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COMPUTING WITH RATIONAL SYMMETRIC FUNCTIONS AND APPLICATIONS TO INVARIANT THEORY AND PI-ALGEBRAS

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Dedicated to Yuri Bahturin on the occasion of his 65th birthday

Abstract. Let K be a field of any characteristic. Let the formal power series

$$f(x_1,\ldots,x_d)=\sum\alpha_n x_1^{n_1}\cdots x_d^{n_d}=\sum m(\lambda)S_\lambda(x_1,\ldots,x_d),\ \ \alpha_n,m(\lambda)\in K,$$

be a symmetric function decomposed as a series of Schur functions. When f is a rational function whose denominator is a product of binomials of the

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form $1-x_1^{a_1}\cdots x_d^{a_d}$, we use a classical combinatorial method of Elliott of 1903 further developed in the Ω-calculus (or Partition Analysis) of MacMahon in 1916 to compute the generating function

$$M(f;x_1,\dots,x_d)=\sum m(\lambda)x_1^{\lambda_1}\cdots x_d^{\lambda_d},\quad \lambda=(\lambda_1,\dots,\lambda_d).$$

M is a rational function with denominator of a similar form as f. We apply the method to several problems on symmetric algebras, as well as problems in classical invariant theory, algebras with polynomial identities, and noncommutative invariant theory.

Introduction. Let K be a field of any characteristic and let $K[[X]]^{S_d}$ be the subalgebra of the symmetric functions in the algebra of formal power series $K[[X]]=K[[x_1,\dots,x_d]]$ in the set of variables $X=\{x_1,\dots,x_d\}$. We study series $f(X)\in K[[X]]^{S_d}$ which can be represented as rational functions whose denominators are products of binomials of the form $1-X^a=1-x_1^{a_1}\cdots x_d^{a_d}$. Following Berele [15], we call such functions *nice rational symmetric functions.* Those functions appear in many places in mathematics. In the examples that have inspired our project, K is of characteristic 0.

If W is a polynomial module of the general linear group $GL_d=GL_d(K)$, then its GL_d-character is a symmetric polynomial, which in turn gives W the structure of a graded vector space. Hence the Hilbert (or Poincaré) series of the symmetric algebra $K[W]$ is a nice rational symmetric function.

Nice rational symmetric functions appear as Hilbert series in classical invariant theory. For example, this holds for the Hilbert series of the pure trace algebra of $n\times n$ generic matrices which is the algebra of invariants of GL_n acting by simultaneous conjugation on several $n\times n$ matrices. The mixed trace algebra also has a meaning in classical invariant theory and has a Hilbert series which is a nice rational symmetric function.

The theorem of Belov [12] gives that for any PI-algebra R the Hilbert series of the relatively free algebra $K\langle Y\rangle/T(R)$, $Y=\{y_1,\dots,y_d\}$, where $T(R)$ is the T-ideal of the polynomial identities in d variables of R, is a rational function. Berele [15] established that the proof of Belov (as presented in the book by Kanel-Belov and Rowen [49]) also implies that this Hilbert series is a nice rational symmetric function.

Every symmetric function $f(X)$ can be presented as a formal series

$$f(X)=\sum_\lambda m(\lambda)S_\lambda(X),\quad m(\lambda)\in K,$$

where $S_\lambda(X) = S_\lambda(x_1, \ldots, x_d)$ is the Schur function indexed with the partition $\lambda = (\lambda_1, \ldots, \lambda_d)$.

Clearly, it is an interesting combinatorial problem to find the multiplicities $m(\lambda)$ of an explicitly given symmetric function $f(X)$. This problem is naturally related to the representation theory of GL_d in characteristic 0 because the Schur functions are the characters of the irreducible polynomial representations of GL_d. Another motivation is that the multiplicities of the Schur functions in the Hilbert series of the relatively free algebra $K\langle Y\rangle/T(R)$, $\mathrm{char} K = 0$, are equal to the multiplicities in the (multilinear) cocharacter sequence of the polynomial identities of R.

Drensky and Genov [34] introduced the *multiplicity series* $M(f; X)$ of $f(X) \in K[[X]]^{S_d}$. If

$$f(X) = \sum_{n_i \geq 0} \alpha(n) X^n = \sum_\lambda m(\lambda) S_\lambda(X), \quad m(\lambda) \in K,$$

then

$$M(f; X) = \sum_\lambda m(\lambda) X^\lambda = \sum_{\lambda_i \geq \lambda_{i+1}} m(\lambda) x_1^{\lambda_1} \cdots x_d^{\lambda_d} \in K[[X]]$$

is the generating function of the multiplicities $m(\lambda)$. Berele, in [16], (and also not explicitly stated in [15]) showed that the multiplicity series of a nice rational symmetric function $f(X)$ is also a nice rational function. This fact was one of the key moments in the recent theorem about the exact asymptotics

$$c_n(R) \simeq a n^{k/2} b^n, \quad a \in \mathbb{R}, \quad k, b \in \mathbb{N},$$

of the codimension sequence $c_n(R)$, $n = 0, 1, 2, \ldots$, of a unital PI-algebra R in characteristic 0 (Berele and Regev [17] for finitely generated algebras and Berele [16] in the general case). Unfortunately, the proof of Berele does not yield an algorithm to compute the multiplicity series of $f(X)$. In two variables, Drensky and Genov [35] developed methods to compute the multiplicity series for nice rational symmetric functions.

The approach of Berele [15, 16] involves classical results on generating functions of nonnegative solutions of systems of linear homogeneous equations, obtained by Elliott [40] and MacMahon [55], as stated in the paper by Stanley [66]. Going back to the originals [40] and [55], we see that the results there provide algorithms to compute the multiplicity series for nice rational symmetric functions in any number of variables. The method of Elliott [40] was further developed by MacMahon [55] in his "Ω-Calculus" or Partition Analysis. The "Ω-Calculus" was improved, with computer realizations, see Andrews, Paule, and

Riese [5, 6], and Xin [76]. The series of twelve papers on MacMahon's partition analysis by Andrews, alone or jointly with Paule, Riese, and Strehl (I – [3], ..., XII – [4]) gave a new life of the methods, with numerous applications to different problems. It seems that for the moment the original approach of [40, 55] and its further developments have not been used very efficiently in invariant theory and theory of PI-algebras. The only results in this direction we are aware of are in the recent paper by Bedratyuk and Xin [11].

Our computations are based on the ideas of Xin [76] and have been performed with standard functions of Maple on a usual personal computer. We illustrate the methods on several problems on symmetric algebras, in classical invariant theory, algebras with polynomial identities, and noncommutative invariant theory. The results of Section 1 hold for any field K of arbitrary characteristic. In the other sections we assume that K is of characteristic 0.

1. Reduction to MacMahon's partition analysis. Recall that one of the ways to define Schur functions (e.g., Macdonald [54]) is as fractions of Vandermonde type determinants

$$S_\lambda(X) = \frac{V(\lambda+\delta, X)}{V(\delta, X)},$$

where $\lambda = (\lambda_1, \dots, \lambda_d)$, $\delta = (d-1, d-2, \dots, 2, 1, 0)$, and

$$V(\mu, X) = \begin{vmatrix} x_1^{\mu_1} & x_2^{\mu_1} & \cdots & x_d^{\mu_1} \\ x_1^{\mu_2} & x_2^{\mu_2} & \cdots & x_d^{\mu_2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\mu_d} & x_2^{\mu_d} & \cdots & x_d^{\mu_d} \end{vmatrix}, \quad \mu = (\mu_1, \dots, \mu_d).$$

If $f(X) \in K[[X]]^{S_d}$ is a symmetric function, it can be presented in a unique way as

$$f(X) = \sum_\lambda m(\lambda) S_\lambda(X),$$

where the "λ-coordinate" $m(\lambda) \in K$ is called the *multiplicity* of $S_\lambda(X)$. Our efforts are concentrated around the problem: *Given $f(X) \in K[[X]]^{S_d}$, find the multiplicity series*

$$M(f;X) = \sum_\lambda m(\lambda) X^\lambda = \sum_{\lambda_i \geq \lambda_{i+1}} m(\lambda) x_1^{\lambda_1} \cdots x_d^{\lambda_d} \in K[[X]]$$

and the multiplicities $m(\lambda)$. It is convenient to introduce new variables

$$v_1 = x_1, v_2 = x_1x_2, \ldots, v_d = x_1 \cdots x_d$$

and to consider the algebra of formal power series $K[[V]] = K[[v_1, \ldots, v_d]]$ as a subalgebra of $K[[X]]$. As in [34], we introduce the function $M'(f;V)$ (also called the multiplicity series of $f(X)$) by

$$M'(f;V) = M(f; v_1, v_1^{-1}v_2, \ldots, v_{d-1}^{-1}v_d) = \sum_{\lambda} m(\lambda) v_1^{\lambda_1-\lambda_2} \cdots v_{d-1}^{\lambda_{d-1}-\lambda_d} v_d^{\lambda_d}.$$

The mapping $M' : K[[X]]^{S_d} \to K[[V]]$ defined by $M' : f(X) \to M'(f;V)$ is a bijection.

The proof of the following easy lemma is given in [15].

Lemma 1.1. *Let $f(X) \in K[[X]]^{S_d}$ be a symmetric function and let*

$$g(X) = f(X) \prod_{i<j} (x_i - x_j) = \sum_{r_i \geq 0} \alpha(r_1, \ldots, r_d) x_1^{r_1} \cdots x_d^{r_d}, \quad \alpha(r_1, \ldots, r_d) \in K.$$

Then the multiplicity series of $f(X)$ is given by

$$M(f;X) = \frac{1}{x_1^{d-1} x_2^{d-2} \cdots x_{d-2}^2 x_{d-1}} \sum_{r_i > r_{i+1}} \alpha(r_1, \ldots, r_d) x_1^{r_1} \cdots x_d^{r_d},$$

where the summation is over all $r = (r_1, \ldots, r_d)$ such that $r_1 > r_2 > \cdots > r_d$.

By the previous lemma, given a nice rational function

$$g(X) = \sum_{r_i \geq 0} \alpha(r) X^r = p(X) \prod \frac{1}{(1 - X^a)^{b_a}}$$

we start by computing "half" of it, i.e., the infinite sum of $\alpha(r) x_1^{r_1} x_2^{r_2} \cdots x_d^{r_d}$ for $r_1 > r_2$, and then we continue in the same way with the other variables. To illustrate the method of Elliott [40], it is sufficient to consider the case of two variables only. Given the series

$$g(x_1, x_2) = \sum_{i,j \geq 0} \alpha_{ij} x_1^i x_2^j$$

we introduce a new variable z and consider the Laurent series

$$g(x_1 z, \frac{x_2}{z}) = \sum_{i,j \geq 0} \alpha_{ij} x_1^i x_2^j z^{i-j} = \sum_{n=-\infty}^{\infty} g_n(x_1, x_2) z^n, \quad g_n(x_1, x_2) \in K[[x_1, x_2]].$$

We want to present $g(x_1z, x_2/z)$ as a sum of two series, one in z and the other in $1/z$:

$$g(x_1z, \frac{x_2}{z}) = \sum_{n\geq 0} g_n(x_1, x_2)z^n + \sum_{n>0} g_{-n}(x_1, x_2)\left(\frac{1}{z}\right)^n,$$

and then take the first summand and replace z with 1 there. If $g(x_1, x_2)$ is a nice rational function, then $g(x_1, x_2)$ and $g(x_1z, x_2/z)$ have the form

$$g(x_1, x_2) = p(x_1, x_2)\prod \frac{1}{1 - x_1^a x_2^b}, \quad p(x_1, x_2) \in K[x_1, x_2],$$

$$g\left(x_1z, \frac{x_2}{z}\right) = p\left(x_1z, \frac{x_2}{z}\right)\prod \frac{1}{1 - x_1^a x_2^b z^{a-b}}.$$

The expression $\prod 1/(1 - x_1^a x_2^b z^{a-b})$ is a product of three factors

$$\prod_{a_0=b_0} \frac{1}{1 - x_1^{a_0} x_2^{b_0}}, \quad \prod_{a_1>b_1} \frac{1}{1 - x_1^{a_1} x_2^{b_1} z^{a_1-b_1}}, \quad \prod_{a_2<b_2} \frac{1}{1 - x_1^{a_2} x_2^{b_2}/z^{b_2-a_2}}.$$

If $\prod 1/(1 - x_1^a x_2^b z^{a-b})$ contains factors of both the second and the third type, Elliott [40] suggests to apply the equality

$$\frac{1}{(1 - Az^a)(1 - B/z^b)} = \frac{1}{1 - ABz^{a-b}}\left(\frac{1}{1 - Az^a} + \frac{1}{1 - B/z^b} - 1\right)$$

to one of the expressions $1/(1 - x_1^{a_1} x_2^{b_1} z^{a_1-b_1})(1 - x_1^{a_2} x_2^{b_2}/z^{b_2-a_2})$ and to represent $\prod 1/(1 - x_1^a x_2^b z^{a-b})$ as a sum of three expressions which are simpler than the original one. Continuing in this way, one represents $\prod 1/(1 - x_1^a x_2^b z^{a-b})$ as a sum of products of two types:

$$\prod_{a\geq b} \frac{1}{1 - x_1^a x_2^b z^{a-b}} \quad \text{and} \quad \prod_{a_0=b_0} \frac{1}{1 - x_1^{a_0} x_2^{b_0}} \prod_{a_2<b_2} \frac{1}{1 - x_1^{a_2} x_2^{b_2}/z^{b_2-a_2}}.$$

With some additional easy arguments we can represent $g(x_1z, x_2/z)$ as a linear combination of monomials A_1z^i, $i \geq 0$, and quotients of the form

$$\frac{A_2}{z^j}, j > 0, \quad B_1z^i \prod_{b\geq 0} \frac{1}{1 - B_2z^b}, i \geq 0, \quad \frac{C_1}{z^j} \prod \frac{1}{1 - C_2} \prod_{c>0} \frac{1}{1 - C_3/z^c}, j \geq 0,$$

with coefficients $A_1, A_2, B_1, B_2, C_1, C_2, C_3$ which are monomials in x_1, x_2. Comparing this form of $g(x_1z, x_2/z)$ with its expansion as a Laurent series in z

$$g(x_1z, x_2/z) = \sum_{n=-\infty}^{\infty} g_n(x_1, x_2)z^n,$$

we obtain that the part $\sum_{n\geq 0} g_n(x_1,x_2)z^n$ which we want to compute is the sum of A_1z^i, $B_1z^i\prod_{b\geq 0}1/(1-Bz^b)$ and the fractions $C_1/z^j\prod 1/(1-C_2)$ with $j=0$.

Generalizing the idea of Elliott, in his famous book [55] MacMahon defined operators $\underset{\geq}{\Omega}$ and $\underset{=0}{\Omega}$. The first operator cuts the negative powers of a Laurent formal power series and then replaces z with 1:

$$\underset{\geq}{\Omega}:\sum_{n_i=-\infty}^{+\infty}\alpha(n)Z^n\to\sum_{n_i=0}^{+\infty}\alpha(n),$$

and the second one takes the constant term of series

$$\underset{=0}{\Omega}:\sum_{n_i=-\infty}^{+\infty}\alpha(n)Z^n\to\alpha(0),$$

where $\alpha(n)=\alpha(n_1,\ldots,n_d)\in K[[X]]$, $\alpha(0)=\alpha(0,\ldots,0)$ and $Z^n=z_1^{n_1}\cdots z_d^{n_d}$.

The next theorem presents the multiplicity series of an arbitrary symmetric function in terms of the Partition Analysis of MacMahon.

Theorem 1.2. *Let $f(X)\in K[[X]]^{S_d}$ be a symmetric function in d variables and let*

$$g(X)=f(X)\prod_{i<j}(x_i-x_j).$$

Then the multiplicity series of $f(X)$ satisfies

$$M(f;X)=\frac{1}{x_1^{d-1}x_2^{d-2}\cdots x_{d-2}^2x_{d-1}}\underset{\geq}{\Omega}\left(g(x_1z_1,x_2z_1^{-1}z_2\cdots x_{d-1}z_{d-2}^{-1}z_{d-1},x_dz_{d-1}^{-1})\right).$$

Proof. Let

$$g(X)=\sum_{r_i\geq 0}\alpha(r)X^r,\quad \alpha(r)\in K, X^r=x_1^{r_1}\cdots x_d^{r_d}.$$

Then

$$g(x_1z_1,x_2z_1^{-1}z_2\cdots x_{d-1}z_{d-2}^{-1}z_{d-1},x_dz_{d-1}^{-1})=\sum_{r_i\geq 0}\alpha(r)X^rz_1^{r_1-r_2}z_2^{r_2-r_3}\cdots z_{d-1}^{r_{d-1}-r_d},$$

$$\underset{\geq}{\Omega}\left(g(x_1z_1,x_2z_1^{-1}z_2\cdots x_{d-1}z_{d-2}^{-1}z_{d-1},x_dz_{d-1}^{-1})\right)=\sum_{r_i\geq r_{i+1}}\alpha(r)X^r.$$

The function $g(X)$ is skew-symmetric because $f(X)$ is symmetric. Hence $\alpha(r)$ is equal to 0, if $r_i = r_j$ for some $i \neq j$ and the summation in the latter equality for $\underset{\geq}{\Omega}$ runs on $r_1 > \cdots > r_d$ (and not on $r_1 \geq \cdots \geq r_d$). Now the proof follows immediately from Lemma 1.1. □

The Ω-operators were applied by MacMahon [55] to Elliott rational functions which share many properties with nice rational functions. He used the Elliott reduction process described above. The computational approach developed by Andrews, Paule, and Riese [5, 6] is based on improving this reduction process. There is another algorithm due to Xin [76] which involves partial fractions. In this paper we shall use an algorithm inspired by the algorithm of Xin [76]. We shall state it in the case of two variables. The case of nice rational symmetric functions in several variables is obtained in an obvious way by multiple application of the algorithm to the function $g(X) = f(X)\prod_{i<j}(x_i - x_j)$ in d variables instead of to the function $g(x_1, x_2) = f(x_1, x_2)(x_1 - x_2)$ in two variables.

Algorithm 1.3. Let $g(x_1, x_2) \in K[[x_1, x_2]]$ be a nice rational function. In $g(x_1 z, x_2/z)$ we replace the factors $1/(1 - C/z^c)$, where C is a monomial in x_1, x_2, with the factor $z^c/(z^c - C)$. Then $g(x_1 z, x_2/z)$ becomes a rational function of the form

$$g(x_1 z, \frac{x_2}{z}) = \frac{p(z)}{z^a} \prod \frac{1}{1 - A} \prod \frac{1}{1 - Bz^b} \prod \frac{1}{z^c - C},$$

where $p(z)$ is a polynomial in z with coefficients which are rational functions in x_1, x_2 and A, B, C are monomials in x_1, x_2. Presenting $g(x_1 z, x_2/z)$ as a sum of partial fractions with respect to z we obtain that

$$g(x_1 z, \frac{x_2}{z}) = p_0(z) + \sum \frac{p_i}{z^i} + \sum \frac{r_{jk}(z)}{q_j(z)^k},$$

where $p_0(z), r_{jk}(z), q_j(z) \in K(x_1, x_2)[z]$, $p_i \in K(x_1, x_2)$, $q_j(z)$ are the irreducible factors over $K(x_1, x_2)$ of the binomials $1 - Bz^b$ and $z^c - C$ in the expression of $g(x_1 z, x_2/z)$, and ${}^{\circ}_{z}r_{jk}(z) <^{\circ}_{z} q_j(z)$. Clearly $p_0(z)$ gives a contribution to the series $\sum_{n\geq 0} g_n(x_1, x_2) z^n$ in the expansion of $g(x_1 z, x_2/z)$ as a Laurent series. Similarly, $r_{jk}(z)/q_j(z)^k$ contributes to the same series for the factors $q_j(z)$ of $1 - Bz^b$. The fraction p_i/z^i is a part of the series $\sum_{n>0} g_n(x_1, x_2)/z^n$. When $q_j(z)$ is a factor of $z^c - C$, we obtain that $q_j(z) = z^d q'_j(1/z)$, where $d =^{\circ}_{z} q_j(z)$ and $q'_j(\zeta) \in K(x_1, x_2)[\zeta]$ is a divisor of $1 - C\zeta^c$. Since ${}^{\circ}_{z}r_{jk}(z) <^{\circ}_{z} q_j(z)$ we derive that $r_{jk}(z)/q_j(z)^k$ contributes to $\sum_{n>0} g_n(x_1, x_2)/z^n$ and does not give any contribution to $\sum_{n\geq 0} g_n(x_1,x_2)z^n$. Hence

$$\sum_{n\geq 0} g_n(x_1,x_2)z^n = p_0(z) + \sum \frac{r_{jk}(z)}{q_j(z)^k},$$

where the sum in the right side of the equation runs on the irreducible divisors $q_j(z)$ of the factors $1 - Bz^b$ of the denominator of $g(x_1 z, x_2/z)$. Substituting 1 for z we obtain the expression for $\underset{\geq}{\Omega}(g(x_1 z, x_2/z))$.

Proof. The process described in the algorithm gives that

$$\underset{\geq}{\Omega}\left(g(x_1 z, \frac{x_2}{z})\right) = \frac{P(x_1,x_2)}{Q(x_1,x_2)}, \quad P(x_1,x_2), Q(x_1,x_2) \in K[x_1,x_2].$$

By the elimination process of Elliott we already know that $\underset{\geq}{\Omega}(g(x_1 z, x_2/z))$ is a nice rational function. Hence the polynomial $Q(x_1,x_2)$ is a divisor of a product of binomials $1 - X^a$. Hence the output is in a form that allows, starting with a nice rational symmetric function $f(X)$ in d variables, to continue the process with the other variables and to compute the multiplicity series $M(f;X)$. □

Remark 1.4. If $f(X)$ is a nice rational symmetric function, we can find the multiplicity series $M(f;X)$ applying Lemma 1.1 and using the above algorithm. On the other hand, it is very easy to check whether the formal power series

$$h(X) = \sum \beta(q) X^q, \quad q_1 \geq \cdots \geq q_d,$$

is equal to the multiplicity series $M(f;X)$ of $f(X)$. This is because $h(X) = M(f;X)$ if and only if

$$f(X)\prod_{i<j}(x_i - x_j) = \sum_{\sigma\in S_d} \operatorname{sign}(\sigma) x_{\sigma(1)}^{d-1} x_{\sigma(2)}^{d-2} \cdots x_{\sigma(d-1)} h(x_{\sigma(1)},\ldots,x_{\sigma(d)}).$$

These arguments can be used to verify most of our computational results on multiplicities.

2. Symmetric algebras. Till the end of the paper we assume that K is a field of characteristic 0. For a background on the representation theory of $GL_d = GL_d(K)$ in the level we need see the book by Macdonald [54] or the paper by Almkvist, Dicks and Formanek [2]. We fix is a polynomial GL_d-module W. Then W is a direct sum of its irreducible components $W(\mu)$, where $\mu = (\mu_1, \dots, \mu_d)$ is a partition in not more than d parts,

$$W = \bigoplus_{\mu} k(\mu) W(\mu),$$

where the nonnegative integer $k(\mu)$ is the multiplicity of $W(\mu)$ in the decomposition of W. The vector space W has a basis of eigenvectors of the diagonal subgroup D_d of GL_d an we fix such a basis $\{w_1, \dots, w_p\}$:

$$g(w_j) = \xi_1^{\alpha_{j1}} \cdots \xi_d^{\alpha_{jd}} w_j, \quad g = \operatorname{diag}(\xi_1, \dots, \xi_d) \in D_d, j = 1, \dots, p,$$

where α_{ji}, $i = 1, \dots, d$, $j = 1, \dots, p$, are nonnegative integers. The action of D_d on W induces a $\mathbb{Z}^d$-grading on W assuming that

$$^\circ(w_j) = (\alpha_{j1}, \dots, \alpha_{jd}), \quad j = 1, \dots, p.$$

The polynomial

$$H(W; X) = \sum_{j=1}^{p} X^{\alpha_j} = \sum_{j=1}^{p} x_1^{\alpha_{j1}} \cdots x_d^{\alpha_{jd}}$$

is the Hilbert series of W and has the form

$$H(W; X) = \sum_{\mu} k(\mu) S_\mu(X).$$

It plays the role of the character of the GL_d-module W. If the eigenvalues of $g \in GL_d$ are equal to $\zeta_1, \dots, \zeta_d$, then

$$\chi_W(g) = \operatorname{tr}_W(g) = H(W; \zeta_1, \dots, \zeta_d) = \sum_{\mu} k(\mu) S_\mu(\zeta_1, \dots, \zeta_d).$$

We identify the symmetric algebra $K[W]$ of W with the polynomial algebra in the variables $w_1, \dots, w_p$. We extend diagonally the action of GL_d to the symmetric algebra $K[W]$ of W by

$$g(f(w)) = f(g(w)), \quad g \in GL_d, f \in K[W], w \in W.$$

The $\mathbb{Z}^d$-grading of W induces a $\mathbb{Z}^d$-grading of $K[W]$ and the Hilbert series of $K[W]$

$$H(K[W]; X) = \prod_{j=1}^{p} \frac{1}{1 - X^{\alpha_j}} = \prod_{j=1}^{p} \frac{1}{1 - x_1^{\alpha_{j1}} \cdots x_d^{\alpha_{jd}}}$$

is a nice rational symmetric function. Clearly, here we have assumed that $k(0) = 0$, (i.e., $|\mu| = \mu_1 + \cdots + \mu_d > 0$ in the decomposition of W). Otherwise the homogeneous component of zero degree of $K[W]$ is infinitely dimensional and the Hilbert series of $K[W]$ is not well defined. If we present $H(K[W];X)$ as a series of Schur functions

$$H(K[W];X) = \sum_{\lambda} m(\lambda) S_{\lambda}(X),$$

then

$$K[W] = \bigoplus_{\lambda} m(\lambda) W(\lambda).$$

Hence the multiplicity series of $H(K[W];X)$ carries the information about the decomposition of $K[W]$ as a sum of irreducible components.

The symmetric function $H(K[W];X)$ is equal to the plethysm $H(K[Y];X) \circ H(W;X)$ of the Hilbert series of $K[Y_\infty]$, $Y_\infty = \{y_1, y_2, \ldots\}$, and the Hilbert series of W. More precisely, $H(K[W];X)$ is the part of d variables of the plethysm

$$\prod_{i\geq 1} \frac{1}{1-x_i} \circ \sum_{\mu} k(\mu) S_{\mu}(X_\infty) = \sum_{n\geq 0} S_{(n)}(X_\infty) \circ \sum_{\mu} k(\mu) S_{\mu}(X_\infty)$$

of the symmetric functions $\sum_{n\geq 0} S_{(n)}$ and $\sum_{\mu} k(\mu) S_{\mu}$ in the infinite set of variables $X_\infty = \{x_1, x_2, \ldots\}$.

Example 2.1. The decomposition of $H(K[W];X)$ as a series of Schur functions is known in few cases only. The well known identities (see [54]; the third is obtained from the second applying the Young rule)

$$\begin{aligned} H(K[W(2)];X) &= \prod_{i\leq j} \frac{1}{1-x_i x_j} \\ &= \sum_{\lambda} S_{(2\lambda_1,\ldots,2\lambda_d)}(X), \end{aligned}$$

$$H(K[W(1^2)];X) = \prod_{i<j} \frac{1}{1-x_i x_j} = \sum_{(\lambda_2,\lambda_4,\ldots)} S_{(\lambda_2,\lambda_2,\lambda_4,\lambda_4,\ldots)}(X),$$

$$\begin{aligned} H(K[W(1) \oplus W(1^2)];X) &= \prod_{i=1}^{d} \frac{1}{1-x_i} \prod_{i<j} \frac{1}{1-x_i x_j} \\ &= \sum_{\lambda} S_{(\lambda_1,\ldots,\lambda_d)}(X) \end{aligned}$$

give the following expressions for the multiplicity series ($\lfloor r\rfloor$ is the floor function, i.e., the integer part of $r\in\mathbb{R}$)

$$\begin{aligned}
M(H(K[W(2)]);X) &= \sum_{n_i\geq 0} x_1^{2n_1}(x_1x_2)^{2n_2}\cdots(x_1\cdots x_d)^{2n_d}\\
&= \prod_{i=1}^{d}\frac{1}{1-(x_1\cdots x_i)^2},\\
M(H(K[W(1^2)]);X) &= \sum_{n_i\geq 0}(x_1x_2)^{n_1}(x_1x_2x_3x_4)^{n_2}\cdots(x_1\cdots x_{2\lfloor d/2\rfloor})^{n_{\lfloor d/2\rfloor}}\\
&= \prod_{i=1}^{\lfloor d/2\rfloor}\frac{1}{1-x_1\cdots x_{2i}},\\
M(H(K[W(1)\oplus W(1^2)]);X) &= \sum_{n_i\geq 0} x_1^{n_1}\cdots(x_1\cdots x_d)^{n_d}\\
&= \prod_{i=1}^{d}\frac{1}{1-x_1\cdots x_i}.
\end{aligned}$$

In the language of the multiplicity series M' we have

$$\begin{aligned}
M'(H(K[W(2)]);V) &= \prod_{i=1}^{d}\frac{1}{1-v_i^2},\\
M'(H(K[W(1^2)]);V) &= \prod_{i=1}^{\lfloor d/2\rfloor}\frac{1}{1-v_{2i}},\\
M'(H(K[W(1)\oplus W(1^2)]);V) &= \prod_{i=1}^{d}\frac{1}{1-v_i}.
\end{aligned}$$

Example 2.2. The multiplicity series of the Hilbert series of the symmetric algebra of the GL_2-module $W(3)$ was computed (in a quite elaborate way) in [34]. The calculations may be simplified using the methods of [35]. Here we illustrate the advantages of Algorithm 1.3. Since

$$S_{(3)}(x_1,x_2)=x_1^3+x_1^2x_2+x_1x_2^2+x_2^3,$$

the Hilbert series of $K[W(3)]$ is

$$H(K[W(3)];x_1,x_2)=\frac{1}{(1-x_1^3)(1-x_1^2x_2)(1-x_1x_2^2)(1-x_2^3)}.$$

We define the function

$$g(x_1,x_2)=(x_1-x_2)H(K[W(3)];x_1,x_2)$$

and decompose $g(x_1z,x_2/z)$ as a sum of partial fractions with respect to z. The result is

$$\frac{1}{3(1-x_1^2x_2^2)(1-x_1^3x_2^3)(1-x_1z)}-\frac{1-x_1z-x_1^2x_2^2-2x_1^3x_2^2z}{3(1-x_1^3x_2^3)(1+x_1^2x_2^2+x_1^4x_2^4)(1+x_1z+x_1^2z^2)}$$
$$-\frac{x_1^2x_2^2}{(1-x_1^3x_2^3)(1-x_1^6x_2^6)(1-x_1^2x_2z)}+\frac{x_2}{3(1-x_1^2x_2^2)(1-x_1^3x_2^3)(x_2-z)}$$
$$+\frac{x_2(-2z-x_2-x_1^2x_2^2z+x_1^2x_2^3)}{3(1-x_1^3x_2^3)(1+x_1^2x_2^2+x_1^4x_2^4)(x_2^2+x_2z+z^2)}-\frac{x_1^3x_2^4}{(1-x_1^3x_2^3)(1-x_1^6x_2^6)(x_1x_2^2-z)}.$$

The factors containing z in the denominators of the first three summands are $(1-x_1z)$, $(1+x_1z+x_1^2z^2)$, and $(1-x_1^2x_2z)$. Hence these summands contribute to $\underset{\geq}{\Omega}(g(x_1z,x_2/z))$. We omit the other three summands because the corresponding factors are (x_2-z), $(x_2^2+x_2z+z^2)$, and $(x_1x_2^2-z)$. Replacing z by 1 we obtain

$$\begin{aligned}\underset{\geq}{\Omega}(g(x_1z,x_2/z)) &= \frac{1}{3(1-x_1^2x_2^2)(1-x_1^3x_2^3)(1-x_1)}\\ &\quad-\frac{1-x_1z-x_1^2x_2^2-2x_1^3x_2^2z}{3(1-x_1^3x_2^3)(1+x_1^2x_2^2+x_1^4x_2^4)(1+x_1+x_1^2)}\\ &\quad-\frac{x_1^2x_2^2}{(1-x_1^3x_2^3)(1-x_1^6x_2^6)(1-x_1^2x_2)}\\ &= \frac{x_1(1-x_1^2x_2+x_1^4x_2^2)}{(1-x_1^3)(1-x_1^2x_2)(1-x_1^6x_2^6)}.\end{aligned}$$

Hence

$$\begin{aligned}M(H(K[W(3)]);x_1,x_2) &= \frac{1-x_1^2x_2+x_1^4x_2^2}{(1-x_1^3)(1-x_1^2x_2)(1-x_1^6x_2^6)},\\ M'(H(K[W(3)]);v_1,v_2) &= \frac{1-v_1v_2+v_1^2v_2^2}{(1-v_1^3)(1-v_1v_2)(1-v_2^6)}.\end{aligned}$$

We can rewrite the expression for M' in the form

$$\begin{aligned}M'(H(K[W(3)]);v_1,v_2) &= \frac{1}{2}\left(\left(\frac{1-v_1v_2+v_1^2v_2^2}{1-v_2^6}+\frac{1+v_1v_2+v_1^2v_2^2}{(1-v_2^3)^2}\right)\frac{1}{1-v_1^3}\right.\\ &\quad\left.+\left(\frac{1}{1-v_2^6}-\frac{1}{(1-v_2^3)^2}\right)\frac{1}{1-v_1v_2}\right)\end{aligned}$$

and to expand it as a power series in v_1 and v_2. The coefficient of $v_1^{a_1}v_2^{a_2}$ is equal to the multiplicity $m(a_1+a_2,a_2)$ of the partition $\lambda=(a_1+a_2,a_2)$.

Example 2.3. In the same way we have computed the multiplicity series for the Hilbert series of all symmetric algebras $K[W]$ for $\dim(W)\leq 7$ and several cases for $\dim(W)=8$. For example

$$\begin{aligned} M'(H(K[W(4)])) &= M'\left(\frac{1}{(1-x_1^4)(1-x_1^3x_2)(1-x_1^2x_2^2)(1-x_1x_2^3)(1-x_2^4)}\right)\\ &= \frac{1-v_1^2v_2+v_1^4v_2^2}{(1-v_1^4)(1-v_1^2v_2)(1-v_2^4)(1-v_2^6)}; \end{aligned}$$

$$\begin{aligned} M'(H(K[W(2)\oplus W(2)])) &= M'\left(\frac{1}{(1-x_1^2)^2(1-x_1x_2)^2(1-x_2^2)^2}\right)\\ &= \frac{1+v_1^2v_2}{(1-v_1^2)^2(1-v_2^2)^3};\\ M'(H(K[W(3)\oplus W(3)])) &= \frac{p(v_1,v_2)}{(1-v_1^3)^2(1-v_1v_2)^2(1-v_2^3)^2(1-v_2^6)^3}, \end{aligned}$$

where

$$\begin{aligned} &p(v_1,v_2)=(1-v_2^3+v_2^6)((1+v_1^6v_2^3)(1+v_2^6)-2v_1^3v_2^3(1+v_2^3))\\ &-v_1v_2(1-v_2^3)(2(1-v_2^3-v_2^9)-v_1v_2(4-v_2^3-v_2^9)-v_1^3(1+v_2^6-4v_2^9)+2v_1^4v_2(1+v_2^6-v_2^9)). \end{aligned}$$

The obtained decompositions can be easily verified using the equation

$$f(x_1,x_2)=\frac{x_1M'(f;x_1,x_1x_2)-x_2M'(f;x_2,x_1x_2)}{x_1-x_2}$$

which for $d=2$ is an M'-version of the equation in Remark 1.4.

Applying our Algorithm 1.3 we have obtained the following decompositions of the Hilbert series of the symmetric algebras of the irreducible GL_3-modules $W(\lambda)$, where $\lambda=(\lambda_1,\lambda_2,\lambda_3)$ is a partition of 3. Again, for the proof one can use Remark 1.4.

Theorem 2.4. *Let $d=3$ and let $v_1=x_1$, $v_2=x_1x_2$, $v_3=x_1x_2x_3$. The*

Hilbert series

$$\begin{aligned} H(K[W(3)];x_1,x_2,x_3) &= \prod_{i\le j\le k}\frac{1}{1-x_ix_jx_k}, \\ H(K[W(2,1)];x_1,x_2,x_3) &= \frac{1}{(1-x_1x_2x_3)^2}\prod_{i\ne j}\frac{1}{1-x_i^2x_j}, \\ H(K[W(1,1,1)];x_1,x_2,x_3) &= \frac{1}{1-x_1x_2x_3} \end{aligned}$$

have the following multiplicity series

$$\begin{aligned} M'(H(K[W(3)]);v_1,v_2,v_3) &= \frac{1}{q(v_1,v_2,v_3)}\sum_{i=0}^{8}v_1^ip_i(v_2,v_3), \\ M'(H(K[W(2,1)]);v_1,v_2,v_3) &= \frac{1+v_1v_2v_3+(v_1v_2v_3)^2}{(1-v_1v_2)(1-v_1^3v_3^2)(1-v_2^3v_3)(1-v_3^2)(1-v_3^3)}, \\ M'(H(K[W(1,1,1)]);v_1,v_2,v_3) &= \frac{1}{1-v_3}, \end{aligned}$$

where

$$\begin{aligned} q &= (1-v_1^3)(1-v_1v_2)(1-v_1^3v_3^2)(1-v_1^3v_3^3)(1-v_2^6)(1-v_2^3v_3)(1-v_2^3v_3^3) \\ &\quad \times(1-v_3^4)(1-v_3^6), \\ p_0 &= 1+v_2^9v_3^6, \\ p_1 &= -v_2(1-v_2^3v_3^2(1+v_3^2+v_3^3)-v_2^6v_3^2(1+v_3^2)+v_2^9v_3^5(1+v_3)), \\ p_2 &= v_2^2((1+v_3^2+v_3^4)-v_2^3v_3^5(1-v_3)-v_2^6v_3^2(1+v_3^2)+v_2^9v_3^5), \\ p_3 &= -((1-v_2^3v_3^3)v_3+v_2^6+v_2^9v_3^4(1+v_3^2+v_3^3))v_3^2, \\ p_4 &= v_2v_3^2((1+2v_3+v_3^3)-v_2^3v_3(v_3+v_2^3)(1+v_3^2)(1+v_3+v_3^2) \\ &\quad +v_2^9v_3^4(1+2v_3^2+v_3^3)), \\ p_5 &= -v_2^2v_3^2(1+v_3+v_3^3+v_2^3v_3^7-v_2^6v_3^3+v_2^9v_3^6), \\ p_6 &= (v_3-v_2^3v_3^2(1+v_3^2)+v_2^6(1-v_3)+v_2^9v_3^2(1+v_3^2+v_3^4))v_3^5, \\ p_7 &= -v_2v_3^5(1+v_3-v_2^3v_3^2(1+v_3^2)-v_2^6v_3(1+v_3+v_3^3)+v_2^9v_3^6), \\ p_8 &= v_2^2v_3^5(1+v_2^9v_3^6). \end{aligned}$$

For a polynomial GL_d-module $W=\sum k(\mu)W(\mu)$, the symmetric algebra $K[W]$ has also another natural $\mathbb{Z}$-grading induced by the assumption that the elements of W are of first degree. Then the homogeneous component $K[W]^{(n)}$ of degree n is the symmetric tensor power $W^{\otimes_s n}$ and

$$K[W] = \bigoplus_{n\geq 0}\bigoplus_{\lambda} m_n(\lambda)W(\lambda), \quad m_n(\lambda)\in\mathbb{Z}.$$

In order to take into account both the $\mathbb{Z}^d$-grading induced by the GL_d-action and the natural $\mathbb{Z}$-grading of $K[W]$, we introduce an additional variable in the Hilbert series of $K[W]$:

$$H(K[W];X,t) = \sum_{n\geq 0} H(K[W]^{(n)};X)t^n.$$

If, as above, the Hilbert series of W is

$$H(W;X) = \sum_{\mu} k(\mu)S_\mu(X) = \sum_{j=1}^{p} X^{\alpha_j},$$

then

$$H(K[W];X,t) = \prod_{j=1}^{p}\frac{1}{1-X^{\alpha_j}t} = \sum_{n\geq 0}\left(\sum_{\lambda} m_n(\lambda)S_\lambda(X)\right)t^n.$$

Hence the multiplicity series

$$M(H(K[W]);X,t) = \sum_{n\geq 0}\left(\sum_{\lambda} m_n(\lambda)X^\lambda\right)t^n$$

carries the information about the multiplicities of the irreducible GL_d-submodules in the homogeneous components $K[W]^{(n)}$ of $K[W]$. A minor difference with the nongraded case is that we allow $|\mu| = 0$ in the decomposition of W: Since W is finite dimensional, the homogeneous components of $K[W]$ are also finite dimensional and the Hilbert series $H(K[W];X,t)$ is well defined even if $|\mu| = 0$ for some of the summands $W(\mu)$ of W. In the next section we shall see the role of this multiplicity series in invariant theory.

Example 2.5. Let $d = 2$ and let $W = W(3)\oplus W(2)$. Then the Hilbert series of W is

$$\begin{aligned} H(W;x_1,x_2) &= S_{(3)}(x_1,x_2) + S_{(2)}(x_1,x_2) \\ &= x_1^3 + x_1^2x_2 + x_1x_2^2 + x_2^3 + x_1^2 + x_1x_2 + x_2^2 \end{aligned}$$

and the Hilbert series of $K[W]$ which takes into account also the $\mathbb{Z}$-grading is

$$H(K[W];x_1,x_2,t)=\prod_{a+b=3}\frac{1}{1-x_1^ax_2^bt}\prod_{a+b=2}\frac{1}{1-x_1^ax_2^bt}.$$

Applying Algorithm 1.3 we obtain that $M'(H(K[W]);x_1,x_2,t)$ is equal to

$$\frac{\sum_{k=0}^{10}p_k(v_1,v_2)t^k}{(1-v_1^2t)(1-v_1^3t)(1-v_1v_2t)(1-v_2^2t^2)(1-v_2^4t^3)(1-v_2^6t^4)(1-v_2^6t^5)},$$

where

$$p_0=1,\quad p_1=-v_1v_2,\quad p_2=v_1v_2(v_2+v_1v_2+v_1^2),\quad p_3=v_1v_2^2(v_2-v_1^3),$$

$$p_4=v_1v_2^5,\quad p_5=v_1v_2^5(1-v_1)(v_2+v_1),\quad p_6=-v_1^3v_2^6,$$

$$p_7=v_2^8(v_2-v_1^3),\quad p_8=-v_1v_2^9(v_2+v_1+v_1^2),\quad p_9=v_1^3v_2^{10},\quad p_{10}=-v_1^4v_2^{11}.$$

Again, we can check directly, without Algorithm 1.3 that the obtained rational function is the multiplicity series of $H(K[W];x_1,x_2,t)$ using Remark 1.4.

We complete this section with the $\mathbb{Z}$-graded version for the multiplicities of the symmetric algebra of $W=W(1)\oplus W(1^2)$ for arbitrary positive integer d.

Proposition 2.6. *Let d be any positive integer. The homogeneous component of degree n of the symmetric algebra $K[W(1)\oplus W(1^2)]$ of the GL_d-module $W=W(1)\oplus W(1^2)$ decomposes as*

$$K[W(1)\oplus W(1^2)]^{(n)}=\bigoplus W(\lambda),$$

where the summation runs on all $\lambda=(\lambda_1,\dots,\lambda_d)$ such that $\lambda_1+\lambda_3+\cdots+\lambda_{2\lfloor(d-1)/2\rfloor+1}=n$. Equivalently, the multiplicity series of $H(K[W(1)\oplus W(1^2)]);X,t)$ is

$$M'(H(K[W(1)\oplus W(1^2)]);V,t)=\prod_{2i\le d}\frac{1}{(1-v_{2i-1}t^i)(1-v_{2i}t^i)},$$

when d is even and

$$M'(H(K[W(1)\oplus W(1^2)]);V,t)=\frac{1}{1-v_dt^{(d+1)/2}}\prod_{2i<d}\frac{1}{(1-v_{2i-1}t^i)(1-v_{2i}t^i)}$$

when d is odd.

P r o o f. Since, as GL_d-module,

$$K[W(1)\oplus W(1^2)]=K[W(1)]\otimes K[W(1^2)],$$

we obtain that

$$\begin{aligned} H(K[W(1)\oplus W(1^2)];X,t) &= H(K[W(1)];X,t)H(K[W(1^2)];X,t)\\ &= \prod_{i=1}^{d}\frac{1}{1-x_it}\prod_{i<j}\frac{1}{1-x_ix_jt} \end{aligned}$$

because the elements of $W(1),W(1^2)\subset W$ are of first degree. The decompositions

$$\begin{aligned} \prod_{i=1}^{d}\frac{1}{1-x_i} &= \sum_{k\geq 0}S_{(k)}(X),\\ \prod_{i<j}\frac{1}{1-x_ix_j} &= \sum_{(\lambda_2,\lambda_4,\dots)}S_{(\lambda_2,\lambda_2,\lambda_4,\lambda_4,\dots)}(X) \end{aligned}$$

give

$$\begin{aligned} \prod_{i=1}^{d}\frac{1}{1-x_it} &= \sum_{m\geq 0}S_{(m)}(X)t^m,\\ \prod_{i<j}\frac{1}{1-x_ix_jt} &= \sum_{(\lambda_2,\lambda_4,\dots)}S_{(\lambda_2,\lambda_2,\lambda_4,\lambda_4,\dots)}(X)t^{\lambda_2+\lambda_4+\cdots} \end{aligned}$$

and hence

$$H(K[W(1)\oplus W(1^2)];X,t)=\sum_{m\geq 0}\sum_{(\lambda_2,\lambda_4,\dots)}S_{(m)}(X)S_{(\lambda_2,\lambda_2,\lambda_4,\lambda_4,\dots)}(X)t^{m+\lambda_2+\lambda_4+\cdots}.$$

The product of the Schur functions $S_{(m)}(X)$ and $S_\mu(X)$ can be decomposed by the Young rule which is a partial case of the Littlewood–Richardson rule:

$$S_{(m)}(X)S_\mu(X)=\sum S_\nu(X),$$

where the summation runs on all partitions $\nu\vdash m+|\mu|$ such that

$$\nu_1\geq\mu_1\geq\nu_2\geq\mu_2\geq\cdots\geq\nu_d\geq\mu_d.$$

Applied to our case this gives

$$S_{(m)}(X)S_{(\lambda_2,\lambda_2,\lambda_4,\lambda_4,\dots)}(X)=\sum S_{(\lambda_1,\lambda_2,\dots,\lambda_d)}(X),$$

where the sum is on all partitions with

$$(\lambda_1-\lambda_2)+(\lambda_3-\lambda_4)+\cdots+(-1)^{d-1}\lambda_d=m.$$

Hence the GL_d-module $W(\lambda)\subset K[W]$ is a submodule of the homogeneous component of degree

$$m+\lambda_2+\lambda_4+\cdots=\lambda_1+\lambda_3+\cdots+\lambda_{2\lfloor (d-1)/2\rfloor+1}.$$

For the statement for the multiplicity series, first let $d=2k+1$. Then

$$\begin{aligned}
M(H(K[W]);X,t) &= \sum_{\lambda_1\geq\cdots\geq\lambda_{2k+1}} x_1^{\lambda_1}\cdots x_{2d+1}^{\lambda_{2k+1}}t^{\lambda_1+\lambda_3+\cdots+\lambda_{2k+1}}\\
&= \sum_{\lambda_1\geq\cdots\geq\lambda_{2k+1}} (x_1t)^{\lambda_1}x_2^{\lambda_2}(x_3t)^{\lambda_3}\cdots x_{2d}^{\lambda_{2k+1}}(x_{2d+1}t)^{\lambda_{2k+1}}\\
&= \sum_{\lambda_1\geq\cdots\geq\lambda_{2k+1}} (x_1t)^{\lambda_1-\lambda_2}(x_1x_2t)^{\lambda_2-\lambda_3}(x_1x_2x_3t^2)^{\lambda_3-\lambda_4}\\
&\quad\times(x_1x_2x_3x_4t^2)^{\lambda_4-\lambda_5}\cdots(x_1\cdots x_{2k-1}t^k)^{\lambda_{2k-1}-\lambda_{2k}}\\
&\quad\times(x_1\cdots x_{2k}t^k)^{\lambda_{2k}-\lambda_{2k+1}}(x_1\cdots x_{2k+1}t^{k+1})^{\lambda_{2k+1}},\\
M'(H(K[W]);X,t) &= \sum_{n_i\geq 0}(v_1t)^{n_1}(v_2t)^{n_2}(v_3t^2)^{n_3}(v_4t^2)^{n_4}\cdots\\
&\quad\times(v_{2k-1}t^k)^{n_{2k-1}}(v_{2k}t^k)^{n_{2k}}(v_{2k+1}t^{k+1})^{n_{2k+1}}\\
&= \frac{1}{1-v_{2k+1}t^{k+1}}\prod_{i=1}^{k}\frac{1}{(1-v_{2i-1}t^i)(1-v_{2i}t^i)}.
\end{aligned}$$

The case $d=2k$ follows immediately from the case $d=2k+1$ by substituting $v_{2k+1}=0$ in the expression of $M'(H(K[W]);X,t)$. □

3. Invariant theory. Without being comprehensive, we shall survey few results related with our topic. One of the main objects in invariant theory in the 19th century is the algebra of SL_2-invariants of binary forms. Let $W_m=W_{m,2}$ be the vector space of all homogeneous polynomials of degree m in two variables with the natural action of SL_2. The computation of the Hilbert series (often also called the Poincaré series) of the algebra of invariants $K[W_m]^{SL_2}$ was a favorite problem actively also studied nowadays. It was computed by Sylvester and Franklin [68, 69] for $m\leq 10$ and $m=12$. In 1980 Springer [65] found an explicit formula for the Hilbert series of $K[W_m]^{SL_2}$. Applying it, Brouwer and Cohen [19] calculated the Hilbert series of $K[W_m]^{SL_2}$ of degree ≤ 17. Littelmann and Procesi [53] suggested an algorithm based on a variation of the result of Springer and computed the Hilbert series for $m = 4k \leq 36$. More recently, Djoković [26] proposed a heuristic algorithm for fast computation of the Hilbert series of the invariants of binary forms, viewed as rational functions, and computed the series for $m \leq 30$.

Not too much is known about the explicit form of the invariants and their Hilbert series when $SL_d(\mathbb{C})$, $d \geq 3$, acts on the vector space of forms of degree $m \geq 3$. Most of the known results are for ternary forms. The generators of the algebra of invariants in the case of forms of degree 3 were found by Gordan [48], see also Clebsch and Gordan [22]; the case of forms of degree 4 has handled by Emmy Noether [58]. The Hilbert series of the algebra of $SL_3(\mathbb{C})$-invariants for forms of degree 4 was calculated by Shioda [64]. Recently Bedratyuk [7, 8] found analogues of Sylvester–Cayley and Springer formulas for invariants also of ternary forms. This allowed him to compute the first coefficients (of the terms of degree ≤ 30) of the Hilbert series of the algebras of $SL_3(\mathbb{C})$-invariants of forms of degree $m \leq 7$.

Computing the Hilbert series of the algebra of SL_d-invariants $\mathbb{C}[W]^{SL_d}$, where W is a direct sum of several vector spaces $W_{m_i,d}$ of forms of degree m_i in d variables, one may use the Molien–Weyl integral formula, evaluating multiple integrals. This type of integrals can be evaluated using the Residue Theorem, see the book by Derksen and Kemper [24] for details. For concrete decompositions of W, the algebra of invariants $K[W]^{SL_d}$ was studied already by Sylvester. Its Hilbert series is also known in some cases. For example, recently Bedratyuk [9] has found a formula for the Hilbert series of the SL_2-invariants $K[W_{m_1,2} \oplus W_{m_2,2}]^{SL_2}$ and has computed these series for $m_1, m_2 \leq 20$. (The results for $m_1, m_2 \leq 5$ are given explicitly in [9].) Very recently, Bedratyuk and Xin [11] applied the MacMahon partition analysis to the Molien–Weyl integral formula and computed the Hilbert series of the algebras of invariants of some ternary and quaternary forms.

Our approach to the Hilbert series of the algebra of invariants $K[W]^{SL_d}$ of the SL_d-module W is based on a theorem of De Concini, Eisenbud and Procesi [23]. This theorem implies that the multiplicities of $S_\lambda(X)$ in the Hilbert series of symmetric algebras $K[W]$ of a GL_d-module W appear in invariant theory of SL_d and of the unitriangular group $UT_d = UT_d(K)$ as subgroups of GL_d, and in invariant theory of a single unitriangular matrix. It is combined with an idea used by Drensky and Genov [35] to compute the Hilbert series of the algebra of invariants of UT_2. We extend the SL_d-action on W to a polynomial action of GL_d. This is possible in the cases that we consider because the SL_d-module $W_{m,d}$ of the forms of degree m can be viewed as a GL_d-module isomorphic to $W(m)$. Then we compute the Hilbert series of the GL_d-module $K[W]$ and its multiplicity series $M'(H(K[W]);V,t)$. The Hilbert series of $K[W]^{SL_d}$ is equal to $M'(H(K[W]);0,\dots,0,1,t)$. Similarly, if W is a polynomial GL_d-module, then the Hilbert series of $K[W]^{UT_d}$ is equal to $M'(H(K[W]);1,\dots,1,t)$. The difference with [35] is that there we use for the evaluation of $M(H(K[W]);x_1,x_2,t)$ the methods developed in [35] and here we use the MacMahon partition analysis for the same purpose and for any number of variables. We shall consider the following problem. *Let W be an arbitrary polynomial GL_d-module. How can one calculate the Hilbert series of the algebras of invariants $K[W]^{SL_d}$ and $K[W]^{UT_d}$?* Clearly, here we assume that SL_d and UT_d are canonically embedded into GL_d. We need the following easy argument. We state it as a lemma and omit the obvious proof.

Lemma 3.1. *Let H be a subgroup of the group G and let W_1, W_2 be G-modules. Then the vector space of invariants $W^H \subset W$ in $W = W_1 \oplus W_2$ satisfy*

$$W^H = W_1^H \oplus W_2^H.$$

Theorem 3.2. *Let W be a polynomial GL_d-module with Hilbert series with respect to the grading induced by the GL_d-action on W*

$$H(W;X) = \sum a_i x_1^{i_1} \cdots x_d^{i_d}, \quad a_i \geq 0, a_i \in \mathbb{Z},$$

and let

$$H(K[W];X,t) = \prod \frac{1}{(1-X^i t)^{a_i}}$$

be the Hilbert series of $K[W]$ which counts also the natural $\mathbb{Z}$-grading. Then the Hilbert series of the algebras of invariants $K[W]^{SL_d}$ and $K[W]^{UT_d}$ are given by

$$\begin{aligned} H(K[W]^{SL_d},t) &= M'(H(K[W]);0,\dots,0,1,t),\\ H(K[W]^{UT_d},t) &= M(H(K[W]);1,\dots,1,t)\\ &= M'(H(K[W]);1,\dots,1,t). \end{aligned}$$

Proof. Let

$$K[W] = \bigoplus_{n\geq 0} \bigoplus_{\lambda} m_n(\lambda) W(\lambda)$$

be the decomposition of the $\mathbb{Z}$-graded GL_d-module $K[W]$. Its Hilbert series is

$$H(K[W];X,t)=\sum_{n\geq 0}\left(\sum_{\lambda}m_n(\lambda)S_\lambda(X)\right)t^n$$

and the multiplicity series of $H(K[W];X,t)$ are

$$\begin{aligned}M(H(K[W]);X,t) &= \sum_{n\geq 0}\left(\sum_{\lambda}m_n(\lambda)X^\lambda\right)t^n,\\ M'(H(K[W]);V,t) &= \sum_{n\geq 0}\left(\sum_{\lambda}m_n(\lambda)v_1^{\lambda_1-\lambda_2}\cdots v_{d-1}^{\lambda_{d-1}-\lambda_d}v_d^{\lambda_d}\right)t^n.\end{aligned}$$

It is a well known fact that the irreducible GL_d-module $W(\lambda)=W(\lambda_1,\dots,\lambda_d)$ contains a one-dimensional SL_d-invariant subspace if $\lambda_1=\cdots=\lambda_d$ (when $\dim(W(\lambda)=1$ and $W(\lambda)^{SL_d}=W(\lambda))$ and contains no invariants if $\lambda_j\neq\lambda_{j+1}$ for some j. Applying Lemma 3.1 we immediately obtain

$$\begin{aligned}K[W]^{SL_d} &= \bigoplus_{n\geq 0}\bigoplus_{\lambda_1=\cdots=\lambda_d}m_n(\lambda)W(\lambda),\\ H(K[W]^{SL_d};t) &= \sum_{n\geq 0}\left(\sum_{\lambda_1=\cdots=\lambda_d}m_n(\lambda)\right)t^n.\end{aligned}$$

Evaluating the monomials in the expansion of $M'(H(K[W]);V,t)$ for $v_1=\cdots=v_{d-1}=0$, $v_d=1$ we obtain

$$v_1^{\lambda_1-\lambda_2}\cdots v_{d-1}^{\lambda_{d-1}-\lambda_d}v_d^{\lambda_d}|_{V=(0,\dots,0,1)}=\begin{cases}1, & \text{if } \lambda_1=\cdots=\lambda_d,\\ 0, & \text{if } \lambda_j\neq\lambda_{j+1} \text{ for some } j\end{cases}$$

which completes the case of SL_d-invariants.

It is also well known that every irreducible GL_d-module $W(\lambda)$ has a one-dimensional UT_d-invariant subspace which is spanned on the only (up to a multiplicative constant) element $w\in W(\lambda)$ with the property that the diagonal subgroup D_d of GL_d acts by

$$g(w)=\xi_1^{\lambda_1}\cdots\xi_d^{\lambda_d}w,\quad g=\operatorname{diag}(\xi_1,\dots,\xi_d).$$

Hence

$$\begin{aligned} K[W]^{UT_d} &= \bigoplus_{n\geq 0}\bigoplus_{\lambda} m_n(\lambda)W(\lambda)^{UT_d}, \\ H(K[W]^{UT_d};t) &= \sum_{n\geq 0}\left(\sum_{\lambda} m_n(\lambda)\right)t^n \\ &= M(H(K[W]);1,\ldots,1,t) = M'(H(K[W]);1,\ldots,1,t). \qquad \square \end{aligned}$$

Below we shall illustrate Theorem 3.2 on the Hilbert series of the SL_2-invariants for the GL_2-modules considered in the examples of Section 2.

Example 3.3. If the polynomial GL_d-module W is homogeneous of degree m, i.e., $g(w) = \xi^m w$ for $w \in W$ and $g = \operatorname{diag}(\xi,\ldots,\xi) \in GL_d$, then

$$\begin{aligned} H(K[W];x_1,\ldots,x_d,t) &= H(K[W];X,t) \\ &= H(K[W];X\sqrt[m]{t}) = H(K[W];x_1\sqrt[m]{t},\ldots,x_d\sqrt[m]{t}) \end{aligned}$$

because the elements of W are of degree 1 with respect to the $\mathbb{Z}$-grading and of degree m with respect to the $\mathbb{Z}^d$-grading. The results of Example 2.1 give

$$\begin{aligned} M'(H(K[W(2)]);V) &= \prod_{i=1}^{d}\frac{1}{1-v_i^2}, \\ M'(H(K[W(2)]);V,t) &= \prod_{i=1}^{d}\frac{1}{1-v_i^2t^i}, \\ H(K[W(2)]^{SL_d};t) &= M'(H(K[W(2)]);0,\ldots,0,1,t) = \frac{1}{1-t^d}, \\ H(K[W(2)]^{UT_d};t) &= M'(H(K[W(2)]);1,\ldots,1,t) = \prod_{i=1}^{d}\frac{1}{1-t^i}; \\ M'(H(K[W(1^2)]);V) &= \prod_{i=1}^{\lfloor d/2\rfloor}\frac{1}{1-v_{2i}}, \\ M'(H(K[W(1^2)]);V,t) &= \prod_{i=1}^{\lfloor d/2\rfloor}\frac{1}{1-v_{2i}t^i}, \end{aligned}$$

$$H(K[W(1^2)]^{SL_d};t) = \begin{cases} \dfrac{1}{1-t^{d/2}}, & \text{if } d \text{ is even,} \\ \\ 1, & \text{if } d \text{ is odd,} \end{cases}$$

$$H(K[W(1^2)]^{UT_d};t) = \prod_{i=1}^{\lfloor d/2 \rfloor} \frac{1}{1-t^i}.$$

Similarly, for $d=2$ Examples 2.2 and 2.3 give

$$M'(H(K[W(3)]);v_1,v_2) = \frac{1-v_1v_2+v_1^2v_2^2}{(1-v_1^3)(1-v_1v_2)(1-v_2^6)},$$

$$M'(H(K[W(3)]);v_1,v_2,t) = M'(H(K[W(3)];v_1\sqrt[3]{t},v_2\sqrt[3]{t^2})$$

$$= \frac{1-v_1v_2t+v_1^2v_2^2t^2}{(1-v_1^3t)(1-v_1v_2t)(1-v_2^6t^4)},$$

$$H(K[W(3)]^{SL_2};t) = M'(H(K[W(3)]);0,1,t) = \frac{1}{1-t^4},$$

$$H(K[W(3)]^{UT_2};t) = M'(H(K[W(3)]);1,1,t) = \frac{1-t+t^2}{(1-t)^2(1-t^4)};$$

$$M'(H(K[W(4)]);v_1,v_2,t) = M'(H(K[W(4)]);v_1\sqrt[4]{t},v_2\sqrt{t}),$$

$$H(K[W(4)]^{SL_2};t) = M'(H(K[W(4)]);0,1,t) = \frac{1}{(1-t^2)(1-t^4)},$$

$$H(K[W(4)]^{UT_2};t) = M'(H(K[W(4)]);1,1,t)$$

$$= \frac{1-t+t^2}{(1-t)^2(1-t^2)(1-t^4)};$$

$$M'(H(K[W(2)\oplus W(2)]);v_1,v_2,t) = M'(H(K[W(2)\oplus W(2)]);v_1\sqrt{t},v_2t),$$

$$H(K[W(2)\oplus W(2)]^{SL_2};t) = \frac{1}{(1-t^2)^3},$$

$$H(K[W(2)\oplus W(2)]^{UT_2};t) = \frac{1+t^2}{(1-t)^2(1-t^2)^3};$$

$$M'(H(K[W(3)\oplus W(3)]) :,v_1,v_2,t) = M'(H(K[W(3)\oplus W(3)]) :,v_1\sqrt[3]{t},v_2\sqrt[3]{t^2}),$$

$$H(K[W(3)\oplus W(3)]^{SL_2};t) = \frac{(1-t^2+t^4)(1+t^4)}{(1-t^2)^5(1+t^2)^3},$$

$$H(K[W(3)\oplus W(3)]^{UT_2};t) = \frac{1+t^{10}+3t^2(1+t^6)+6t^3(1+t+t^2+t^3+t^4)}{(1-t)^2(1-t^2)^5(1+t^2)^3}.$$

Finally, for $d=3$ Theorem 2.4 gives that

$$\begin{aligned}
M'(H(K[W(3)]);v_1,v_2,v_3,t) &= M'(H(K[W(3)]);v_1\sqrt[3]{t},v_2\sqrt[3]{t^2},v_3t),\\
(H(K[W(3)]^{SL_3};t) &= \frac{1}{(1-t^4)(1-t^6)},\\
(H(K[W(3)]^{UT_3};t) &= \frac{(1+t^3)(1+t^9)+2t^4(1+t^4)+3t^5(1+t+t^2)}{(1-t)(1-t^2)(1-t^3)^2(1-t^4)^2(1-t^5)}
\end{aligned}$$

and similarly

$$\begin{aligned}
H(K[W(2,1)]^{SL_3};t) &= \frac{1}{(1-t^2)(1-t^3)},\\
H(K[W(2,1)]^{UT_3};t) &= \frac{1-t+t^2}{(1-t)^2(1-t^2)(1-t^3)^2};\\
H(K[W(1^3)]^{SL_3};t) &= H(K[W(1^3)]^{UT_3};t)=\frac{1}{1-t}.
\end{aligned}$$

Example 3.4. The translation of Example 2.5 to the language of SL_2- and UT_2-invariants gives

$$\begin{aligned}
H(K[W(3)\oplus W(2)]^{SL_2};t) &= \frac{1+t^9}{(1-t^2)(1-t^3)(1-t^4)(1-t^5)},\\
H(K[W(3)\oplus W(2)]^{UT_2};t) &= \frac{(1-t)(1-t^7)+4t^2(1+t^4)-t^3(1+t^2)+5t^4}{(1-t)^3(1-t^3)(1-t^4)(1-t^5)}.
\end{aligned}$$

For an arbitrary d Proposition 2.6 gives

$$H(K[W(1)\oplus W(1^2)]^{SL_d};t) = \begin{cases} \dfrac{1}{1-t^k}, \text{ if } d=2k,\\[2ex] \dfrac{1}{1-t^{k+1}}, \text{ if } d=2k+1,\end{cases}$$

$$H(K[W(1)\oplus W(1^2)]^{UT_d};t) = \begin{cases} \displaystyle\prod_{i=1}^{k}\frac{1}{(1-t^i)^2}, \text{ if } d=2k,\\[2ex] \displaystyle\frac{1}{1-t^{k+1}}\prod_{i=1}^{k}\frac{1}{(1-t^i)^2}, \text{ if } d=2k+1.\end{cases}$$

In the above examples our results coincide with the known ones, see e.g., [68, 25, 26, 9, 7].

The invariant theory of UT_2 may be restated in the language of linear locally nilpotent derivations. Recall that a derivation of a (not necessarily commutative or associative) algebra R is a linear operator δ with the property that

$$\delta(u_1u_2) = \delta(u_1)u_2 + u_1\delta(u_2), \quad u_1, u_2 \in R.$$

The derivation δ is locally nilpotent if for any $u \in R$ there exists a p such that $\delta^p(u) = 0$. Locally nilpotent derivations are interesting objects with relations to invariant theory, the 14th Hilbert problem, automorphisms of polynomial algebras, the Jacobian conjecture, etc., see the monographs by Nowicki [59], van den Essen [41], and Freudenburg [46]. Linear locally nilpotent derivations δ of the polynomial algebra $K[Y]$ (acting as linear operators on the vector space KY with basis $Y = \{y_1, \ldots, y_d\}$) were studied by Weitzenböck [75] who proved that the algebra of constants $K[Y]^\delta$, i.e., the kernel of δ, is finitely generated. Nowadays linear locally nilpotent derivations of $K[Y]$ are known as *Weitzenböck derivations* and are subjects of intensive study.

The algebra $K[Y]^\delta$ coincides with the algebra $K[Y]^G$ of invariants of the cyclic group G generated by

$$\exp(\delta) = 1 + \frac{\delta}{1!} + \frac{\delta^2}{2!} + \cdots$$

and with the algebra of invariants of the additive group K_a of the field K with its d-dimensional representation

$$\alpha \to \exp(\alpha\delta), \quad \alpha \in K_a,$$

which allows to involve invariant theory. Historically, it seems that this relation was used quite rarely and some of the results on Weitzenböck derivations rediscover classical results in invariant theory. For example, the modern proof of the theorem of Weitzenböck given by Seshadri [63] is equivalent to the results of Roberts [62] that for an SL_2-module W the algebra $K[W]^{UT_2}$ is isomorphic to the algebra of covariants. There is an elementary version of the proof by Seshadri given by Tyc [72] which is in the language of representations of the Lie algebra $sl_2(K)$ and can be followed without serious algebraic knowledge. Let δ be a Weitzenböck derivation. All eigenvalues of δ (acting on KY) are equal to 0 and, up to a linear change of the coordinates of $K[Y]$, δ is determined by its Jordan normal form. Hence, for each fixed dimension d there is only a finite number of Weitzenböck derivations. The only derivation which corresponds to a single

Jordan cell is called *basic*. Onoda [60] presented an algorithm which calculates the Hilbert series in the case of a basic Weitzenböck derivation. He calculated the Hilbert series for the basic derivation δ and $d = 6$ and, as a consequence showed that the algebra of constants $\mathbb{C}[Y]^\delta$ is not a complete intersection. (By the same paper [60], $\mathbb{C}[Y]^\delta$ is Gorenstein for any Weitzenböck derivation δ which agrees with a general fact in invariant theory of classical groups.) Other methods to compute the Hilbert series of $K[W]^\delta$ for any Weitzenböck derivation δ are developed by Bedratyuk; see [10] and the references there. Below we show how the MacMahon partition analysis can be used to compute the Hilbert series of $K[W]^\delta$. The following theorem and its corollary were announced in [35].

Theorem 3.5. *Let δ be a Weitzenböck derivation of $K[Y]$ with Jordan normal form consisting of k cells of sizes $d_1+1,\dots,d_k+1$, respectively. Let*

$$f_\delta(x_1,x_2,t) = \frac{1}{q_{d_1}(x_1,x_2,t)\cdots q_{d_k}(x_1,x_2,t)},$$

where

$$q_d(x_1,x_2,t) = (1-x_1^dt)(1-x_1^{d-1}x_2t)\cdots(1-x_1x_2^{d-1}t)(1-x_2^dt).$$

Then the Hilbert series of the algebra of constants $K[Y]^\delta$ is given by

$$H(K[Y]^\delta;t) = M(f_\delta;1,1),$$

where $M(f_\delta;x_1,x_2)$ is the multiplicity series of the symmetric with respect to x_1,x_2 function $f_\delta(x_1,x_2,t) \in K(t)[[x_1,x_2]]^{S_2}$.

Proof. If δ has k Jordan cells and the ith cell is of size d_i+1, $i=1,\dots,k$, we identify the vector space KY with the GL_2-module

$$W = W(d_1)\oplus\cdots\oplus W(d_k)$$

and the algebra $K[Y]$ with the symmetric algebra $K[W]$. Then the algebra of constants $K[Y]^\delta$ coincides with the algebra $K[W]^{UT_2}$ of UT_2-invariants. Obviously, the function $f_\delta(x_1,x_2,t)$ is equal to the Hilbert series of the $\mathbb{Z}$-graded GL_2-module $K[W]$. Hence Theorem 3.2 completes the proof. $\square$

Example 3.6. Let $\delta = \delta(d_1,\dots,d_k)$ be the Weitzenböck derivation with k Jordan cells of size d_i+1, $i=1,\dots,k$, respectively. If the matrix of δ contains a Jordan cell of size 1 corresponding to x_d, then

$$K[x_1,\dots,x_{d-1},x_d]^\delta = K[x_1,\dots,x_{d-1}]^\delta[x_d]$$

and the Hilbert series of the algebras of constants $K[x_1,\dots,x_d]^\delta$ and $K[x_1,\dots,x_{d-1}]^\delta$ are related by

$$H(K[x_1,\dots,x_d]^\delta;t)=\frac{1}{1-t}H(K[x_1,\dots,x_{d-1}]^\delta;t).$$

Hence it is sufficient to consider only δ with Jordan matrices without 1-cells. Below we extend the results in Examples 3.3 and 3.4 and give the Hilbert series of $K[Y]^\delta$ for all possible δ with $d\leq 7$. Originally the computations were performed in [35] illustrating the methods developed there for symmetric functions in two variables. Here we repeated the computations with the methods of the MacMahon partition analysis. Clearly, the results coincide with those from [10].
$d=2$:

$$H(K[Y]^{\delta(1)};t)=\frac{1}{1-t};$$

$d=3$:

$$H(K[Y]^{\delta(2)};t)=\frac{1}{(1-t)(1-t^2)};$$

$d=4$:

$$H(K[Y]^{\delta(3)};t)=\frac{1-t+t^2}{(1-t)^2(1-t^4)}=\frac{1+t^3}{(1-t)(1-t^2)(1-t^4)},$$

$$H(K[Y]^{\delta(1,1)};t)=\frac{1}{(1-z)^2(1-z^2)};$$

$d=5$:

$$H(K[Y]^{\delta(4)};t)=\frac{1-t+t^2}{(1-t)^2(1-t^2)(1-t^3)}=\frac{1+t^3}{(1-t)(1-t^2)^2(1-t^3)},$$

$$H(K[Y]^{\delta(2,1)};t)=\frac{1}{(1-z)^2(1-z^2)(1-z^3)};$$

$d=6$:

$$H(K[Y]^{\delta(5)};t)=\frac{p(z)}{(1-z)(1-z^2)(1-z^4)(1-z^6)(1-z^8)},$$

$$p(z)=1+z^2+3z^3+3z^4+5z^5+4z^6+6z^7+6z^8+4z^9+5z^{10}+3z^{11}+3z^{12}+z^{13}+z^{15},$$

$$H(K[Y]^{\delta(3,1)};t)=\frac{1+z^2+3z^3+z^4+z^6}{(1-z)^2(1-z^2)(1-z^4)^2},$$

$$H(K[Y]^{\delta(2,2)};t)=\frac{1+z^2}{(1-z)^2(1-z^2)^3},$$

$$H(K[Y]^{\delta(1,1,1)};t)=\frac{1-z^3}{(1-z)^3(1-z^2)^3}=\frac{1+z+z^2}{(1-z)^2(1-z^2)^3};$$

$d=7$:

$$H(K[Y]^{\delta(6)};t)=\frac{1+z^2+3z^3+4z^4+4z^5+4z^6+3z^7+z^8+z^{10}}{(1-z)(1-z^2)^2(1-z^3)(1-z^4)(1-z^5)},$$

$$H(K[Y]^{\delta(4,1)};t)=\frac{1+2z^2+2z^3+4z^4+2z^5+2z^6+z^8}{(1-z)^2(1-z^2)(1-z^3)^2(1-z^5)},$$

$$H(K[Y]^{\delta(3,2)};t)=\frac{1-z+4z^2-z^3+5z^4-z^5+4z^6-z^7+z^8}{(1-z)^3(1-z^3)(1-z^4)(1-z^5)}$$

$$=\frac{1+3z^2+3z^3+4z^4+4z^5+3z^6+3z^7+z^9}{(1-z)^2(1-z^2)(1-z^3)(1-z^4)(1-z^5)},$$

$$H(K[Y]^{\delta(2,1,1)};t)=\frac{1+3z^2+z^4}{(1-z)^3(1-z^2)(1-z^3)^2}.$$

Corollary 3.7. *For $d\le 7$ the algebra of constants $K[Y]^\delta$ of the Weitzenböck derivation $\delta=\delta(d_1,\ldots,d_k)$ is not a complete intersection for*

$$(d_1,\ldots,d_k)=(5),(3,1),(6),(4,1),(3,2),(2,1,1).$$

Proof. Using, as in [60], that the zeros of the nominator of the Hilbert series of a complete intersection are roots of unity (see [67]), the proof follows immediately from Example 3.6. (The case $(d_1,\ldots,d_k)=(5)$ was established in [60].) □

4. PI-algebras and noncommutative invariant theory. In this section we assume that all algebras are unital (and $\mathrm{char}(K)=0$). For a background on PI-algebras we refer, e.g., to [31]. Let $Y_\infty=\{y_1,y_2,\ldots\}$ and let $K\langle Y_\infty\rangle$ be the free associative algebra of countable rank freely generated by Y_∞. This is the algebra of polynomials in infinitely many noncommutative variables. Let $K\langle Y\rangle=K\langle y_1,\ldots,y_d\rangle$ be its subalgebra of rank d. Recall that $f(y_1,\ldots,y_m)\in K\langle Y_\infty\rangle$ is called a polynomial identity for the associative algebra R if $f(r_1,\ldots,r_m)=0$ for all $r_1,\ldots,r_m\in R$. If R satisfies a nonzero polynomial identity, it is called a *PI-algebra*. We denote by $T_\infty(R)$ the ideal of all polynomial identities of R (called the *T-ideal of* R) and

$$T(R) = K\langle Y\rangle \cap T_\infty(R)$$

is the T-ideal of the polynomial identities in d variables for R. Since we work over a field of characteristic 0, all polynomial identities of R follow from the multilinear ones. The vector space of the multilinear polynomials of degree n

$$P_n = \text{span}\{y_{\sigma(1)}\cdots y_{\sigma(n)} \mid \sigma \in S_n\} \subset K\langle Y_\infty\rangle$$

has a natural structure of a left S_n-module and the factor space

$$P_n(R) = P_n/(P_n \cap T_\infty(R))$$

is its S_n-factor module. One of the main problems in the quantitative study of PI-algebras is to compute the *cocharacter sequence of* R

$$\chi_n(R) = \chi_{S_n}(P_n(R)) = \sum_{\lambda\vdash n} m_\lambda(R)\chi_\lambda,$$

where χ_λ, $\lambda \vdash n$, is the irreducible S_n-character indexed by the partition λ. A possible way to compute the multiplicities $m_\lambda(R)$ is the following. One considers the diagonal GL_d-action on $K\langle Y\rangle$ extending the natural action of GL_d on the d-dimensional vector space KY with basis Y. Then the factor algebra

$$F(R) = K\langle Y\rangle/T(R)$$

called the *relatively free algebra of rank* d *in the variety of associative algebras generated by* R, inherits the GL_d-action of $K\langle Y\rangle$. Its Hilbert series as a GL_d-module coincides with its Hilbert series as a $\mathbb{Z}^d$-graded vector space with grading defined by

$$^\circ(y_i) = (\underbrace{0,\ldots,0}_{i-1 \text{ times}}, 1, \underbrace{0,\ldots,0}_{d-i \text{ times}}).$$

It is a symmetric function in d variables and

$$H(F(R);X) = \sum m_\lambda(R)S_\lambda(X),$$

where the sum is on all $(\lambda_1,\ldots,\lambda_d)$ and the multiplicities $m_\lambda(R)$ are exactly the same as in the cocharacter sequence of R. Hence, if we know the Hilbert series of $F(R)$, we can compute the multiplicities $m_\lambda(R)$ for partitions λ in $\leq d$

parts. The theorem of Belov [12] gives that for any PI-algebra R the Hilbert series of $F(R)$ is a rational function. Berele [15] found that the proof of Belov also implies that this Hilbert series is a nice rational symmetric function. Hence we can apply our methods to calculate the multiplicity series of $H(F(R);X)$ and to find the multiplicities of R. See the introduction of Boumova and Drensky [20] for a survey of results on the multiplicities of concrete algebras.

The most important algebras in PI-theory are the so called *T-prime algebras* whose T-ideals are the building blocks of the structure theory of T-ideals developed by Kemer, see his book [50] for the account. There are few cases only when the Hilbert series of the relatively free algebras $F(R)$ are explicitly known. For T-prime algebras these are the base filed K, the Grassmann (or exterior) algebra E, the 2×2 matrix algebra $M_2(K)$, and the algebra $M_{1,1}\subset M_2(E)$ which has the same polynomial identities as the tensor square $E\otimes_K E$ of the Grassmann algebra. In all these cases the multiplicities are also known. The case $R=K$ is trivial because $F(K)=K[Y]$:

$$m_\lambda(K)=\begin{cases}1, & \text{if } \lambda=(n),\\ 0, & \text{otherwise.}\end{cases}$$

The multiplicities for $M_2(K)$ were obtained by Formanek [42] and Drensky [29], see also [31]:

$$m_\lambda(M_2(K))=\begin{cases}0, \text{ if } \lambda_5>0,\\ 1, \text{ if } \lambda=(n),\\ (\lambda_1-\lambda_2+1)\lambda_2, \text{ if } \lambda=(\lambda_1,\lambda_2), \lambda_2>0,\\ \lambda_1(2-\lambda_4)-1, \text{ if } \lambda=(\lambda_1,1,1,\lambda_4),\\ (\lambda_1-\lambda_2+1)(\lambda_2-\lambda_3+1)(\lambda_3-\lambda_4+1) \text{ in all other cases.}\end{cases}$$

Hence the multiplicity series are

$$\begin{aligned} M(H(F(K));X) &= \frac{1}{1-x_1},\\ M'(H(F(K));V) &= \frac{1}{1-v_1};\\ M'(H(F(M_2(K)));V) &= \frac{1}{(1-v_1)^2(1-v_2)^2(1-v_3)^2(1-v_4)}\\ &\quad -\frac{v_2+v_1(1-v_2)}{(1-v_1)^2(1-v_2)}-\frac{v_3+v_4}{1-v_1}. \end{aligned}$$

For any d, there are partitions $\lambda = (\lambda_1, \dots, \lambda_d)$ with $\lambda_d > 0$ and nonzero multiplicities $m_\lambda(E)$ and $m_\lambda(E \otimes_K E)$. Hence for these cases the multiplicity series $M(H(E); X)$ and $M(H(E \otimes_K E); X)$ do not carry all the information about the cocharacter sequences of E and $E \otimes_K E$. For this purpose Berele [16] suggested to use hook Schur functions instead of ordinary ones.

Another case when the Hilbert series of the relatively free algebras may be computed and used to find the multiplicities is for algebras R with T-ideals which are products of two T-ideals, $T(R) = T(R_1)T(R_2)$. See again [20] for details. Formanek [43] found the following simple formula for the Hilbert series of $T(R)$ in terms of the Hilbert series of $T(R_1)$ and $T(R_2)$:

$$H(T(R)) = \frac{H(T(R_1))H(T(R_2))}{H(K\langle Y\rangle)} = (1 - (x_1 + \cdots + x_d))H(T(R_1))H(T(R_2)).$$

Translated for the corresponding relatively free algebras this gives

$$H(F(R)) = H(F(R_1)) + H(F(R_2)) + ((x_1 + \cdots + x_d) - 1)H(F(R_1))H(F(R_2)).$$

It is known that $T(U_m(K)) = T^m(K)$ (Maltsev [56]) and $T(U_m(E)) = T^m(E)$ (this follows from the results of Abakarov [1]), where $U_k(K)$ and $U_k(E)$ are the algebras of $k \times k$ upper triangular matrices with entries from K and E, respectively. The multiplicities of $U_k(K)$ were studied by Boumova and Drensky [20], with explicit results for "large" partitions $\lambda = (\lambda_1, \dots, \lambda_n)$ (such that $\lambda_{k+1} + \cdots + \lambda_n = k - 1$). The multiplicities of $U_2(E)$ were determined by Centrone [21]. In both cases the results were obtained using the Young rule only, without the MacMahon partition analysis. Here we shall illustrate once again Algorithm 1.3.

Example 4.1. Let C be the commutator ideal of the free associative algebra $K\langle Y\rangle$. Then by Maltsev [56] the T-ideal C^k coincides with the T-ideal of $U_k(K)$. Since $K\langle Y\rangle/C$ is the polynomial algebra in d variables and

$$H(K[Y]; X) = \prod_{i=1}^{d} \frac{1}{1 - x_i} = \sum_{n \geq 0} S_{(n)}(X),$$

the formula

$$H(F(R)) = H(F(R_1)) + H(F(R_2)) + ((x_1 + \cdots + x_d) - 1)H(F(R_1))H(F(R_2)).$$

for the Hilbert series of relatively free algebras corresponding to products of T-ideals gives

$$H(F(U_2(K)); X) = 2\prod_{i=1}^{d} \frac{1}{1 - x_i} + ((x_1 + \cdots + x_d) - 1)\prod_{i=1}^{d} \frac{1}{(1 - x_i)^2}.$$

The decomposition of the product $S_\mu(X)S_{(n)}(X) = \sum S_\lambda(X)$ is given by the Young rule. If μ is a partition in k parts, then λ is a partition in k or $k+1$ parts. Hence $H(F(U_2(K));X)$ decomposes into a series of Schur functions $S_\lambda(X)$, where λ is a partition in no more than three parts. Therefore, it is sufficient to consider the multiplicity series of $H(F(U_2(K));X)$ for $d=3$ only. Clearly,

$$M'(H(K[Y]);V) = M'\left(\prod_{i=1}^{d}\frac{1}{1-x_i};V\right) = \frac{1}{1-v_1}.$$

Algorithm 1.3 gives

$$g_1(x_1,x_2,x_3) = \frac{(x_1-x_2)(x_1-x_3)(x_2-x_3)((x_1+x_2+x_3)-1)}{(1-x_1)^2(1-x_2)^2(1-x_3)^2},$$

$$g_1(x_1z_1,x_2/z_1,x_3) = \frac{-x_3+x_1x_2-2x_1x_2x_3-x_1x_3z_1}{(1-x_3)^2}$$

$$+\frac{x_3+x_3^2-x_3^3-x_1x_2+x_1x_2x_3-x_1^2x_2^2x_3}{(1-x_1x_2)(1-x_3)^2(1-x_1z_1)}+\frac{-x_3^2+x_1^2x_2^2}{(1-x_1x_2)(1-x_3)(1-x_1z_1)^2}$$

$$+\frac{x_2x_3}{(1-x_3)^2z_1}+\frac{(x_3-x_3^2+x_3^3-x_1x_2+x_1x_2x_3+2x_1^2x_2^2-3x_1^2x_2^2x_3)x_2}{(1-x_1x_2)(1-x_3)^2(x_2-z_1)}$$

$$+\frac{(x_3^2-x_1^2x_2^2)x_2^2}{(1-x_1x_2)(1-x_3)(x_2-z_1)^2}.$$

We omit the last three summands which give negative degrees of z_1 in the expansion of $g_1(x_1z_1,x_2/z_1,x_3)$ as a Laurent series and, substituting $z_1=1$, we obtain

$$g_2(x_1,x_2,x_3) = \frac{-x_3+x_1x_2-2x_1x_2x_3-x_1x_3}{(1-x_3)^2}$$

$$+\frac{x_3+x_3^2-x_3^3-x_1x_2+x_1x_2x_3-x_1^2x_2^2x_3}{(1-x_1)(1-x_1x_2)(1-x_3)^2}+\frac{-x_3^2+x_1^2x_2^2}{(1-x_1)^2(1-x_1x_2)(1-x_3)}.$$

Repeating the procedure with $g_2(x_1,x_2z_2,x_3/z_2)$ we obtain

$$\underset{\geq}{\Omega}(g_1(x_1z_1,x_2z_2/z_1,x_3/z_2)) = \frac{x_1^2x_2(-1+x_1+2x_1x_2-x_1^2x_2+x_1x_2x_3)}{(1-x_1)^2(1-x_1x_2)},$$

$$M'\left(\frac{(x_1+x_2+x_3)-1}{(1-x_1)^2(1-x_2)^2(1-x_3)^2};V\right) = \frac{-1+v_1+2v_2-v_1v_2+v_3}{(1-v_1)^2(1-v_2)}$$

$$= -\frac{1}{1-v_1}+\frac{v_2+v_3}{(1-v_1)(1-v_2)},$$

$$M'(H(F(U_2(K));V) = \frac{1}{1-v_1} + \frac{v_2+v_3}{(1-v_1)^(1-v_2)}$$

$$= \sum_{n\geq 0} v_1^n + \sum_{p\geq 0}\sum_{q\geq 1}(p+1)v_1^p v_2^q + \sum_{p\geq 0}\sum_{q\geq 0}(p+1)v_1^p v_2^q v_3.$$

Hence the multiplicities in the cocharacter sequence of $U_2(K)$ are

$$m_\lambda(U_2(K)) = \begin{cases} 1, \text{ if } \lambda = (\lambda_1), \\ \lambda_1 - \lambda_2 + 1, \text{ if } \lambda = (\lambda_1, \lambda_2), \lambda_2 > 0, \\ \lambda_1 - \lambda_2 + 1, \text{ if } \lambda = (\lambda_1, \lambda_2, 1), \\ 0 \text{ in all other cases.} \end{cases}$$

Compare our approach with the approach on the multiplicities of $U_2(K)$ given by Mishchenko, Regev, and Zaicev [57] and in [20].

A case of products of T-ideals when we do need the MacMahon partition analysis is of block triangular matrices with entries from the field. Let $d_1, \ldots, d_m$ be positive integers and let $U(d_1, \ldots, d_m)$ be the algebra of matrices of the form

$$\begin{pmatrix} M_{d_1}(K) & * & \ldots & * & * \\ 0 & M_{d_2}(K) & \ldots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \ldots & M_{d_{m-1}}(K) & * \\ 0 & 0 & \ldots & 0 & M_{d_m}(K) \end{pmatrix}.$$

It is known, see Giambruno and Zaicev [47], that

$$T(U(d_1, \ldots, d_m)) = T(M_{d_1}(K)) \cdots T(M_{d_m}(K)).$$

The only cases when we know the Hilbert series of $T(M_k(K))$ are $k = 1, 2$, and we can compute the Hilbert series of $F(U(d_1, \ldots, d_m))$. The multiplicities of $U(d_1, \ldots, d_m)$ when all d_i are equal to 1 and 2 were studied in the master thesis of Kostadinov [52], see also his paper with Drensky [39]. If $d_1 = \cdots = d_m = 1$, the algebra $U(d_1, \ldots, d_m)$ is equal to $U_k(K)$, handled in [20]. If only one d_i is equal to 2 and the others are equal to 1, we still can use the Young rule. The MacMahon partition analysis was applied in [52] in the case when several d_i are equal to 2. In particular exact formulas for the multiplicity series and the multiplicities as well as the asymptotics of the multiplicities were found for a small number of blocks.

Studying the polynomial identities of the matrix algebra $M_k(K)$, there is another object which behaves much better than the relatively free algebra $F(M_k(K))$. Let

$$K[Z] = K[z_{pq}^{(i)} \mid p, q = 1, \ldots, k, i = 1, \ldots, d]$$

be the polynomial algebra in $k^2 d$ commuting variables and let R_{kd} be *the generic matrix algebra* generated by the d generic $k \times k$ matrices

$$z_i = \begin{pmatrix} z_{11}^{(i)} & \cdots & z_{1k}^{(i)} \\ \vdots & \ddots & \vdots \\ z_{k1}^{(i)} & \cdots & z_{kk}^{(i)} \end{pmatrix}, \quad i = 1, \ldots, d.$$

It is well known that $R_{kd} \cong F(M_k(K))$. Let C_{kd} be *the pure* (or *commutative*) *trace algebra* generated by all traces of products $\mathrm{tr}(z_{i_1} \cdots z_{i_n})$, $i_1, \ldots, i_n = 1, \ldots, d$. It coincides with the algebra of invariants $K[Z]^{GL_k}$ where the action of GL_k on $K[Z]$ is induced by the action of GL_k on the generic matrices $z_1, \ldots, z_d$ by simultaneous conjugation. Hence one may study C_{kd} with methods of the classical invariant theory. *The mixed* (or *noncommutative*) *trace algebra* $T_{kd} = C_{kd}R_{kd}$ also has a meaning in invariant theory. See the books by Formanek [45], and also with Drensky [33], for a background on trace algebras. The mixed trace algebra approximates quite well the algebra $F(M_k(K))$. In particular, one may consider the multilinear components of the pure and mixed trace algebras $C_k = C_{k,\infty}$ and $T_k = T_{k,\infty}$ of infinitely many generic $k \times k$ matrices and the related sequences

$$\chi_n(C_k) = \sum_{\lambda \vdash n} m_\lambda(C_k)\chi_\lambda, \quad \chi_n(T_k) = \sum_{\lambda \vdash n} m_\lambda(T_k)\chi_\lambda, \quad n = 0, 1, 2, \ldots,$$

of S_n-characters called *the pure and mixed cocharacter sequences*, respectively. Formanek [44] showed that the multiplicities $m_\lambda(T_k)$ in the mixed cocharacter sequence and $m_\lambda(M_k(K))$ in the ordinary cocharacter sequence for $M_k(K)$ coincide for all partitions $\lambda = (\lambda_1, \ldots, \lambda_{k^2})$ with $\lambda_{k^2} \geq 2$. The only case when the pure and mixed cocharacter sequences are known is for $n = 2$ due to Procesi [61] and Formanek [42] (besides the trivial case of $k = 1$). We state the result for T_2 only.

$$m_\lambda(T_2) = \begin{cases} (\lambda_1 - \lambda_2 + 1)(\lambda_2 - \lambda_3 + 1)(\lambda_3 - \lambda_4 + 1), & \text{if } \lambda = (\lambda_1, \lambda_2, \lambda_3, \lambda_4), \\ 0 \text{ otherwise}, \end{cases}$$

The situation with the Hilbert series of C_{kd} and T_{kd} is better. The case $k = 2$ was handled by Procesi [61] and Formanek [42]:

$$\begin{aligned} H(C_{2d};X) &= \prod_{i=1}^{d} \frac{1}{1-x_i} \sum_{p,q,r\geq 0} S_{(2p+2q+r,2q+r,r)}(X), \\ H(T_{2d};X) &= \prod_{i=1}^{d} \frac{1}{1-x_i} \sum_{(\lambda_1,\lambda_2,\lambda_3)} S_{(\lambda_1,\lambda_2,\lambda_3)}(X) \\ &= \prod_{i=1}^{d} \frac{1}{(1-x_i)^2} \sum_{n\geq 0} S_{(n,n)}(X). \end{aligned}$$

The Molien–Weyl formula gives that the Hilbert series of C_{kd} and T_{kd} can be expressed as multiple integrals but for $k \geq 3$ their direct evaluation is quite difficult and was performed by Teranishi [70, 71] for C_{32} and C_{42} only. Van den Bergh [73] found a graph-theoretical approach for the calculation of $H(C_{kd})$ and $H(T_{kd})$. Berele and Stembridge [18] calculated the Hilbert series of C_{kd} and T_{kd} for $k = 3$, $d \leq 3$ and of T_{42}, correcting also some typographical errors in the expression of $H(C_{42})$ in [71]. Recently Djoković [27] computed the Hilbert series of C_{k2} and T_{k2} for $k = 5$ and 6.

Using the Hilbert series of C_{32}, Berele [14] found an asymptotic expression of $m_{(\lambda_1,\lambda_2)}(C_3)$. The explicit form of multiplicity series of the Hilbert series of C_{32} was found by Drensky and Genov [34] correcting also a minor technical error (a missing summand) in [14]. The quite technical method of [34] was improved in [35] and applied by Drensky, Genov and Valenti [37] to compute the multiplicity series of $H(T_{32})$ and by Drensky and Genov [36] for the multiplicity series of $H(C_{42})$ and $H(T_{42})$. In principle, the same methods work for the multiplicities of the Hilbert series of $H(C_{k2})$ and $H(T_{k2})$, $k = 5, 6$.

Example 4.2. We shall apply the MacMahon partition analysis to find the multiplicities of $H(T_{32})$. By Berele and Stembridge [18]

$$H(T_{32}, x_1, x_2) = \frac{1}{(1-x_1)^2(1-x_2)^2(1-x_1^2)(1-x_2^2)(1-x_1x_2)^2(1-x_1^2x_2)(1-x_1x_2^2)}.$$

As in Example 2.2 we define the function

$$g(x_1, x_2) = (x_1 - x_2)H(T_{32}; x_1, x_2)$$

and decompose $g(x_1z, x_2/z)$ as a sum of partial fractions with respect to z. The result is

$$\frac{1}{2(1-x_1x_2)^6(1+x_1x_2)^2(1-x_1z)^3} + \frac{1+2x_1x_2-5x_1^2x_2^2}{4(1-x_1x_2)^7(1+x_1x_2)^3(1-x_1z)^2}$$

$$-\frac{1+2x_1x_2-10x_1^2x_2^2+10x_1^3x_2^3-7x_1^4x_2^4}{8(1-x_1x_2)^8(1+x_1x_2)^4(1-x_1z)}$$

$$-\frac{1}{8(1-x_1x_2)^2(1+x_1x_2)^4(1+x_1^2x_2^2)(1+x_1z)}$$

$$-\frac{x_1^3x_2^3}{(1-x_1x_2)^8(1+x_1x_2)^4(1+x_1^2x_2^2)(1-x_1^2x_2z)}+\frac{x_2^3}{2(1-x_1x_2)^6(1+x_1x_2)^2(x_2-z)^3}$$

$$+\frac{x_2^2(-5+2x_1x_2+x_1^2x_2^2)}{4(1-x_1x_2)^7(1+x_1x_2)^3(x_2-z)^2}+\frac{x_2(7-10x_1x_2+10x_1^2x_2^2-2x_1^3x_2^3-x_1^4x_2^4)}{8(1-x_1x_2)^8(1+x_1x_2)^4(x_2-z)}$$

$$-\frac{x_2}{8(1-x_1x_2)^2(1+x_1^2x_2^2)(1+x_1x_2)^4(x_2+z)}$$

$$-\frac{x_1^4x_2^5}{(1-x_1x_2)^8(1+x_1x_2)^4(1+x_1^2x_2^2)(x_1x_2^2-z)}.$$

We remove the last five summands because their expansions as Laurent series contain negative degrees of z only. Then we replace z by 1 and obtain

$$\underset{\geq}{\Omega}(g(x_1z,x_2/z))=\frac{1}{2(1-x_1x_2)^6(1+x_1x_2)^2(1-x_1)^3}$$

$$+\frac{1+2x_1x_2-5x_1^2x_2^2}{4(1-x_1x_2)^7(1+x_1x_2)^3(1-x_1)^2}-\frac{1+2x_1x_2-10x_1^2x_2^2+10x_1^3x_2^3-7x_1^4x_2^4}{8(1-x_1x_2)^8(1+x_1x_2)^4(1-x_1)}$$

$$-\frac{1}{8(1-x_1x_2)^2(1+x_1x_2)^4(1+x_1^2x_2^2)(1+x_1)}$$

$$-\frac{x_1^3x_2^3}{(1-x_1x_2)^8(1+x_1x_2)^4(1+x_1^2x_2^2)(1-x_1^2x_2)}.$$

Dividing $\underset{\geq}{\Omega}(g(x_1z,x_2/z))$ by x_1 and after the substitution $v_1=x_1$, $v_2=x_1x_2$ we obtain

$$M'(H(T_{32});v_1,v_2)=\frac{v_1^3h_3(v_2)+v_1^2h_2(v_2)+v_1h_1(v_2)+h_0(v_2)}{(1-v_1)^3(1+v_1)(1-v_1v_2)(1-v_2)^7(1+v_2)^4(1+v_2^2)},$$

$$h_3(v_2)=v_2^2(v_2^4-v_2^3+3v_2^2-v_2+1),\quad h_2(v_2)=v_2(2v_2^4-4v_2^3+v_2^2-v_2-1),$$

$$h_1(v_2)=v_2(-v_2^4-v_2^3+v_2^2-4v_2+2),\quad h_0(v_2)=v_2^4-v_2^3+3v_2^2-v_2+1,$$

which coincides with the result of [37]. The multiplicity series has also the form

$$M'(H(T_{32});v_1,v_2)=\frac{a_3(v_2)}{(1-v_1)^3}+\frac{a_2(v_2)}{(1-v_1)^2}+\frac{a_1(v_2)}{(1-v_1)}+\frac{b(v_2)}{1+v_1}+\frac{c(v_2)}{1-v_1v_2},$$

where

$$a_3(v_2) = \frac{1}{2(1-v_2)^6(1+v_2)^2}, \quad a_2(v_2) = \frac{(3v_2^2 - 2v_2 + 1)}{2^2(1-v_2)^7(1+v_2)^3},$$

$$a_1(v_2) = \frac{(v_2^4 - 6v_2^3 + 14v_2^2 - 6v_2 + 1)}{2^3(1-v_2)^8(1+v_2)^4},$$

$$b(v_2) = \frac{1}{2^3(1-v_2)^2(1+v_2)^4(1+v_2^2)}, \quad c(v_2) = \frac{-v_2^4}{(1-v_2)^8(1+v_2)^4(1+v_2^2)}.$$

In a forthcoming paper by Benanti, Boumova and Drensky [13] we shall apply our methods to find the multiplicities in the pure and mixed cocharacter sequence of three generic 3×3 matrices.

One of the directions of noncommutative invariant theory is to study subalgebras of invariants of linear groups acting on free and relatively free algebras. For a background see the surveys by Formanek [43] and Drensky [30]. Recall that we consider the action of GL_d on the vector space KY with basis $Y = \{y_1, \ldots, y_d\}$ and extend this action diagonally on the free algebra $K\langle Y\rangle$ and the relatively free algebras $F(R)$, where R is a PI-algebra. Let G be a subgroup of GL_d. Then G acts on $F(R)$ and the algebra of G-invariants is

$$F(R)^G = \{f(Y) \in F(R) \mid g(f) = f \text{ for all } g \in G\}.$$

Comparing with commutative invariant theory, when $K[Y]^G$ is finitely generated for all "nice" groups (e.g., finite and reductive), the main difference in the noncommutative case is that $F(R)^G$ is finitely generated quite rarely. For a survey on invariants of finite groups G see [30, 43] and the survey by Kharlampovich and Sapir [51]. For a fixed PI-algebra R there are many conditions which are equivalent to the fact that the algebra $F(R)^G$ is finitely generated for all finite groups G. Maybe the simplest one is that this happens if and only if $F(R)^G$ is finitely generated for $d = 2$ and the cyclic group $G = \langle g\rangle$ of order 2 generated by the matrix

$$g = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Clearly, in this case $F(R)^{\langle g\rangle}$ is spanned on all monomials in y_1, y_2 which are of even degree with respect to y_1. For the algebras of invariants $F(R)^G$ of reductive groups G see Vonessen [74] and Domokos and Drensky [28].

Concerning the Hilbert series of $F(R)^G$, for G finite there is an analogue of the Molien formula, see Formanek [43]: If $\xi_1(g), \dots, \xi_d(g)$ are the eigenvalues of $g \in G$, then the Hilbert series of the algebra of invariants $F(R)^G$ is

$$H(F(R)^G; t) = \frac{1}{|G|} \sum_{g \in G} H(F(R); \xi_1(g)t, \dots, \xi_d(g)t).$$

Combined with the theorem of Belov [12] for the rationality of $H(F(R); X)$ (as specified by Berele [15]) this gives that the Hilbert series of $F(R)^G$ is a nice rational function for every finite group G. By a result of Domokos and Drensky [28] the Hilbert series of $F(R)^G$ for a reductive group G is a nice rational function if R satisfies a nonmatrix polynomial identity (i.e., an identity which does not hold for the algebra $M_2(K)$ of 2×2 matrices). The proof uses that for algebras R with nonmatrix identity the relatively free algebra has a finite series of graded ideals with factors which are finitely generated modules of polynomial algebras. This allows to reduce the considerations to the commutative case when the rationality of the Hilbert series is well known. We believe that the careful study of the proof of Belov [12] would give that $H(F(R)^G; t)$ is a nice rational function for every reductive group G and an arbitrary PI-algebra R.

Let W be a p-dimensional GL_d-module with basis $Y_p = \{y_1, \dots, y_p\}$. Consider the related representation $\rho : GL_d \to GL_p$ of GL_d in the p-dimensional vector space with this basis. If $F_p(R)$ is a relatively free algebra of rank p freely generated by Y_p, then the representation ρ induces an action of GL_d on $F_p(R)$. The following theorem is a noncommutative analogue of Theorem 3.2.

Theorem 4.3. *Let W be a p-dimensional polynomial GL_d-module with Hilbert series with respect to the grading induced by the GL_d-action on W*

$$H(W; X) = \sum a_i x_1^{i_1} \cdots x_d^{i_d}, \quad a_i \geq 0, a_i \in \mathbb{Z}, \sum a_i = p.$$

Let R be a PI-algebra with the corresponding relatively free algebra $F_p(R)$ of rank p freely generated by Y_p, with the natural structure of a GL_d-module induced by the GL_d-action on W. Let

$$f(X, t) = H(F_p(R); X^{i^{(1)}} t, \dots, X^{i^{(p)}} t)$$

be the formal power series obtained from the Hilbert series $H(F_p(R); x_1, \dots, x_p)$ of $F_p(R)$ by substitution of the variables x_j with $x_1^{i_1} \cdots x_d^{i_d} t$ in such a way that each $x_1^{i_1} \cdots x_d^{i_d} t$ appears exactly a_i times. Then the Hilbert series of the algebras $F_p(R)^{SL_d}$ and $F_p(R)^{UT_d}$ of SL_d- and UT_d-invariants are, respectively,

$$H(F_p(R)^{SL_d}; t) = M'(f; 0, \dots, 0, 1, t),$$

$$H(F_p(R)^{UT_d};t) = M(f;1,\dots,1,t) = M'(f;1,\dots,1,t),$$

where $M(f;X,t)$ and $M'(f;V,t)$ are the multiplicity series of the symmetric in X function $f(X,t)$.

Proof. We may choose the basis Y_p of W to consist of eigenvectors of the diagonal group D_d. Then for a fixed d-tuple $i=(i_1,\dots,i_d)$ exactly a_i of the elements y_j satisfy

$$g(y_j) = \xi_1^{i_1}\cdots\xi_d^{i_d}y_j,\quad g=\mathrm{diag}(\xi_1,\dots,\xi_d)\in D_d.$$

The monomials in $y_1,\dots,y_p$ are eigenvectors of D_d and $H(F_p(R);X^{i^{(1)}}t,\dots, X^{i^{(p)}}t)$ is the Hilbert series of the GL_d-module $F_p(R)$ which counts also the $\mathbb{Z}$-grading of $F_p(R)$. Now the proof is completed as the proof of Theorem 3.2 because the irreducible GL_d-submodule $W(\lambda)$ contains a one-dimensional SL_d-invariant if $\lambda_1=\cdots=\lambda_d$ and does not contain any GL_d-invariants otherwise. Similarly, $W(\lambda)$ contains a one-dimensional UT_d-invariant for every λ. □

Combined with the nice rationality of the Hilbert series of relatively free algebras Theorem 4.3 immediately gives:

Corollary 4.4. *Let W be a p-dimensional polynomial GL_d-module with basis $Y_p=\{y_1,\dots,y_p\}$ and let $F_p(R)$ be the relatively free algebra freely generated by Y_p and related to the PI-algebra R. Then the Hilbert series of the algebras of invariants $H(F_p(R)^{SL_d};t)$ and $H(F_p(R)^{UT_d};t)$ are nice rational functions.*

Example 4.5. We shall apply Theorem 4.3 to Example 4.1. Let $d\geq 2$ and let SL_d and UT_d act as subgroups of GL_d on the relatively free algebra $F(U_2(K))$ with d generators. Then the generators $y_i\in Y$ of $F(U_2(K))$ are of first degree with respect to the GL_d-action. Hence

$$\begin{aligned}
H(F(U_2(K));X,t) &= H(F(U_2(K));Xt) = H(F(U_2(K));x_1t,\dots,x_dt),\\
M'(H(F(U_2(K));X,t);V,t) &= M'(H(F(U_2(K));X,t);v_1t,v_2t^2,\dots,v_dt^d)\\
&= \frac{1}{1-v_1t}+\frac{v_2t^2+v_3t^3}{(1-v_1t)^2(1-v_2)}.
\end{aligned}$$

And therefore

$$\begin{aligned} H(F(U_2(K))^{SL_d};t) &= M'(H(F(U_2(K));X,t);0,\dots,0,t^d) \\ &= \begin{cases} \dfrac{1}{1-t^2}, & \text{if } d=2, \\ 1+t^3, & \text{if } d=3, \\ 1, & \text{if } d>3; \end{cases} \\ H(F(U_2(K))^{UT_d};t) &= M'(H(F(U_2(K));X,t);t,t^2,\dots,t^d) \\ &= \begin{cases} \dfrac{1-t+t^3}{(1-t)^2(1-t^2)}, & \text{if } d=2, \\ \\ \dfrac{1-2t+2t^2}{(1-t)^3}, & \text{if } d\geq 3. \end{cases} \end{aligned}$$

Example 4.6. Again, let $R = U_2(K)$ and let $W = W(1^2)$ be the irreducible GL_3-module indexed by the partition $(1^2) = (1,1,0)$. We consider the relatively free algebra $F_3(U_2(K))$ with the GL_3-action induced by the action on W. The Hilbert series of $F_3(U_2(K))$ which counts both the action of GL_3 and the $\mathbb{Z}$-grading is

$$\begin{aligned} f(X,t) &= H(F_3(U_2(K));x_1x_2t,x_2x_3t,x_2x_3t) \\ &= \frac{1}{(1-x_1x_2t)(1-x_1x_3t)(1-x_2x_3t)} \\ &\quad + \frac{x_1x_2+x_1x_3+x_2x_3-1}{(1-x_1x_2t)^2(1-x_1x_3t)^2(1-x_2x_3t)^2}. \end{aligned}$$

Applying Theorem 4.3 and Algorithm 1.3 we obtain

$$M'(f;V,t) = \frac{1-v_2t+(v_1v_2+v_3)v_3t^3}{(1-v_2t)^2(1-v_1v_3t^2)}.$$

Hence

$$\begin{aligned} H(F_3(U_2(K))^{SL_3};t) &= M'(f;0,0,1,t) = 1+t^3, \\ H(F_3(U_2(K))^{UT_3};t) &= M'(f;1,1,1,t) = \frac{1-2t+2t^2}{(1-t)^3}. \end{aligned}$$

Similarly, if we consider the GL_3-module $W = W(2)$, then GL_3 acts on $F_6(U_2(K))$ extending the action on W,

$$f(X,t) = H(F_6(U_2(K));x_1^2t,x_2^2t,x_3^2t,x_1x_2t,x_2x_3t,x_2x_3t)$$

and, applying again Theorem 4.3 and Algorithm 1.3, we obtain

$$\begin{aligned}
H(F_6(U_2(K))^{ST_3};t) &= \frac{1-3t^3+6t^6-2t^9}{(1-t^3)^4},\\
H(F_6(U_2(K))^{UT_3};t) &= \frac{p(t)}{((1-t)(1-t^2)(1-t^3))^3},
\end{aligned}$$

where

$$p(t) = 1-2t+7t^3+11t^4+6t^5-10t^6+t^7+6t^8+4t^9-2t^{10}-4t^{11}+2t^{12}.$$

Example 4.7. Let R_{2p} be the algebra generated by p generic 2×2 matrices $z_1,\dots,z_p$ with the canonical GL_p-action. We extend the action of the pure and mixed trace algebras by

$$g(\mathrm{tr}(z_{i_1}\cdots z_{i_n})z_{j_1}\cdots z_{j_m}) = \mathrm{tr}(g(z_{i_1}\cdots z_{i_n}))g(z_{j_1}\cdots z_{j_m}),$$

$z_{i_1}\cdots z_{i_n}, z_{j_1}\cdots z_{j_m} \in R_{2p}$, $g\in GL_p$. For a p-dimensional GL_d-module W, we consider the induced GL_d-action on R_{2p}, C_{2p} and T_{2p}. Let $d=2$. Then $W = W(2)\oplus W(0)$ is a 4-dimensional GL_2-module with Hilbert series

$$H(W;x_1,x_2) = x_1^2+x_1x_2+x_2^2+1.$$

The Hilbert series of T_{24} is

$$\begin{aligned}
H(T_{24};x_1,x_2,x_3,x_4) &= \prod_{i=1}^4\frac{1}{(1-x_i)^2}\sum_{n\geq 0}S_{(n,n)}(X)\\
&= (1-x_1x_2x_3x_4)\prod_{i=1}^4\frac{1}{(1-x_i)^2}\prod_{1\leq i<j\leq 4}\frac{1}{1-x_ix_j}.
\end{aligned}$$

Hence the Hilbert series of the $\mathbb{Z}$-graded GL_2-module T_{24} is

$$\begin{aligned}
f(x_1,x_2,t) &= H(T_{24};x_1^2t,x_1x_2t,x_2^2t,t)\\
&= \frac{1-x_1^3x_2^3t^4}{((1-t)(1-x_1^2t)(1-x_1x_2t)(1-x_2^2t))^2}\\
&\quad\times\frac{1}{(1-x_1^2t^2)(1-x_1x_2t^2)(1-x_2^2t^2)(1-x_1^3x_2t^2)(1-x_1^2x_2^2t^2)(1-x_1x_2^3t^2)}.
\end{aligned}$$

Computing the multiplicity series $M'(f;v_1,v_2,t)$ and replacing (v_1,v_2) with $(0,1)$ and $(1,1)$ we obtain, respectively, the Hilbert series of $T_{24}^{SL_2}$ and $T_{24}^{UT_2}$:

$$H(T_{24}^{SL_2};t) = \frac{1-t+t^2+2t^4+t^6-t^7+t^8}{(1-t)^3(1-t^2)^2(1-t^3)^3(1-t^4)^2},$$

$$H(T_{24}^{UT_2};t) = \frac{(1-t+t^2)(1+3t^2+4t^3+6t^4+4t^5+3t^6+t^8)}{(1-t)^5(1-t^2)^2(1-t^3)^3(1-t^4)^2}.$$

By considering the three-dimensional GL_2-module $W(2)$ and the induced GL_2-action on T_{23}, we obtain

$$H(T_{23};x_1,x_2,x_3) = \frac{1}{(1-x_1)^2(1-x_2)^2(1-x_3)^2(1-x_1x_2)(1-x_1x_3)(1-x_2x_3)},$$

$$f(x_1,x_2,t) = H(T_{23};x_1^2t,x_1x_2t,x_2^2t)$$

$$= \frac{1}{((1-x_1^2t)(1-x_1x_2t)(1-x_2^2t))^2(1-x_1^3x_2t^2)(1-x_1^2x_2^2t^2)(1-x_1x_2^3t^2)},$$

$$H(T_{23}^{SL_2};t) = \frac{1+t^4}{(1-t^2)^3(1-t^3)^2(1-t^4)},$$

$$H(T_{23}^{UT_2};t) = \frac{1+2t^2+2t^3+2t^4+t^6}{(1-t)^2(1-t^2)^3(1-t^3)^2(1-t^4)}.$$

As in the commutative case $K[Y]$ one may consider linear locally nilpotent derivations of the free algebra $K\langle Y\rangle$ and of any relatively free algebra $F(R)$. Again, we call such derivations Weitzenböck derivations. There is a very simple condition when the algebra of constants $F(R)^\delta$ of a nonzero Weitzenböck derivation δ is finitely generated. By a result of Drensky and Gupta [38] if the T-ideal $T(R)$ of the polynomial identities of R is contained in the T-ideal $T(U_2(K))$, then $F(R)^\delta$ is not finitely generated. The main result of Drensky [32] states that if $T(R)$ is not contained in $T(U_2(K))$, then $F(R)^\delta$ is finitely generated. For various properties and applications of Weitzenböck derivations acting on free and relatively free algebras see [38]. The following theorem and its corollary combine Theorems 3.5 and 4.3. We omit the proofs which repeat the main steps of the proofs of these two theorems.

Theorem 4.8. *Let δ be a Weitzenböck derivation of the relatively free algebra $F(R)$ with Jordan normal form consisting of k cells of size $d_1+1,\dots,d_k+1$, respectively. Let*

$$f_\delta(x_1,x_2,t) = H(F(R);x_1^{d_1}t,x_1^{d_1-1}x_2t,\dots,x_2^{d_1}t,\dots,x_1^{d_k}t,\dots,x_1x_2^{d_k-1}t,x_2^{d_k}t)$$

be the function obtained from the Hilbert series of $F(R)$ by substitution of the first group of d_1+1 variables $x_1,x_2,\dots,x_{d_1+1}$ with $x_1^{d_1}t, x_1^{d_1-1}x_2t,\dots,x_2^{d_k}t$, the second group of d_2+1 variables $x_{d_1+2},x_{d_1+3},\dots,x_{d_1+d_2+2}$ with $x_1^{d_2}t,x_1^{d_2-1}x_2t,\dots,x_2^{d_2}t$, $\dots$, the k-th group of d_k+1 variables $x_{d-d_k},\dots,x_{d-1},x_d$ with $x_1^{d_k}t,\dots,x_1x_2^{d_k-1}t,x_2^{d_k}t$. Then the Hilbert series of the algebra of constants $F(R)^\delta$ is given by

$$H(F(R)^\delta;t)=M(f_\delta;1,1),$$

where $M(f_\delta;x_1,x_2)$ is the multiplicity series of the symmetric with respect to x_1,x_2 function $f_\delta(x_1,x_2,t)\in K(t)[[x_1,x_2]]^{S_2}$. Hence the Hilbert series $H(F(R)^\delta;t)$ is a nice rational function.

Corollary 4.9. *Let δ be a Weitzenböck derivation of the relatively free algebra $F(R)$ with Jordan normal form consisting of k cells of size $d_1+1,\dots,d_k+1$, respectively. Let us identify the vector space KY spanned by the free generators of $F(R)$ with the GL_2-module*

$$W=W(d_1)\oplus\cdots\oplus W(d_k).$$

Then the Hilbert series $H(F(R)^\delta;t)$ and $H(F(R)^{UT_2};t)$ of the algebras of constants $F(R)^\delta$ and of UT_2-invariants coincide.

Example 4.10. By Example 4.1 the Hilbert series of the relatively free algebra $F(U_2(K))$ is

$$H(F(U_2(K));X)=2\prod_{i=1}^{d}\frac{1}{1-x_i}+((x_1+\cdots+x_d)-1)\prod_{i=1}^{d}\frac{1}{(1-x_i)^2}.$$

Let $d=3$ and let δ be the Weitzenböck derivation with one three-dimensional cell acting on $F(U_2(K))$. Following the procedure of Theorem 4.8, we define the function

$$f(x_1,x_2,t)=\frac{2}{(1-x_1^2t)(1-x_1x_2t)(1-x_2^2t)}+\frac{(x_1^2+x_1x_2+x_2^2)t-1}{(1-x_1^2t)^2(1-x_1x_2t)^2(1-x_2^2t)^2}.$$

As in Example 4.6 we compute

$$M'(f;v_1,v_2)=\frac{1-(v_1^2+v_2)t+(2v_1^2-v_2)v_2t^2+2(v_1^2+v_2)v_2^2t^3-2v_1^2v_2^3t^4}{(1-v_1^2t)^2(1-v_2t)(1-v_2^2t^2)^2},$$

$$H(F_3(U_2(K))^\delta;t)=M'(f;1,1)=\frac{1-2t+t^2+4t^3-2t^4}{(1-t)^3(1-t^2)^2}.$$

If $d = 4$ and δ is a Weitzenböck derivation with two 2×2 cells, then

$$f(x_1, x_2, t) = \frac{2}{(1 - x_1 t)^2 (1 - x_2 t)^2} + \frac{2(x_1 + x_2)t - 1}{(1 - x_1 t)^4 (1 - x_2 t)^4},$$

$$H(F_4(U_2(K))^\delta; t) = \frac{1 + 10t^3 + 23t^4 + 2t^5 - 8t^6 + 2t^8}{(1 - t)^2 (1 - t^2)^5}.$$

Example 4.11. As in the case of invariants we can extend the derivations of the generic trace algebra R_{kp} to the pure and mixed trace algebras C_{kp} and T_{kp}. Let δ_{20} be the Weitzenböck derivation with a three-dimensional and a one-dimensional Jordan cell acting on the mixed trace algebra T_{23}. Then Corollary 4.9 and Example 4.7 give that

$$H(T_{24}^\delta; t) = H(T_{24}^{UT_2}; t) = \frac{(1 - t + t^2)(1 + 3t^2 + 4t^3 + 6t^4 + 4t^5 + 3t^6 + t^8)}{(1 - t)^5 (1 - t^2)^2 (1 - t^3)^3 (1 - t^4)^2}.$$

If δ has one three-dimensional cell only, then again Example 4.7 gives

$$H(T_{23}^\delta; t) = H(T_{23}^{UT_2}; t) = \frac{1 + 2t^2 + 2t^3 + 2t^4 + t^6}{(1 - t)^2 (1 - t^2)^3 (1 - t^3)^2 (1 - t^4)}.$$

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