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ALGEBRAS, DIALGEBRAS, AND POLYNOMIAL IDENTITIES*

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Communicated by V. Drensky

Dedicated to Yuri Bahturin on his 65th birthday

Abstract. This is a survey of some recent developments in the theory of associative and nonassociative dialgebras, with an emphasis on polynomial identities and multilinear operations. We discuss associative, Lie, Jordan, and alternative algebras, and the corresponding dialgebras; the KP algorithm for converting identities for algebras into identities for dialgebras; the BSO algorithm for converting operations in algebras into operations in dialgebras; Lie and Jordan triple systems, and the corresponding disystems; and a noncommutative version of Lie triple systems based on the trilinear operation $abc - bca$. The paper concludes with a conjecture relating the KP and BSO algorithms, and some suggestions for further research. Most of the original results are joint work with Raúl Felipe, Luiz A. Peresi, and Juana Sánchez-Ortega.

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1. Algebras. Throughout this talk the base field $\mathbb{F}$ will be arbitrary, but we usually exclude low characteristics, especially $p \leq n$ where n is the degree of the polynomial identities under consideration. The assumption $p > n$ allows us to assume that all polynomial identities are multilinear and that the group algebra $\mathbb{F}S_n$ is semisimple.

In this first section we summarize some basic concepts and well-known facts about algebras. They have been included to indicate the parallels between these classical results and the corresponding concepts and facts for dialgebras.

Definition 1.1. *An **algebra** is a vector space A with a bilinear operation*

$$\mu\colon A \times A \to A.$$

*Unless otherwise specified, we write $ab = \mu(a,b)$ for $a, b \in A$. We say that A is **associative** if it satisfies the polynomial identity*

$$(ab)c \equiv a(bc).$$

Throughout this paper we will use the symbol $\equiv$ to indicate an equation that holds for all values of the arguments; in this case, all $a, b, c \in A$.

Theorem 1.2. *The free unital associative algebra on a set X of generators has basis consisting of all words of degree $n \geq 0$,*

$$x = x_1 x_2 \cdots x_n, \quad \textit{where} \quad x_1, x_2, \ldots, x_n \in X,$$

with the product defined on basis elements by concatenation and extended bilinearly,

$$(x_1 x_2 \cdots x_m)(y_1 y_2 \cdots y_n) = x_1 x_2 \cdots x_m y_1 y_2 \cdots y_n.$$

Definition 1.3. *The **commutator** in an algebra is the bilinear operation*

$$[a,b] = ab - ba.$$

*This operation is **anticommutative**: it satisfies $[a,b] + [b,a] \equiv 0$.*

Lemma 1.4. *In an associative algebra, the commutator satisfies the identity*

$$[[a,b],c] + [[b,c],a] + [[c,a],b] \equiv 0 \qquad \text{(Jacobi)}$$

Definition 1.5. *A **Lie algebra** is an algebra which satisfies anticommutativity and the Jacobi identity.*

Theorem 1.6 (Poincaré-Birkhoff-Witt). *Every Lie algebra L has a universal associative enveloping algebra $U(L)$ for which the canonical map $L \to U(L)$ is injective. It follows that every polynomial identity satisfied by the commutator in every associative algebra is a consequence of anticommutativity and the Jacobi identity.*

Remark 1.7. Most texts on Lie algebras include a proof of the PBW Theorem. The most beautiful proof is that of Bergman [2] using noncommutative Gröbner bases; see also de Graaf [18, Ch. 6]. For the history of the PBW Theorem, see Grivel [23]. For a survey on Gröbner-Shirshov bases, see Bokut and Kolesnikov [4].

Definition 1.8. *The **anticommutator** in an algebra is the bilinear operation*

$$a \circ b = ab + ba;$$

*we omit the scalar $\frac{1}{2}$. This operation is **commutative**: it satisfies $a \circ b - b \circ a \equiv 0$.*

Lemma 1.9. *In an associative algebra, the anticommutator satisfies the identity*

$$((a \circ a) \circ b) \circ a - (a \circ a) \circ (b \circ a) \equiv 0 \qquad \text{(Jordan)}$$

Definition1.10. *A **Jordan algebra** is an algebra which satisfies commutativity and the Jordan identity.*

Theorem 1.11. *There exist polynomial identities satisfied by the anticommutator in every associative algebra which do not follow from commutativity and the Jordan identity. The lowest degree in which such identities exist is* 8.

Remark 1.12. A Jordan algebra is called *special* if it is isomorphic to a subspace of an associative algebra closed under the anticommutator. A polynomial identity for Jordan algebras is called *special* if it is satisfied by all special Jordan algebras but not by all Jordan algebras. The first special identities for Jordan algebras were found by Glennie [20, 21]. For a computational approach, see Hentzel [26]. Another s-identity was obtained by Thedy [48]; see also McCrimmon [38] and [40, Appendix B.5]. For a survey on identities in Jordan algebras, see McCrimmon [39].

Remark 1.13. From the perspective of polynomial identities, there is a clear dichotomy between the two bilinear operations, commutator and anticommutator. Both operations satisfy simple identities in low degree; for the commutator, these identities imply all the identities satisfied by the operation, but for the anticommutator, there exist special identities of higher degree.

2. Dialgebras. We now recall the concept of a dialgebra: a vector space with two multiplications. Associative dialgebras were originally defined by Loday in the 1990s, and the results quoted in this section were proved by him; see especially his original paper [33] and his survey article [34]. Associative dialgebras provide the natural setting for Leibniz algebras, a "non-anticommutative" generalization of Lie algebras; see Loday [32].

Definition 2.1. *A **dialgebra** is a vector space A with two bilinear operations,*

$$\dashv\colon A \times A \to A, \qquad \vdash\colon A \times A \to A,$$

*called the **left** and **right** products. We say that A is a **0-dialgebra** if it satisfies the **left** and **right bar identities**,*

$$(a \dashv b) \vdash c \equiv (a \vdash b) \vdash c, \qquad a \dashv (b \dashv c) \equiv a \dashv (b \vdash c).$$

*An **associative dialgebra** is a 0-dialgebra satisfying **left**, **right**, and **inner associativity**:*

$$(a \dashv b) \dashv c \equiv a \dashv (b \dashv c), \quad (a \vdash b) \vdash c \equiv a \vdash (b \vdash c), \quad (a \vdash b) \dashv c \equiv a \vdash (b \dashv c).$$

Definition 2.2. *Let $x = x_1 x_2 \cdots x_n$ be a monomial in an associative dialgebra, with some placement of parentheses and choice of operations. The **center** of x, denoted $c(x)$, is defined by induction on n:*

- *If $n = 1$ then $x = x_1$ and $c(x) = x_1$.*
- *If $n \geq 2$ then $x = y \dashv z$ or $x = y \vdash z$, and $c(x) = c(y)$ or $c(x) = c(z)$ respectively.*

Lemma 2.3. *Let $x = x_1 x_2 \cdots x_n$ be a monomial in an associative dialgebra with $c(x) = x_i$. Then the following expression does not depend on the placement of parentheses:*

$$x = x_1 \vdash \cdots \vdash x_{i-1} \vdash x_i \dashv x_{i+1} \dashv \cdots \dashv x_n.$$

Definition 2.4. *The expression in Lemma 2.3 is called the **normal form** of the monomial x, and is abbreviated using the **hat notation**:*

$$x = x_1 \cdots \widehat{x}_i \cdots x_n.$$

Theorem 2.5. *The free associative dialgebra on a set X of generators has basis consisting of all monomials in normal form:*

$$x = x_1 \cdots \widehat{x}_i \cdots x_n \qquad (1 \leq i \leq n,\ x_1, x_2, \ldots, x_n \in X).$$

Two such monomials are equal if and only if they have the same permutation of the generators and the same position of the center. The left and right products are defined on monomials as follows and extended bilinearly:

$$x \dashv y = (x_1 \cdots \widehat{x}_i \cdots x_n) \dashv (y_1 \cdots \widehat{y}_j \cdots y_p) = x_1 \cdots \widehat{x}_i \cdots x_n y_1 \cdots y_p,$$
$$x \vdash y = (x_1 \cdots \widehat{x}_i \cdots x_n) \vdash (y_1 \cdots \widehat{y}_j \cdots y_p) = x_1 \cdots x_n y_1 \cdots \widehat{y}_j \cdots y_p.$$

Definition 2.6. *The* ***dicommutator*** *in a dialgebra is the bilinear operation*

$$\langle a, b \rangle = a \dashv b - b \vdash a.$$

In general, this operation is not anticommutative.

Lemma 2.7. *In an associative dialgebra, the dicommutator satisfies the identity*

$$\langle \langle a, b \rangle, c \rangle \equiv \langle \langle a, c \rangle, b \rangle + \langle a, \langle b, c \rangle \rangle \qquad \text{(Leibniz)}$$

Definition 2.8. *A* ***Leibniz algebra*** *(or Lie dialgebra) is an algebra satisfying the Leibniz identity.*

Remark 2.9. If we set $b = c$ in the Leibniz identity then we obtain $\langle a, \langle b, b \rangle \rangle \equiv 0$, and the linearized form of this identity (assuming characteristic not 2) is

$$\langle a, \langle b, c \rangle \rangle + \langle a, \langle c, b \rangle \rangle \equiv 0 \qquad \text{(right anticommutativity)}$$

Theorem 2.10 (Loday-Pirashvili)**.** *Every Leibniz algebra L has a universal associative enveloping dialgebra $U(L)$ for which the canonical map $L \to U(L)$ is injective. Hence every polynomial identity satisfied by the dicommutator in every associative dialgebra is a consequence of the Leibniz identity.*

Remark 2.11. The Loday-Pirashvili Theorem is the generalization to dialgebras of the PBW Theorem. For the original proof, see [35]. For different approaches, see Aymon and Grivel [1], Insua and Ladra [28].

Remark 2.12. The definition of associative dialgebra can be motivated in terms of the Leibniz identity. If we expand the Leibniz identity in a nonassociative dialgebra using the dicommutator as the operation, then we obtain

$$(a \dashv b - b \vdash a) \dashv c - c \vdash (a \dashv b - b \vdash a) \equiv$$
$$(a \dashv c - c \vdash a) \dashv b - b \vdash (a \dashv c - c \vdash a) + a \dashv (b \dashv c - c \vdash b) - (b \dashv c - c \vdash b) \vdash a.$$

Equating terms with the same permutation of a, b, c gives the following identities:

$$\begin{aligned} (a \dashv b) \dashv c &\equiv a \dashv (b \dashv c), & 0 &\equiv (a \dashv c) \dashv b - a \dashv (c \vdash b), \\ (b \vdash a) \dashv c &\equiv b \vdash (a \dashv c), & 0 &\equiv b \vdash (c \vdash a) - (b \dashv c) \vdash a, \\ c \vdash (a \dashv b) &\equiv (c \vdash a) \dashv b, & c \vdash (b \vdash a) &\equiv (c \vdash b) \vdash a. \end{aligned}$$

These are equivalent to the identities defining associative dialgebras.

Definition 2.13. *The **antidicommutator** in a dialgebra is the bilinear operation*

$$a \star b = a \dashv b + b \vdash a.$$

In general, this operation is not commutative.

Lemma 2.14. *In an associative dialgebra, the antidicommutator satisfies*

$$\begin{aligned} a \star (b \star c) &\equiv a \star (c \star b) && \text{(right commutativity)} \\ (b \star a^2) \star a &\equiv (b \star a) \star a^2 && \text{(right Jordan identity)} \\ \langle a, b, c^2 \rangle &\equiv 2\langle a \star c, b, c \rangle && \text{(right Osborn identity)} \end{aligned}$$

where $a^2 = a \star a$ and $\langle a, b, c \rangle = (a \star b) \star c - a \star (b \star c)$.

Remark 2.15. These identities were obtained independently by different authors: Velásquez and Felipe [49], Kolesnikov [29], Bremner [5]. A generalization of the TKK construction from Lie and Jordan algebras to Lie and Jordan dialgebras has been given by Gubarev and Kolesnikov [24]. For further work on the structure of Jordan dialgebras, see Felipe [19]. I have named the last identity in Lemma 2.14 after Osborn [41]; it is a noncommutative version of the identity stating that a commutator of multiplications is a derivation.

Definition 2.16. *A **Jordan dialgebra** (or quasi-Jordan algebra) is an algebra satisfying right commutativity and the right Jordan and Osborn identities.*

Remark 2.17. Strictly speaking, Leibniz algebras and Jordan dialgebras have two operations, but they are opposite, so we consider only one. This will become clear when we discuss the KP algorithm for converting identities for algebras into identities for dialgebras.

Theorem 2.18. *There exist special identities for Jordan dialgebras; that is, polynomial identities satisfied by the antidicommutator in every associative dialgebra which are not consequences of right commutativity and the right Jordan and Osborn identities.*

Remark 2.19. This result was obtained using computer algebra by Bremner and Peresi [11]. The lowest degree for such identities is 8; some but not all of these identities are noncommutative versions of the Glennie identity. For a theoretical approach to similar results, including generalizations of the classical theorems of Cohn, Macdonald, and Shirshov, see Voronin [50].

3. From algebras to dialgebras. We now discuss a general approach to the following problem.

Problem 3.1. *Given a polynomial identity for algebras, how do we obtain the corresponding polynomial identity (or identities) for dialgebras?*

An algorithm has been developed by Kolesnikov and Pozhidaev for converting multilinear identities for algebras into multilinear identities for dialgebras. For binary algebras, see [29]; for the generalization to n-ary algebras, see [45]. The underlying structure from the theory of operads is discussed by Chapoton [16].

Kolesnikov-Pozhidaev (KP) algorithm. The input is a multilinear polynomial identity of degree d for an n-ary operation denoted by the symbol $\{-, \cdots, -\}$ with n arguments. The output of Part 1 is a collection of d multilinear identities of degree d for n new n-ary operations denoted $\{-, \cdots, -\}_i$ for $1 \leq i \leq n$. The output of Part 2 is a collection of multilinear identities of degree $2n-1$ for the same new operations.

Part 1. Given a multilinear identity of degree d in the n-ary operation $\{-, \cdots, -\}$, we describe the application of the algorithm to one monomial, and extend this by linearity to the entire identity. Let $a_1 a_2 \ldots a_d$ be a multilinear monomial of degree d with some placement of n-ary operation symbols. For each $i = 1, \ldots, d$ we convert the monomial $a_1 a_2 \ldots a_d$ in the original n-ary operation into a new monomial of the same degree in the n new n-ary operations, according to the following rule, based on the position of the variable a_i, called the central variable of the monomial. For each occurrence of the original n-ary operation in the monomial, either a_i occurs in one of the n arguments or not, and we have two cases:

(a) If a_i occurs in the j-th argument then we convert $\{-,\cdots,-\}$ to the j-th new operation symbol $\{-,\cdots,-\}_j$.

(b) If a_i does not occur in any of the n arguments, then either

- a_i occurs to the left of $\{-,\cdots,-\}$: we convert $\{-,\cdots,-\}$ to the first new operation symbol $\{-,\cdots,-\}_1$, or
- a_i occurs to the right of $\{-,\cdots,-\}$: we convert $\{-,\cdots,-\}$ to the last new operation symbol $\{-,\cdots,-\}_n$.

Part 2. We also include the following identities, generalizing the bar identities for associative dialgebras, for all $i,j=1,\dots,n$ with $i\neq j$ and all $k,\ell=1,\dots,n$:

$$\{a_1,\dots,a_{i-1},\{b_1,\cdots,b_n\}_k,a_{i+1},\dots,a_n\}_j\equiv$$
$$\{a_1,\dots,a_{i-1},\{b_1,\cdots,b_n\}_\ell,a_{i+1},\dots,a_n\}_j.$$

This identity says that the n new operations are interchangeable in the i-th argument of the j-th new operation when $i\neq j$.

Example 3.2. The definition of associative dialgebra can be obtained by applying the KP algorithm to the associativity identity, which we write in the form

$$\{\{a,b\},c\}\equiv\{a,\{b,c\}\}.$$

The operation $\{-,-\}$ produces two new operations $\{-,-\}_1$, $\{-,-\}_2$. Part 1 of the algorithm produces three identities by making a,b,c in turn the central variable:

$$\{\{a,b\}_1,c\}_1\equiv\{a,\{b,c\}_1\}_1,\qquad \{\{a,b\}_2,c\}_1\equiv\{a,\{b,c\}_1\}_2,$$
$$\{\{a,b\}_2,c\}_2\equiv\{a,\{b,c\}_2\}_2.$$

Part 2 of the algorithm produces two identities:

$$\{a,\{b,c\}_1\}_1\equiv\{a,\{b,c\}_2\}_1,\qquad \{\{a,b\}_1,c\}_2\equiv\{\{a,b\}_2,c\}_2.$$

If we write $a\dashv b$ for $\{a,b\}_1$ and $a\vdash b$ for $\{a,b\}_2$ then these are the three associativity identities and the two bar identities.

Example 3.3. The definition of Leibniz algebra can be obtained by applying the KP algorithm to the identities defining Lie algebras: anticommutativity (in its bilinear form) and the Jacobi identity,

$$[a,b]+[b,a] \equiv 0, \qquad [[a,b],c]+[[b,c],a]+[[c,a],b] \equiv 0.$$

Part 1 of the algorithm produces five identities:

$$\begin{aligned} &[a,b]_1+[b,a]_2 \equiv 0, && [[a,b]_1,c]_1+[[b,c]_2,a]_2+[[c,a]_2,b]_1 \equiv 0, \\ &[a,b]_2+[b,a]_1 \equiv 0, && [[a,b]_2,c]_1+[[b,c]_1,a]_1+[[c,a]_2,b]_2 \equiv 0, \\ & && [[a,b]_2,c]_2+[[b,c]_2,a]_1+[[c,a]_1,b]_1 \equiv 0. \end{aligned}$$

The two identities of degree 2 are equivalent to $[a,b]_2 \equiv -[b,a]_1$, so the second operation is superfluous. Eliminating the second operation from the three identities of degree 3 shows that each of them is equivalent to the identity

$$[[a,b]_1,c]_1+[a,[c,b]_1]_1-[[a,c]_1,b]_1 \equiv 0.$$

If we write $\langle a,b\rangle = [a,b]_1$ then we obtain a form of the Leibniz identity. Part 2 of the algorithm produces two identities:

$$[a,[b,c]_1]_1 \equiv [a,[b,c]_2]_1, \qquad [[a,b]_1,c]_2 \equiv [[a,b]_2,c]_2.$$

Eliminating the second operation gives right anticommutativity:

$$\langle a,\langle b,c\rangle\rangle + \langle a,\langle c,b\rangle\rangle \equiv 0.$$

However, as we have already seen in Remark 2.9, the Leibniz identity implies right anticommutativity, so it suffices to retain only the Leibniz identity.

Example 3.4. To apply the KP algorithm to the defining identities for Jordan algebras, we write commutativity and the multilinear form of the Jordan identity using the operation symbol $\{-,-\}$:

$$\begin{aligned} &\{a,b\}-\{b,a\} \equiv 0, \\ &\{\{\{a,c\},b\},d\}+\{\{\{a,d\},b\},c\}+\{\{\{c,d\},b\},a\} \\ &\quad -\{\{a,c\},\{b,d\}\}-\{\{a,d\},\{b,c\}\}-\{\{c,d\},\{b,a\}\} \equiv 0. \end{aligned}$$

From commutativity, Part 1 of the algorithm gives two identities of degree 2:

$$\{a,b\}_1-\{b,a\}_2 \equiv 0, \qquad \{a,b\}_2-\{b,a\}_1 \equiv 0,$$

These two identities are equivalent to $\{a,b\}_2 \equiv \{b,a\}_1$: the second operation is the opposite of the first, and so we may eliminate $\{-,-\}_2$. From the linearized Jordan identity, Part 1 of the algorithm gives four identities of degree 4:

$$
\begin{aligned}
&\{\{\{a,c\}_1,b\}_1,d\}_1 + \{\{\{a,d\}_1,b\}_1,c\}_1 + \{\{\{c,d\}_2,b\}_2,a\}_2 \\
&\quad - \{\{a,c\}_1,\{b,d\}_1\}_1 - \{\{a,d\}_1,\{b,c\}_1\}_1 - \{\{c,d\}_2,\{b,a\}_2\}_2 \equiv 0, \\
&\{\{\{a,c\}_2,b\}_2,d\}_1 + \{\{\{a,d\}_2,b\}_2,c\}_1 + \{\{\{c,d\}_2,b\}_2,a\}_1 \\
&\quad - \{\{a,c\}_2,\{b,d\}_1\}_2 - \{\{a,d\}_2,\{b,c\}_1\}_2 - \{\{c,d\}_2,\{b,a\}_1\}_2 \equiv 0, \\
&\{\{\{a,c\}_2,b\}_1,d\}_1 + \{\{\{a,d\}_2,b\}_2,c\}_2 + \{\{\{c,d\}_1,b\}_1,a\}_1 \\
&\quad - \{\{a,c\}_2,\{b,d\}_1\}_1 - \{\{a,d\}_2,\{b,c\}_2\}_2 - \{\{c,d\}_1,\{b,a\}_1\}_1 \equiv 0, \\
&\{\{\{a,c\}_2,b\}_2,d\}_2 + \{\{\{a,d\}_2,b\}_1,c\}_1 + \{\{\{c,d\}_2,b\}_1,a\}_1 \\
&\quad - \{\{a,c\}_2,\{b,d\}_2\}_2 - \{\{a,d\}_2,\{b,c\}_1\}_1 - \{\{c,d\}_2,\{b,a\}_1\}_1 \equiv 0.
\end{aligned}
$$

We replace every instance of $\{-,-\}_2$ by the opposite of $\{-,-\}_1$:

$$
\begin{aligned}
&\{\{\{a,c\}_1,b\}_1,d\}_1 + \{\{\{a,d\}_1,b\}_1,c\}_1 + \{a,\{b,\{d,c\}_1\}_1\}_1 \\
&\quad - \{\{a,c\}_1,\{b,d\}_1\}_1 - \{\{a,d\}_1,\{b,c\}_1\}_1 - \{\{a,b\}_1,\{d,c\}_1\}_1 \equiv 0, \\
&\{\{b,\{c,a\}_1\}_1,d\}_1 + \{\{b,\{d,a\}_1\}_1,c\}_1 + \{\{b,\{d,c\}_1\}_1,a\}_1 \\
&\quad - \{\{b,d\}_1,\{c,a\}_1\}_1 - \{\{b,c\}_1,\{d,a\}_1\}_1 - \{\{b,a\}_1,\{d,c\}_1\}_1 \equiv 0, \\
&\{\{\{c,a\}_1,b\}_1,d\}_1 + \{c,\{b,\{d,a\}_1\}_1\}_1 + \{\{\{c,d\}_1,b\}_1,a\}_1 \\
&\quad - \{\{c,a\}_1,\{b,d\}_1\}_1 - \{\{c,b\}_1,\{d,a\}_1\}_1 - \{\{c,d\}_1,\{b,a\}_1\}_1 \equiv 0, \\
&\{d,\{b,\{c,a\}_1\}_1\}_1 + \{\{\{d,a\}_1,b\}_1,c\}_1 + \{\{\{d,c\}_1,b\}_1,a\}_1 \\
&\quad - \{\{d,b\}_1,\{c,a\}_1\}_1 - \{\{d,a\}_1,\{b,c\}_1\}_1 - \{\{d,c\}_1,\{b,a\}_1\}_1 \equiv 0.
\end{aligned}
$$

We simplify the notation and write $\{a,b\}_1$ as ab. The last four identities become:

$$
\begin{aligned}
&((ac)b)d + ((ad)b)c + a(b(dc)) - (ac)(bd) - (ad)(bc) - (ab)(dc) \equiv 0, \\
&(b(ca))d + (b(da))c + (b(dc))a - (bd)(ca) - (bc)(da) - (ba)(dc) \equiv 0, \\
&((ca)b)d + c(b(da)) + ((cd)b)a - (ca)(bd) - (cb)(da) - (cd)(ba) \equiv 0, \\
&d(b(ca)) + ((da)b)c + ((dc)b)a - (db)(ca) - (da)(bc) - (dc)(ba) \equiv 0.
\end{aligned}
$$

The first is equivalent to the third and to the fourth, so we retain only the first and second. Part 2 of the algorithm produces two identities:

$$\{a,\{b,c\}_1\}_1 \equiv \{a,\{b,c\}_2\}_1, \qquad \{\{a,b\}_1,c\}_2 \equiv \{\{a,b\}_2,c\}_2.$$

Rewriting these using only the first operation gives

$$\{a,\{b,c\}_1\}_1 \equiv \{a,\{c,b\}_1\}_1, \qquad \{c,\{a,b\}_1\}_1 \equiv \{c,\{b,a\}_1\}_1.$$

These two identities are equivalent to right commutativity: $a(bc) \equiv a(cb)$. We rearrange the two retained identities of degree 4 and apply right commutativity:

$$((ac)b)d - (ac)(bd) + ((ad)b)c - (ad)(bc) - (ab)(cd) + a(b(cd)) \equiv 0,$$
$$(b(ac))d + (b(ad))c + (b(cd))a - (bd)(ac) - (bc)(ad) - (ba)(cd) \equiv 0.$$

The first identity can be reformulated in terms of associators as follows,

$$(ac,b,d) + (ad,b,c) - (a,b,cd) \equiv 0,$$

and assuming characteristic $\neq 2$ this is equivalent to

$$(a,b,c^2) \equiv 2(ac,b,c).$$

Setting $a=c=d$ in the second identity and assuming characteristic $\neq 3$ gives

$$(ba^2)a \equiv (ba)a^2,$$

Thus we obtain right commutativity and the right Osborn and Jordan identities.

Example 3.5. The multilinear forms of the left and right alternative identities defining alternative algebras are:

$$(a,b,c) + (b,a,c) \equiv 0, \qquad (a,b,c) + (a,c,b) \equiv 0.$$

Expanding the associators gives

$$(ab)c - a(bc) + (ba)c - b(ac) \equiv 0, \qquad (ab)c - a(bc) + (ac)b - a(cb) \equiv 0.$$

We apply the KP algorithm to these identities, writing $\{-,-\}$ for the original bilinear operation. Part 1 gives six identities relating the two new operations $\{-,-\}_1$ and $\{-,-\}_2$: in each of the two original identities we make either a, b, or c the central argument. In this case, we retain both operations, since there is no identity of degree 2 relating $\{-,-\}_1$ and $\{-,-\}_2$. We obtain six identities defining alternative dialgebras; in the first (second) group of three, the only differences are in the subscripts 1 and 2 indicating the position of the central variable:

$$\{\{a,b\}_1,c\}_1 - \{a,\{b,c\}_1\}_1 + \{\{b,a\}_2,c\}_1 - \{b,\{a,c\}_1\}_2 \equiv 0,$$
$$\{\{a,b\}_2,c\}_1 - \{a,\{b,c\}_1\}_2 + \{\{b,a\}_1,c\}_1 - \{b,\{a,c\}_1\}_1 \equiv 0,$$
$$\{\{a,b\}_2,c\}_2 - \{a,\{b,c\}_2\}_2 + \{\{b,a\}_2,c\}_2 - \{b,\{a,c\}_2\}_2 \equiv 0,$$
$$\{\{a,b\}_1,c\}_1 - \{a,\{b,c\}_1\}_1 + \{\{a,c\}_1,b\}_1 - \{a,\{c,b\}_1\}_1 \equiv 0,$$
$$\{\{a,b\}_2,c\}_1 - \{a,\{b,c\}_1\}_2 + \{\{a,c\}_2,b\}_2 - \{a,\{c,b\}_2\}_2 \equiv 0,$$
$$\{\{a,b\}_2,c\}_2 - \{a,\{b,c\}_2\}_2 + \{\{a,c\}_2,b\}_1 - \{a,\{c,b\}_1\}_2 \equiv 0.$$

We revert to standard notation: $\dashv$ for $\{-,-\}_1$ and $\vdash$ for $\{-,-\}_2$:

$$(a \dashv b) \dashv c - a \dashv (b \dashv c) + (b \vdash a) \dashv c - b \vdash (a \dashv c) \equiv 0,$$
$$(a \vdash b) \dashv c - a \vdash (b \dashv c) + (b \dashv a) \dashv c - b \dashv (a \dashv c) \equiv 0,$$
$$(a \vdash b) \vdash c - a \vdash (b \vdash c) + (b \vdash a) \vdash c - b \vdash (a \vdash c) \equiv 0,$$
$$(a \dashv b) \dashv c - a \dashv (b \dashv c) + (a \dashv c) \dashv b - a \dashv (c \dashv b) \equiv 0,$$
$$(a \vdash b) \dashv c - a \vdash (b \dashv c) + (a \vdash c) \vdash b - a \vdash (c \vdash b) \equiv 0,$$
$$(a \vdash b) \vdash c - a \vdash (b \vdash c) + (a \vdash c) \dashv b - a \vdash (c \dashv b) \equiv 0.$$

We rewrite these in terms of the left, right and inner associators:

$$(a,b,c)_\dashv + (b,a,c)_\times \equiv 0, \qquad (a,b,c)_\times + (b,a,c)_\dashv \equiv 0,$$
$$(a,b,c)_\vdash + (b,a,c)_\vdash \equiv 0, \qquad (a,b,c)_\dashv + (a,c,b)_\dashv \equiv 0,$$
$$(a,b,c)_\times + (a,c,b)_\vdash \equiv 0, \qquad (a,b,c)_\vdash + (a,c,b)_\times \equiv 0.$$

These six identities show how the associators change under various transpositions of the arguments. In particular, the identities in the second row show that the right operation $a \vdash b$ is left alternative, and the left operation $a \dashv b$ is right alternative. (We do not have two alternative operations.) Part 2 of the algorithm simply gives the left and right bar identities. To summarize, we define an alternative dialgebra to be a 0-dialgebra satisfying

$$(a,b,c)_\dashv + (c,b,a)_\vdash \equiv 0, \quad (a,b,c)_\dashv - (b,c,a)_\vdash \equiv 0, \quad (a,b,c)_\times + (a,c,b)_\vdash \equiv 0,$$

where the left, right, and inner associators are defined by

$$(a,b,c)_\dashv = (a \dashv b) \dashv c - a \dashv (b \dashv c), \qquad (a,b,c)_\vdash = (a \vdash b) \vdash c - a \vdash (b \vdash c),$$
$$(a,b,c)_\times = (a \vdash b) \dashv c - a \vdash (b \dashv c).$$

This definition was originally obtained in a different way by Liu [31].

Example 3.6. Malcev algebras [44] can be defined by the polynomial identities of degree ≤ 4 satisfied by the commutator in every alternative algebra. Bremner, Peresi and Sánchez-Ortega [12] used computer algebra to study the identities satisfied by the dicommutator in every alternative dialgebra, and proved that every such identity of degree ≤ 6 is a consequence of the identities of degree ≤ 4. They showed that the identities of degree ≤ 4 are equivalent to those obtained by applying the KP algorithm to linearized forms of anticommutativity the Malcev identity, namely right anticommutativity and a "noncommutative" version of the Malcev identity:

$$a(bc) + a(cb) \equiv 0, \qquad ((ab)c)d - ((ad)b)c - (a(cd))b - (ac)(bd) - a((bc)d) \equiv 0.$$

These two identities define the variety of Malcev dialgebras.

4. Multilinear operations.

We now consider generalizations of the commutator $ab - ba$ and anticommutator $ab + ba$ to operations of arbitrary "arity" (number of arguments). The following definitions and examples are based primarily on Bremner and Peresi [10].

Definition 4.1. *A* ***multilinear*** n***-ary operation*** $\omega(a_1, a_2, \ldots, a_n)$*, or more concisely an* n*-linear operation, is a linear combination of permutations of the monomial* $a_1 a_2 \cdots a_n$ *regarded as an element of the free associative algebra on* n *generators:*

$$\omega(a_1, a_2, \ldots, a_n) = \sum_{\sigma \in S_n} x_\sigma \, a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} \qquad (x_\sigma \in \mathbb{F}).$$

We identify $\omega(a_1, a_2, \ldots, a_n)$ *with an element of* $\mathbb{F}S_n$*, the group algebra of the symmetric group* S_n *which acts by permuting the subscripts of the generators.*

Definition 4.2. *Two multilinear operations are* ***equivalent*** *if each is a linear combination of permutations of the other; this is the same as saying that the two operations generate the same left ideal in* $\mathbb{F}S_n$*.*

Example 4.3. For $n = 2$, we have the Wedderburn decomposition $\mathbb{F}S_2 \approx \mathbb{F} \oplus \mathbb{F}$, where the two simple ideals correspond to partitions 2 and $1+1$ and have bases $ab + ba$ and $ab - ba$ respectively (writing a, b instead of a_1, a_2). There are four equivalence classes, corresponding to the commutator, the anticommutator, the zero operation, and the original associative operation ab.

Example 4.4. For $n = 3$, we have the Wedderburn decomposition

$$\mathbb{F}S_3 \approx \mathbb{F} \oplus M_2(\mathbb{F}) \oplus \mathbb{F},$$

where the simple ideals correspond to partitions 3, $2+1$ and $1+1+1$. As representatives of the equivalence classes of trilinear operations we take ordered triples of matrices in row canonical form:

$$\left[\; x, \; \begin{bmatrix} y_{11} & y_{12} \\ y_{21} & y_{22} \end{bmatrix}, \; z \;\right]$$

The first and third components are either 0 or 1; the second can be one of

$$\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & q \\ 0 & 0 \end{bmatrix} \; (q \in \mathbb{F}), \qquad \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

There are infinitely many equivalence classes: four infinite families (for which the 2×2 matrix has rank 1) and six isolated operations (for which the 2×2 matrix has rank 0 or 2). In order to classify these operations, we consider two bases for the group algebra $\mathbb{F}S_3$, assuming that the characteristic of $\mathbb{F}$ is not 2 or 3. The first basis consists of the permutations in lexicographical order:

$$abc, \quad acb, \quad bac, \quad bca, \quad cab, \quad cba.$$

The second basis consists of the matrix units for the Wedderburn decomposition:

$$S = \frac{1}{6}(abc + acb + bca + bca + cab + cba),$$

$$E_{11} = \frac{1}{3}(abc + bca - bca - cba), \qquad E_{12} = \frac{1}{3}(acb - bca + bca - cab),$$

$$E_{21} = \frac{1}{3}(acb - bca + cab - cba), \qquad E_{22} = \frac{1}{3}(abc - bca - cab + cba),$$

$$A = \frac{1}{6}(abc - acb - bca + bca + cab - cba).$$

Table 1. Simplified trilinear operations

	reduced matrix form	permutation form
1	$\left[0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 0\right]$	$abc - bac - cab + cba$
2	$\left[0, \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}, 0\right]$	$abc + acb - bca - cba$
3	$\left[1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 0\right]$	$abc + cba$
4	$\left[1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 0\right]$	$abc + bac$
5	$\left[1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, 0\right]$	$abc + acb$
6	$\left[1, \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}, 0\right]$	$2abc + acb + 2bac + bca$
7	$\left[0, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 1\right]$	$2abc - acb - 2bac + bca$
8	$\left[0, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, 1\right]$	$abc - acb$
9	$\left[0, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, 1\right]$	$abc - bac$
10	$\left[0, \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}, 1\right]$	$abc - cba$
11	$\left[1, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, 1\right]$	$abc - bac + bca$
12	$\left[1, \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, 1\right]$	$abc + cab - cba$
13	$\left[1, \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, 1\right]$	$abc + bca - cba$
14	$\left[1, \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}, 1\right]$	$abc + bac + cab$
15	$\left[1, \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}, 1\right]$	$abc + acb + bca$
16	$\left[1, \begin{bmatrix} 1 & 1/2 \\ 0 & 0 \end{bmatrix}, 1\right]$	$abc + acb + bac$
17	$\left[0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 0\right]$	$abc - bca$
18	$\left[1, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 0\right]$	$abc + acb + bac - cba$
19	$\left[0, \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, 1\right]$	$abc + acb - bca - cab$

The change of basis matrices are

$$M = \frac{1}{6}\begin{bmatrix} 1 & 2 & 0 & 0 & 2 & 1 \\ 1 & 0 & 2 & 2 & 0 & -1 \\ 1 & 2 & -2 & 0 & -2 & -1 \\ 1 & -2 & 2 & -2 & 0 & 1 \\ 1 & 0 & -2 & 2 & -2 & 1 \\ 1 & -2 & 0 & -2 & 2 & -1 \end{bmatrix}, \quad M^{-1} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & -1 & -1 \\ 0 & 1 & 0 & 1 & -1 & -1 \\ 0 & 1 & -1 & -1 & 1 & 0 \\ 1 & 0 & -1 & -1 & 0 & 1 \\ 1 & -1 & -1 & 1 & 1 & -1 \end{bmatrix}$$

Except for the associative operation abc, all these operations satisfy polynomial identities in degree 3. Bremner and Peresi [10] identified 19 of these operations which satisfy polynomial identities in degree 5 which do not follow from the identities in degree 3. These operations are given in Table 1, which contains the representative of the equivalence class in matrix form and the simplest operation in that class written as a linear combination of permutations. (The simplified forms of the operations were found by enumerating all $5^6 = 15625$ linear combinations of the permutations with coefficients $\{0, \pm 1, \pm 2\}$, computing the reduced matrix form of each of the resulting group algebra elements, and recording those which belong to the same equivalence class as one of the operations from [10].) This list includes the Lie and anti-Lie triple products,

$$abc - bac - cab + cba, \qquad abc + bac - cab - cba,$$

and the Jordan and anti-Jordan triple products,

$$abc + cba, \qquad abc - cba.$$

The list does not include the symmetric, alternating, and cyclic sums,

$$abc + acb + bca + bca + cab + cba, \qquad abc - acb - bca + bca + cab - cba,$$
$$abc + bca + cab,$$

since every polynomial identity of degree 5 satisfied by these operations is a consequence of the identities in degree 3. In other words, there are no new identities until degree 7; see Bremner and Hentzel [8].

We now discuss a general approach to the following problem.

Problem 4.5. *Given a multilinear operation for algebras, how do we obtain the corresponding operation (or operations) for dialgebras?*

A simple algorithm which converts a multilinear operation of degree n in an associative algebra into n multilinear operations of degree n in an associative dialgebra was introduced by Bremner and Sánchez-Ortega [13].

Bremner–Sánchez-Ortega (BSO) algorithm. The input is a multilinear n-ary operation ω in an associative algebra:

$$\omega(a_1, a_2, \dots, a_n) = \sum_{\sigma \in S_n} x_\sigma \, a_{\sigma(1)} a_{\sigma(2)} \cdots a_{\sigma(n)} \qquad (x_\sigma \in \mathbb{F}).$$

For each $i = 1, 2, \dots, n$ we partition the set of all permutations into subsets according to the position of i:

$$S_n^{j,i} = \{ \sigma \in S_n \mid \sigma(j) = i \}.$$

For each $i = 1, 2, \dots, n$ we collect the terms of ω in which a_i is in position j:

$$\omega_i(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} \sum_{S_n^{j,i}} x_\sigma \, a_{\sigma(1)} \cdots a_{\sigma(j-1)} a_i a_{\sigma(j+1)} \cdots a_{\sigma(n)}.$$

The output consists of n new multilinear n-ary operations $\widehat{\omega}_1, \dots, \widehat{\omega}_n$ in an associative dialgebra, obtained from ω by making a_i the center of each term:

$$\widehat{\omega}_i(a_1, a_2, \dots, a_n) = \sum_{j=1}^{n} \sum_{S_n^{j,i}} x_\sigma \, a_{\sigma(1)} \cdots a_{\sigma(j-1)} \, \widehat{a}_i \, a_{\sigma(j+1)} \cdots a_{\sigma(n)}.$$

Example 4.6. The commutator $ab - ba$ produces two dicommutators; the second is the negative of the opposite of the first, $\langle a, b \rangle_2 = -\langle b, a \rangle_1$:

$$\langle a, b \rangle_1 = \widehat{a}b - b\widehat{a}, \qquad \langle a, b \rangle_2 = a\widehat{b} - \widehat{b}a.$$

The anticommutator $ab + ba$ produces two antidicommutators; the second is the opposite of the first, $\langle a, b \rangle_2 = \langle b, a \rangle_1$:

$$\langle a, b \rangle_1 = \widehat{a}b + b\widehat{a}, \qquad \langle a, b \rangle_2 = a\widehat{b} + \widehat{b}a.$$

Example 4.7. We apply the BSO algorithm to the Lie triple product,

$$\omega(a, b, c) = abc - bac - cab + cba.$$

We obtain these three dialgebra operations:

$$\widehat{\omega}_1(a,b,c) = \widehat{a}bc - b\widehat{a}c - c\widehat{a}b + cb\widehat{a}, \qquad \widehat{\omega}_2(a,b,c) = a\widehat{b}c - \widehat{b}ac - ca\widehat{b} + c\widehat{b}a,$$
$$\widehat{\omega}_3(a,b,c) = ab\widehat{c} - ba\widehat{c} - \widehat{c}ab + \widehat{c}ba.$$

We have

$$\widehat{\omega}_2(a,b,c) = -\widehat{\omega}_1(b,a,c), \qquad \widehat{\omega}_3(a,b,c) = \widehat{\omega}_1(c,b,a) - \widehat{\omega}_1(c,a,b),$$

so we only retain $\widehat{\omega}_1(a,b,c)$.

Example 4.8. We apply the BSO algorithm to the Jordan triple product,

$$\omega(a,b,c) = abc + cba.$$

We obtain these three dialgebra operations:

$$\widehat{\omega}_1(a,b,c) = \widehat{a}bc + cb\widehat{a}, \qquad \widehat{\omega}_2(a,b,c) = a\widehat{b}c + c\widehat{b}a, \qquad \widehat{\omega}_3(a,b,c) = ab\widehat{c} + \widehat{c}ba.$$

We have $\widehat{\omega}_3(a,b,c) = \widehat{\omega}_1(c,b,a)$, so we only retain $\widehat{\omega}_1(a,b,c)$ and $\widehat{\omega}_2(a,b,c)$. The second operation is symmetric in its first and third arguments: $\widehat{\omega}_2(c,b,a) = \widehat{\omega}_2(a,b,c)$.

5. Leibniz triple systems. We consider the dialgebra analogue of Lie triple systems. We apply the KP algorithm to the defining polynomial identities, and then find the identities satisfied by the operations obtained from the BSO algorithm applied to the Lie triple product. We then use computer algebra to verify that the results are equivalent. This section is a summary of Bremner and Sánchez-Ortega [14]. We assume that the base field $\mathbb{F}$ does not have characteristic 2, 3 or 5.

Definition 5.1. *A **Lie triple system** is a vector space T with a trilinear operation $T \times T \times T \to T$ denoted $[a,b,c]$ satisfying these multilinear identities:*

$$[a,b,c] + [b,a,c] \equiv 0,$$
$$[a,b,c] + [b,c,a] + [c,a,b] \equiv 0,$$
$$[a,b,[c,d,e]] - [[a,b,c],d,e] - [c,[a,b,d],e] - [c,d,[a,b,e]] \equiv 0.$$

These identities are satisfied by the Lie triple product in any associative algebra.

5.1. KP algorithm. Applying Part 1 of the algorithm to the identities of degree 3 in Definition 5.1 gives

$$
\begin{aligned}
&[a,b,c]_1 + [b,a,c]_2 \equiv 0, && [a,b,c]_1 + [b,c,a]_3 + [c,a,b]_2 \equiv 0,\\
&[a,b,c]_2 + [b,a,c]_1 \equiv 0, && [a,b,c]_2 + [b,c,a]_1 + [c,a,b]_3 \equiv 0,\\
&[a,b,c]_3 + [b,a,c]_3 \equiv 0, && [a,b,c]_3 + [b,c,a]_2 + [c,a,b]_1 \equiv 0.
\end{aligned}
$$

The first two identities on the left are equivalent and show that $[-,-,-]_2$ is superfluous; the three on the right are equivalent and show that $[-,-,-]_3$ is superfluous:

$$
[a,b,c]_2 \equiv -[b,a,c]_1, \qquad [a,b,c]_3 \equiv -[b,c,a]_2 - [c,a,b]_1 \equiv [c,b,a]_1 - [c,a,b]_1.
$$

We retain only the first operation which we write as $\langle -,-,- \rangle$. Applying Part 1 of the algorithm to the identity of degree 5 in Definition 5.1 gives five identities, two of which are redundant. We use the previous equations to eliminate $[-,-,-]_2$ and $[-,-,-]_3$ from the remaining three identities and obtain:

$$
(1) \quad \left\{
\begin{aligned}
&\langle a,b,\langle c,d,e\rangle\rangle - \langle\langle a,b,c\rangle,d,e\rangle + \langle\langle a,b,d\rangle,c,e\rangle - \langle\langle a,b,e\rangle,d,c\rangle\\
&\quad + \langle\langle a,b,e\rangle,c,d\rangle \equiv 0,\\
&\langle\langle c,d,e\rangle,b,a\rangle - \langle\langle c,d,e\rangle,a,b\rangle - \langle\langle c,b,a\rangle,d,e\rangle + \langle\langle c,a,b\rangle,d,e\rangle\\
&\quad - \langle c,\langle a,b,d\rangle,e\rangle - \langle c,d,\langle a,b,e\rangle\rangle \equiv 0,\\
&\langle\langle e,d,c\rangle,b,a\rangle - \langle\langle e,c,d\rangle,b,a\rangle - \langle\langle e,d,c\rangle,a,b\rangle + \langle\langle e,c,d\rangle,a,b\rangle\\
&\quad - \langle e,d,\langle c,b,a\rangle\rangle + \langle e,d,\langle c,a,b\rangle\rangle + \langle e,\langle c,b,a\rangle,d\rangle - \langle e,\langle c,a,b\rangle,d\rangle\\
&\quad - \langle e,\langle d,b,a\rangle,c\rangle + \langle e,\langle d,a,b\rangle,c\rangle + \langle e,c,\langle d,b,a\rangle\rangle - \langle e,c,\langle d,a,b\rangle\rangle\\
&\quad - \langle\langle e,b,a\rangle,d,c\rangle + \langle\langle e,a,b\rangle,d,c\rangle + \langle\langle e,b,a\rangle,c,d\rangle - \langle\langle e,a,b\rangle,c,d\rangle \equiv 0.
\end{aligned}
\right.
$$

Part 2 produces 12 identities; after eliminating $[-,-,-]_2$ and $[-,-,-]_3$ we obtain:

$$
(2) \quad \left\{
\begin{aligned}
&\langle a,\langle b,c,d\rangle,e\rangle + \langle a,\langle c,b,d\rangle,e\rangle \equiv 0,\\
&\langle a,\langle b,c,d\rangle,e\rangle + \langle a,\langle c,d,b\rangle,e\rangle + \langle a,\langle d,b,c\rangle,e\rangle \equiv 0,\\
&\langle a,b,\langle c,d,e\rangle\rangle + \langle a,b,\langle d,c,e\rangle\rangle \equiv 0,\\
&\langle a,b,\langle c,d,e\rangle\rangle + \langle a,b,\langle d,e,c\rangle\rangle + \langle a,b,\langle e,c,d\rangle\rangle \equiv 0.
\end{aligned}
\right.
$$

These identities show that the inner triple in a monomial of the second or third association types, $\langle -,\langle -,-,-\rangle,-\rangle$ and $\langle -,-,\langle -,-,-\rangle\rangle$, has properties analogous to the identities of degree 3 in the definition of Lie triple system: the ternary analogues of skew-symmetry and the Jacobi identity.

5.2. BSO algorithm. We saw in Example 4.7 that we need only one operation,

$$\langle a,b,c\rangle = \widehat{a}bc - b\widehat{a}c - c\widehat{a}b + cb\widehat{a}.$$

Every identity of degree at most 5 satisfied by this operation follows from the two identities in the next definition. Furthermore, the seven identities (1) and (2) are equivalent to the next two identities.

Definition 5.2. *A* ***Leibniz triple system*** *(or Lie triple disystem) is a vector space T with a trilinear operation $\langle -,-,-\rangle\colon T\times T\times T\to T$ satisfying these identities:*

$$\langle a,\langle b,c,d\rangle,e\rangle \equiv \langle\langle a,b,c\rangle,d,e\rangle - \langle\langle a,c,b\rangle,d,e\rangle - \langle\langle a,d,b\rangle,c,e\rangle + \langle\langle a,d,c\rangle,b,e\rangle,$$
$$\langle a,b,\langle c,d,e\rangle\rangle \equiv \langle\langle a,b,c\rangle,d,e\rangle - \langle\langle a,b,d\rangle,c,e\rangle - \langle\langle a,b,e\rangle,c,d\rangle + \langle\langle a,b,e\rangle,d,c\rangle.$$

In the right sides of these identities, the signs and permutations of b,c,d and c,d,e correspond to the expansion of the Lie triple products $[[b,c],d]$ and $[[c,d],e]$.

Theorem 5.3. *Any subspace of a Leibniz algebra which is closed under the iterated Leibniz bracket is a Leibniz triple system.*

Proof. This follows from $\langle a,b,c\rangle = (a \dashv b - b \vdash a) \dashv c - c \vdash (a \dashv b - b \vdash a)$. □

Theorem 5.4. *Every identity satisfied by the iterated Leibniz bracket $\langle\langle a,b\rangle,c\rangle$ in every Leibniz algebra is a consequence of the defining identities for Leibniz triple systems.*

Proof. This follows from the construction in [14] of universal Leibniz envelopes for Leibniz triple systems. □

The next result from [14] generalizes the classical result that the associator in a Jordan algebra satisfies the defining identities for Lie triple systems.

Theorem 5.5. *Let L be a subspace of a Jordan dialgebra which is closed under the associator (a,b,c). Then L is a Leibniz triple system with the trilinear operation defined to be the permuted associator $\langle a,b,c\rangle = (a,c,b)$.*

6. Jordan triple disystems.

6. Jordan triple disystems. We consider the dialgebra analogue of the variety of Jordan triple systems. This section is a summary of Bremner, Felipe and Sánchez-Ortega [7]. We assume that the base field $\mathbb{F}$ does not have characteristic 2, 3 or 5.

Definition 6.1. *A **Jordan triple system** is a vector space T with a trilinear operation $T \times T \times T \to T$ denoted $\{-,-,-\}$ satisfying these identities:*

$$\{a,b,c\} - \{c,b,a\} \equiv 0,$$
$$\{a,b,\{c,d,e\}\} - \{\{a,b,c\},d,e\} + \{c,\{b,a,d\},e\} - \{c,d,\{a,b,e\}\} \equiv 0.$$

These identities are satisfied by the Jordan triple product in any associative algebra.

6.1. KP algorithm. We first consider Part 1 of the algorithm. In the identity of degree 3, we make a, b, c in turn the central argument and obtain

$$\{a,b,c\}_1 - \{c,b,a\}_3 \equiv 0, \quad \{a,b,c\}_2 - \{c,b,a\}_2 \equiv 0, \quad \{a,b,c\}_3 - \{c,b,a\}_1 \equiv 0.$$

The third operation is superfluous and the second is symmetric in its first and third arguments. In the identity of degree 5, we make a,b,c,d,e in turn the central argument. Replacing $\{a,b,c\}_3$ by $\{c,b,a\}_1$ in these five identities gives

$$\{a,b,\{c,d,e\}_1\}_1 - \{\{a,b,c\}_1,d,e\}_1 + \{c,\{b,a,d\}_2,e\}_2 - \{\{a,b,e\}_1,d,c\}_1 \equiv 0,$$
$$\{a,b,\{c,d,e\}_1\}_2 - \{\{a,b,c\}_2,d,e\}_1 + \{c,\{b,a,d\}_1,e\}_2 - \{\{a,b,e\}_2,d,c\}_1 \equiv 0,$$
$$\{\{c,d,e\}_1,b,a\}_1 - \{\{c,b,a\}_1,d,e\}_1 + \{c,\{b,a,d\}_1,e\}_1 - \{c,d,\{a,b,e\}_1\}_1 \equiv 0,$$
$$\{\{c,d,e\}_2,b,a\}_1 - \{\{c,b,a\}_1,d,e\}_2 + \{c,\{d,a,b\}_1,e\}_2 - \{c,d,\{a,b,e\}_1\}_2 \equiv 0,$$
$$\{\{e,d,c\}_1,b,a\}_1 - \{e,d,\{c,b,a\}_1\}_1 + \{e,\{d,a,b\}_1,c\}_1 - \{\{e,b,a\}_1,d,c\}_1 \equiv 0.$$

Part 2 of the algorithm produces the following identities, in which we have replaced $\{a,b,c\}_3$ by $\{c,b,a\}_1$:

$$\{a,\{b,c,d\}_1,e\}_1 \equiv \{a,\{b,c,d\}_2,e\}_1 \equiv \{a,\{d,c,b\}_1,e\}_1,$$
$$\{a,b,\{c,d,e\}_1\}_1 \equiv \{a,b,\{c,d,e\}_2\}_1 \equiv \{a,b,\{e,d,c\}_1\}_1,$$
$$\{\{a,b,c\}_1,d,e\}_2 \equiv \{\{a,b,c\}_2,d,e\}_2 \equiv \{\{c,b,a\}_1,d,e\}_2,$$
$$\{a,b,\{c,d,e\}_1\}_2 \equiv \{a,b,\{c,d,e\}_2\}_2 \equiv \{a,b,\{e,d,c\}_1\}_2.$$

Definition 6.2. *A **Jordan triple disystem** is a vector space with trilinear operations $\{-,-,-\}_1$ and $\{-,-,-\}_2$ satisfying these eight identities:*

$$\{a,b,c\}_2 \equiv \{c,b,a\}_2, \qquad \{\{a,b,c\}_1,d,e\}_2 \equiv \{\{a,b,c\}_2,d,e\}_2,$$

$$\{a,\{b,c,d\}_1,e\}_1 \equiv \{a,\{b,c,d\}_2,e\}_1, \qquad \{a,b,\{c,d,e\}_1\}_1 \equiv \{a,b,\{c,d,e\}_2\}_1,$$
$$\{\{e,d,c\}_1,b,a\}_1 \equiv \{\{e,b,a\}_1,d,c\}_1 - \{e,\{d,a,b\}_1,c\}_1 + \{e,d,\{c,b,a\}_1\}_1,$$
$$\{\{e,d,c\}_2,b,a\}_1 \equiv \{\{e,b,a\}_1,d,c\}_2 - \{e,\{d,a,b\}_1,c\}_2 + \{e,d,\{c,b,a\}_1\}_2,$$
$$\{a,b,\{c,d,e\}_1\}_1 \equiv \{\{a,b,c\}_1,d,e\}_1 - \{c,\{b,a,d\}_2,e\}_2 + \{\{a,b,e\}_1,d,c\}_1,$$
$$\{a,b,\{c,d,e\}_1\}_2 \equiv \{\{a,b,c\}_2,d,e\}_1 - \{c,\{b,a,d\}_1,e\}_2 + \{\{a,b,e\}_2,d,c\}_1.$$

6.2. BSO algorithm. We saw in Example 4.8 that we need only two operations,
$$(a,b,c)_1 = \widehat{a}bc + cb\widehat{a}, \qquad (a,b,c)_2 = a\widehat{b}c + c\widehat{b}a.$$
We use computer algebra to find the identities of low degree for these operations.

Lemma 6.3. *Operation $(-,-,-)_1$ satisfies no polynomial identity of degree 3.*

Proof. An identity is a linear combination of the six permutations of $(a,b,c)_1$:

$$x_1(a,b,c)_1 + x_2(a,c,b)_1 + x_3(b,a,c)_1 + x_4(b,c,a)_1 + x_5(c,a,b)_1 + x_6(c,b,a)_1 \equiv 0.$$

We expand each ternary monomial to obtain a linear combination of the 18 multilinear dialgebra monomials of degree 3 ordered as follows:

$\widehat{a}bc$, $\widehat{a}cb$, $\widehat{b}ac$, $\widehat{b}ca$, $\widehat{c}ab$, $\widehat{c}ba$, $a\widehat{b}c$, $a\widehat{c}b$, $b\widehat{a}c$, $b\widehat{c}a$, $c\widehat{a}b$, $c\widehat{b}a$, $ab\widehat{c}$, $ac\widehat{b}$, $ba\widehat{c}$, $bc\widehat{a}$, $ca\widehat{b}$, $cb\widehat{a}$.

We construct an 18×6 matrix E whose (i,j) entry is the coefficient of the i-th dialgebra monomial in the expansion of the j-th diproduct monomial:

$$E^t = \left[\begin{array}{cccccccccccccccccc}
1 & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 \\
. & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . & . \\
. & . & 1 & . & . & . & . & . & . & . & . & . & . & . & . & . & 1 & . \\
. & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . & . \\
. & . & . & . & 1 & . & . & . & . & . & . & . & . & . & 1 & . & . & . \\
. & . & . & . & . & 1 & . & . & . & . & . & . & 1 & . & . & . & . & .
\end{array}\right]$$

The coefficient vectors of the polynomial identities satisfied by $(-,-,-)_1$ are the vectors in the nullspace of E, which is zero. $\square$

Lemma 6.4. *Every polynomial identity of degree* 3 *satisfied by operation* $(-,-,-)_2$ *is a consequence of* $(a,b,c)_2 \equiv (c,b,a)_2$.

P r o o f. Following the same method as in the previous Lemma gives

$$E^t = \begin{bmatrix} . & . & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . & . \\ . & . & . & . & . & . & . & . & 1 & . & 1 & . & . & . & . & . & . & . \\ . & . & . & . & . & . & 1 & . & . & . & . & 1 & . & . & . & . & . & . \end{bmatrix}$$

The canonical basis of the nullspace consists of three vectors representing the three permutations of the stated identity. □

These computations were extended to degree 5 to produce a list of identities satisfied by $(-,-,-)_1$ and $(-,-,-)_2$ separately and together, such that every identity of degree at most 5 satisfied by these operations follows from the identities in the list. It can then be verified that these identities are equivalent to the defining identities for Jordan triple disystems. In this way we obtain a large class of examples of special Jordan triple disystems.

Theorem 6.5. *If D is a subspace of an associative dialgebra which is closed under the Jordan diproducts $(-,-,-)_1$ and $(-,-,-)_2$, then D is a Jordan triple disystem with respect to these operations.*

6.3. Jordan dialgebras and Jordan triple disystems. A Jordan algebra with product $a \circ b$ becomes a Jordan triple system by means of the trilinear operation

$$\langle a, b, c \rangle = (a \circ b) \circ c - (a \circ c) \circ b + a \circ (b \circ c).$$

Similarly, a Jordan dialgebra with operation ab becomes a Jordan triple disystem by means of two trilinear operations; the first is obtained by replacing $a \circ b$ by ab:

$$\langle a, b, c \rangle_1 = (ab)c - (ac)b + a(bc), \qquad \langle a, b, c \rangle_2 = (ba)c + (bc)a - b(ac).$$

In a special Jordan dialgebra, we have $ab = a \dashv b + b \vdash a$, and these two operations reduce (up to a scalar multiple) to the first and second dialgebra operations in Example 4.8, namely $2(\widehat{a}bc + cb\widehat{a})$ and $2(a\widehat{b}c + c\widehat{b}a)$. This construction provides a larger class of examples of Jordan triple disystems.

Theorem 6.6. *If D is a subspace of a Jordan dialgebra which is closed under the trilinear operations $\langle -,-,- \rangle_1$ and $\langle -,-,- \rangle_2$, then D is a Jordan triple disystem with respect to these operations.*

P r o o f. This is a sketch of a computational proof of this result, starting with degree 3. We must show that every polynomial identity of degree 3 satisfied by $\langle a,b,c\rangle_1$ and $\langle a,b,c\rangle_2$ follows from the symmetry of $\langle a,b,c\rangle_2$ in its first and third arguments. We construct an 18×24 matrix E in which columns 1–12 correspond to the 12 multilinear monomials of degree 3 in the free nonassociative algebra,

$$(ab)c,\ (ac)b,\ (ba)c,\ (bc)a,\ (ca)b,\ (cb)a,\ a(bc),\ a(cb),\ b(ac),\ b(ca),\ c(ab),\ c(ba),$$

and columns 13–24 correspond to the 12 trilinear monomials of degree 3 in the trilinear operations $\langle\cdots\rangle_1$ and $\langle\cdots\rangle_2$,

$$\begin{array}{cccccc} \langle a,b,c\rangle_1, & \langle a,c,b\rangle_1, & \langle b,a,c\rangle_1, & \langle b,c,a\rangle_1, & \langle c,a,b\rangle_1, & \langle c,b,a\rangle_1, \\ \langle a,b,c\rangle_2, & \langle a,c,b\rangle_2, & \langle b,a,c\rangle_2, & \langle b,c,a\rangle_2, & \langle c,a,b\rangle_2, & \langle c,b,a\rangle_2. \end{array}$$

The matrix E has the following block structure,

$$E = \begin{bmatrix} R & O \\ X & I \end{bmatrix},$$

and its entries are determined as follows:

- the upper left 6×12 block R contains the coefficient vectors of the permutations of the right commutative identity;
- the lower left 12×12 block X contains the coefficient vectors of the expansions of the operations $\langle -,-,-\rangle_1$ and $\langle -,-,-\rangle_2$;
- the upper right 6×12 block O contains the zero matrix;
- the lower right 12×12 block I contains the identity matrix.

This matrix is displayed in Table 2 using $\cdot, +, -$ for $0, 1, -1$. The row canonical form is displayed in Table 3 using $*$ for $\frac{1}{2}$; the rank is 15. The dividing line between the upper and lower parts of the row canonical form lies immediately above row 13: the uppermost row whose leading 1 is in the right part of the matrix. The rows below this line represent the dependence relations among the expansions of the trilinear monomials which hold as a result of the right commutative identities. The rows of the lower right 3×12 block represent the permutations of $\langle a,b,c\rangle_2 - \langle c,b,a\rangle_2 \equiv 0$. $\square$

Table 2. The matrix E

$$
\left[
\begin{array}{cccccccccccc|cccccccccccc}
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & - & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\hline
+ & - & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
- & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & - & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & - & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & - & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot & \cdot \\
+ & + & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot & \cdot \\
\cdot & \cdot & \cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot & \cdot \\
+ & + & \cdot & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & + & \cdot \\
\cdot & \cdot & + & + & \cdot & \cdot & \cdot & \cdot & \cdot & - & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & +
\end{array}
\right]
$$

We can extend these computations to degree 5; the matrix E has the same block structure but is much larger. In degree 5, there are 5! permutations of the variables, and 14 association types for a nonassociative binary operation,

$$
\begin{array}{lllll}
(((ab)c)d)e, & ((a(bc))d)e, & ((ab)(cd))e, & (a((bc)d))e, & (a(b(cd)))e, \\
((ab)c)(de), & (a(bc))(de), & (ab)((cd)e), & (ab)(c(de)), & a(((bc)d)e), \\
a((b(cd))e), & a((bc)(de)), & a(b((cd)e)), & a(b(c(de))), &
\end{array}
$$

giving 1680 monomials labeling the columns in the left part. There are 10 association types in degree 5 for two trilinear operations, assuming that the second operation is symmetric in its first and third arguments:

$$
\begin{array}{llll}
\langle\langle a,b,c\rangle_1,d,e\rangle_1, & \langle a,\langle b,c,d\rangle_1,e\rangle_1, & \langle a,b,\langle c,d,e\rangle_1\rangle_1, & \langle\langle a,b,c\rangle_2,d,e\rangle_2, \\
\langle a,\langle b,c,d\rangle_2,e\rangle_2, & \langle\langle a,b,c\rangle_2,d,e\rangle_1, & \langle a,\langle b,c,d\rangle_2,e\rangle_1, & \langle a,b,\langle c,d,e\rangle_2\rangle_1, \\
\langle\langle a,b,c\rangle_1,d,e\rangle_2, & \langle a,\langle b,c,d\rangle_1,e\rangle_2. & &
\end{array}
$$

Using the symmetry of $\langle -,-,-\rangle_2$ we obtain the number of multilinear monomials in each type, giving $120+120+120+60+60+60+120+60+60+30 = 810$ monomials labeling the columns in the right part.

Table 3. The row canonical form of E

$$
\left[\begin{array}{cccccccccccc|cccccccccccc}
+ & . & . & . & . & . & . & . & . & . & . & . & * & . & . & . & . & . & . & . & . & . & * & . \\
. & + & . & . & . & . & . & . & . & . & . & . & . & * & . & . & . & . & . & . & . & . & * & . \\
. & . & + & . & . & . & . & . & . & . & . & . & . & . & * & . & . & . & . & . & . & . & . & * \\
. & . & . & + & . & . & . & . & . & . & . & . & . & . & . & * & . & . & . & . & . & . & . & * \\
. & . & . & . & + & . & . & . & . & . & . & . & . & . & . & . & * & . & . & . & . & * & . & . \\
. & . & . & . & . & + & . & . & . & . & . & . & . & . & . & . & . & * & . & . & . & * & . & . \\
. & . & . & . & . & . & + & . & . & . & . & . & * & * & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & + & . & . & . & . & * & * & . & . & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & + & . & . & . & . & . & * & * & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & + & . & . & . & . & * & * & . & . & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & + & . & . & . & . & . & * & * & . & . & . & . & . & . \\
. & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & * & * & . & . & . & . & . & . \\
\hline
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & . & . & . & - \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & - & . & . \\
. & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & . & + & . & - & .
\end{array}\right]
$$

We next generate all the consequences in degree 5 of the defining identities for Jordan dialgebras. A multilinear identity $I(a_1, \dots, a_n)$ of degree n produces $n{+}2$ identities of degree $n{+}1$; we have n substitutions and two multiplications:

$$
I(a_1 a_{n+1}, \dots, a_n), \; \dots, \; I(a_1, \dots, a_n a_{n+1}), \; I(a_1, \dots, a_n) a_{n+1}, \; a_{n+1} I(a_1, \dots, a_n).
$$

The right commutative identity of degree 3 produces 5 identities of degree 4, and each of these produces 6 identities of degree 5, for a total of 30. The linearized versions of the right Osborn and right Jordan identities of degree 4 each produce 6 identities of degree 5, for a total of 12. Altogether we have 42 identities of degree 5, and each allows 5! permutations of the variables, for a total of 5040. The upper left block R of the matrix E has size 5040×1680.

The lower left block X has size 810×1680 and contains the coefficients of the expansions of the ternary monomials. The upper right block O is the 5040×810 zero matrix, and the lower right block I is the 810×810 identity matrix:

$$
E = \left[\begin{array}{c|c}
\begin{array}{c}\text{consequences in degree 5 of} \\ \text{the Jordan dialgebra identities}\end{array} & \text{zero matrix} \\
\hline
\begin{array}{c}\text{expansions of the monomials} \\ \text{in degree 5 for } \langle \cdots \rangle_1 \text{ and } \langle \cdots \rangle_2\end{array} & \text{identity matrix}
\end{array}\right]
$$

We compute the row canonical form and find that the rank is 2215. We ignore the first 1655 rows since their leading 1s are in the left part; we retain only the 560 rows which have their leading 1s in the right part. We sort these rows by increasing number of nonzero components. These rows represent the identities in degree 5 satisfied by the Jordan triple diproducts in a Jordan dialgebra.

We construct another matrix M with an upper block of size 810×810 and a lower block of size 120×810. For each of the 560 identities satisfied by the operations $\langle -,-,- \rangle_1$ and $\langle -,-,- \rangle_2$, we apply all 5! permutations of the variables, store the permuted identities in the lower block, and compute the row canonical form. We record the index numbers of the identities which increase the rank:

identity	1	121	241	301	331	342	451	454
rank	120	240	360	390	450	470	530	560

We then verify directly that these eight identities generate the same S_5-module as the defining identities for Jordan triple disystems obtained from the KP algorithm.

7. The cyclic commutator. In this section, we present some new results about a trilinear operation called the **cyclic commutator**, $(a, b, c) = abc - bca$. This operation provides a "noncommutative" version of Lie triple systems different from Leibniz triple systems.

7.1. Polynomial identities. The next result appears in Bremner and Peresi [10] in a slightly different form.

Lemma 7.1. *Every multilinear polynomial identity of degree* 3 *satisfied by the cyclic commutator follows from the ternary Jacobi identity,*

$$(a, b, c) + (b, c, a) + (c, a, b) \equiv 0.$$

Every multilinear polynomial identity of degree 5 *satisfied by the cyclic commutator follows from the ternary Jacobi identity and the (right) ternary derivation identity,*

$$((a, b, c), d, e) \equiv ((a, d, e), b, c) + (a, (b, d, e), c) + (a, b, (c, d, e)).$$

We now extend these computations to degree 7. For a general trilinear operation, the number of association types in (odd) degree n equals the number of ternary trees with n leaf nodes; see sequence A001764 in Sloane [47] and Example 5 on page 360 of Graham et al. [22]. There is a simple formula for this number:

$$t(k) = \frac{1}{2k+1}\binom{3k}{k} \qquad (n = 2k + 1).$$

The first few values are as follows:

k	1	2	3	4	5	6	7	8	9	10
n	3	5	7	9	11	13	15	17	19	21
$t(k)$	1	3	12	55	273	1428	7752	43263	246675	1430715

We order the 12 ternary association types in degree 7 as follows:

$$
\begin{array}{lll}
(-,-,(-,-,(-,-,-))), & (-,-,(-,(-,-,-),-)), & (-,-,((-,-,-),-,-)), \\
(-,(-,-,-),(-,-,-)), & (-,(-,-,(-,-,-)),-), & (-,(-,(-,-,-),-),-), \\
(-,((-,-,-),-,-),-), & ((-,-,-),-,(-,-,-)), & ((-,-,-),(-,-,-),-), \\
((-,-,(-,-,-)),-,-), & ((-,(-,-,-),-),-,-), & (((-,-,-),-,-),-,-).
\end{array}
$$

Using the ternary Jacobi identity, we can eliminate types 3, 7, 9, 10, 11, 12 by means of the following equations and retain only types 1, 2, 4, 5, 6, 8:

$$
(3) \qquad \begin{cases}
(a,b,((c,d,e),f,g)) = -(a,b,(f,g,(c,d,e))) - (a,b,(g,(c,d,e),f)), \\
(a,((b,c,d),e,f),g) = -(a,(e,f,(b,c,d)),g) - (a,(f,(b,c,d),e),g), \\
((a,b,c),(d,e,f),g) = -((d,e,f),g,(a,b,c)) - (g,(a,b,c),(d,e,f)), \\
((a,b,(c,d,e)),f,g) = -(f,g,(a,b,(c,d,e))) - (g,(a,b,(c,d,e)),f), \\
((a,(b,c,d),e),f,g) = -(f,g,(a,(b,c,d),e)) - (g,(a,(b,c,d),e),f), \\
(((a,b,c),d,e),f,g) = (f,g,(d,e,(a,b,c))) + (f,g,(e,(a,b,c),d)) \\
\qquad\qquad +(g,(d,e,(a,b,c)),f) + (g,(e,(a,b,c),d),f).
\end{cases}
$$

Using the ternary Jacobi identity again, we can further reduce multilinear monomials in the remaining 6 types by means of the following equations:

$$
(4) \qquad \begin{cases}
(a,b,(c,d,(g,e,f))) = -(a,b,(c,d,(e,f,g))) - (a,b,(c,d,(f,g,e))), \\
(a,b,(c,(f,d,e),g)) = -(a,b,(c,(d,e,f),g)) - (a,b,(c,(e,f,d),g)), \\
(a,(d,b,c),(g,e,f)) = -(a,(b,c,d),(g,e,f)) - (a,(c,d,b),(g,e,f)), \\
\qquad = -(a,(d,b,c),(e,f,g)) - (a,(d,b,c),(f,g,e)), \\
(a,(b,c,(f,d,e)),g) = -(a,(b,c,(d,e,f)),g) - (a,(b,c,(e,f,d)),g), \\
(a,(b,(e,c,d),f),g) = -(a,(b,(c,d,e),f),g) - (a,(b,(d,e,c),f),g), \\
((c,a,b),d,(g,e,f)) = -((a,b,c),d,(g,e,f)) - ((b,c,a),d,(g,e,f)), \\
\qquad = -((c,a,b),d,(e,f,g)) - ((c,a,b),d,(f,g,e)).
\end{cases}
$$

The basic principle is that when all three arguments have degree 1, the first argument should not lexicographically follow both the second and third arguments. It follows that the total number of multilinear monomials in degree 7 equals

$$\left(\frac{2}{3} + \frac{2}{3} + \left(\frac{2}{3} \right)^2 + \frac{2}{3} + \frac{2}{3} + \left(\frac{2}{3} \right)^2 \right) \cdot 7! = 17920.$$

In order to prove that these multilinear monomials are linearly independent, we first write the ternary Jacobi identity as follows:

$$I(a,b,c) = (a,b,c) + (b,c,a) + (c,a,b).$$

We consider the following consequences of $I(a,b,c)$ in degree 5:

$$(5) \qquad \left\{ \begin{array}{lll} I((a,d,e),b,c), & I(a,(b,d,e),c), & I(a,b,(c,d,e)), \\ (I(a,b,c),d,e), & (d,I(a,b,c),e), & (d,e,I(a,b,c)). \end{array} \right.$$

Every consequence of $I(a,b,c)$ in degree 5 is a linear combination of permutations of these 6 identities. We write $J(a,b,c,d,e)$ for one of these identities. We consider the following 8 consequences of $J(a,b,c,d,e)$ in degree 7:

$$(6) \qquad \left\{ \begin{array}{lll} J((a,f,g),b,c,d,e), & J(a,(b,f,g),c,d,e), & J(a,b,(c,f,g),d,e), \\ J(a,b,c,(d,f,g),e), & J(a,b,c,d,(e,f,g)), & (J(a,b,c,d,e),f,g), \\ (f,J(a,b,c,d,e),g), & (f,g,J(a,b,c,d,e)). & \end{array} \right.$$

Every consequence of $I(a,b,c)$ in degree 7 is a linear combination of permutations of the resulting 48 identities. We now reduce each of these identities in degree 7 using equations (3) and (4), and verify that in every case the result collapses to 0. This proves that the multilinear monomials are linearly independent, and hence form a basis for the multilinear subspace of degree 7 in the free ternary algebra in the variety defined by the ternary Jacobi identity.

We now write the ternary derivation identity in the form

$$J(a,b,c,d,e) = ((a,b,c),d,e) - ((a,d,e),b,c) - (a,(b,d,e),c) - (a,b,(c,d,e)),$$

and consider its consequences in degree 7 using (6). Every consequence in degree 7 is a linear combination of permutations of these 8 identities; we reduce each of them using equations (3) and (4). We create a matrix of size 22960×17920 with an upper block of size 17920×17920 and a lower block of size 5040×17920. In order to control memory allocation, we use modular arithmetic with $p = 101$.

(Since the group algebra $\mathbb{F}S_n$ is semisimple for $p > n$, the structure constants from characteristic 0 are well-defined for any $p > n$. It follows that we can do these computations using modular arithmetic with any $p > n$ and then use rational reconstruction to recover the correct results for characteristic 0.) For each of the 8 consequences of the ternary derivation identity, we apply all 5040 permutations of the variables, store the coefficient vectors of the resulting identities in the lower block, and compute the row canonical form. At the end of this calculation, the matrix has rank 13372; the row space of this matrix consists of the coefficient vectors of all polynomial identities in degree 7 for the ternary commutator which are consequences of the ternary derivation identity.

We construct another matrix of size 5040×17920; in each column we put the coefficient vector of the expansion of the corresponding ternary monomial into the free associative algebra using the ternary commutator. The expansions for association types 1, 2, 4, 5, 6, 8 with the identity permutation are as follows:

$$
\begin{aligned}
(a,b,(c,d,(e,f,g))) &= abcdefg - bcdefga - abdefgc + bdefgca \\
&\quad - abcdfge + bcdfgea + abdfgec - bdfgeca, \\
(a,b,(c,(d,e,f),g)) &= abcdefg - bcdefga - abdefgc + bdefgca \\
&\quad - abcefdg + bcefdga + abefdgc - befdgca, \\
(a,(b,c,d),(e,f,g)) &= abcdefg - bcdefga - abcdfge + bcdfgea \\
&\quad - acdbefg + cdbefga + acdbfge - cdbfgea, \\
(a,(b,c,(d,e,f)),g) &= abcdefg - bcdefga - acdefbg + cdefbga \\
&\quad - abcefdg + bcefdga + acefdbg - cefdbga, \\
(a,(b,(c,d,e),f),g) &= abcdefg - bcdefga - acdefbg + cdefbga \\
&\quad - abdecfg + bdecfga + adecfbg - decfbga, \\
((a,b,c),d,(e,f,g)) &= abcdefg - defgabc - abcdfge + dfgeabc \\
&\quad - bcadefg + defgbca + bcadfge - dfgebca.
\end{aligned}
$$

Still using arithmetic modulo $p = 101$, we compute the row canonical form of this matrix and extract the canonical basis of the nullspace. The rank is 4128 and hence the dimension of the nullspace is 13792. Comparing this result with that of the previous paragraph, we see that there is a quotient space of dimension $13792 - 13372 = 420$ consisting of polynomial identities in degree 7 for the ternary commutator which are not consequences of the identities of lower degree. We sort these identities by increasing number of nonzero entries in the coefficient vector. Starting with the matrix of rank 13372 from the previous paragraph, we process each identity in this sorted list by applying all 5040 permutations to the variables, storing the results in the lower block, and reducing the matrix. Only two identities increase the rank: an identity with 20 terms increases the rank to 13722, and an identity with 45 terms increases the rank to 13792. Further calculations show that the first identity is a consequence of the second.

The second identity is given in the following Theorem. The fact that this identity is satisfied by the cyclic commutator can be verified directly by expanding each term into the free associative algebra. But to prove that this identity is not a consequence of the identities of lower degree requires a computation such as that just described.

Theorem 7.2. *Every multilinear polynomial identity of degree 7 satisfied by the cyclic commutator is a consequence of the ternary Jacobi identity, the ternary derivation identity, and the following identity with 45 terms and coefficients* ± 1*:*

$$
\begin{aligned}
&(ab(cd(efg))) - (ab(cf(deg))) - (ab(cf(egd))) - (ab(ce(dgf))) - (ab(cg(fed))) \\
&- (ab(dc(feg))) + (ab(df(cge))) + (ab(de(cfg))) + (ab(de(fgc))) - (ab(fc(dge))) \\
&- (ab(fc(edg))) - (ab(fd(egc))) + (ab(fg(dec))) + (ab(ec(fgd))) + (ab(ed(cgf))) \\
&+ (ab(ed(fcg))) - (ab(eg(cfd))) - (ab(eg(dcf))) + (ab(gf(dce))) - (ab(ge(cdf))) \\
&+ (ac(fb(deg))) - (ac(gb(dfe))) - (ac(gb(fed))) - (ad(eb(cfg))) - (ad(gb(fce))) \\
&- (ag(db(cfe))) - (ag(db(fec))) + (ag(fb(ced))) - (a(bcd)(efg)) + (a(bcf)(deg)) \\
&- (a(bcg)(dfe)) - (a(bde)(cfg)) - (a(bdg)(fce)) + (a(beg)(dcf)) - (a(bgd)(cfe)) \\
&- (a(bgd)(fec)) + (a(bgf)(ced)) - (a(cbd)(feg)) + (a(cbe)(dfg)) + (a(cbe)(fgd)) \\
&- (a(cdb)(efg)) + (a(cgb)(fed)) - (a(dbf)(ceg)) - (a(dbf)(egc)) + (a(egb)(dcf)) \\
&\equiv 0.
\end{aligned}
$$

Remark 7.3. The following identity with 20 terms and coefficients ± 1 is the simplest identity in degree 7 for the cyclic commutator which increased the rank in the computation described above:

$$
\begin{aligned}
&(ab(cd(efg))) + (ab(cd(gef))) - (ab(ed(gfc))) - (ab(gd(ecf))) - (ab(fd(ceg))) \\
&- (ad(cb(egf))) - (ad(cb(gfe))) + (ad(eb(gcf))) + (ad(gb(efc))) + (ad(fb(cge)))
\end{aligned}
$$

$$- (a(bdc)(egf)) - (a(bdc)(gfe)) + (a(bde)(gcf)) + (a(bdg)(efc)) + (a(bdf)(cge))$$
$$+ (a(dbc)(efg)) + (a(dbc)(gef)) - (a(dbe)(gfc)) - (a(dbg)(ecf)) - (a(dbf)(ceg))$$
$$\equiv 0.$$

Theorem 7.4. *There are no new identities for the cyclic commutator in degree* 9*: every multilinear polynomial identity of degree* 9 *satisfied by the cyclic commutator is a consequence of the identities in degrees* 3, 5 *and* 7.

Proof. Owing to the large size of this problem, we use the representation theory of the symmetric group to decompose the computation into smaller pieces corresponding to the irreducible representations. A summary of the theory and algorithms underlying this method has been given by Bremner and Peresi [11]. We briefly explain this computation in the present case; see Table 4. A partition will be denoted $\lambda = (n_1, \dots, n_k)$, $n \geq n_1 \geq \cdots \geq n_k \geq 1$, $n_1 + \cdots + n_k = 9$. These partitions label the irreducible representations of S_9; the dimension of the representation corresponding to λ will be denoted d_λ. Given a partition λ and a permutation $\pi \in S_9$, the algorithm of Clifton [17] shows how to efficiently compute a matrix $A_\lambda(\pi)$. Furthermore, the formula

$$R_\lambda(\pi) = A_\lambda(1)^{-1} A_\lambda(\pi),$$

where 1 is the identity permutation, gives the matrix representing π in the representation corresponding to λ.

We have already seen in (5) and (6) how to generate, for $n = 3$ and $n = 5$, the consequences in degree $n+2$ of a ternary identity in degree n. A similar process generates the consequences in degree 9 of a ternary identity in degree 7: from $K(a, b, \dots, g)$ we perform (i) seven substitutions, replacing x by (x, h, i) for $x = a, b, \dots g$, and (ii) three multiplications, namely (K, h, i), (h, K, i) and (h, i, K). In this way we generate all consequences in degree 9 of the ternary Jacobi identity, the (right) ternary derivation identity, and the 45-term identity of Theorem 7.2; the total number of these identities is $6 \cdot 8 \cdot 10 + 8 \cdot 10 + 10 = 570$. Every identity in degree 9, which is satisfied by the cyclic commutator and is a consequence of the identities of lower degree, is a linear combination of permutations of these 570 identities. For each representation λ of dimension $d = d_\lambda$, we construct a matrix of size $570d \times 55d$ consisting of $d \times d$ blocks. In the (i, j) block we put the representation matrix, computed by Clifton's method, of the terms of identity i with association type j. (Note that we are using all 55 ternary association types in degree 9.) The rank of this matrix of "lifted identities" is denoted "lifrank" in Table 4.

Table 4. Ranks of identities in degree 9 for cyclic commutator

#	partition	dimension	lifrank	exprank	toprank	allrank
1	9	1	55	55	0	55
2	81	8	435	440	5	435
3	72	27	1464	1485	21	1464
4	711	28	1519	1540	21	1519
5	63	48	2600	2640	40	2600
6	621	105	5686	5775	89	5686
7	6111	56	3034	3080	46	3034
8	54	42	2272	2310	38	2272
9	531	162	8768	8910	142	8768
10	522	120	6496	6600	104	6496
11	5211	189	10231	10395	164	10231
12	51111	70	3790	3850	60	3790
13	441	84	4546	4620	74	4546
14	432	168	9091	9240	149	9091
15	4311	216	11689	11880	191	11689
16	4221	216	11689	11880	191	11689
17	42111	189	10231	10395	164	10231
18	411111	56	3034	3080	46	3034
19	333	42	2274	2310	36	2274
20	3321	168	9091	9240	149	9091
21	33111	120	6496	6600	104	6496
22	3222	84	4546	4620	74	4546
23	32211	162	8768	8910	142	8768
24	321111	105	5686	5775	89	5686
25	3111111	28	1519	1540	21	1519
26	22221	42	2272	2310	38	2272
27	222111	48	2600	2640	40	2600
28	2211111	27	1464	1485	21	1464
29	21111111	8	435	440	5	435
30	111111111	1	55	55	0	55

For each representation λ of dimension $d = d_\lambda$, we construct a second matrix of size $55d \times 56d$ consisting of $d \times d$ blocks. In the $(i, 1)$ block we put the representation matrix of the terms of the expansion in the free associative algebra of the ternary monomial with association type i and the identity permutation of the variables; in the $(i, i+1)$ block we put the identity matrix; the other blocks are zero. The rank of this "expansion matrix" is always $55d$; this number is denoted "exprank" in Table 4. We compute the row canonical form of this matrix and identify the rows whose leading 1s occur within the first column of blocks; the number of these rows is denoted "toprank". The number of remaining rows, whose leading 1s occur to the right of the first column of blocks, is denoted "allrank"; these rows represent all the identities satisfied by the cyclic commutator in this representation.

For every representation, we find that "lifrank = allrank"; every identity in degree 9 satisfied by the cyclic commutator is a consequence of identities of lower degree. This completes the proof. $\square$

Definition 7.5. *A* ***noncommutative Lie triple system*** *is a vector space T with a trilinear operation $(-,-,-)\colon T \times T \times T \to T$ satisfying the ternary Jacobi identity, the (right) ternary derivation identity, and the 45-term identity of Theorem* 7.2.

An open problem is the existence of special identities for noncommutative Lie triple systems: polynomial identities satisfied by the cyclic commutator in every associative algebra, but which do not follow from the identities of Defintion 7.5.

7.2. Universal associative envelopes. We can obtain more information about a nonassociative structure by studying its irreducible finite dimensional representations. For a structure defined by a multilinear operation, the first step toward classifying the representations is to construct the universal associative enveloping algebra. This generalizes the familiar construction of the universal enveloping algebras of Lie and Jordan algebras, where an important dichotomy arises: a finite dimensional simple Lie algebra has an infinite dimensional universal envelope and infinitely many isomorphism classes of irreducible finite dimensional representations, but a finite dimensional simple Jordan algebra has a finite dimensional envelope and only finitely many irreducible representations.

The general definition of the universal associative envelope is as follows. Suppose that B is a subspace, of an associative algebra A over the field $\mathbb{F}$, closed under the n-ary multilinear operation

$$(a_1, \ldots, a_n) = \sum_{\sigma \in S_n} x_\sigma a_{\sigma(1)} \cdots a_{\sigma(n)} \quad (x_\sigma \in \mathbb{F}).$$

Write $d = \dim B$ and let $b_1, \ldots, b_d$ be a basis of B over $\mathbb{F}$; we then have the structure constants for the resulting n-ary algebra structure on B:

$$(b_{i_1}, \dots, b_{i_n}) = \sum_{j=1}^{d} c^{j}_{i_1 \cdots i_n} b_j \quad (1 \le i_1, \dots, i_n \le d).$$

Let $F\langle B\rangle$ be the free associative algebra generated by the symbols $b_1, \dots, b_d$ (this ambiguity should not cause confusion). Consider the ideal $I \subseteq F\langle B\rangle$ generated by the d^n elements

$$\sum_{\sigma \in S_n} x_\sigma b_{i_{\sigma(1)}} \cdots b_{i_{\sigma(n)}} - \sum_{j=1}^{d} c^{j}_{i_1 \cdots i_n} b_j \quad (1 \le i_1, \dots, i_n \le d).$$

The quotient algebra $U(B) = F\langle B\rangle / I$ is the universal associative enveloping algebra of the n-ary structure on B; by assumption, the natural map $B \to U(B)$ will be injective, since the n-ary structure on B is defined in terms of the associative structure on A. This generalizes the construction of the enveloping algebras of Lie algebras, where I is generated by the elements $b_i b_j - b_j b_i - [b_i, b_j]$, and of Jordan algebras, where I is generated by $b_i b_j + b_j b_i - b_i \circ b_j$. If B is a finite-dimensional Lie (resp. Jordan) algebra, then $U(B)$ is infinite-dimensional (resp. finite-dimensional).

More generally, the same construction applies to any n-ary algebra which satisfies the same low-degree polynomial identities as the n-ary operation $(a_1, \dots, a_n)$. This gives rise to a universal associative enveloping algebra; however, the natural map $B \to U(B)$ is no longer necessarily injective: for example, the universal enveloping algebra of an exceptional Jordan algebra. Once a set of generators for the ideal I is known, one can compute a noncommutative Gröbner basis for this ideal, and then use this to obtain a basis and structure constants for $U(B)$.

7.3. An example. We make the vector space of $n \times n$ matrices of trace 0 into a ternary algebra C_n with the cyclic commutator $\omega(x, y, z) = xyz - yzx$ as the trilinear operation. In the simplest case, $n = 2$, we have this basis for C_2:

$$a = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \qquad b = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad c = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}.$$

The universal associative envelope $U(C_2)$ is the quotient of the free associative algebra with three generators (also denoted a, b, c) modulo the ideal generated by the elements $xyz - yzx - \omega(x, y, z)$ for $x, y, z \in \{a, b, c\}$. This gives the following set of 24 ideal generators, in reverse degree lexicographical order:

$$
(7) \quad \left\{ \begin{array}{llll}
c^2b - bc^2, & c^2b - cbc + c, & c^2a - ac^2, & c^2a - cac, \\
cbc - bc^2 - c, & cb^2 - bcb + b, & cb^2 - b^2c, & cba - acb, \\
cba - bac - a, & cac - ac^2, & cab - bca + a, & cab - abc + a, \\
ca^2 - aca - 2c, & ca^2 - a^2c, & bcb - b^2c - b, & bca - abc, \\
b^2a - ab^2, & b^2a - bab, & bac - acb + a, & bab - ab^2, \\
ba^2 - aba - 2b, & ba^2 - a^2b, & aca - a^2c + 2c, & aba - a^2b + 2b.
\end{array} \right.
$$

We compute a Gröbner basis for this ideal following the ideas of Bergman [2] and the exposition by de Graaf [18]. We self-reduce the set of generators (7) by performing noncommutative division with remainder in order to eliminate terms which contain leading monomials of other terms. This leaves a set of 16 ideal generators:

$$
(8) \quad \left\{ \begin{array}{llll}
c^2b - bc^2, & c^2a - ac^2, & cbc - bc^2 - c, & cb^2 - b^2c, \\
cba - acb, & cac - ac^2, & cab - abc + a, & ca^2 - a^2c, \\
bcb - b^2c - b, & bca - abc, & b^2a - ab^2, & bac - acb + a, \\
bab - ab^2, & ba^2 - a^2b, & aca - a^2c + 2c, & aba - a^2b + 2b.
\end{array} \right.
$$

We find all compositions of these generators, obtaining 93 elements, and then compute the normal forms of the compositions by reducing them modulo the ideal generators; we obtain 18 elements which must be included as new ideal generators:

$$
(9) \quad \left\{ \begin{array}{llll}
\multicolumn{2}{l}{a^3cb - a^3bc + 2abc + a^3 - 2a,} & \multicolumn{2}{l}{a^3cb - a^3bc - 2acb - a^3 + 2a,} \\
\multicolumn{2}{l}{a^2cb - a^2bc - 2cb + a^2,} & \multicolumn{2}{l}{a^2cb - a^2bc + 2bc - a^2,} \\
c^3, & bc^2, & b^2c, & b^2c - a^2b + b, \\
b^3, & ac^2, & acb + abc - a, & ab^2, \\
a^2c - c, & a^2b - b, & c^2, & ca + ac, \\
b^2, & ba + ab. & &
\end{array} \right.
$$

We combine the generators (8) with the compositions (9), and self-reduce the resulting set, obtaining a new set of 8 ideal generators:

$$
(10) \qquad a^2c - c, \;\; a^2b - b, \;\; a^3 - a, \;\; c^2, \;\; cb + bc - a^2, \;\; ca + ac, \;\; b^2, \;\; ba + ab.
$$

We repeat the same process once more: finding all compositions of the generators, and computing the normal forms of the compositions modulo the generators. Every composition reduces to 0, and hence (10) is a Gröbner basis for the ideal. From this we easily obtain a vector space basis for the universal envelope $U(C_2)$: the cosets of the monomials which do not contain the leading monomial of any element of the Gröbner basis. Hence $U(C_2)$ is finite dimensional and has this basis:

$$1, \qquad a, \qquad b, \qquad c, \qquad a^2, \qquad ab, \qquad ac, \qquad bc, \qquad abc. \tag{11}$$

The multiplication for this monomial basis is given in Table 5, where we write monomials but mean cosets. If the product of two basis monomials is not a basis monomial, then we must compute its normal form modulo the Gröbner basis.

Table 5. Multiplication table for monomial basis of $U(C_2)$

$\cdot$	1	a	b	c	a^2	ab	ac	bc	abc
1	1	a	b	c	a^2	ab	ac	bc	abc
a	a	a^2	ab	ac	a	b	c	abc	bc
b	b	$-ab$	0	bc	b	0	$-abc$	0	0
c	c	$-ac$	$a^2 - bc$	0	c	$-a + abc$	0	c	$-ac$
a^2	a^2	a	b	c	a^2	ab	ac	bc	abc
ab	ab	$-b$	0	abc	ab	0	$-bc$	0	0
ac	ac	$-c$	$a - abc$	0	ac	$-a^2 + bc$	0	ac	$-c$
bc	bc	abc	b	0	bc	ab	0	bc	abc
abc	abc	bc	ab	0	abc	b	0	abc	bc

We now compute the Wedderburn decomposition of $U(C_2)$ using the algorithms described in the author's survey paper [6]. The radical of $U(C_2)$ consists of the elements whose coefficient vectors with respect to the ordered basis (11) belong to the nullspace of the Dickson matrix (Table 6), but this matrix has full rank. It follows that $U(C_2)$ is semisimple, and hence a direct sum of full matrix algebras.

A basis for the center of $U(C_2)$ is easily found, and consists of these three elements: 1, $a - 2abc$, a^2. From this we obtain a basis of orthogonal primitive idempotents for the center:

$$1 - a^2, \qquad \frac{1}{2}a + \frac{1}{2}a^2 - abc, \qquad -\frac{1}{2}a + \frac{1}{2}a^2 + abc.$$

Table 6. Dickson matrix for $U(C_2)$

$$\begin{bmatrix} 9 & 0 & 0 & 0 & 8 & 0 & 0 & 4 & 0 \\ 0 & 8 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \\ 0 & 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 \\ 8 & 0 & 0 & 0 & 8 & 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -4 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 4 & 0 & 0 & 4 & 0 \\ 0 & 4 & 0 & 0 & 0 & 0 & 0 & 0 & 4 \end{bmatrix}$$

Table 7. The change of basis matrix for $U(C_2)$

$$M = \frac{1}{2}\begin{bmatrix} 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 \\ -2 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 & 1 & -1 & 0 & 0 & 1 \\ 0 & -1 & 0 & 0 & -1 & 1 & 0 & 0 & 1 \end{bmatrix}$$

The first idempotent generates a 1-dimensional ideal, and the second and third each generate a 4-dimensional ideal. From this it follows that

$$U(C_2) \approx \mathbb{F} \oplus M_2(\mathbb{F}) \oplus M_2(\mathbb{F}). \tag{12}$$

Hence C_2 has exactly three distinct irreducible finite dimensional representations: the 1-dimensional trivial representation, the 2-dimensional natural representation, and another 2-dimensional representation which is in fact the negative of the natural representation. Moreover, $U(C_2)$ satisfies the standard identity for 2×2 matrices. From this point of view C_2 is more like a Jordan structure than a Lie structure.

Further calculations give the matrix units in the 4-dimensional ideals, and so we obtain another basis for $U(C_2)$:

$$E^{(1)}_{11}, \quad E^{(2)}_{11}, \quad E^{(2)}_{12}, \quad E^{(2)}_{21}, \quad E^{(2)}_{22}, \quad E^{(3)}_{11}, \quad E^{(3)}_{12}, \quad E^{(3)}_{21}, \quad E^{(3)}_{22}.$$

The columns of the matrix M in Table 7 give the coefficients of these basis elements in terms of the original basis elements (11). The inverse matrix M^{-1} gives the coefficients of the original basis in terms of the matrix units, and from the columns of the inverse we extract the representation matrices.

Remark 7.6. The theory of noncommutative Gröbner bases has been extended recently to associative dialgebras by Bokut et al. [3]. An interesting open problem is to use these results to construct the universal associative enveloping dialgebras of certain finite dimensional nonassociative dialgebras. In the case that the enveloping dialgebra is finite dimensional, then it would be useful to have a generalization to dialgebras of the classical Wedderburn structure theory for associative algebras. A first step in this direction has been taken recently by Mártin-González [37].

7.4. Dialgebra analogues of the cyclic commutator. Applying the KP algorithm to the ternary Jacobi identity gives three identities, each of which is equivalent to the following identity relating the three new operations:

$$(a,b,c)_3 + (b,c,a)_2 + (c,a,b)_1 \equiv 0.$$

Hence the third new operation can be eliminated using the equation

$$(a,b,c)_3 \equiv -(c,a,b)_1 - (b,c,a)_2.$$

Applying the KP algorithm to the ternary Jacobi identity gives five identities:

$$(a,b,(c,d,e)_1)_1 \equiv ((a,b,c)_1,d,e)_1 + (c,(a,b,d)_1,e)_2 + (c,d,(a,b,e)_1)_3,$$
$$(a,b,(c,d,e)_1)_2 \equiv ((a,b,c)_2,d,e)_1 + (c,(a,b,d)_2,e)_2 + (c,d,(a,b,e)_2)_3,$$
$$(a,b,(c,d,e)_1)_3 \equiv ((a,b,c)_3,d,e)_1 + (c,(a,b,d)_1,e)_1 + (c,d,(a,b,e)_1)_1,$$
$$(a,b,(c,d,e)_2)_3 \equiv ((a,b,c)_3,d,e)_2 + (c,(a,b,d)_3,e)_2 + (c,d,(a,b,e)_1)_2,$$
$$(a,b,(c,d,e)_3)_3 \equiv ((a,b,c)_3,d,e)_3 + (c,(a,b,d)_3,e)_3 + (c,d,(a,b,e)_3)_3.$$

Eliminating the third operation gives five identities relating the first two operations:

$$(a,b,(c,d,e)_1)_1 - ((a,b,c)_1,d,e)_1 - (c,(a,b,d)_1,e)_2 + ((a,b,e)_1,c,d)_1 + (d,(a,b,e)_1,c)_2 \equiv 0,$$

$$(a,b,(c,d,e)_1)_2 - ((a,b,c)_2,d,e)_1 - (c,(a,b,d)_2,e)_2 + ((a,b,e)_2,c,d)_1 + (d,(a,b,e)_2,c)_2 \equiv 0,$$

$$
\begin{aligned}
&((c,d,e)_1,a,b)_1 + (b,(c,d,e)_1,a)_2 - ((c,a,b)_1,d,e)_1 - ((b,c,a)_2,d,e)_1 \\
&\quad + (c,(a,b,d)_1,e)_1 + (c,d,(a,b,e)_1)_1 \equiv 0, \\
&((c,d,e)_2,a,b)_1 + (b,(c,d,e)_2,a)_2 - ((c,a,b)_1,d,e)_2 - ((b,c,a)_2,d,e)_2 \\
&\quad - (c,(d,a,b)_1,e)_2 - (c,(b,d,a)_2,e)_2 + (c,d,(a,b,e)_1)_2 \equiv 0, \\
&((e,c,d)_1,a,b)_1 + ((d,e,c)_2,a,b)_1 + (b,(e,c,d)_1,a)_2 + (b,(d,e,c)_2,a)_2 \\
&\quad - (e,(c,a,b)_1,d)_1 - (e,(b,c,a)_2,d)_1 - (d,e,(c,a,b)_1)_2 - (d,e,(b,c,a)_2)_2 \\
&\quad - (e,c,(d,a,b)_1)_1 - (e,c,(b,d,a)_2)_1 - ((d,a,b)_1,e,c)_2 - ((b,d,a)_2,e,c)_2 \\
&\quad - ((e,a,b)_1,c,d)_1 - ((b,e,a)_2,c,d)_1 - (d,(e,a,b)_1,c)_2 - (d,(b,e,a)_2,c)_2 \equiv 0.
\end{aligned}
$$

Applying the KP algorithm to the 45-term identity of Theorem 7.2 will produce seven identities from which we can eliminate the third operation. All these identities together will define the dialgebra analogue of noncommutative Lie triple systems.

If we apply the BSO algorithm to the cyclic commutator, $\omega(a,b,c) = abc - bca$, then we obtain these three dialgebra operations:

$$
\widehat{\omega}_1(a,b,c) = \widehat{a}bc - bc\widehat{a}, \qquad \widehat{\omega}_2(a,b,c) = a\widehat{b}c - \widehat{b}ca, \qquad \widehat{\omega}_3(a,b,c) = ab\widehat{c} - b\widehat{c}a.
$$

We have the linear dependence relation,

$$
\widehat{\omega}_1(a,b,c) + \widehat{\omega}_2(c,a,b) + \widehat{\omega}_3(b,c,a) = 0,
$$

so we only retain $\widehat{\omega}_1(a,b,c)$ and $\widehat{\omega}_2(a,b,c)$. It is an open problem to determine the polynomial identities of degrees 3, 5 and 7 satisfied by these operations in every associative dialgebra, and to check whether these identities are equivalent to those produced by the KP algorithm.

8. Conjecture relating the KP and BSO algorithms. In this section we state a conjecture first formulated by Bremner, Felipe, and Sánchez-Ortega [7]. Let $\mathbb{F}$ be a field, and let ω be a multilinear n-ary operation over $\mathbb{F}$. Fix a degree d and consider the multilinear polynomial identities of degree $e \le d$ satisfied by ω. Precisely, let A_e be the multilinear subspace of degree e in the free nonassociative n-ary algebra on e generators. Let $I_e \subseteq A_e$ be the subspace of polynomials which vanish identically when the n-ary operation is replaced by ω. The multilinear identities of degree $\le d$ satisfied by ω are then

$$
I_d(\omega) = \bigoplus_{1 \le e \le d} I_e.
$$

Applying the KP algorithm to the identities in $I_d(\omega)$ produces multilinear identities for n new n-ary operations. Precisely, let B_e be the multilinear subspace of degree e in the free nonassociative algebra with n operations of arity n. Let $\mathrm{KP}(I_e) \subseteq B_e$ be the subspace obtained by applying the KP algorithm to I_e, and define

$$\mathrm{KP}_d(\omega) = \bigoplus_{1 \le e \le d} \mathrm{KP}(I_e).$$

We now consider a different path to the same goal. Applying the BSO algorithm to ω produces n multilinear n-ary operations $\widehat{\omega}_1, \dots, \widehat{\omega}_n$. Consider the multilinear polynomial identities of degree $e \le d$ satisfied by $\widehat{\omega}_1, \dots, \widehat{\omega}_n$. Precisely, let $J_e \subseteq B_e$ be the subspace of polynomials which vanish identically when the n operations are replaced by $\widehat{\omega}_1, \dots, \widehat{\omega}_n$ and define

$$J_d(\widehat{\omega}_1, \dots, \widehat{\omega}_n) = \bigoplus_{1 \le e \le d} J_e.$$

Conjecture 8.1. *If $\mathbb{F}$ has characteristic 0 or $p > d$ then*

$$\mathrm{KP}_d(\omega) = J_d(\widehat{\omega}_1, \dots, \widehat{\omega}_n).$$

The conjecture states that these two processes give the same results when the group algebra $\mathbb{F}S_d$ is semisimple:

- Find the multilinear polynomial identities satisfied by ω, and apply the KP algorithm.
- Apply the BSO algorithm, and find the multilinear polynomial identities satisfied by $\widehat{\omega}_1, \dots, \widehat{\omega}_n$.

The conjecture is equivalent to the commutativity of this diagram:

$$\begin{tikzcd}
\omega \arrow[r, "\mathrm{BSO}"] \arrow[d] & \widehat{\omega}_1, \dots, \widehat{\omega}_n \arrow[d] \\
I_d(\omega) \arrow[r, "\mathrm{KP}"] & \begin{array}{c} J_d(\widehat{\omega}_1, \dots, \widehat{\omega}_n) \\ \stackrel{?}{=} \mathrm{KP}_d(\omega) \end{array}
\end{tikzcd}$$

The vertical arrows indicate the process of determining the multilinear polynomial identities satisfied by the given operations.

Remark 8.2. A proof of this conjecture has recently been announced by Kolesnikov and Voronin [30].

9. Open problems. In this final section we list some open problems related to generalizing well-known varieties of algebras to the setting of dialgebras.

The next step beyond Lie and Malcev algebras leads to the notion of Bol algebras. Just as Lie algebras (respectively Malcev algebras) can be defined by the polynomial identities satisfied by the commutator in every associative algebra (respectively alternative algebra), so also Bol algebras can be defined by the identities satisfied by the commutator and associator in every right alternative algebra; see Pérez-Izquierdo [42], Hentzel and Peresi [27]. One can find the defining identities for right alternative dialgebras by an application of the KP algorithm, and then use computer algebra to determine the identities satisfied by the dicommutator and the left, right, and inner associators in every right alternative dialgebra. On the other hand, one can apply the KP algorithm to the defining identities for Bol algebras. Are these two sets of identities equivalent?

Beyond Bol algebras, one obtains structures with binary, ternary and quaternary operations, which are closely related to the tangent algebras of monoassociative loops; see Bremner and Madariaga [9]. These structures can be defined by the identities satisfied by the commutator, associator and quaternator $\langle a,b,c,d\rangle = (ab,c,d) - (a,c,d)b - a(b,c,d)$ in every power associative algebra. What is the dialgebra analogue of these structures?

The tangent algebras of analytic loops have binary and ternary operations, which correspond to the commutator and associator in a free nonassociative algebra; these operations are related by the Akivis identity:

$$
\begin{aligned}
&[[a,b],c] + [[b,c],a] + [[c,a],b] \equiv \\
&(a,b,c) - (a,c,b) - (b,a,c) + (b,c,a) + (c,a,b) + (c,b,a).
\end{aligned}
$$

To obtain the correct generalization of Lie's third theorem to an arbitrary analytic loop, one must consider the infinite family of multilinear operations whose polynomial identities define the variety of Sabinin algebras. The basic references on Akivis and Sabinin algebras are Pérez-Izquierdo [43], Shestakov and Umirbaev [46]. What can one say about Akivis dialgebras and Sabinin dialgebras?

In a different direction, a generalization of dialgebras to structures with three associative operations has been considered by Loday and Ronco [36]; see also Casas [15]. It would be interesting to generalize the KP algorithm to the setting of trialgebras: that is, for any variety of nonassociative structures, give a functorial definition of the corresponding variety of trialgebras. For recent work on this problem, see Gubarev and Kolesnikov [25]. One can also consider the application of the KP algorithm to the variety of associative dialgebras: this would produce a variety of structures with four associative operations satisfying various identities. This procedure can clearly be iterated n times to produce a variety of structures with 2^n associative operations related by certain natural identities.

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