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GROUP GRADINGS ON FREE ALGEBRAS OF NILPOTENT VARIETIES OF ALGEBRAS*

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Abstract. The main result is the classification, up to isomorphism, of all gradings by arbitrary abelian groups on the finitely generated algebras that are free in a nilpotent variety of algebras over an algebraically closed field of characteristic zero.

1. Introduction. In this short note I put together two areas that have been of interest to me for an extended period of time: varieties of algebras and group gradings on algebras. Namely, we look at the gradings by abelian groups on relatively free algebras of finite rank in nilpotent varieties of algebras. These algebras do not need to be associative or Lie but it is important that they are finite-dimensional and the varieties in question must be defined by multihomogeneous identities. If the base field of coefficients is infinite, then this is always true. Among algebras to which our results apply are free nilpotent associative or Lie algebras, free metabelian nilpotent Lie algebras and many others. At the

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same time, our Theorem 1 may be of interest also beyond these two areas, for instance in the study of nil-manifolds equipped with some groups of symmetries. An example of such applications of the gradings on algebras can be found in [1].

While writing this note, we had an opportunity of using the forthcoming monograph [2], which we believe will soon become a standard source in the theory of gradings of algebras. We would like to thank the authors for this.

2. Some basics. We recall that given an algebra A, two gradings $\Gamma : A = \bigoplus_{g\in G} A_g$ and $\Gamma' : A = \bigoplus_{h\in H} A'_h$ are called *equivalent* if there is an automorphism $\varphi \in \operatorname{Aut} A$ and a bijection $\beta : \operatorname{Supp}\Gamma \to \operatorname{Supp}\Gamma'$ such that for all $g \in G$ we have $\varphi(A_g) = A'_{\beta(g)}$. Here $\operatorname{Supp}\Gamma$ denotes the *support*: $\operatorname{Supp}\Gamma = \{g \in G \mid A_g \neq 0\}$. We call φ a *weak isomorphism* if β can be extended to an isomorphism of groups G and H. If $\varphi(A_g) = A'_g$ for all $g \in G$ (that is, β is the identity map) then we call Γ and Γ' *isomorphic*.

Using the same notation as just above, we call Γ' a *refinement* of Γ if for any $h \in H$ there is $g \in G$ such that $A'_h \subset A_g$. (In this situation also Γ is called the *coarsening* of Γ'.) If at least for one $h \in H$, the latter containment is proper, we call the refinement proper. A grading Γ is called *fine* if it does not admit proper refinements.

Let k be a field and F_n a finite-dimensional algebra which is free of finite rank n in a variety of algebras given by multihomogeneous identities (of degree > 1). Let $X = \{x_1, \ldots, x_n\}$ stand for the set of free generators of F_n. Given $\alpha = (d_1, \ldots, d_n) \in \mathbb{Z}^n$, we consider the span F_n^α of all the monomials in X whose degree with respect to each variable x_i equals d_i, $i = 1, \ldots, n$. The subspaces F_n^α form a $\mathbb{Z}^n$-grading of F_n:

$$F_n = \bigoplus_{\alpha\in\mathbb{Z}^n} F_n^\alpha. \tag{1}$$

Since F_n is finite-dimensional, there is c such that $F_n^\alpha = \{0\}$ as soon as $|\alpha| > c$. Thus F_n is a nilpotent algebra.

The automorphism group $\operatorname{Aut} F_n$ is an algebraic group and it contains an n-dimensional toral subgroup D_n whose elements are defined as follows. Pick $\overline{t} = (t_1, \ldots, t_n) \in (k^\times)^n$. The map $x_1 \mapsto t_1x_1, \ldots, x_n \mapsto t_nx_n$ uniquely extends to an automorphism $\delta(\overline{t})$ of F_n. We set $D_n = \{\delta(\overline{t}) \mid \overline{t} \in (k^\times)^n\}$. Obviously, $D_n \cong (k^\times)^n$.

3. Quasitori in Aut F_n. In the theory of gradings by abelian groups an important role is played by the toral subgroups and their normalizers in $\operatorname{Aut} F_n$. Let us define a subgroup P_n in $\operatorname{Aut} F_n$ as follows. Given a permutation σ on

$\{1,2,\ldots,n\}$ and $\overline{t}$, as before, we consider the map $x_1 \mapsto t_1x_{\sigma(1)},\ldots,x_n \mapsto t_nx_{\sigma(n)}$. The extension of this map to the whole of F_n will be denoted by $\delta(\sigma,\overline{t})$. Clearly, $\delta(\overline{t}) = \delta(\mathrm{id},\overline{t})$. The set of all $\delta(\sigma,\overline{t})$ where $\sigma \in S_n$ and $\overline{t} \in (k^\times)^n$ is a subgroup of $\mathrm{Aut}\,F_n$. We denote this subgroup by P_n.

Proposition 1. *If the ground field k is infinite, the normalizer of D_n in $\mathrm{Aut}\,F_n$ equals P_n.*

Proof. Suppose an automorphism μ belongs to the normalizer of D_n. On the elements of the generating set X the action of μ can be written as

$$\mu(x_i) = \sum_{j=1}^{n} a_i^j x_j + \sum_{\alpha,|\alpha|>1} w_i^\alpha \text{ where for all } i,j,\alpha \text{ we have } a_i^j \in k,\ w_i^\alpha \in F_n^\alpha.$$

Pick $\delta(\overline{t}) \in D_n$. Then $\mu^{-1}\delta(\overline{t})\mu \in D_n$. In other words, there is $\delta(\overline{s}) \in D_n$ such that $\delta(\overline{t})\mu = \mu\delta(\overline{s})$. Using the definition of μ we determine that

$$\mu\delta(\overline{s})(x_i) = \sum_{j=1}^{n} s_i a_i^j x_j + \sum_{\alpha,|\alpha|>1} s_i w_i^\alpha$$

while

$$\delta(\overline{t})\mu(x_i) = \sum_{j=1}^{n} a_i^j t_j x_j + \sum_{\alpha,|\alpha|>1} t_1^{d_1}\cdots t_n^{d_n} w_i^\alpha.$$

It follows that we must have $a_i^j t_j = s_i a_i^j$ for all i,j and also $t_1^{d_1}\cdots t_n^{d_n} w_i^\alpha = s_i w_i^\alpha$ for any α with $|\alpha| > 1$.

From the equations of the first kind, provided that all t_i are chosen pairwise different, it follows that for each $i = 1,\ldots,n$ there is only one j such that $a_i^j \neq 0$ and we must have $s_i = t_{\sigma(i)}$. Since μ is an automorphism, σ is a permutation on the set $\{1,\ldots,n\}$.

Now if $w_i^\alpha \neq 0$ we must have $t_{\sigma(i)} = \overline{t}^\alpha$. Since $t_1,\ldots,t_n$ are chosen different from 0, it follows that $\overline{t}^\alpha \neq 0$. If this is the case for some α and i, we have, for any $(t_1,\ldots,t_n) \in (k^\times)^n$ that $t_{\sigma(i)} = t_1^{d_1}\cdots t_n^{d_n}$ where $|\alpha| = d_1+\cdots+d_n > 1$. If $d_j \neq 0$ for some $j \neq \sigma(i)$ then, replacing all t_k, $k \neq j$, by 1 we obtain a contradiction $1 = t_j^{d_j}$, for all $t_j \in k^\times$, k infinite. Otherwise, we have that $t_{\sigma(i)} = t_{\sigma(i)}^{|\alpha|}$, for any $t_{\sigma(i)} \in k^\times$, which is again a contradiction. Therefore, we must have $w_i^\alpha = 0$, for all σ and i.

Our conclusion is that for each μ in the normalizer of D there exists a permutation $\sigma \in S_n$ on the set $\{1,\ldots,n\}$ such that

$$\mu(x_i) = a_i^{\sigma(i)} x_{\sigma(i)} \text{ for } i = 1,\ldots,n.$$

Thus we obtain $\mu = \delta(\sigma, \bar{t})$ with $t_1 = a_1^{\sigma(1)}, \ldots, t_n = a_n^{\sigma(n)}$, hence $\mu \in P_n$, as claimed. The converse statement being obvious, our proof is complete. $\square$

Corollary 1. *If k is an infinite field, then D_n is a maximal torus in* $\mathrm{Aut}\, F_n$.

P r o o f. We consider the subspace $V = \bigoplus_{|\alpha|=1} F_n^\alpha$. Every non-singular linear transformation of V uniquely extends to an automorphism of F_n. All such automorphisms form a subgroup H of $\mathrm{Aut}\, F_n$ isomorphic $\mathrm{GL}(V)$. Choosing X as a basis of V allows us to assume the existence of an isomorphism $\iota : H \to \mathrm{GL}_n(k)$ such that $\iota(D_n)$ is the group of all non-singular diagonal matrices and $\iota(P_n)$ the subgroup of "monomial" matrices, that is, all matrices with exactly one nonsingular entry in each row and in each column. It is well known that the group of all diagonal matrices is a maximal torus in $\mathrm{GL}_n(k)$. It follows that the centralizer of $\iota(D_n)$ in $\mathrm{GL}_n(k)$ is itself. But then also the centralizer of $\iota(D_n)$ in $\iota(P_n)$ is itself. Since ι is an isomorphism, the centralizer of D_n in P_n is itself, proving that D_n is indeed a maximal torus in $\mathrm{Aut}\, F_n$. $\square$

The next corollary deals with quasitori. Given an algebraic group, a *quasitorus* is a subgroup consisting of semisimple elements, which is isomorphic to the direct product of a torus and a finite abelian group. A typical example of importance to us is the following. Let G be an abelian group and $\Gamma : F = \bigoplus_{g\in G} (F_n)_g$ a grading of F_n by G. Let us consider the "diagonal" subgroup $D(\Gamma) \subset \mathrm{Aut}\, F_n$. This subgroup consists of the automorphisms of F_n whose restriction to any component $(F_n)_g$ of the grading Γ is a scalar linear transformation. Clearly, $D(\Gamma)$ is a quasitorus in $\mathrm{Aut}\, F_n$.

Corollary 2. *If k is an algebraically closed field, any quasitorus in* $\mathrm{Aut}\, F_n$ *is conjugate to a subgroup of D_n.*

P r o o f. Let Q be the quasitorus in question. According to [3] (see also [4]), any quasitorus is a subgroup in the normalizer of a maximal torus. Since in an algebraic group any two maximal tori are conjugate, the same is true for their normalizers and thus using Proposition 1, we may assume that there exists $\pi \in \mathrm{Aut}\, F_n$ such $\pi Q \pi^{-1} \subset P_n$. Denote $Q' = \pi Q \pi^{-1}$.

Next we use the isomorphism ι from the proof of Corollary 1. We observe that $\iota(Q')$ is a quasitorus in $\mathrm{GL}_n(k)$. Since k is algebraically closed, there exists $\psi \in \mathrm{GL}_n(k)$ such that $\psi\iota(Q')\psi^{-1} \subset \iota(D_n)$. Set $\rho = \iota^{-1}(\psi) \in H$. Then $\iota(\rho Q \rho^{-1}) \subset \iota(D_n)$ and thus $\rho Q \rho^{-1} \subset D_n$. Setting $\tau = \rho\pi \in \mathrm{Aut}\, F_n$, we obtain that $\tau Q \tau^{-1} \subset D_n$, as claimed. $\square$

4. Gradings of F_n. From now on we assume that k is an algebraically closed field and finally turn our attention to the gradings on F_n by arbitrary abelian groups. Consider one of such gradings, $\Gamma : F = \bigoplus_{g\in G}(F_n)_g$, by an abelian group G. Since F_n is finite-dimensional, only finitely many elements of G are used for the grading. This allows us, without loss of generality, to assume that G is a finitely generated group. We consider $\widehat{G}$, the group of characters of G with values in $k^\times$. This group is a quasitorus. As is well known, $\widehat{G}$ acts by semisimple automorphisms on F_n, according to the rule $\chi * w = \chi(g)w$, for any $\chi \in \widehat{G}$ and $w \in (F_n)_g$. As a result, we have a homomorphism $\varphi : \widehat{G} \to \operatorname{Aut} F_n$, which maps $\widehat{G}$ into the quasitorus $D(\Gamma)$. By Corollary 2, there is an automorphism $\tau : F_n \to F_n$ such that $\tau D(\Gamma)\tau^{-1} \subset D_n$. We can also write that $D(\Gamma) \subset \tau^{-1}D_n\tau$.

Now let us consider the decomposition of F_n as the sum of subspaces $\tau^{-1}(F^\alpha)$. This is a grading of F_n by $\mathbb{Z}^n$ isomorphic to our standard $\mathbb{Z}^n$-grading. At the same time, this is the eigenspace decomposition of F_n under the action of the maximal torus $\tau^{-1}D_n\tau$. Since $D(\Gamma) \subset \tau^{-1}D_n\tau$, we have that each eigenspace of the decomposition of F_n with respect to the action of the quasitorus $D(\Gamma)$ is the sum of several subspaces $\tau^{-1}(F^\alpha)$.

The eigenspace decomposition of F_n with respect to $\varphi(\widehat{G})$ is Γ if k is of characteristic 0. If k has characteristic p then this is still Γ provided that G has no elements of order p. Otherwise, this is the coarsening of Γ induced by the quotient map $G \to G/S_p$ where S_p is the Sylow p-subgroup of G and p is the characteristic of k.

Now given $\alpha \in \mathbb{Z}^n$, there is a character $\omega(\alpha) : \tau^{-1}D_n\tau \to k^\times$ such that $\tau^{-1}(F^\alpha)$ is the eigenspace of the action of $\tau^{-1}D_n\tau$ corresponding to $\omega(\alpha)$. The restriction of $\omega(\alpha)$ to $\varphi(\widehat{G})$ is a character of this quasitorus, which can be identified with an element $\overline{g} = gH$, $g \in G$, of G/H where H is either trivial or S_p, as explained above. Since the restriction map is a homomorphism, it follows that $\nu : \alpha \to \overline{g}$ is a homomorphism of $\mathbb{Z}^n$ to G/H. A simple calculation shows that $(F_n)_{\overline{g}} = \sum_{\nu(\alpha)=\overline{g}} F^\alpha$ where $(F_n)_{\overline{g}} = \sum_{h\in H}(F_n)_{gh}$.

The above discussion brings us to the following main result of this note.

Theorem 1. *Let k be an algebraically closed field, F_n a (nonzero) finite-dimensional algebra which is free of rank n in a variety of algebras over k. Let $\Gamma : F = \bigoplus_{g\in G}(F_n)_g$ be a grading of F_n by an abelian group G. If k is of characteristic zero or if G has no elements of order p where p is the characteristic of k then Γ is isomorphic to a G-grading induced from the standard $\mathbb{Z}^n$-grading* (1) *by a homomorphism $\mathbb{Z}^n \to G$. Otherwise, the same is true for the coarsening of Γ obtained by taking the quotient of G with respect to its Sylow p-subgroup.*

To obtain all gradings, up to isomorphism, of F_n by an abelian group

G (assuming $\operatorname{char} k = 0$ or $\operatorname{char} k = p$ and G has no elements of order p) we simply have to choose n elements $g_1, \ldots, g_n$ from G and assign to each $w \in F^\alpha$ the degree $d(w) = g_1^{d_1} \cdots g_n^{d_n}$ where $\alpha = (d_1, \ldots, d_n) \in \mathbb{Z}^n$. In particular, $d(x_1) = g_1, \ldots, d(x_n) = g_n$. The isomorphism class of such grading is determined by the elements of G that occur in the n-tuple $(g_1, \ldots, g_n)$, counting multiplicity.

Our last result deals with the fine gradings on F_n by abelian groups. We note that, in characteristic zero, Theorem 1 implies that any such grading is isomorphic to a coarsening of the standard grading, hence we obtain the following:

Corollary 3. *Let k be an algebraically closed field of characteristic zero, F_n a (nonzero) finite-dimensional algebra which is free of rank n in a variety of algebras over k. Then, up to equivalence, the standard grading* (1) *is the only fine abelian group grading of F_n.*

REFERENCES

[1] Y. Bahturin, M. Goze. $\mathbb{Z}_2 \times \mathbb{Z}_2$-symmetric spaces. *Pacific J. Math.* **236** (2008), 1–21.

[2] A. Elduque, M. Kochetov. Gradings on Simple Lie Algebras, manuscript.

[3] V. P. Platonov. The theory of algebraic linear groups and periodic groups. *Trans Amer. Math. Soc. Ser.* (*2*) **69** (1968), 61–159.

[4] T. A. Springer, R. Steinberg. Conjugacy classes. In: Seminar on Algebraic Groups and Related Finite Groups (The Institute for Advanced Study, Princeton, N.J., 1968/69), Lecture Notes in Mathematics, vol. **131**, Springer, Berlin, 1970, 167–266.

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