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PANCYCLIC CAYLEY GRAPHS

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Dedicated to Yuri Bahturin on the occasion of his 65th birthday

Abstract. Let $\mathrm{Cay}(G;S)$ denote the Cayley graph on a finite group G with connection set S. We extend two results about the existence of cycles in $\mathrm{Cay}(G;S)$ from cyclic groups to arbitrary finite Abelian groups when S is a "natural" set of generators for G.

1. Introduction. Let G be a finite group with identity 1 and let S be a subset of G satisfying $1 \notin S$ and $S = S^{-1}$. The Cayley graph on G with connection set S, denoted $\mathrm{Cay}(G;S)$, is defined as follows:

(i) the vertices of $\mathrm{Cay}(G;S)$ are the elements of G

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(ii) there is an edge joining vertices g, h if and only if $h = sg$ for some $s \in S$.

Properties of Cayley graphs have been well studied and an excellent survey can be found in Alspach [2].

The following easily proved lemma will be needed throughout this note.

Lemma 1.1. $\mathrm{Cay}(G; S)$ *is connected if and only if* S *is a generating set for* G.

A much deeper result which is central to our work is the following. Recall that a graph is Hamiltonian if it contains a cycle passing through all of its vertices.

Theorem 1.2. *If* A *is a finite Abelian group of order* ≥ 3 *then* $\mathrm{Cay}(A; S)$ *is Hamiltonian if and only if* S *is a generating set for* A.

Alspach [2] obtains Theorem 1.2 as a corollary of a theorem of Chen and Quimpo, while an alternative direct argument given by Marušič [5] seems to be missing essential details. It would be very interesting to see a reasonably short self-contained proof of Theorem 1.2.

It is an open question (see [2]) as to whether Theorem 1.2 can be extended to arbitrary finite groups.

Recall that a graph with n vertices is pancyclic if it contains cycles of all possible lengths $i, 3 \leq i \leq n$. While pancyclic is clearly a much stronger property than Hamiltonian, in practice it seems that conditions which are sufficient to conclude that a graph is Hamiltonian are also sufficient to conclude that a graph is pancyclic (except for a few easily listed exceptions). For example Bondy [4] proved that if a Hamiltonian graph with n vertices has at least $\dfrac{n^2}{4}$ edges then either the graph is pancyclic or it is complete bipartite of type $K_{\frac{n}{2},\frac{n}{2}}$, so it follows that well known Hamiltonian results like Dirac's Theorem or Ore's Theorem extend to the pancyclic case. Because of this, and keeping in mind Theorem 1.2, it is very natural to investigate the pancyclic property for Cayley graphs of finite Abelian groups.

Since Cayley graphs of finite cyclic groups form precisely the same class as circulant graphs, this investigation has already been carried out for cyclic groups by Bogdanowicz [3]. His results are as follows.

Theorem 1.3. *Assume* A *is finite cyclic of order* ≥ 3 *and* S *is a generating set for* A. *If* $|A| \geq 5$ *assume in addition that* $\mathrm{Cay}(A; S)$ *is not a single cycle. Then*

(i) $\mathrm{Cay}(A;S)$ *contains cycles of all possible* <u>*even*</u> *lengths* $i, 4 \le i \le |A|$.

(ii) $\mathrm{Cay}(A;S)$ *is pancyclic if and only if* $\mathrm{Cay}(A;S)$ *contains a* 3*-cycle.*

The assumption that $\mathrm{Cay}(A;S)$ is not a single cycle is a condition on the connection set S which is clearly required (just as S being a generating set is required for Theorems 1.2 and 1.3 to hold). It is equivalent to saying that if x is any generator of A and $x \in S$ then there exists $s \in S$ such that $s \neq x, x^{-1}$.

Note that $\mathrm{Cay}(A;S)$ contains a 3-cycle if and only if there exist $a, b \in S$ such that $ab \in S$.

We are interested in the following.

Question. *Does Theorem* 1.3 *extend to arbitrary finite Abelian groups?*

Note that if A is noncyclic Abelian and S is a generating set for A then $\mathrm{Cay}(A;S)$ cannot be a cycle, so this particular assumption no longer concerns us. Also note that if $\mathrm{Cay}(A;S)$ satisfies one of the properties we are interested in and $S \subseteq S'$ then $\mathrm{Cay}(A;S')$ will satisfy the same property. It follows from the last two sentences that if we can prove a property holds for $\mathrm{Cay}(A;S)$ when S is a minimal generating set satisfying $S = S^{-1}$, then we will have proved the result in general.

Any decomposition of a finite Abelian group into a direct product of cyclic groups gives a very natural generating set. In the next section we will show that Theorem 1.3 does extend to this case.

2. Results.

Our first main result is the following.

Theorem 2.1. *Let* $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ *be a finite Abelian group, where* $n > 1$ *and each* $A_i = \langle a_i \rangle$ *is cyclic. If* $S = \{a_1, a_1^{-1}, a_2, a_2^{-1}, \cdots a_n, a_n^{-1}\}$ *then* $\mathrm{Cay}(A;S)$ *contains even cycles of all possible lengths* $\le |A|$.

Proof. If $n = 2$ and $|A_1| = |A_2| = 2$ then $\mathrm{Cay}(A;S)$ is a 4-cycle and the result is clear. So, setting $B = A_2 \oplus \cdots \oplus A_n$, we may assume (relabelling if necessary) that $|B| \ge 3$. Setting $S' = \{a_2, a_2^{-1}, \cdots a_n, a_n^{-1}\}$, it follows from Theorem 1.2 that $\mathrm{Cay}(B;S')$ has a Hamiltonian cycle $1, b_1, b_2, \cdots b_{k-1} (k = |B|)$.

Observe that $1, a_1, a_1b_1, b_1$ is a 4-cycle in $\mathrm{Cay}(A; S)$. This can be expanded to the 6-cycle $1, a, a_1b_1, a_1b_2, b_2, b_1$ (since b_1 and b_2 are joined in $\mathrm{Cay}(B, S'), b_2b_1^{-1} \in S' \subseteq S$) and more generally, to the $(2s+2)$-cycle.

$$1, a_1, a_1b_1, a_1b_2, \cdots a_1b_s, b_s, b_{s-1}, \cdots b_2, b_1$$

for all $s, 2 \leq s \leq k-1$ (recall $k \geq 3$).

So we have constructed cycles of all possible even lengths up to and including $2k$, the longest being

$$1, a_1, a_1b_1, \cdots a_1b_{k-1}, b_{k-1}, b_{k-2}, \cdots b_2, b_1.$$

Let $|a_1| = \ell$. If $\ell = 2$ we're done, so assume $\ell > 2$. Since $k-1 \geq 2$ the above chain can be extended to

$$1, a_1, a_1b_1, \cdots a_1b_{k-2}, a_1^2b_{k-2}, \cdots a_1^rb_{k-2}, a_1^rb_{k-1},$$
$$a^{r-1}b_{k-1}, \cdots a_1b_{k-1}, b_{k-1}, b_{k-2}, \cdots b_1$$

for all $r, 2 \leq r \leq \ell - 1$.

This gives cycles of all possible even lengths up to and including $2k + 2(\ell-2) = 2k + 2\ell - 4$, the longest being

$$1, a_1, a_1b_1, \cdots a_1b_{k-2}, a_1^2b_{k-2}, \cdots a_1^{\ell-1}b_{k-2}, a_1^{\ell-1}b_{k-1},$$
$$a_1^{\ell-2}b_{k-1}, \cdots a_1b_{k-1}, b_{k-1}, b_{k-2}, \cdots b_1$$

We can continue extending the above chain by changing the edge a_1b_{k-4}, a_1b_{k-3} to $a_1b_{k-4}, \cdots a_1^tb_{k-4}, a_1^tb_{k-3}, a_1^{t-1}b_{k-3}, \cdots a_1b_{k-3}$ for any $t, 2 \leq t \leq \ell-1$, and then repeat with a_1b_{k-6}, a_1b_{k-5} etc. (assuming k is large enough).

Two cases need to be considered.

Case I. k is even.

In this case the last edge to which the above process applies is a_1b_2, a_1b_3. But it can also be applied to the edge a_1, a_1b_1. The longest cycle obtained in this way is

$$1, a_1, a_1^2, \cdots a_1^{\ell-1}, a_1^{\ell-1}b_1, a_1^{\ell-2}b_1, \cdots a_1b_1, a_1b_2, a_1^2b_2, \cdots$$
$$a_1^{\ell-1}b_2, a_1^{\ell-1}b_3, \cdots a_1b_3, a_1b_4, \cdots, a_1b_{k-2}, a_1^2b_{k-2}, \cdots$$
$$a_1^{\ell-1}b_{k-2}, a_1^{\ell-1}b_{k-1}, \cdots a_1b_{k-1}, b_{k-1}, \cdots b_1$$

Since this cycle is Hamiltonian our result is proved.

Case II. k is odd.

Observe first that if ℓ is even and $\ell > 2$ then a suitable relabelling of $A_1, A_2, \cdots A_n$ would put us in the case where k is even. So we may assume from now on that both ℓ and k are odd.

Since k is odd the last edge to which the previously described process applies is a_1b_1, a_1b_2. This gives cycles of all possible even lengths up to and including $\ell k - (\ell - 2)$. The longest cycle constructed so far is

$$1, a_1, a_1b_1, a_1^2b_1, \cdots a_1^{\ell-1}b_1, a_1^{\ell-1}b_2, \cdots a_1b_2, a_1b_3, \cdots$$
$$a_1^{\ell-1}b_3, a_1^{\ell-1}b_4, \cdots a_1b_4, a_1b_5, \cdots a_1b_{k-2}, \cdots a_1^{\ell-1}b_{k-2}, a_1^{\ell-1}b_{k-1}, \cdots$$
$$a_1b_{k-1}, b_{k-1}, b_{k-2}, \cdots 1.$$

If $\ell = 3$ this cycle is of maximal even length and we are again done.

So we may assume ℓ is odd and $\ell > 3$. Note now that if q is any odd integer, $1 < q < \ell$, the above cycle can be extended by changing $1, a_1, a_1b_1, \cdots a_1^{\ell-1}b_1$ to

$$1, a_1, a_1b_1, \cdots a_1^qb_1, a_1^q, a_1^{q+1}, a_1^{q+1}b_1, a_1^{q+2}b_1, a_1^{q+2}, a_1^{q+3}, a_1^{q+3}b_1, \cdots$$
$$a_1^{\ell-2}b_1, a_1^{\ell-2}, a_1^{\ell-1}, a_1^{\ell-1}b_1$$

This new cycle is of length $\ell k - (q-2)$, so we have now constructed cycles of all possible even lengths up to and including $\ell k - 1$.

The proof is complete. $\square$

Our second main result shows, as in Theorem 1.3, that the existence of a 3-cycle guarantees that $\mathrm{Cay}(A; S)$ is pancyclic. We remark that this result is also obtained (in more generality) in [1], but we include the proof here because reference [1] may be somewhat difficult for readers to obtain.

Theorem 2.2. *Let $A = A_1 \oplus A_2 \oplus \cdots \oplus A_n$ be a finite Abelian group of order ≥ 3, where each $A_i = \langle a_i \rangle$ is cyclic. If $S = \{a_1, a_1^{-1}, a_2, a_2^{-1}, \cdots a_n, a_n^{-1}\}$ then $\mathrm{Cay}(A; S)$ is pancyclic if and only if $|A_i| = 3$ for some i.*

P r o o f. First assume $\mathrm{Cay}(A; S)$ is pancyclic, so in particular it must contain a 3-cycle. As noted earlier, this means there must exist $a, b \in S$ such that $ab \in S$. Because of the nature of S, this forces $b = a$ and $a^2 = a^{-1}$, and the result follows.

Conversely assume $|A_i| = 3$ for some i. We may assume $i = 1$. If $|A| = 3$ we're done – otherwise, we have $A = \langle a \rangle \oplus B$ where $|a| = 3$ and $B = A_2 \oplus \cdots \oplus A_n$.

It is easy to check directly that the result holds if $|B| = 2$, so assume $|B| > 2$. As in the proof of Theorem 2.1, we set $S' = \{a_2, a_2^{-1}, \cdots a_n, a_n^{-1}\}$ and observe, using Theorem 1.2, that $\mathrm{Cay}(B; S')$ has a Hamiltonian cycle $1, b_1, b_2, \cdots b_{k-1}$.

We will now give a direct method for constructing cycles of all possible lengths.

Start with the 3-cycle $1, a, a^{-1}(= a^2)$. We add on to get a 6-cycle $1, a, a^{-1}$, $a^{-1}b_1, ab_1, b_1$. A 9-cycle can now be obtained by inserting $ab_2, a^{-1}b_2, b_2$ between ab_1 and b_1, and then a 12-cycle can be obtained by inserting $a^{-1}b_3, ab_3, b_3$ between $a^{-1}b_2$ and b_2. This process continues – to get from one cycle to the next insert $a^{\pm 1}b_{i+1}, a^{\mp 1}b_{i+1}, b_{i+1}$ between $a^{\pm 1}b_i$ and b_i. In this way we obtain cycles of all possible lengths $3x$.

To get the remaining cycles, we carry out exactly the same procedure starting with the 4-cycle $1, a, ab_1, b_1$ and the 5-cycle $1, a, a^{-1}, a^{-1}b_1, b_1$. This gives all cycles of length $3x + 1$ and $3x + 2$, completing the proof. sq

REFERENCES

[1] M. Akelbek. Hamiltonian properties of Cayley graphs on Abelian groups. *J. Xinjiang Univ. Natur. Sci.* **20** (2003), 14–21.

[2] B. Alspach. Cayley Graphs. In: Topics in Algebraic Graph Theory (Eds L. Beineke, R. Wilson), Cambridge University Press, 2004, 1–27.

[3] Z. R. Bogdanowicz. Pancyclicity of connected circulant graphs. *J. Graph Theory* **22** (1996), 167–174.

[4] J. A. Bondy. Pancyclic graphs 1. *J. Combin. Theory* **11** (1971), 80–84.

[5] D. Marušič. Hamiltonian circuits in Cayley graphs. *Discrete Math.* **46** (1983), 49–54.

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