Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **37** (2011), 283–294

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE GENERALIZED KATO SPECTRUM

Mohammed Benharrat, Bekkai Messirdi

Communicated by J. Albert

Abstract. We show that the symmetric difference between the generalized Kato spectrum and the essential spectrum defined in [7] by $\sigma_{ec}(T) = \{\lambda \in \mathbb{C}\ ;\ R(\lambda I - T) \text{ is not closed}\}$ is at most countable and we also give some relationship between this spectrum and the SVEP theory.

1. Introduction. Let X be a Banach space and $\mathcal{L}(X)$ be the set of all bounded linear operators from X into X. For $T \in \mathcal{L}(X)$ we denote by $R(T)$ its range, $N(T)$ its null space, $\sigma(T)$ its spectrum and T^* the adjoint of T. Let I denote the identity operator in X. An operator $T \in \mathcal{L}(X)$ is said to be semi-regular if $R(T)$ is closed and $N(T^n) \subseteq R(T)$, for all $n \geq 0$. T admits a generalized Kato decomposition, abbreviated as GKD, if we can write $T = T_1 \oplus T_0$ where T_0 is a quasi-nilpotent operator and T_1 is a semi-regular one. If we assume in the definition above that T_0 is nilpotent, T is said to be of Kato type.

2010 *Mathematics Subject Classification:* 47A10.

Key words: Semi-regular operators, Kato type operators, generalized Kato spectrum, essential spectrum.

The Kato decomposition for bounded operator on Banach spaces arises from the classical treatment of perturbation theory of Kato [9], and its flourishing has greatly benefited from the work of many authors in the last ten years, in particular from the work of Mbekhta [13, 14, 16], Aiena [1] and Q. Jiang-H. Zhong [8]. The operators which satisfy this property form a class which includes the class of quasi-Fredholm operators, semi-regular, Kato type, semi-Fredholm and B-Fredholm operators. This concept leads in a natural way to the generalized Kato spectrum $\sigma_{gk}(T)$, an important subset of the ordinary spectrum which is defined as the set of all $\lambda \in \mathbb{C}$ for which $\lambda I - T$ does not admit a generalized Kato decomposition. It was shown in [8, Corollary 2.3] that $\sigma_{gk}(T)$ is a compact subset of $\mathbb{C}$. In the present paper, the relationship between the generalized Kato spectrum and the essential spectrum defined in [7] by $\sigma_{ec}(T) = \{\lambda \in \mathbb{C} \ ; \ R(\lambda I - T) \text{ is not closed } \}$ is examined. It is shown in [12], in the Hilbert space case, that the symmetric difference between $\sigma_{ec}(T)$ and the essential quasi-Fredholm spectrum which is the set of all complex λ such that $\lambda I - T$ is not quasi-Fredholm operator, is at most countable, which is of course, in this case, a quasi-Fredholm operators equivalent to T is of Kato type, but in the case of Banach spaces the Kato type operator is also quasi-Fredholm, the inverse is not true. This results examined in Banach space in [3], for the Kato essential spectrum, which is the set of all complex λ such that $\lambda I - T$ is not of Kato type operator.

Our paper is organized as follows:

In section 2, we give some preliminary results upon which our investigation will be based.

In section 3, we extend the results proved in [3] for the generalized Kato spectrum, we give some relationship between the generalized Kato spectrum and the SVEP theory.

Finally, in section 4 we apply the results obtained in section 3 to study the generalized Kato spectrum for tow classes operators, first class is the class of operators which satisfy a polynomial growth condition, and the second is the class of of Cesaro operators.

2. Preliminary results. In this section, we collect some technical results which we will use in the sequel.

The reduced minimum modulus of a non-zero operator T is defined by

$$\gamma(T) = \inf_{x \notin N(T)} \frac{\|Tx\|}{\operatorname{dist}(x, N(T))}$$

where $\mathrm{dist}(x, N(T)) = \inf_{y \in N(T)} \|x - y\|$. If $T = 0$ then we take $\gamma(T) = \infty$.
Note that (see [10]):

$$\gamma(T) > 0 \Leftrightarrow R(T) \text{ is closed.}$$

Let M, N be two closed linear subspaces of the Banach space X and set

$$\delta(M, N) = \sup\{\mathrm{dist}(x, N) : x \in M, \|x\| = 1\},$$

in the case that $M \neq \{0\}$, otherwise we define $\delta(\{0\}, N) = 0$ for any subspace N.
The gap between M and N is defined by

$$\widehat{\delta}(M, N) = \max\{\delta(M, N), \delta(N, M)\}$$

$\widehat{\delta}$ is a metric on the set $\mathcal{F}(X)$ of all linear closed subspaces of X, and the convergence $M_n \longrightarrow M$ in $\mathcal{F}(X)$ is obviously defined by $\widehat{\delta}(M_n, M) \longrightarrow 0$ as $n \longrightarrow \infty$ in $\mathbb{R}$. Moreover, $(\mathcal{F}(X), \widehat{\delta})$ is complete metric space (see [10]).

Proposition 2.1 [1]. *For every operator $T \in \mathcal{L}(X)$ and for arbitrary $\lambda, \mu \in \mathbb{C}$, we have:*

(1) $\gamma(\lambda I - T)\delta(N(\mu I - T), N(\lambda I - T)) \leq |\mu - \lambda|$.

(2) $\min\{\gamma(\mu I - T), \gamma(\lambda I - T)\}\widehat{\delta}(N(\mu I - T), N(\lambda I - T)) \leq |\mu - \lambda|$.

Proposition 2.2 [1]. *Let $M, N \in \mathcal{F}(X)$. For every $x \in X$ and $0 < \epsilon < 1$ there exists $x_0 \in X$ such that $(x - x_0) \in M$ and*

$$\mathrm{dist}(x_0, N) \geq \left((1 - \epsilon)\frac{1 - \delta(M, N)}{1 + \delta(M, N)}\right) \|x_0\| . \tag{1}$$

Definition 2.3 [17]. *Let $T \in \mathcal{L}(X)$. T is said to be semi-regular if $R(T)$ is closed and $N(T^n) \subseteq R(T)$, for all $n \geq 0$.*

Trivial examples of semi-regular operators are surjective operators as well as injective operators with closed range, Fredholm operators and semi-Fredholm operators with zero jump (for more details see [1]). Some other examples of semi-regular operators may be found in Mbekhta and Ouahab [17] and Labrousse [11]. The following theorem shows that the semi-regularity of an operator may be characterized in terms of the continuity of certain maps.

Theorem 2.4 [18]. *For $T \in \mathcal{L}(X)$ and $\lambda_0 \in \mathbb{C}$, the following statements are equivalent:*

(1) *$\lambda_0 I - T$ is semi-regular.*

(2) *$\gamma(\lambda_0 I - T) > 0$ and the mapping $\lambda \to \gamma(\lambda I - T)$ is continuous at λ_0.*

(3) *$\gamma(\lambda_0 I - T) > 0$ and the mapping $\lambda \to N(\lambda I - T)$ is continuous at λ_0 in the gap topology.*

(4) *$R(\lambda_0 I - T)$ is closed in a neighborhood of λ_0 and the mapping $\lambda \to R(\lambda I - T)$ is continuous at λ_0 in the gap topology.*

For an essential version of semi-regular operators we use the following notation. For subspaces $M, L \subset X$ write $M \subset_e L$ if there exists a finite-dimensional subspace F of X for which $M \subset L + F$. Obviously

$$M \subset_e L \Leftrightarrow \dim \frac{M}{M \cap L} < \infty$$

An operator $T \in \mathcal{L}(X)$ is called essentially semi-regular if $R(T)$ is closed and $N(T^n) \subset_e R(T)$, for all $n \geq 0$.

The semi-regular spectrum of a bounded operator T is defined by

$$\sigma_{se}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not semi-regular}\}$$

and its essential version by

$$\sigma_{es}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not essentially semi-regular}\}$$

The semi-regular spectrum was studied by Apostol [2], Rakocević [19], Müller [18] , Mbekhta and Ouahab [17] and Mbekhta [15]. The sets $\sigma_{se}(T)$ and $\sigma_{es}(T)$ are always non-empty compact subsets of the complex plane, $\sigma_{se}(f(T)) = f(\sigma_{se}(T))$ and $\sigma_{es}(f(T)) = f(\sigma_{es}(T))$ for any analytic function f in a neighborhood of $\sigma(T)$. If T is a semi-regular operator, then the limit $\lim_{n\to\infty}(\gamma(T)^n)^{\frac{1}{n}}$ exists and

$$\lim_{n\to\infty}(\gamma(T)^n)^{\frac{1}{n}} = \operatorname{dist}(0, \sigma_{se}(T)) = \sup\{r; \lambda I - T \text{ is semi-regular for } |\lambda| < r\}$$

Now we recall some results about $\sigma_{se}(T)$ and $\sigma_{es}(T)$

Theorem 2.5 [19]. *Let $T \in \mathcal{L}(X)$.*

(1) *$\sigma_{se}(T) = \sigma_{se}(T^*)$ and $\sigma_{es}(T) = \sigma_{es}(T^*)$;*

(2) *$\partial\sigma(T) \subseteq \sigma_{se}(T)$; where $\partial\sigma(T)$ is the boundary of the spectrum of T.*

(3) *$\lambda \in \sigma_{se}(T) \setminus \sigma_{es}(T)$ if and only if λ is an isolated point of $\sigma_{se}(T)$, $\sup_n \dim \dfrac{N(\lambda I - T) + N((\lambda I - T)^n)}{N(\lambda I - T)} < \infty$ and $R(T - \lambda I)$ is closed.*

Now, we introduce an important class of bounded operators which involves the concept of semi-regularity.

Definition 2.6. *An operator $T \in \mathcal{L}(X)$, is said to admit a generalized Kato decomposition, if there exists a pair of closed subspaces (M, N) of X such that:*

(1) $X = M \oplus N$.

(2) $T(M) \subset M$ *and* $T_{/M}$ *is semi-regular.*

(3) $T(N) \subset N$ *and* $T_{/N}$ *is quasi-nilpotent (i.e* $\sigma(T_{/N}) = \{0\}$).

(M, N) is said to be a generalized Kato decomposition of T, abbreviated as GKD(M, N).

If we assume in the definition above that $T_{/N}$ is nilpotent, then there exists $d \in \mathbb{N}$ for which $(T_{/N})^d = 0$. In this case T is said to be of Kato type of order d. Clearly, every semi-regular operator is of Kato type with $M = X$ and $N = \{0\}$ and a quasi-nilpotent operator has a GKD with $M = \{0\}$ and $N = X$. Note that if T is essentialy semi-regular then N is finite-dimensional and $T_{/N}$ is nilpotent, since every quasi-nilpotent operator on a finite-dimensional space is nilpotent. Discussions of operators which admit a generalized decomposition may be found in [14], [16].

For every operator $T \in \mathcal{L}(X)$, let us define the Kato type spectrum and the generalized Kato spectrum as follows respectively:

$\sigma_k(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not of Kato type}\}$

$\sigma_{gk}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ does not admit a generalized Kato decomposition}\}$

$\sigma_{gk}(T)$ is not necessarily non-empty. For example, each quasi-nilpotent operator T has empty generalized Kato spectrum.

The following result shows that the generalized Kato spectrum of a bounded operator is a closed subset of the spectra $\sigma(T)$ of T. The next theorem is due to Q. Jiang , H. Zhong [8, Theorem 2.2]:

Theorem 2.7. *Suppose that $T \in \mathcal{L}(X)$, admits a GKD(M, N). Then there exists an open disc $\mathbb{D}(0, \epsilon)$ for which $\lambda I - T$ is semi-regular for all $\lambda \in \mathbb{D}(0, \epsilon) \setminus \{0\}$*

Since $\sigma_{gk}(T) \subseteq \sigma_k(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{se}(T)$, as a straightforward consequence of Theorem 2.7, we easily obtain that these spectra differ from each other on at most countably many isolated points.

Proposition 2.8 ([1], [8]). *The sets $\sigma_{se}(T) \setminus \sigma_{gk}(T)$, $\sigma_{se}(T) \setminus \sigma_k(T)$, $\sigma_{es}(T) \setminus \sigma_k(T)$, $\sigma_{es}(T) \setminus \sigma_{gk}(T)$ and $\sigma_k(T) \setminus \sigma_{gk}(T)$ are at most countable.*

3. Main results. The essential spectra were studied by many authors (see [1, 13, 18]). Now, the main question is the relationship between them. Motivated by a problem concerning the essential quasi-Fredholm spectrum posed in [12], J. P. Labrousse characterized in the case of Hilbert spaces, a relation of the essential quasi-Fredholm spectrum and another not closed essential spectrum defined in [7] by

$$\sigma_{ec}(T) = \{\lambda \in \mathbb{C} \ ; \ R(\lambda I - T) \text{ is not closed}\}$$

This result is extended to Banach space in [3].

Now, we study this relation in the case of the generalized Kato spectrum. Let $T \in \mathcal{L}(X)$. For α a nonzero positive real number, we introduce the following set

$$\mathcal{R}(\alpha) = \{\lambda \in \mathbb{C}; \ \gamma(\lambda I - T) \geq \alpha\}$$

We begin with the following preparatory result proved in [3, Theorem 3] which is crucial for our purposes.

Theorem 3.1. *Let $(\lambda_n)_n \subset \mathcal{R}(\alpha)$ non stationary sequence and $\lambda_n \longrightarrow \lambda_0$ in $\mathbb{C}$, then*

(1) $\widehat{\delta}(N(\lambda_n I - T), N(\lambda_0 I - T)) \leq \frac{1}{\alpha} |\lambda_n - \lambda_0|$.

(2) $\lambda_0 \in \mathcal{R}(\alpha)$.

(3) $\lambda_0 I - T$ *is semi-regular.*

Note that this theorem is extended to the Banach space, the result was shown by J. P. Labrousse [12] in the case of Hilbert spaces.

Proposition 3.2. *If $\lambda \in \sigma_{ec}(T)$ is non-isolated point then $\lambda \in \sigma_{gk}(T)$.*

Proof. Let $\lambda \in \sigma_{ec}(T)$ be a non-isolated point. Assume that $\lambda I - T$ admits a GKD(M, N). Then by Theorem 2.7 there exists an open disc $\mathbb{D}(\lambda, \epsilon)$ such that $\mu I - T$ is semi-regular in $\mathbb{D}(\lambda, \epsilon) \setminus \{\lambda\}$, so that $R(\mu I - T)$ is closed if $\mu \in \mathbb{D}(\lambda, \epsilon) \setminus \{\lambda\}$. This contradicts our assumption that λ is a non-isolated point. □

Theorem 3.3. *The symmetric difference $\sigma_{gk}(T) \Delta \sigma_{ec}(T)$ is at most countable.*

Proof. We have

$$\sigma_{gk}(T)\Delta\sigma_{ec}(T) = (\sigma_{gk}(T) \cap (\mathbb{C} \setminus \sigma_{ec}(T))) \cup (\sigma_{ec}(T) \cap (\mathbb{C} \setminus \sigma_{gk}(T)))$$

From Proposition 3.2 the set $\sigma_{ec}(T) \setminus \sigma_{gk}(T)$ is at most countable. We have $\mathbb{C} \setminus \sigma_{ec}(T) = \bigcup_{m=1}^{\infty} \mathcal{R}(\frac{1}{m})$ and

$$\sigma_{gk}(T) \cap (\mathbb{C} \setminus \sigma_{ec}(T)) = \bigcup_{m=1}^{\infty} \left(\sigma_{gk}(T) \cap \mathcal{R}\left(\frac{1}{m}\right)\right).$$

To finish the proof we prove that the set $\sigma_{gk}(T) \cap \mathcal{R}\left(\frac{1}{m}\right)$ is at most countable. Let λ_0 be a non-isolated point of $\sigma_{gk}(T) \cap \mathcal{R}\left(\frac{1}{m}\right)$. Then there exists $(\lambda_n)_n \subset \mathcal{R}\left(\frac{1}{m}\right) \cap \sigma_{gk}(T)$ such that $\lambda_n \to \lambda_0$, by Theorem 3.1 $\lambda_0 \notin \sigma_{gk}(T)$. This contradicts the closedness of $\sigma_{gk}(T)$. □

Proposition 3.4. *$\sigma_{se}(T) \setminus (\sigma_{gk}(T) \cap \sigma_{ec}(T))$ is at most countable.*

Proof. We have

$$\sigma_{se}(T) \setminus (\sigma_{gk}(T) \cap \sigma_{ec}(T)) = (\sigma_{gk}(T)\Delta\sigma_{ec}(T)) \cup \sigma_{se}(T) \setminus (\sigma_{gk}(T) \cup \sigma_{ec}(T))$$

Since the sets $\sigma_{se}(T) \setminus \sigma_{gk}(T)$, $\sigma_{se}(T) \setminus \sigma_{ec}(T)$ are at most countable, Theorem 3.3 implies that $\sigma_{gk}(T)\Delta\sigma_{ec}(T)$ is at most countable, establishing the result. □

The fact that $\sigma_k(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{se}(T)$ then we have

Corollary 3.5. *$\sigma_{es}(T) \setminus (\sigma_{gk}(T) \cap \sigma_{ec}(T))$ and $\sigma_k(T) \setminus (\sigma_{gk}(T) \cap \sigma_{ec}(T))$ are at most countable.*

Definition 3.6. *Let $T \in \mathcal{L}(X)$. The operator T is said to have the single-valued extension property at $\lambda_0 \in \mathbb{C}$, abbreviated T has the SVEP at λ_0, if for every neighborhood $\mathcal{U}$ of λ_0 the only analytic function $f : \mathcal{U} \to X$ which satisfies the equation*

$$(\lambda I - T)f(\lambda) = 0$$

is the constant function $f \equiv 0$.

The operator T is said to have the SVEP if T has the SVEP at every $\lambda \in \mathbb{C}$.

We collect some basic properties of the SVEP (see [1]):

(1) Every operator T has the SVEP at an isolated point of the spectrum.

(2) If $p(\lambda I - T) < \infty$, then T has the SVEP at λ.

(3) If $q(\lambda I - T) < \infty$, then T^* has the SVEP at λ where $p(\lambda I - T) = \min\{p \in \mathbb{N} : N((\lambda I - T)^p) = N((\lambda I - T)^{p+1})\}$ and $q(\lambda I - T) = \min\{q \in \mathbb{N} : R((\lambda I - T)^q) = R((\lambda I - T)^{q+1})\}$ are respectively the ascent and the descent of $(\lambda I - T)$.

Proposition 3.7. *If $\lambda \in \partial\sigma(T)$ is a non-isolated point, then $\lambda \in \sigma_{gk}(T)$.*

The approximate point spectrum is defined by

$$\sigma_{ap}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not bounded below}\}$$

and the surjectivity spectrum is defined by

$$\sigma_{su}(T) := \{\lambda \in \mathbb{C} : \lambda I - T \text{ is not surjective}\}.$$

By the closed range theorem we know that the approximate point spectrum and the surjectivity spectrum are dual to each other, in the sense that $\sigma_{ap}(T) = \sigma_{su}(T^*)$ and $\sigma_{ap}(T^*) = \sigma_{su}(T)$. All results established above have a numerous of interesting applications. In the next theorem we consider a situation which occurs in some concrete cases.

Theorem 3.8. *Let $T \in \mathcal{L}(X)$ be an operator for which $\sigma_{ap}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is non-isolated in $\sigma(T)$. Then $\sigma_{ec}(T) \subseteq \sigma_{gk}(T) = \sigma_{es}(T) = \sigma_{se}(T)$.*

Proof. Since $\lambda \in \partial\sigma(T)$ is non-isolated, according to Proposition 3.7,

$$\sigma_{ap}(T) = \partial\sigma(T) \subseteq \sigma_{gk}(T) \subseteq \sigma_k(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{se}(T) \subseteq \sigma_{ap}(T),$$

that is,

$$\sigma_{gk}(T) = \sigma_k(T) = \sigma_{es}(T) = \sigma_{se}(T) = \sigma_{ap}(T) = \sigma_p(T) \cup \sigma_{ec}(T)$$

and

$$\sigma_{ec}(T) \subseteq \sigma_{gk}(T) = \sigma_{es}(T) = \sigma_{se}(T). \qquad \square$$

Dually we have

Theorem 3.9. *Let $T \in \mathcal{L}(X)$ an operator for which $\sigma_{su}(T) = \partial\sigma(T)$ and every $\lambda \in \partial\sigma(T)$ is non-isolated in $\sigma(T)$. Then $\sigma_{ec}(T) \subseteq \sigma_{gk}(T) = \sigma_{es}(T) = \sigma_{se}(T)$.*

P r o o f. Since $\lambda \in \partial\sigma(T)$ is non-isolated, then $\sigma_{su}(T)$ cluster in λ. Observe that T^* has the SVEP at $\lambda \in \partial\sigma(T)$, then $\lambda I - T$ does not admit a generalized Kato decomposition and thus $\lambda \in \sigma_{gk}(T)$. So

$$\sigma_{su}(T) = \partial\sigma(T) \subseteq \sigma_{gk}(T) \subseteq \sigma_k(T) \subseteq \sigma_{es}(T) \subseteq \sigma_{se}(T) \subseteq \sigma_{ap}(T)$$

and

$$\sigma_{gk}(T) = \sigma_k(T) = \sigma_{es}(T) = \sigma_{se}(T) = \sigma_{su}(T).$$

Thus we have

$$\sigma_{ec}(T) \subseteq \sigma_{gk}(T) = \sigma_{es}(T) = \sigma_{se}(T). \qquad \square$$

4. Examples.

Example 1. Let $X = l^2$ the space of complex square-summable sequences and the linear operator T defined by

$$Tx = \left(0, x_1, 0, \frac{1}{3}x_2, 0, \frac{1}{5}x_3, 0, \dots\right), \qquad x = (x_n) \in \ell^2$$

The operator T is compact and $R(T)$ is not closed, then $0 \in \sigma_{ec}(T)$. It easy to see that $T^2 = 0$, so $0 \notin \sigma_{gk}(T)$, $\sigma_{ec}(T) = \{0\}$ and $\sigma_{gk}(T) = \emptyset$.

Example 2. Let $\mathcal{P}_g(X)$ be the class of operators on the Banach space X which satisfy a polynomial growth condition. An operator T satisfies this condition if there exists $K > 0$, and $\delta > 0$ for which

$$\|\exp(i\lambda T)\| \leq K(1 + |\lambda|^{\delta}) \text{ for all } \lambda \in \mathbb{R},$$

Examples of operators which satisfy a polynomial growth condition are Hermitian operators on Hilbert spaces, nilpotent and projection operators, algebraic operators with real spectra. It is shown that $\mathcal{P}_g(X)$ coincides with the class of all generalized scalar operators having real spectra. We first note that the polynomial growth condition may be reformulated as follows (see [1]) : $T \in \mathcal{P}_g(X)$ if and only if $\sigma(T) \subseteq \mathbb{R}$ and there is a constant $K > 0$, and $\delta > 0$ such that

$$\left\|(\lambda I - T)^{-1}\right\| \leq K(1 + |\mathrm{Im}\lambda|^{-\delta}) \text{ for all } \lambda \in \mathbb{C} \text{ with } \mathrm{Im}\lambda \neq 0, \tag{2}$$

The finiteness of the ascent and of the descent of a linear operator T is related to a certain decomposition of X.

Theorem 4.1 [10]. *Let $T \in \mathcal{L}(X)$. If both $p(T)$ and $q(T)$ are finite then $p(T) = q(T) = p$, and we have the decomposition*

$$X = R(T^p) \oplus N(T^p)$$

Conversely, if for a natural number p we have the decomposition $X = R(T^p) \oplus N(T^p)$ *then* $p(T) = q(T) \leq p$. *In this case* $T_{/R(T^p)}$ *is bijective.*
Moreover, $\lambda \in \sigma(T)$ *is a pole of the resolvent* $(\lambda I - T)^{-1}$ *if and only if* $0 < p(\lambda I - T) = q(\lambda I - T) < \infty$.

The following proposition establishes the finiteness of the ascent of a linear operator $T \in \mathcal{P}_g(X)$.

Proposition 4.2 [1]. *Assume that* $T \in \mathcal{P}_g(X)$, *for every* $\lambda \in \sigma(T)$ *we have:*

(1) $p(\lambda I - T) < \infty$.

(2) $\overline{R((\lambda I - T)^p)} = \overline{R((\lambda I - T)^{p+k})}$; $k \in \mathbb{N}$. *and* $p = p(\lambda I - T)$.

Proposition 4.3. *Let* $T \in \mathcal{P}_g(X)$, *we have:*

(1) *If* $\lambda \notin \sigma_{ec}(T)$, *then* λ *is an isolated point in* $\sigma(T)$.

(2) *If* $\lambda \in \sigma_{ec}(T)$ *and* $R((\lambda I - T)^p)$ *is closed for some* $p \in \mathbb{N}$, *then* λ *is a pole of the resolvent* T.

P r o o f. 1. If we assume that $T \in \mathcal{P}_g(X)$ and $R((\lambda I - T))$ is closed for some $\lambda \in \mathbb{C}$ then also $p = p(\lambda I - T)$ is finite, $R((\lambda I - T)) + N((\lambda I - T)^p)$ is closed and $R((\lambda I - T)) + N((\lambda I - T)^p) = R((\lambda I - T)) + N((\lambda I - T)^n)$ for all $n \geq p$. Since $p(\lambda I - T) < \infty$, T has the SVEP at λ. It follows by [4, Theorem 2.5] that λ is an isolated point in $\sigma(T)$.

2. If $R((\lambda I - T)^p)$ is closed, then $R((\lambda I - T)^p) = R((\lambda I - T)^{p+k})$; $k \in \mathbb{N}$, so $q(\lambda I - T) < \infty$, it follows that λ is a pole of the resolvent of T. □

Corollary 4.4. *Let* $T \in \mathcal{P}_g(X)$, *then* $\sigma_{gk}(T)\Delta\sigma_{ec}(T) = \sigma_{ec}(T) \setminus \sigma_{gk}(T)$ *is at most countable.*

P r o o f. From Proposition 4.3, if $\lambda \notin \sigma_{ec}(T)$, then λ is a an isolated point in $\sigma(T)$. This implies By [6, Theorem 6.7] that T admits GKD and $\lambda \notin \sigma_{gk}(T)$ and the set $\sigma_{gk}(T)\setminus\sigma_{ec}(T)$ is empty. Now if $\lambda \in \sigma_{ec}(T)$, we have two cases. First if there exists $p \in \mathbb{N}$ such that $R((\lambda I - T)^p)$ is closed, by Proposition 4.3 part 2, λ is a pole of the resolvent and $\lambda \notin \sigma_{gk}(T)$, thus $\sigma_{ec}(T) \setminus \sigma_{gk}(T)$ is at most countable. Now if $R((\lambda I - T)^p)$ is not closed for every $p \in \mathbb{N}$, then $R(((\lambda I - T)_{/M})^p)$ is not closed for every T-invariant closed subset M and $p \in \mathbb{N}$, so $\lambda I - T$ does not admits GKD and $\lambda \in \sigma_{gk}(T)$. The set $\sigma_{ec}(T) \setminus \sigma_{gk}(T)$ is then empty. □

Example 3. Let $H_p(\mathbb{D})$, $1 < p < \infty$, the classical Hardy space, where $\mathbb{D}$ is the open unit disc of $\mathbb{C}$. The Cesaro operator C_p is defined by

$$(C_p f)(\lambda) = \frac{1}{\lambda}\int_0^\lambda \frac{f(\lambda)}{1-\lambda}\mathrm{d}\mu, \text{ for all } f \in H_p(\mathbb{D}) \text{ and } \lambda \in \mathbb{D}.$$

The spectrum of the operator C_p is the closed disc Γ_p centered at $\frac{p}{2}$ with radius $\frac{p}{2}$, see [1], and $\sigma_{ef}(C_p) \subseteq \sigma_{ap}(C_p) = \partial\Gamma_p$. From Theorem 3.8 we also have

$$\sigma_{ec}(C_p) \subseteq \sigma_{gk}(C_p) = \sigma_{ap}(C_p) = \sigma_{se}(C_p) = \sigma_k(C_p) = \sigma_{es}(C_p) = \partial\Gamma_p$$

REFERENCES

[1] P. Aiena. Fredholm and Local Spectral Theory, with applications to Multipliers. Kluwer Academic Publishres, 2004.

[2] C. Apostol. The reduced minimum modulus. *Michigan Math. J.* **32** (1985), 279–294.

[3] M. Benharrat, B. Messirdi. Relationship between the Kato essential spectrum and a variant of essential spectrum. *General Mathematics* (to appear).

[4] M. Berkani, N. Castro, S. V. Djordjevič. Single valued extension property and generalized Weyl's theorem. *Math. Bohem.* **131**, *1* (2006) 29–38.

[5] S. Grabiner. Ascent, descent, and compact perturbations. *Proc. Amer. Math. Soc.* **71** (1978), 79–80.

[6] A. Dajic, J. J. Koliha. The σg-Drazin inverse and the generalized Mbekhta decomposition. *Integral Equations Operator Theory* **99** (2006), 1–26.

[7] S. Goldberg. Unbounded Linear Operators. McGraw-Hill, New-York, 1966.

[8] Q. Jiang, H. Zhong. Generalized Kato decomposition, single-valued extension property and approximate point spectrum. *J. Math. Anal. Appl.* **356** (2009) 322–327.

[9] T. Kato. Perturbation theory for nullity, deficiency and other quantities of linear operators. *J. Anal. Math.* **6** (1958), 261–322.

[10] T. Kato. Perturbation Theory for Linear Operators. Springer-Verlag, New York, 1966.

[11] J.-P. Labrousse. Les opérateurs quasi Fredholm: une généralisation des opérateurs semi Fredholm. *Rend. Circ. Mat. Palermo (2)* **29** (1980), 161–258.

[12] J.-P. Labrousse. Relation entre deux définitions possible de spectre essentiel d'un opérateur. *Rev. Roumaine Math. Pures Appl.* **XXV**, *9* (1980), 1391–1394.

[13] M. Mbekhta. Résolvant généralisé et théorie spectrale. *J. Operator Theory* **21** (1989), 69–105.

[14] M. Mbekhta. Généralisation de la décomposition de Kato aux opérateurs paranormaux et spectraux. *Glasgow Math. J.* **29** (1987) 159–175.

[15] M. Mbekhta. On the generalized resolvent in Banach spaces. *J. Math. Anal. Appl.* **189** (1995), 362–377.

[16] M. Mbekhta. Sur l'unicité de la décomposition de Kato généralisée. *Acta Sci. Math. (Szeged)* **54** (1990), 367–377.

[17] M. Mbekhta, A. Ouahab. Opérateur s-régulier dans un espace de Banach et théorie spectrale. *Acta Sci. Math. (Szeged)* **59** (1994) 525–543.

[18] V. Müller. On the regular spectrum. *J. Operator Theory* **31** (1994), 363–380.

[19] V. Rakocević. Generalized spectrum and commuting compact perturbations. *Proc. Edinburgh Math. Soc.* **36** (1993), 197–209.

Mohammed Benharrat
Department of Mathematics and Informatics
ENSET Oran
B.P. 1523 Oran El M'Naouar, Algeria
e-mail: `Benharrat2006@yahoo.fr/mohammed.benharrat@gmail.com`

Bekkai Messirdi
Department of Mathematics
University of Oran Es-Senia
31000 Oran, Algeria
e-mail: `messirdi.bekkai@univ-oran.dz`

Received September 16, 2011