta, \theta_2\theta) \in \sigma$ and σ is right compatible. It remains to check that σ is left compatible. One can easily show that $\theta\exp(\operatorname{ad} u) = \exp(\operatorname{ad}(\theta(u)))\theta$ and $\theta(u) \in F'_m$. Then

$$\theta\theta_1 = \theta\exp(\operatorname{ad} u)\theta_2 = \exp(\operatorname{ad}(\theta(u)))(\theta\theta_2)$$

which gives that $(\theta\theta_1, \theta\theta_2) \in \sigma$. Hence σ is left compatible and σ is a congruence. For details see e.g. [14]. Consequently the multiplication

$$(\mathrm{Inn}(F_m)\theta_1) \cdot (\mathrm{Inn}(F_m)\theta_2) = \mathrm{Inn}(F_m)(\theta_1\theta_2)$$

is well defined. □

We denote the semigroup defined in Lemma 1 as $\mathrm{IE}(F_m)/\,\mathrm{Inn}(F_m)$. As we mentioned in the introduction, the IA-component of the outer automorphism group of F_m is canonically embedded into the semigroup $\mathrm{IE}(F_m)/\,\mathrm{Inn}(F_m)$. Hence, it is natural to work in the semigroup $\mathrm{IE}(F_m)/\,\mathrm{Inn}(F_m)$ of outer IA-endomorphisms of F_m in order to derive information for the automorphisms in $\mathrm{Out}(F_m)$.

Now we collect the necessary information about wreath products and Jacobian matrices. For details and references see e.g. [6]. Let $K[t_1, \dots, t_m]$ be the (commutative) polynomial algebra over K freely generated by the variables $t_1, \dots, t_m$ and let A_m and B_m be abelian Lie algebras with bases $\{a_1, \dots, a_m\}$ and $\{b_1, \dots, b_m\}$, respectively. Let C_m be the free right $K[t_1, \dots, t_m]$-module with free generators $a_1, \dots, a_m$. We give it the structure of a Lie algebra with trivial multiplication. The abelian wreath product $A_m \,\mathrm{wr}\, B_m$ is equal to the semidirect sum $C_m \leftthreetimes B_m$. The elements of $A_m \,\mathrm{wr}\, B_m$ are of the form

$$\sum_{i=1}^{m} a_i f_i(t_1, \dots, t_m) + \sum_{i=1}^{m} \beta_i b_i,$$

where f_i are polynomials in $K[t_1, \dots, t_m]$ and $\beta_i \in K$. The multiplication in $A_m \,\mathrm{wr}\, B_m$ is defined by

$$[C_m, C_m] = [B_m, B_m] = 0,$$

$$[a_i f_i(t_1, \dots, t_m), b_j] = a_i f_i(t_1, \dots, t_m) t_j, \quad i, j = 1, \dots, m.$$

Hence $A_m \,\mathrm{wr}\, B_m$ is a metabelian Lie algebra and every mapping $\{x_1, \dots, x_m\} \to A_m \,\mathrm{wr}\, B_m$ can be extended to a homomorphism $F_m \to A_m \,\mathrm{wr}\, B_m$. As a special case of the embedding theorem of Shmel'kin [18], the homomorphism $\varepsilon : F_m \to A_m \,\mathrm{wr}\, B_m$ defined by $\varepsilon(x_i) = a_i + b_i$, $i = 1, \dots, m$, is a monomorphism. If

$$f = \sum [x_i, x_j] f_{ij}(\mathrm{ad}\, x_1, \dots, \mathrm{ad}\, x_m), \quad f_{ij}(t_1, \dots, t_m) \in K[t_1, \dots, t_m],$$

then

$$\varepsilon(f) = \sum (a_i t_j - a_j t_i) f_{ij}(t_1, \dots, t_m).$$

Let us define $T_j = \{t_j, \dots, t_m\}$ for each $j = 1, \dots, m$. In the sequel we shall need the following obvious property. If $f_j(T_j) \in K[T_j]$, $1 \le j \le m$, and $f_j(T_j) \neq 0$ for some j, then $t_j f_j(T_j) + \cdots + t_m f_m(T_m) \neq 0$, because $t_j f_j(T_j)$ is the only summand which depends on x_j.

The next lemma follows from [18], see also [6].

Lemma 2. *The element* $\sum_{i=1}^{m} a_i f_i(t_1, \dots, t_m)$ *of* C_m *belongs to* $\varepsilon(F'_m)$ *if and only if* $\sum_{i=1}^{m} t_i f_i(t_1, \dots, t_m) = 0$.

The embedding of F_m into $A_m \operatorname{wr} B_m$ allows to introduce partial derivatives in F_m with values in $K[t_1, \dots, t_m]$. If $f \in F_m$ and

$$\varepsilon(f) = \sum_{i=1}^{m} \beta_i b_i + \sum_{i=1}^{m} a_i f_i(t_1, \dots, t_m), \quad \beta_i \in K, f_i \in K[t_1, \dots, t_m],$$

then

$$\frac{\partial f}{\partial x_i} = f_i(t_1, \dots, t_m).$$

The Jacobian matrix $J(\phi)$ of an endomorphism ϕ of F_m is defined as

$$J(\phi) = \left(\frac{\partial \phi(x_j)}{\partial x_i}\right) = \begin{pmatrix} \dfrac{\partial \phi(x_1)}{\partial x_1} & \cdots & \dfrac{\partial \phi(x_m)}{\partial x_1} \\ \vdots & \ddots & \vdots \\ \dfrac{\partial \phi(x_1)}{\partial x_m} & \cdots & \dfrac{\partial \phi(x_m)}{\partial x_m} \end{pmatrix} \in M_m(K[t_1, \dots, t_m]),$$

where $M_m(K[t_1, \dots, t_m])$ is the associative algebra of $m \times m$ matrices with entries from $K[t_1, \dots, t_m]$. Let I_m be the identity $m \times m$ matrix and let S be the subspace of $M_m(K[t_1, \dots, t_m])$ defined by

$$S = \left\{ (f_{ij}) \in M_m(K[t_1, \dots, t_m]) \mid \sum_{i=1}^{m} t_i f_{ij} = 0, j = 1, \dots, m \right\}.$$

Clearly $I_m + S$ is a subsemigroup of the multiplicative group of $M_m(K[t_1, \dots, t_m])$. If $\phi \in \mathrm{IE}(F_m)$, then $J(\phi) = I_m + (s_{ij})$, where $(s_{ij}) \in S$. It is easy to check that if $\phi, \psi \in \mathrm{IE}(F_m)$ then $J(\phi\psi) = J(\phi)J(\psi)$. The following proposition is well known, see e.g. [6].

Proposition 3. *The map* $J : \mathrm{IE}(F_m) \to I_m + S$ *defined by* $\phi \to J(\phi)$ *is an isomorphism of the semigroups* $\mathrm{IE}(F_m)$ *and* $I_m + S$.

The following well known lemma gives the Jacobian matrix of the inner automorphisms of F_m. We include the proof for completeness of the exposition.

Lemma 4. *Let* $u \in F'_m$ *such that*

$$u = \sum_{p>q} [x_p, x_q] h_{pq}(\operatorname{ad} x_q, \dots, \operatorname{ad} x_m).$$

where $h_{pq}(t_q, \dots, t_m) \in K[t_q, \dots, t_m]$. *Then*

$$J(\exp(\operatorname{ad} u)) = I_m + D, \quad D = \left(\frac{\partial [x_j, u]}{\partial x_i}\right),$$

More precisely

$$D = \begin{pmatrix} -t_1 f_1 & -t_2 f_1 & \cdots & -t_m f_1 \\ -t_1 f_2 & -t_2 f_2 & \cdots & -t_m f_2 \\ \vdots & \vdots & \ddots & \vdots \\ -t_1 f_m & -t_2 f_m & \cdots & -t_m f_m \end{pmatrix},$$

where

$$f_i = \sum_{p>q} \frac{\partial \left([x_p, x_q] h_{pq}(\operatorname{ad} x_q, \dots, \operatorname{ad} x_m)\right)}{\partial x_i}$$

$$= \sum_{q=1}^{i-1} t_q h_{iq}(t_q, \dots, t_m) - \sum_{p=i+1}^{m} t_p h_{pi}(t_i, \dots, t_m).$$

P r o o f. By definition,

$$\exp(\operatorname{ad} u)(x_j) = x_j + [x_j, u], \quad j = 1, \dots, m.$$

By direct calculations we obtain

$$u = \sum_{p>q} [x_p, x_q] h_{pq}(\operatorname{ad} x_q, \dots, \operatorname{ad} x_m),$$

$$[x_j, u] = -\left(\sum_{p>q} [y_p, y_q] h_{pq}(\operatorname{ad} x_q, \dots, \operatorname{ad} x_m)\right) \operatorname{ad} x_j,$$

$$\frac{\partial[x_j,u]}{\partial x_i}=-t_j\sum_{p>q}\frac{\partial[x_p,x_q]}{\partial x_i}h_{pq}(t_q,\ldots,t_m),$$

$$\frac{\partial[x_p,x_q]}{\partial x_i}=\begin{cases}t_q & p=i,\\ -t_p & q=i,\\ 0 & p,q\neq i,\end{cases}$$

$$\frac{\partial[x_j,u]}{\partial x_i}=-t_jf_i(t_1,\ldots,t_m),$$

$$f_i(t_1,\ldots,t_m)=\sum_{q=1}^{i-1}t_qh_{iq}(t_q,\ldots,t_m)-\sum_{p=i+1}^{m}t_ph_{pi}(t_i,\ldots,t_m)$$

and in this way we obtain the explicit form of the matrix D. $\square$

2. Main results. In this section we give the explicit form of the Jacobian matrix of the coset representatives of the outer endomorphisms in $\mathrm{IE}(F_m)/\operatorname{Inn}(F_m)$, i.e., we shall find a set of IA-endomorphisms θ of F_m such that the factor semigroup $\mathrm{IE}(F_m)/\operatorname{Inn}(F_m)$ of the outer IA-endomorphisms of F_m is presented as the disjoint union of the cosets $\operatorname{Inn}(F_m)\theta$.

Recall that the augmentation ideal of the polynomial algebra $K[t_1,\ldots,t_m]$ consists of the polynomials without constant terms. We denote this ideal as ω.

Theorem 5. *Let Θ be the set of endomorphisms θ of F_m with Jacobian matrix of the form*

$$J(\theta)=I_m+\begin{pmatrix} s(t_2,\ldots,t_m) & f_{12} & \cdots & f_{1m}\\ t_1q_2(t_2,t_3,\ldots,t_m)+r_2(t_2,\ldots,t_m) & f_{22} & \cdots & f_{2m}\\ t_1q_3(t_3,\ldots,t_m)+r_3(t_2,\ldots,t_m) & f_{32} & \cdots & f_{3m}\\ \vdots & \vdots & \ddots & \vdots\\ t_1q_m(t_m)+r_m(t_2,\ldots,t_m) & f_{m2} & \cdots & f_{mm}\end{pmatrix},$$

where $s,r_i,f_{ij}\in\omega^2$, $q_i\in\omega$ are polynomials satisfying the conditions

$$s+\sum_{i=2}^{m}t_iq_i=0,\quad \sum_{i=2}^{m}t_ir_i=0,\quad \sum_{i=1}^{m}t_if_{ij}=0,\quad j=2,\ldots,m,$$

$r_i = r_i(t_2, \dots, t_m)$, $i = 2, \dots, m$, *does not depend on* t_1, $q_i(t_i, \dots, t_m)$, $i = 2, \dots, m$, *does not depend on* $t_1, \dots, t_{i-1}$. *Then* Θ *consists of coset representatives of the subgroup* $\operatorname{Inn}(F_m)$ *of the semigroup* $\mathrm{IE}(F_m)$ *and* $\mathrm{IE}(F_m)/\operatorname{Inn}(F_m)$ *is a disjoint union of the cosets* $\operatorname{Inn}(F_m)\theta$, $\theta \in \Theta$.

Proof. Let $A = I_m + (f_{ij}) \in I_m + S$,

$$f_{11} = s, \quad f_{i1} = t_1 q_i + r_i, \quad i = 2, \dots, m,$$

be an $m \times m$ matrix satisfying the conditions of the theorem. The equation

$$s + \sum_{i=2}^{m} t_i q_i = 0$$

implies that

$$t_1 s + \sum_{i=2}^{m} t_i (t_1 q_i) = 0.$$

Hence Lemma 2 gives that there exists an f_1 in the commutator ideal of F_m such that

$$\frac{\partial f_1}{\partial x_1} = s, \quad \frac{\partial f_1}{\partial x_i} = t_1 q_i, \quad i = 2, \dots, m.$$

Similarly, the conditions

$$\sum_{i=2}^{m} t_i r_i = 0, \quad \sum_{i=1}^{m} t_i f_{ij} = 0, \quad j = 2, \dots, m,$$

imply that there exist f_1', f_j, $j = 2, \dots, m$, in F_m with

$$\frac{\partial f_1'}{\partial x_1} = 0, \quad \frac{\partial f_1'}{\partial x_i} = r_i, \quad i = 2, \dots, m,$$

$$\frac{\partial f_j}{\partial x_i} = f_{ij}, \quad i = 1, \dots, j, \quad j = 2, \dots, m.$$

This means that A is the Jacobian matrix of a certain IA-endomorphism of F_m.

Now we shall show that for any $\psi \in \mathrm{IE}(F_m)$ there exists an inner automorphism $\phi = \exp(\operatorname{ad} u) \in \operatorname{Inn}(F_m)$ and an endomorphism θ in Θ such that $\psi = \exp(\operatorname{ad} u) \cdot \theta$. Let ψ be an arbitrary element of $\mathrm{IE}(F_m)$ and let

$$\psi(x_1) = x_1 + \sum_{k>l} [x_k, x_l] f_{kl}(\operatorname{ad} x_l, \dots, \operatorname{ad} x_m),$$

$$\psi(x_2) = x_2 + \sum_{k>l} [x_k, x_l] g_{kl}(\operatorname{ad} x_l, \ldots, \operatorname{ad} x_m),$$

where $f_{kl} = f_{kl}(t_l, \ldots, t_m), g_{kl} = g_{kl}(t_l, \ldots, t_m) \in K[t_1, \ldots, t_m]$.

Let us denote the $m \times 2$ matrix consisting of the first two columns of $J(\psi)$ by $J(\psi)_2$. Then $J(\psi)_2$ is of the form

$$J(\psi)_2 = \begin{pmatrix} 1 - t_2 f_{21} - t_3 f_{31} - \cdots - t_m f_{m1} & -t_2 g_{21} - t_3 g_{31} - \cdots - t_m g_{m1} \\ t_1 f_{21} - t_3 f_{32} - \cdots - t_m f_{m2} & 1 + t_1 g_{21} - t_3 g_{32} - \cdots - t_m g_{m2} \\ t_1 f_{31} + t_2 f_{32} - \cdots - t_m f_{m3} & * \\ \vdots & \vdots \\ t_1 f_{m1} + \cdots + t_{(m-1)} f_{m(m-1)} & * \end{pmatrix},$$

where we have denoted by $*$ the corresponding entries of the second column of the Jacobian matrix of ψ. We can rewrite $J(\psi)_2$ as

$$J(\psi)_2 = \begin{pmatrix} 1 + t_1 s_1(t_1, \ldots, t_m) + s_2(t_2, \ldots, t_m) & * \\ t_1^2 p_2(t_1, \ldots, t_m) + t_1 q_2(t_2, \ldots, t_m) + r_2(t_2, \ldots, t_m) & * \\ t_1^2 p_3(t_1, \ldots, t_m) + t_1 q_3(t_2, \ldots, t_m) + r_3(t_2, \ldots, t_m) & * \\ \vdots & \vdots \\ t_1^2 p_m(t_1, \ldots, t_m) + t_1 q_m(t_2, \ldots, t_m) + r_m(t_2, \ldots, t_m) & * \end{pmatrix},$$

where we have collected the components $t_1^2 p_i$ divisible by t_1^2, the components $t_1 q_i$ divisible by t_1 only (but not by t_1^2) and finally the components r_i which do not depend on t_1, $i = 2, \ldots, m$. By Lemma 2 we obtain

$$t_1^2(s_1 + t_2 p_2 + \cdots + t_m p_m) = 0,$$

$$t_1(s_2 + t_2 q_2 + \cdots + t_m q_m) = 0,$$

$$t_2 r_2 + \cdots + t_m r_m = 0.$$

Recalling the fact that $T_s = \{t_s, \ldots, t_m\}$, we can rewrite $J(\psi)_2$ as

$$J(\psi)_2 = \begin{pmatrix} 1 - t_1 t_2 p_2 - \cdots - t_1 t_m p_m - t_2 q_2 - \cdots - t_m q_m & * \\ t_1^2 p_2 + t_1 q_2(T_2) + r_2(T_2) & * \\ t_1^2 p_3 + t_1 q_3(T_2) + r_3(T_2) & * \\ \vdots & \vdots \\ t_1^2 p_m + t_1 q_m(T_2) + r_m(T_2) & * \end{pmatrix},$$

Now we define

$$\phi_1 = \exp(\operatorname{ad} u_1), \quad u_1 = \sum_{i=2}^{m} [x_i, x_1] p_i(\operatorname{ad} x_1, \dots, \operatorname{ad} x_m).$$

The Jacobian matrix of ϕ_1 has the form

$$J(\phi_1) = I_m + \begin{pmatrix} t_1 \sum\limits_{i\neq 1} t_i p_i & t_2 \sum\limits_{i\neq 1} t_i p_i & \cdots & t_m \sum\limits_{i\neq 1} t_i p_i \\ -t_1^2 p_2 & -t_1 t_2 p_2 & \cdots & -t_1 t_m p_2 \\ \vdots & \vdots & \ddots & \vdots \\ -t_1^2 p_m & -t_1 t_2 p_m & \cdots & -t_1 t_m p_m \end{pmatrix}.$$

The element u_1 belongs to the commutator ideal of F_m and the linear operator $\operatorname{ad} u_1$ acts trivially on F_m'. Hence $\exp(\operatorname{ad} u_1)$ is the identity map restricted on F_m'. Since the endomorphism ψ is IA, we obtain that

$$\psi(x_j) \equiv x_j \pmod{F_m'}, \quad \phi_1\psi(x_j) = \psi(x_j) + x_j \operatorname{ad} u_1.$$

Easy calculations give that

$$J(\phi_1\psi)_2 = \begin{pmatrix} 1 - t_2p_2 - \cdots - t_mp_m & * \\ t_1q_2(T_2) + r_2(T_2) & * \\ t_1q_3(T_2) + r_3(T_2) & * \\ \vdots & \vdots \\ t_1q_m(T_2) + r_m(T_2) & * \end{pmatrix}.$$

Now we write $q_i(T_2)$ in the form

$$q_i(T_2) = t_2 q_i'(T_2) + q_i''(T_3), \quad i = 3, \dots, m,$$

and define

$$\phi_2 = \exp(\operatorname{ad} u_2), \quad u_2 = \sum_{i=3}^{m} [x_i, x_2] q_i'(\operatorname{ad} x_2, \dots, \operatorname{ad} x_m).$$

Then we obtain that

$$J(\phi_2\phi_1\psi)_2 = \begin{pmatrix} 1 - t_2p_2 - \cdots - t_mp_m & * \\ t_1Q_2(T_2) + r_2(T_2) & * \\ t_1q''_3(T_3) + r_3(T_2) & * \\ \vdots & \vdots \\ t_1q''_m(T_3) + r_m(T_2) & * \end{pmatrix},$$

$$Q_2(T_2) = q_2(T_2) - \sum_{i=3}^{m} t_iq'_i(T_2).$$

Repeating this process we construct inner automorphisms $\phi_3, \ldots, \phi_{m-1}$ such that

$$\theta = \phi_{m-1} \cdots \phi_2\phi_1\phi_0\psi,$$

$$J(\phi_{m-1} \cdots \phi_2\phi_1\psi)_2 = \begin{pmatrix} 1 + s(T_2) & * \\ t_1Q_2(T_2) + r_2(T_2) & * \\ t_1Q_3(T_3) + r_3(T_2) & * \\ \vdots & \vdots \\ t_1Q_m(T_m) + r_m(T_2) & * \end{pmatrix},$$

$$s(T_2) = -t_2p_2(T_2) - \cdots - t_mp_m(T_2).$$

Hence, starting from an arbitrary coset of IA-endomorphisms $\mathrm{Inn}(F_m)\psi$, we have found that it contains an endomorphism $\theta \in \Theta$ with Jacobian matrix prescribed in the theorem. Now, let θ_1 and θ_2 be two different endomorphisms in Θ with $\mathrm{Inn}(F_m)\theta_1 = \mathrm{Inn}(F_m)\theta_2$. Hence, there exists a nonzero element $u \in F'_m$ such that $\theta_1 = \exp(\mathrm{ad}\, u)\theta_2$. Since $\theta_2(x_1) \equiv x_1$ modulo F'_m, as above we obtain

$$\theta_1(x_1) = \exp(\mathrm{ad}\, u)\theta_2(x_1) = \theta_2(x_1) + x_1 \,\mathrm{ad}\, u.$$

Hence

$$J(\mathrm{ad}\, u)_2 = J(\theta_1)_2 - J(\theta_2)_2.$$

If u is of the form

$$u = \sum_{p>q} [x_p, x_q] h_{pq}(\mathrm{ad}\, x_q, \ldots, \mathrm{ad}\, x_m)$$

then, by Lemma 4, $J(\operatorname{ad} u)_2$ is of the form

$$J(\operatorname{ad} u)_2 = \begin{pmatrix} +t_1t_2h_{21} + t_1t_3h_{31} + \cdots + t_1t_mh_{m1} & * \\ -t_1^2h_{21} + t_1t_3h_{32} + \cdots + t_1t_mh_{m2} & * \\ -t_1^2h_{31} - t_1t_2h_{32} + \cdots + t_1t_mh_{m3} & * \\ \vdots & \vdots \\ -t_1^2h_{m1} - t_1t_2h_{m2} - \cdots - t_1t_{m-1}h_{m,m-1} & * \end{pmatrix},$$

$$h_{pq} = h_{pq}(T_q) \in K[T_q] = K[t_q, \ldots, t_m], \quad p > q.$$

On the other hand, $J(\theta_1)_2 - J(\theta_2)_2$ is of the form

$$J(\theta_1)_2 - J(\theta_2)_2 = \begin{pmatrix} s(t_2, \ldots, t_m) & * \\ t_1q_2(t_2, t_3, \ldots, t_m) + r_2(t_2, \ldots, t_m) & * \\ t_1q_3(t_3, \ldots, t_m) + r_3(t_2, \ldots, t_m) & * \\ \vdots & \vdots \\ t_1q_m(t_m) + r_m(t_2, \ldots, t_m) & * \end{pmatrix},$$

where the polynomials s, r_i, f_{ij}, q_i satisfy the conditions in the statement of the theorem. Comparing the degrees of t_1 in the monomials of the entries of the first columns of both matrices we derive that

$$s = 0, \quad h_{p1} = 0, \quad r_p = 0, \quad p = 2, \ldots, m.$$

By similar arguments we conclude that $h_{pq} = 0$ for all $p > q$ which implies that $u = 0$ and $\theta_1 = \theta_2$. $\square$

Example 6. When $m = 2$ the results of Lemma 4 and Theorem 5 have the following simple form. If

$$u = [x_2, x_1]h(\operatorname{ad} x_1, \operatorname{ad} x_2), \quad h = h(t_1, t_2) \in K[t_1, t_2],$$

then the Jacobian matrix of the inner automorphism $\exp(\operatorname{ad} u)$ is

$$J(\exp(\operatorname{ad} u)) = I_2 + \begin{pmatrix} t_1t_2h & t_2^2h \\ -t_1^2h & -t_1t_2h \end{pmatrix}.$$

The Jacobian matrix of the outer IA-endomorphism $\theta \in \Theta$ is

$$J(\theta) = \begin{pmatrix} 1 + t_2 f_1(t_2) & t_2 f_2(t_1, t_2) \\ -t_1 f_1(t_2) & 1 - t_1 f_2(t_1, t_2) \end{pmatrix}, \quad f_1(0,0) = f_2(0,0) = 0.$$

This allows to show easily the result of Shmel'kin [18] that $\mathrm{IA}(F_2) = \mathrm{Inn}(F_2)$. If $\theta \in \Theta$ is an IA-automorphism, then its Jacobian matrix

$$J(\theta) = \begin{pmatrix} 1 + t_2 f_1(t_2) & t_2 f_2(t_1, t_2) \\ -t_1 f_1(t_2) & 1 - t_1 f_2(t_1, t_2) \end{pmatrix}$$

is invertible and

$$\det(J(\theta)) = (1 + t_2 f_1(t_2))(1 - t_1 f_2(t_1, t_2)) + (t_1 f_1(t_2))(t_2 f_2(t_1, t_2)) = 1$$

which gives $t_2 f_1(t_2) = t_1 f_2(t_1, t_2)$ and hence $f_1(t_2) = f_2(t_1, t_2) = 0$. Therefore θ is the identity automorphism which means that all IA-automorphisms of F_2 are inner.

Finally, we want to raise the following natural problem.

Problem 7. *Describe the group* $\mathrm{IOut}(F_m) = \mathrm{IA}(F_m)/\mathrm{Inn}(F_m)$, $m \geq 3$, *of outer IA-automorphisms of* F_m. *This would give immediately the description of the group* $\mathrm{Out}(F_m) = \mathrm{Aut}(F_m)/\mathrm{Inn}(F_m)$ *of outer automorphisms of* F_m.

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REFERENCES

[1] S. Bachmuth. Automorphisms of free metabelian groups. *Trans. Amer. Math. Soc.* **118** (1965), 93–104.

[2] S. Bachmuth, H. Y. Mochizuki. The nonfinite generation of $\mathrm{Aut}(G)$, G free metabelian of rank 3. *Trans. Amer. Math. Soc.* **270** (1982), 693–700.

[3] S. Bachmuth, H. Y. Mochizuki. $\mathrm{Aut}(F) \to \mathrm{Aut}(F/F'')$ is surjective for free group F of rank ≥ 4. *Trans. Amer. Math. Soc.* **292** (1985), 81–101.

[4] Yu. A. Bahturin. Identical Relations in Lie Algebras. Nauka, Moscow, 1985 (in Russian); English translation: VNU Science Press, Utrecht, 1987.

[5] Y. Bahturin, S. Nabiyev. Automorphisms and derivations of abelian extensions of some Lie algebras. *Abh. Math. Sem. Univ. Hamburg* **62** (1992), 43–57.

[6] R. M. Bryant, V. Drensky. Dense subgroups of the automorphism groups of free algebras. *Canad. J. Math.* **45** (1993), 1135–1154.

[7] P. M. Cohn. Subalgebras of free associative algebras. *Proc. London. Math. Soc.* **14**, *3* (1964), 618–632.

[8] E. Yu. Daniyarova, I. V. Kazachkov, V. N. Remeslennikov. Semidomains and metabelian product of metabelian Lie algebras. *Sovrem. Mat. Prilozh. 14* (2004), 3–10 (in Russian); English translation: *J. Math. Sci., New York* **131**, *6* (2005), 6015–6022.

[9] E. Yu. Daniyarova, I. V. Kazachkov, V. N. Remeslennikov. Algebraic geometry over free metabelian Lie algebras. I: U-algebras and universal classes. *Fundam. Prikl. Mat.* **9**, *3* (2003), 37–63 (in Russian); English translation: *J. Math. Sci., New York* **135**, *5* (2006), 3292–3310.

[10] E. Yu. Daniyarova, I. V. Kazachkov, V. N. Remeslennikov. Algebraic geometry over free metabelian Lie algebras. II: Finite-field case. *Fundam. Prikl. Mat.* **9**, *3* (2003), 65-87 (in Russian). English translation: *J. Math. Sci., New York* **135**, *5* (2006), 3311–3326.

[11] V. Drensky, Ş. Findik. Inner and outer automorphisms of free metabelian nilpotent Lie algebras. *Commun. Algebra* (to appear).

[12] Ş. Findik. Normal and normally outer automorphisms of free metabelian nilpotent Lie algebras. *Serdica Math. J.* **36**, *2* (2010), 171–201.

[13] L. Gerritzen. Taylor expansion of noncommutative power series with an application to the Hausdorff series. *J. Reine Angew. Math.* **556** (2003), 113–125.

[14] J. M. Howie. Fundamentals of Semigroup Theory. Cleradon Press, Oxford, 1995.

[15] V. Kurlin. The Baker-Campbell-Hausdorff formula in the free metabelian Lie algebra. *J. Lie Theory* **17**, *3* (2007), 525–538.

[16] V. A. Roman'kov. The automorphism groups of free metabelian Lie algebras, In: Abstracts: The International Conference "Algebra and Its Applications", Krasnoyarsk, August 12–18, 2007, Siberian Federal Univ., IM SORAN, IBM SORAN, Novosibirsk, 2007, 114–115.

[17] V. Roman'kov. On the automorphism group of a free metabelian Lie algebra. *Int. J. Algebra Comput.* **18**, *2* (2008), 209–226.

[18] A. L. Shmel'kin. Wreath products of Lie algebras and their application in the theory of groups. *Trudy Moskov. Mat. Obshch.* **29** (1973), 247–260 (in Russian). English translation: *Trans. Moscow Math. Soc.* **29** (1973), 239–252.

[19] U. U. Umirbaev. Defining relations for automorphism groups of free algebras. *J. Algebra* **314** (2007), 209–225.

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