Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **38** (2012), 211–236

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

CHARACTERIZATION OF CERTAIN T-IDEALS FROM THE VIEW POINT OF REPRESENTATION THEORY OF THE SYMMETRIC GROUPS

$\boxed{\text{I. B. Volichenko}}$, A. E. Zalesskii

Communicated by M. Zaicev

Dedicated to Yu. Bahturin in occasion of his 65th birthday

Abstract. Let $K[X]$ be a free associative algebra (without identity) over a field K of characteristic 0 with free generators $X = (X_1, X_2, \ldots)$, and let P_n be the set of all multilinear elements of degree n in $K[X]$. Then P_n is a KS_n-module, where KS_n is the group algebra of the symmetric group S_n. An ideal of $K[X]$ stable under all endomorphisms of $K[X]$ is called a T-ideal. If L is an arbitrary T-ideal of $K[X]$ then $L_n := P_n \cap L$ is a KS_n-module too. An important task in the theory of varieties of algebras is to reveal general regularities in the behavior of the sequence A_n for various T-ideals A. In certain cases, given a property $\mathcal{P}$, say, of the sequence, one can find a T-ideal $L(\mathcal{P})$ such that a T-ideal L' satisfies $\mathcal{P}$ if and only if L' contains $L(\mathcal{P})$. The results of this paper have to be regarded from this point of view.

2010 *Mathematics Subject Classification:* 08B20, 16R10, 16R40, 20C30.
Key words: T-ideals, Free associative algebras.

Let m be a natural number, and let $R_n^{(m)}$ (respectively, $R_n^{(-m)}$), $n > m$, be the set of all irreducible KS_n-modules whose restriction to the subgroup S_{n-m} contains an irreducible KS_{n-m}-module labeled by the partition $[n-m]$ (respectively, $[1^{n-m}]$) of $n-m$. We define the property $\mathcal{P}^m$ (respectively, $\mathcal{P}^{-m}$) by the condition that L_n contains no submodule isomorphic to a module in the set $R_n^{(m)}$ (respectively, $R_n^{(-m)}$). Set $[a,b] = ab - ba$ and $[a,b,c] = [a,[b,c]]$ for $a,b,c \in K[X]$. We proof that the T-ideal $L(\mathcal{P}^m)$ (respectively, $L(\mathcal{P}^{-m})$) coincides with the T-ideal generated by the polynomial $d_{m+1}(X) := [X_1,X_2]\cdots[X_{2m+1},X_{2m+2}]$, (respectively $t_{m+1}(X) = [X_1,X_2,X_3]\cdots[X_{3m+1},X_{3m+2},X_{3m+3}]$. One can interpret the result as a characterization of the T-ideal generated by $d_{m+1}(X)$ (respectively $t_{m+1}(X)$) by the property $\mathcal{P}^m$ (respectively, $\mathcal{P}^{-m}$).

1. Introduction. Let K be a field of characteristic 0, and let $K[X]$ be a free associative algebra over K with free generators $X = (X_1, X_2, \ldots)$. An ideal of $K[X]$ stable under all endomorphisms of $K[X]$ is called a **T-ideal**. These ideals are of certain interest in their own, but the significance of T-ideals is derived from the fact that they play a key role in the theory of varieties of algebras. Namely, there is an inverse isomorphism between the lattice of T-ideals and the lattice of varieties (with respect to inclusion).

An arbitrary T-ideal $A \subset K[X]$ is completely determined by the sequence A_n of multilinear polynomials of degree n in $X_1, \ldots, X_n$. Reorderings of the indeterminates $X_1, \ldots, X_n$ produce automorphisms of $K[X]$, which endows A_n with the structure of a KS_n-module. Here S_n stands for the symmetric group of degree n and KS_n for the group algebra of S_n over K. The study of T-ideals does not reduce to the representation theory of symmetric groups, as the latter makes no difference between isomorphic modules, whereas in the theory of T-ideals A_n are treated individually. Nonetheless the question on how the sequence A_n looks like from the viewpoint of the representation theory of S_n seems to be essential for the theory of T-ideals. Various aspects of this question arise in the analysis of certain concrete problems (see for instance [1, 7]), but there are publications completely devoted to this. One can mention a result by Regev [13] saying that for every T-ideal A there is n and an irreducible KS_n-module M that does not occur as a constituent of P_n/A_n. Equivalently, the multiplicity of M in A_n equals $\dim M$. In this paper we discuss the dual situation (in a sense): what are the T-ideals A such that A_n contains no submodule of a certain kind for every n. For instance, every T-ideal A such that A_n for every n contains no copy of the trivial KS_n-module E_n is contained in the T-ideal generated by $[X_1, X_2] = X_1X_2 - X_2X_1$.

Let R be arbitrary sequence $\{R_n\}_{n=1,2,\dots}$ of KS_n-modules (some R_n may be empty). As the sum of T-ideals is a T-ideal again, the set of T-ideals A such that no irreducible constituent of a module from R_n occurs as a constituent of A_n has a unique maximal element $L(R)$. (Note that $L(R)$ may be the zero ideal.) In this paper we take for $\{R_n\}$ the sets of the following two types. Fix an integer $m \geq 0$. For $n > m$ denote by $R_n^{(m)}$ (respectively, $R_n^{(-m)}$) the set of all irreducible KS_n-modules whose Young diagrams contain at least $m+1$ boxes off the 1-st row (respectively, off the 1-st column). If $m > n$ we set $R_n^{(m)}$ and $R_n^{(-m)}$ to be empty. For brevity set $L^{(m)} = L(R^{(m)})$ (respectively, $L^{(-m)} = L(R^{(-m)})$). (Note that $L^{(m)}$ (respectively, $L^{(-m)}$) is the smallest T-ideal A such that the irreducible constituents of the KS_n-module P_n/A_n are labeled by Young diagrams with at most m boxes off the first row (respectively, the first column).

We characterize $L^{(m)}$ and $L^{(-m)}$ as follows.

Theorem 1.1. *The T-ideal $L^{(m)}$ $(m \geq 0)$ is generated by the polynomial*

$$d_{m+1}(X) = [X_1, X_2] \cdots [X_{2m+1}, X_{2m+2}].$$

Theorem 1.2. *The T-ideal $L^{(-m)}$ $(m \geq 0)$ is generated by the polynomial*

$$t_{m+1}(X) = [X_1, X_2, X_3] \cdots [X_{3m+1}, X_{3m+2}, X_{3m+3}].$$

The T-ideals generated by the polynomials d_m and t_m were studied in details, see [3, 4, 8, 9, 10, 12]. Theorems 1.1, 1.2 can be also regarded as new characterizations of these T-ideals.

Latyshev [9] discussed extremal properties of the T-ideal generated by d_m. Let $\mathcal{P}$ be a property of T-ideals. A T-ideal A is said to be $\mathcal{P}$-extremal if A does not have $\mathcal{P}$ but every T-ideal $A_1 \neq A$ containing A does have $\mathcal{P}$. Studying exteremality properties of T-ideals turns out to be rather fruitful and one can find a number of results in this direction in [5].

The T-ideals $L(R)$ are extremal by the very definition. Note that the corresponding extremal properties of $L^{(m)}$ and $L^{(-m)}$ can be described in terms of polynomials, namely:

Theorem 1.3. *The T-ideal generated by d_{m+1} is the largest T-ideal that contains no polynomial $f(X)$ symmetric in $(^\circ f) - m$ indeterminates.*

Theoren 1.4. *The T-ideal generated by t_{m+1} is the largest T-ideal that contains no polynomial $f(X)$ alternating in $(^\circ f) - m$ indeterminates.*

If $m = 0$ then a polynomial f alternating in $^\circ f$ indeterminates is called standard. A special case of Theorem 1.4 for $m = 0$ states therefore that the T-ideal generated by $t_1 = [X_1, X_2, X_3]$ is the largest T-ideal that contains no standard polynomial. This is equivalent to [5, Theorem 7.1.2]. (Recall that the T-ideal generated by t_1 coincides with the T-ideal of identities of the Grassmann algebra [5, Theorem 4.1.8].)

This paper is based on an unpublished preprint (Preprint No. 10, 1979 (in Russian), Institute of Mathematics, National Academy of Sciences of Belarus, Minsk). The first named author, I. B. Volichenko, passed away in 1988 at age 33. Although some experts owned the preprint, the results obtained remained unknown to a wider cicle of those working in the area. Since the preprint was written, the theory devoted to the understanding of T-ideals in terms of their cocharacters has had huge advances. However, to our knowledge, no part of the results of the preprint can be deduced from more recent publications. Therefore, the results deserve to be available to the mathematical community in an English version. So the second named author edited, polished the original text, and translated it into English. He is very thankful to the guest-editors of the special issue of Serdica Mathematical Journal for the willingness to publish the paper and especially to M. Zaicev and A. Giambruno for their encouragement.

2. Notation and definitions. We denote by $\mathbb{N}$ the set of the natural numbers and by $\mathbb{N}(n)$ the set $\{1, \dots, n\}$. Fix an arbitrary field K of characteristic 0; in this paper the term "algebra" always means "an associative algebra over K". We denote by $K[X]$ a free algebra (without identity) with generators (indeterminates) $X = \{X_i : i \in \mathbb{N}\}$. In some cases we add additional indeterminates and consider the algebra $K[X;Y]$. By $\overline{K[X]}$ we denote the free algebra with identity with the same generating set X, so $K[X] \subset \overline{K[X]}$. The elements of $\overline{K[X]}$ are also called polynomials (in the indeterminates X_i ($i \in \mathbb{N}$)).

Ideals of $K[X]$ and $\overline{K[X]}$ stable under all endomorphisms are called T-**ideals**. There is a well known natural bijective correspondence between varieties of algebras (respectively, varieties of algebras with identity) and T-ideals in $K[X]$ (respectively, in $\overline{K[X]}$).

If F is a polynomial or a set of polynomials then we denote by $\langle F\rangle$ (respectively, $\overline{\langle F\rangle}$) the least T-ideal of $K[X]$ (respectively, $\overline{K[X]}$) containing F. If $f(X) \in \langle g(X)\rangle$ then $g(X)$ is said to be a consequence of $f(X)$, and in this case we write $g(X) \Longrightarrow f(X)$.

One can define in a natural way differential operators on $K[X]$. If $M = \{i_1, \dots, i_k\}$ is a set of distinct natural numbers then we denote by D_M the operator $\partial^k/\partial X_{i_1}\cdots\partial X_{i_k}$. A T-ideal A of $K[X]$ is a T-ideal of $\overline{K[X]}$ if and only if A is stable under all operators D_M.

We denote by P_n the set of all multilinear polynomials in $X_1,\dots,X_n$ of degree n in $K[X]$. Let $\overline{P_n}$ be the subspace of so called commutator polynomials in P_n; by definition, $\overline{P_n}$ is the K-span of the multilinear products of the expressions $[X_{i_1},\dots,X_{i_k}]$. (We use the word "multilinear product" to indicate that the terms of the product has no common indeterminates, so the resulting polynomial is multilinear.) Here and below $[X_{i_1},\dots,X_{i_k}]$ is meant to be $[X_{i_1},[X_{i_2},[\dots[X_{i_{k-1}},X_{i_k}]\cdots]]]$.

If A is a T-ideal of $K[X]$ (respectively, of $\overline{K[X]}$) then we set $A_n = P_n\cap A$ (respectively, $\overline{A_n} = A\cap\overline{P_n}$). It is known that the sequence $\{A_n\}_{n\in\mathbb{N}}$ (respectively, $\{\overline{A_n}\}_{n\in\mathbb{N}}$) determines A (respectively, $\overline{A}$).

The symmetric group S_n acts naturally on P_n: if $\sigma\in S_n$ and $f(X_1,\dots,X_n)\in P_n$ then $\sigma f(X_1,\dots,X_n) = f(X_{\sigma(1)},\dots,X_{\sigma(n)})$. So P_n is a KS_n-module, as well as $\overline{P_n}$. If A is a T-ideal then A_n is a submodule of P_n and $\overline{A_n}$ is a submodule in $\overline{P_n}$. Obviously, there is a KS_n-module isomorphism between P_n and the regular KS_n-module KS_n.

3. Preliminaries. We recall some known facts of the representation theory of S_n, see [2, §28]. Let D be a Young diagram of type $(n_1,,\dots,n_s)$, where $n_1\geq\cdots\geq n_s>0$ and $\sum_{i=1}^{s} n_i = n$. So D may be identified with a partition of $\{1,\dots,n\}$. Let $\overline{D}$ be a Young tableau obtained from D by filling in the boxes of D by numbers $1,\dots,n$. Denote by $R(\overline{D})$ the set of row permutations, that is, the permutations $\sigma\in S_n$ that preserve the sets of the numbers in each row. Similarly, one defines the set $Q(\overline{D})$ of column permutations. Obviously, $R(\overline{D})$ and $Q(\overline{D})$ are subgroups of S_n. Set

$$r(\overline{D}) = \sum_{\sigma\in R(\overline{D})}\sigma \quad\text{and}\quad q(\overline{D}) = \sum_{\sigma\in Q(\overline{D})}(-1)^\sigma\sigma.$$

Then the element $r(\overline{D})q(\overline{D})$ of KS_n generates a left ideal of KS_n, which is known to be an irreducible KS_n-module. This yields a bijection between the Young diagrams and the isomorphism classes of irreducible KS_n-modules. Thus, distinct Young tableaux of the same diagram yield isomorphic irreducible KS_n-modules, whereas distinct Young diagrams D yield non-isomorphic KS_n-modules.

The Young diagram (n) yields the trivial KS_n-module 1_{S_n}, and $(1^n) := (1, \dots, 1)$ yields a one-dimensional module $1^-_{S_n}$ called the **alternating** (or sign) module.

Lemma 3.1. *Let N be an irreducible KS_n-module of type $(n_1, \dots, n_s)$, and let $1 < m < n$.*

(1) *The restriction $N|_{KS_{n-m}}$ contains $1_{S_{n-m}}$ if and only if $n_1 \geq n - m$.*

(2) *The restriction $N|_{KS_{n-m}}$ contains the sign module $1^-_{S_n}$ if and only if $s \geq n - m$.*

Proof. This is an immediate consequence of the branching rule [6, 9.2]. □

Definition 3.2. *Let M be an irreducible KS_n-module corresponding to a Young diagram $(n_1, \dots, n_s)$. The number $h(M) = n_2 + \cdots + n_s = n - n_1$ is called the* **depth of** *M. For an arbitrary KS_n-module N we set*

$$h(N) = \min_{M \subset N}\{h(M) : \ M \text{ is an irreducible submodule in } N\}.$$

Finally, for $f \in P_n$ the **depth** *$h(f)$ is the number $h(N)$, where $N = \langle f \rangle_n = KS_n \cdot f$.*

The notion of depth was introduced by Murnahgan [11].

A polynomial $f \in K[X]$ is said to be **symmetric in** $X_{i_1}, \dots, X_{i_k}$ if f is invariant under any permutation of $X_{i_1}, \dots, X_{i_k}$.

Proposition 3.3. *Let $f \in P_n$ and let $M = KS_n \cdot f$ be a KS_n-module generated by f. Then the following conditions are equivalent:*

(1) *$h(f) \leq m$;*

(2) *the trivial KS_{n-m}-module $1_{S_{n-m}}$ is a constituent of the restriction $M|_{S_{n-m}}$ of M to KS_{n-m};*

(3) *M contains a polynomial f' symmetric in $n - m$ indeterminates;*

(4) *the T-ideal of $K[X]$ generated by f contains a polynomial $g(X_1, \dots, X_m, X_{m+1})$ that is linear in $X_1, \dots, X_m$ and of degree $n - m$ in X_{m+1}.*

Proof. It is obvious that (2) and (3) are equivalent. It is easy to observe that (3) and (4) are equivalent. Indeed, by reordering $X_1, \dots, X_n$ we can assume that f' is symmetric on $X_{m+1}, \dots, X_n$. Then the endomorphism of $K[X]$ defined by $X_i \Longrightarrow X_{m+1}$ for $i > m$, $X_i \Longrightarrow X_i$ for $i \leq m$ yields $g(X)$, and the linearization of $g(X_1, \dots, X_m, X_{m+1})$ is f'. So it remains to prove that (1) and (2) are equivalent.

(1) $\Longrightarrow$ (2) Suppose that $h(f) \leq m$. By definition, this means that M contains an irreducible submodule N which corresponds to a Young diagram $(n_1, \dots, n_s)$, where $n_1 \geq n - m$. So the claim follows from Lemma 3.1.

(2) $\Longrightarrow$ (1) Here M contains an irreducible submodule N such that $1_{S_{n-m}}$ is a constituent of the restriction $M|_{S_{n-m}}$. By Lemma 3.1, if N corresponds to a diagram of type $(n_1, \dots, n_s)$ then $n_1 \geq n - m$, as desired. □

Let M be an irreducible KS_n-module of type $(n_1, \dots, n_s)$. We call the number $h'(M) := n - s$ the **skew depth of** M. If N is an arbitrary KS_n-module, we set $h'(N) = \min_{M \subset N} h'(M)$, where M runs over all irreducible submodules of N. If $f \in P_n$ then we set $h'(f) = h'(KS_n \cdot f)$.

A polynomial $f \in P_n$ is called **alternating in** $X_{i_1}, \dots, X_{i_n}$ if $\sigma f = (-1)^{\sigma} f$ for every $\sigma \in S(I)$, where $S(I)$ is the symmetric group of the set $I = \{i_1, \dots, i_k\}$, see [5, 1.5.1]. For instance $f = X_1X_2X_3 - X_3X_2X_1$ is alternating in X_1, X_3.

Proposition 3.4. *Let $f \in P_n$ and $M = KS_n \cdot f$. Then the following assertions are equivalent:*

(1) *$h'(f) \leq m$;*

(2) *The sign KS_{n-m}-module $1^-_{S_{n-m}}$ is a constituent of the restriction $M|_{S_{n-m}}$ of M to S_{n-m}.*

(3) *f has a consequences in P_n that is alternating in some $n - m$ indeterminates.*

The proof is similar to that of Proposition 3.3.

4. Proof of Theorem 1.1.

Let $D^{(m)}$ be the T-ideal in $K[X, Y]$ generated by the polynomial d_m, and denote by $A^{(m)}$ the T-ideal of all polynomials having no consequences $g(X_1, \dots, X_{m-1}, Y)$ that are linear in $X_1, \dots, X_{m-1}$ (here Y is a single indeterminate). By Proposition 3.3, $A^{(m)}$ coincides with $L^{(m)}$, and hence Theorem 1.1 is equivalent to Theorem1.3.

It is rather obvious that

$$d_m(X) = [X_1, X_2] \cdots [X_{2m-1}, X_{2m}] \in A^{(m)},$$

and hence $D^{(m)}$ is contained in $A^{(m)}$. Indeed, substitute f_i for X_i for $i \leq 2m$, where f_i is a monomial in $X_1, \dots, X_{m-1}, Y$ such that the product $f_1 \cdots f_{2m}$ is a linear polynomial in $X_1, \dots, X_{m-1}$. The latter condition implies that at most

$m-1$ polynomials f_i differ from Y^j for some j (where j depends on i). Then none of $X_1, \dots, X_{m-1}$ occurs in $[f_{2i-1}, f_{2i}]$ for some i with $1 \le i \le m-1$, and hence $d_m(f_1, \dots, f_{2m}) = 0$. It follows that no polynomial g as above is a consequence of d_m, and the claim follows.

To prove the equality $D^{(m)} = A^{(m)}$, we argue by induction on m in order to show that $A^{(m)} = D^{(m)}$. The case $m = 1$ is trivial, as f has no consequences of the form Y^n ($n \in \mathbb{N}$) if and only if f follows from $[X_1, X_2]$.

Suppose that our claim is true for $m = \ell - 1$. Let $m = \ell$ and $f \in A_n^{(\ell)}$ ($n \in \mathbb{N}$). Observe that $A^{(\ell)} \subset A^{(\ell-1)}$; it follows from the induction assumption that $f \in D^{(\ell-1)}$. Every polynomial from $D^{(\ell-1)}$ is a linear combination of polynomials of the form

$$(1) \qquad v_1[X_{i_1}, X_{j_1}]v_2 \cdots [X_{i_{\ell-1}}, X_{j_{\ell-1}}]v_\ell,$$

where $v_1, \dots, v_\ell$ are monomials, some of them can be equal to 1. We write $v_i < X_j$ if $k < j$ for all indeterminates X_k occurring in the monomial v_i. The following equality holds for every triple $i, j, k \in \mathbb{N}$:

$$X_i[X_j, X_k] = [X_j, X_k]X_i + [X_i, X_j]X_k - X_k[X_i, X_j] + X_j[X_i, X_k] - [X_i, X_k]X_j.$$

Using this formula, one can write every element (1) as a linear combination of polynomials of the form (1) with the additional property:

$$(2) \qquad v_\nu < X_{i_\nu} \quad \text{for} \quad \nu = 1, 2, \dots, \ell - 1.$$

Denote by $M(n_1, \dots, n_m)$ the set of all elements of the form (1) satisfying (2) and such that $^\circ v_\nu = n_\nu$ for all $\nu \in \{1, \dots, \ell\}$. Let R be the set of all sequences $P = (n_1, \dots, n_\ell)$, where $n_1, \dots, n_\ell$ are non-negative and $n_1 + \cdots + n_\ell = n - 2\ell + 2$. We endow R with the lexicographic ordering.

Thus,

$$f = \sum_{P \in R} a_P \pmod{D^{(\ell)}},$$

where $a_P \in M(n_1, \dots, n_\ell)$ and $P = (n_1, \dots, n_\ell)$. We show that $a_P = 0$ for all $P \in R$. Indeed, suppose the contrary. Let P_0 be the maximal element in R such that $a_{P_0} \neq 0$. Suppose that an element of the form (1) occurs with a non-zero coefficient b at the expression of a_P as a sum of polynomials from $M(n_1, \dots, n_\ell)$, where $(n_1, \dots, n_\ell) \in P_0$. Perform the substitution $\varepsilon : X_i \Longrightarrow Y^{2^i}$, where $i \in \{1,\dots,n\} \setminus \{j_1,\dots,j_{\ell-1}\}$. Then $\varepsilon\left(\sum_{P\in R} a_P\right)$ is a linear combination of monomials of the the form $Y^{\alpha_1}X_{j_1}Y^{\alpha_2}\cdots X_{j_{\ell-1}}Y^{\alpha_\ell}$, and the monomial $\varepsilon(v_1X_{i_1})X_{j_1}\varepsilon(v_2X_{i_2})X_{j_2}\cdots X_{j_{\ell-1}}\varepsilon(v_\ell)$ occurs in this linear combination with coefficient b. As $f \in A^{(\ell)}$, it follows that $\varepsilon(f(X)) = 0$. This contradiction proves Theorems 1.3 and 1.1.

In the above arguing we have shown the following:

Proposition 4.1. *The elements of the form* (1) *satisfying* (2) *constitute a basis of the vector space $D^{(\ell-1)}$ modulo $D^{(\ell)}$.*

5. The T-ideal L^{-m} and proof of Theorem 1.2. In this section we study the T-ideal $T^{(m)}$ generated by the polynomial

$$t_m := [X_1, X_2, X_3]\cdots[X_{3m-2}, X_{3m-1}X_{3m}].$$

Our aim is to prove Theorem 1.2. By Proposition 3.4, Theorem 1.2 is equivalent to Theorem 1.4.

Along with the free algebra $K[X]$ we shall use the free algebra $K[X;Y]$ with free generators $X \cup Y$, where $Y = \{Y_i : i \in \mathbb{N}\}$.

Let $I = \{t_1,\dots,t_k\} \subseteq \mathbb{N}(n)$, where $t_1 < \cdots < t_k$, and let $\phi : I \Longrightarrow \mathbb{N}$ be an injective mapping. Let $\phi(t_\nu) = r_\nu$ for $\nu \in \{1,\dots,k\}$. Denote by $\varepsilon_0(I,\phi)$ the ring homomorphism $K[X] \Longrightarrow K[Y]$ defined by

$$\varepsilon_0(I,\phi) : X_{t_\nu} \Longrightarrow Y_{r_1+\cdots+r_{\nu-1}+1}\cdots Y_{r_1+\cdots+r_\nu}$$

(if $\nu = 1$ then $\varepsilon_0(I,\phi)(X_{t_1}) = Y_1Y_2\cdots Y_{r_1}$). Denote by $\varepsilon(I,\phi) = \varepsilon(t_1,\dots,t_k,r_1,\dots,r_k)$ the linear mapping $K[X] \Longrightarrow K[X;Y]$ defined for $f \in K[X]$ as follows:

$$\varepsilon(I,\phi)f(X) = \sum_{\sigma\in S_r} (-1)^\sigma \cdot \varepsilon_0(I,\phi)f,$$

where $r = r_1 + \cdots + r_k$. That is, the right hand side is the alternating sum of $\varepsilon_0(I,\phi)f$ over $Y_1,\dots,Y_r$. (Note that if f is multilinear then so is $\varepsilon(I,\phi)f$.)

Polynomials of the form $\varepsilon(I,\phi)(X_{\sigma(1)}\cdots X_{\sigma(n)})$ for $\sigma \in S_n$ are called **Y-words**. For $\varepsilon(I,\phi)f(X_1,\dots,X_n)$ we write $f(X_1,\dots,X_{i_1-1},Y^{(r_1)},\dots,X_{i_k-1},Y^{(r_k)},\dots,X_n)$. For instance, $X_1\cdots X_{i_1-1}Y^{(r_1)}\cdots X_{i_k-1}Y^{(r_k)}\cdots X_n$ denotes the Y-word $\varepsilon(I,\phi)(X_1\cdots X_n)$ and $[Y^{(r_1)},Y^{(r_2)}]$ is $\varepsilon(i,j,r_1,r_2)[X_i,X_j]$, and so on.

(Observe that $\varepsilon(i,j,r_1,r_2)[X_i,X_j]$ does not depend on i,j, so the notation $[Y^{(r_1)},Y^{(r_2)}]$ is unambiguous.)

For two disjoint subsets I,J of $\mathbb{N}(n)$ we define the composition $\varepsilon(I,\phi) * \varepsilon(J,\psi)$ as follows:

$$\varepsilon(I,\phi) * \varepsilon(J,\psi) = \varepsilon(I \cup J, \phi \cup \psi), \text{ where } (\phi \cup \psi)(x) = \begin{cases} \phi(x) & if \;\; x \in I; \\ \psi(x) & if \;\; x \in J. \end{cases}$$

In the sequel we shall frequently use the following obvious lemma:

Lemma 5.1. (1) *If $r,s \in \mathbb{N}$ are odd then $[Y^{(r)},Y^{(s)}] = 2Y^{(r+s)}$, otherwise, $[Y^{(r)},Y^{(s)}] = 0$.*

(2) *Let $k \geq 3$. Then $[Y^{(r_1)},\dots,Y^{(r_k)}] = 0$ for any choice of $r_1,\dots,r_k \in \mathbb{N}$.*

We denote by $[X_1,\underbrace{X_2][X_3},X_4]$ the polynomial $[X_1,X_2][X_3,X_4]+[X_1,X_3]$ $[X_2,X_4]$. Let $J = \{i,j,k,l \in \mathbb{N}\}$. Then $W(J)$ denotes the set of all polynomials of the form

$$\sum_{\sigma \in S(J)} \alpha_\sigma [X_{\sigma(i)},\underbrace{X_{\sigma(j)}][X_{\sigma(k)}},X_{\sigma(l)}],$$

where $\alpha_\sigma \in K$ and σ runs over the symmetric group $S(J)$ of the set J. Obviously, $W(J)$ is a vector space over K. The elements of $W(J)$ will be called w-**elements**, and the polynomials of the form $[X_i,\underbrace{X_j][X_k},X_l]$ are called w-**words**.

The next lemma follows from Lemma 5.1.

Lemma 5.2. *Let $I \subset \mathbb{N}$, where $|I| = 4$, and let $r_1,r_2,r_3,r_4 \in \mathbb{N}$. Then*

(i) $\varepsilon(I,r_1,r_2,r_3,r_4) \cdot W(I) = (0)$;

(ii) If $J \subset I$ and $|J| = 3$ then $\varepsilon(I,r_1,r_2,r_3) \cdot W(J) \neq 0$ if and only if there is at most one even number in $\{r_1,r_2,r_3\}$.

One can easily verify the following lemma.

Lemma 5.3. *The following formulas are true:*

$$[X_1,\underbrace{X_2][X_3},X_4] - [X_1,\underbrace{X_3][X_2},X_4] = 0; \tag{3}$$

$$[X_1,\underbrace{X_2][X_3},X_4] + [X_2,\underbrace{X_3][X_1},X_4] + [X_3,\underbrace{X_1][X_2},X_4] = 0; \tag{4}$$

(5) $$[X_1,\underbrace{X_2][X_3},X_4]+[X_1,\underbrace{X_3][X_4},X_2]+[X_1,\underbrace{X_4][X_2},X_3]=0;$$

(6) $$[X_1,\underbrace{X_2][X_4},X_3]+[X_2,\underbrace{X_4][X_1},X_3]+[X_4,\underbrace{X_2][X_1},X_3]=0;$$

(7) $$\sum_{\sigma\in S_3}[X_{\sigma(1)},\underbrace{X_{\sigma(2)}][X_4},X_{\sigma(3)}]=0;$$

(8) $$[X_1,X_2][X_3,\underbrace{X_4][X_5},X_6]+[X_1,X_3][X_2,\underbrace{X_4][X_5},X_6]$$
$$+[X_1,\underbrace{X_2][X_3},X_4][X_5,X_6]+[X_1,\underbrace{X_2][X_3},X_5][X_4,X_6]=0.$$

Using the formulas (3)–(7), one can observe that every w-element in $W(J)$, where $J=\{k,l,m,p\}$, $k<l<m<p$, is a linear combination of polynomials of the form:

$$w_J^{(1)}=[X_k,\underbrace{X_l][X_m},X_p],$$

$$w_J^{(2)}=[X_p,\underbrace{X_k][X_l},X_m],$$

$$w_J^{(3)}=[X_m,\underbrace{X_k][X_l},X_p]-[X_k,\underbrace{X_m][X_p},X_l],$$

$$w_J^{(4)}=[X_p,\underbrace{X_m][X_k},X_l],$$

$$w_J^{(5)}=[X_k,\underbrace{X_m][X_p},X_l].$$

We call the polynomials $w_J^{(\nu)}$ ($\nu\in\{1,2,3,4,5\}$ **basic w-elements** (of $W(J)$).

Multilinear products of double commutators

(9) $$[X_{i_1},X_{i_2}]\cdots[X_{i_{2k-1}},X_{i_{2k}}]$$

will be called **v-polynomials**. A v-polynomial is called **canonical** if $i_1<\cdots<i_{2k}$. (The polynomial $f(X)=1$ is viewed as a canonical polynomial of degree 0.)

Remark 1. It is known (see, for instance [7]) that $[X_1, \underbrace{X_2][X_3}, X_4] \in T^{(1)}$. Therefore, if v_1, v_2 are v-polynomials in the same set of indeterminates then either $v_1 \equiv v_2 \pmod{T^{(1)}}$ or $v_1 \equiv -v_2 \pmod{T^{(1)}}$. In particular, every v-polynomial is congruent (up to a sign) modulo $T^{(1)}$ to a canonical v-polynomial.

A polynomial $g(X)$ of the form $g_1(X)g_2(X)$ is called a vw-**polynomial**, if $g_1(X)$ is a v-polynomial and $g_2(X)$ is a w-polynomial.

Definition 5.4. *Let I be a finite set of natural numbers and $J = \{k, l, m, p\}$ be a subset of I such that $i < k < l < m < p$ for all $i \in I \setminus J$. Let $g \in K[X]$ be a multilinear polynomial in X_i with $i \in I$. Then g is called a* **canonical** vw-**polynomial** *if $g = g_1 g_2$, where g_1 is a canonical v-polynomial and g_2 is a w-polynomial of one of the following the forms:*

(i) $w_J^{(\nu)}$ $(\nu \in \{1, 2, 3, 4, 5\})$;

(ii) $[X_m, \underbrace{X_l][X_j}, X_p] = [X_m, \underbrace{X_j][X_l}, X_p]$, $[X_p, \underbrace{X_l][X_j}, X_m] = [X_p, \underbrace{X_j][X_l}, X_m]$ *or* $[X_p, \underbrace{X_m][X_j}, X_l] = [X_p, \underbrace{X_j][X_m}, X_l]$, *where $j \in I$ (so $j \neq k$).*

(iii) $[X_m, \underbrace{X_p][X_j}, X_i] = [X_m, \underbrace{X_j][X_p}, X_i] = -[X_m, \underbrace{X_i][X_p}, X_j] - [X_m, \underbrace{X_i][X_j}, X_p]$, *where $i, j \in I$, $i < j$ and $j \neq l$. (The former equality holds by* (3)*, and the latter does by* (5)*.)*

If (i) (respectively, (ii), (iii)) holds then $g_1 g_2$ is called a **canonical** vw-**polynomial of the first type** *(respectively,* **of the second type, of the third type***).*

Note that the polynomials in *(ii)* above can be described as $[X_{i_2}, X_{i_1}][X_{i_3}, X_{i_4}] = [X_{i_2}, X_{i_3}][X_{i_1}, X_{i_4}]$, where $k \neq i_3 \in I$, $\{i_1, i_2, i_4\} = \{l, m, p\}$ and $i_2 > i_1$. This will be used in Definition 5.6 below.

In this notation we have:

Lemma 5.5. *Let f be a vw-polynomial in X_i $(i \in I)$. Let $J = \{k, l, m, p\} \subseteq I$, where $i < k < l < m < p$ for all $i \in I \setminus J$. Then*

$$f \equiv g(X) + h(X) \pmod{T^{(2)}},$$

where $h(X)$ is a linear combination of polynomials of the form $h_1(X)h_2(X)$, $h_1(X)$ is a vw-polynomial, $h_2(X)$ is a v-polynomial of degree > 0, and $g(X)$ is a linear combination of canonical vw-polynomials.

P r o o f. Denote by R the set of all polynomials h of the form described in the statement of the lemma. We have to show that $f \equiv g \,(\mathrm{mod}\, R + T^{(2)})$, where g is a linear combination of canonical vw-polynomials. Let

$$f = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}][X_{j_1}, \underbrace{X_{j_2}][X_{j_3}}, X_{j_4}],$$

where $i_1, \ldots, i_{2q}, j_1, j_2, j_3, j_4 \in I$ and $2q = |I| - 4$. We can assume that $i_1 < \cdots < i_{2q}$ by Remark 1 (which will be frequently used below).

Claim 1. The lemma is true if $\{j_1, j_2, j_3, j_4\} = \{k, l, m, p\}$.

Indeed, using the formulas (3) – (7), one can express f as a linear combination of canonical vw-polynomials of the first type.

Claim 2. The lemma is true if $j_2, j_3, j_4 \in \{k, l, m, p\}$.

If $j_1, j_2, j_3, j_4 \in \{k, l, m, p\}$, we are done by Claim 1. Otherwise, let $\nu \in \{k, l, m, p\}$ and $\nu \notin \{j_1, j_2, j_3, j_4\}$. Then we may assume that $\nu = l_{2q}$, see Remark 1. By (8), we get

$$f \equiv -[X_{i_1}, X_{i_2}] \cdots [X_{2q-1}, X_{j_1}][X_\nu, \underbrace{X_{j_2}][X_{j_3}}, X_{j_4}] \,(\mathrm{mod}\, T^{(2)} + R),$$

and the claim follows by Claim 1.

Claim 3. It suffices to prove the lemma in the case where $j_3 = p$.

(i) Let $p \in \{j_1, j_2, j_3, j_4\}$. Using (3), (4), (5), we can express $[X_{j_1}, \underbrace{X_{j_2}][X_{j_3}}, X_{j_4}]$ as a linear combination of w-polynomials of the form $[X_{k_1}, \underbrace{X_{k_2}][X_p}, X_{k_3}]$ (where $\{k_1, k_2, k_3\} \subset \{j_1, j_2, j_3, j_4\}$).

(ii) Let $p \notin \{j_1, j_2, j_3, j_4\}$. Then $p = i_{2q}$. By (8), we get

$$f \equiv -[X_{i_1}, X_{i_2}] \cdots [X_{2q-1}, X_{j_1}][X_p, \underbrace{X_{j_2}][X_{j_3}}, X_{j_4}] \,(\mathrm{mod}\, T^{(2)} + R).$$

So the claim follows by (i).

Claim 4. The lemma is true if $(j_3, j_4) = (p, m)$.

If $j_2 \in \{k, l\}$ then the lemma follows by Claim 2. So $j_2 \notin \{k, l\}$, and hence either $l = j_1$ or $l = i_{2q}$. If $l = j_1$ then

$$[X_l, \underbrace{X_{j_2}][X_p}, X_m] = [X_l, \underbrace{X_p][X_{j_2}}, X_m] = -[X_{j_2}, \underbrace{X_l][X_m}, X_p] - [X_p \underbrace{X_l][X_{j_2}}, X_m].$$

So (setting $v(X) = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}]$) we have

$$f = v(X)[X_l, \underbrace{X_{j_2}][X_p}, X_m] = -v(X)[X_{j_2}, \underbrace{X_l][X_m}, X_p] - v(X)[X_p \underbrace{X_l][X_{j_2}}, X_m],$$

and the claim follows by applying Claim 2 to the first summand, as the second summand is a canonical vw-polynomial of the second type.

If $l = i_{2q}$ then by (8):

$$f \equiv -[X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{j_1}][X_l, \underbrace{X_{j_2}][X_p}, X_m] \,(\operatorname{mod} T^{(2)} + R),$$

and the result follows from the above.

Claim 5. The lemma is true if $j_2 = m$.

By Claim 2, $j_4 \notin \{k, l\}$. Then either $j_1 = l$ or $i_{2q} = l$. If $j_1 = l$ then, by (5) and (6)

$$[X_l, \underbrace{X_m][X_p}, X_{j_4}] = -[X_l, \underbrace{X_p][X_{j_4}}, X_m] - [X_l, \underbrace{X_{j_4}][X_m}, X_p]$$

$$= [X_p, \underbrace{X_{j_4}][X_l}, X_m] + [X_{j_4}, \underbrace{X_p][X_l}, X_m] + [X_{j_4}, \underbrace{X_m][X_l}, X_p] + [X_m, \underbrace{X_{j_4}][X_l}, X_p].$$

So (setting $v(X) = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}]$) we have

$$v(X)[X_p, \underbrace{X_{j_4}][X_l}, X_m] + v(X)[X_{j_4}, \underbrace{X_p][X_l}, X_m]$$

$$+ v(X)[X_{j_4}, \underbrace{X_m][X_l}, X_p] + v(X)[X_m, \underbrace{X_{j_4}][X_l}, X_p].$$

Here the first and the third summands are vw-words of the second type, and the second and the forth summands can be ignored by Claim 2.

Let $i_{2q} = l$. Then by (8), we have

$$f \equiv -[X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{j_1}] \,([X_l, \underbrace{X_m][X_p}, X_{j_4}] \,(\operatorname{mod} T^{(2)} + R),$$

and the result follows from the previous paragraph.

Claim 6. It suffices to prove the lemma when $(j_1, j_3) = (m, p)$.

Indeed, by Claims 4 and 5, $m \notin \{j_2, j_4\}$. So either $m = j_1$ or $m = i_{2q}$, but the latter case reduces to the former one by (8).

Thus, we can assume that $m = j_1$, so (keeping $v(X)$ as in Claim 5) we have

$$f = v(X)\,[X_m, \underbrace{X_{j_2}][X_p}, X_{j_4}].$$

If $j_4 = l$ then $[X_m, \underbrace{X_{j_2}][X_p}, X_{j_4}] = -[X_{j_2}, \underbrace{X_p][X_m}, X_l] - [X_p, \underbrace{X_m][X_{j_2}}, X_l]$ by (4). Therefore,

$$f = v(X)\,[X_m, \underbrace{X_{j_2}][X_p}, X_{j_4}] = -v(X)[X_{j_2}, \underbrace{X_p][X_m}, X_l] - v(X)[X_p, \underbrace{X_m][X_{j_2}}, X_l].$$

The second term at the right hand side is a vw-polynomial of the second type. The first term can be ignored by Claim 2.

Thus, $j_4 \neq l$. Suppose that $j_2 = l$. As

$$\begin{aligned} [X_m, \underbrace{X_l][X_p}, X_{j_4}] &= -[X_m, \underbrace{X_{j_4}][X_p}, X_l] - [X_m, \underbrace{X_{j_4}][X_l}, X_p] \\ &= [X_{j_4}, \underbrace{X_p][X_m}, X_l] + [X_p, \underbrace{X_m][X_{j_4}}, X_l] - [X_m, \underbrace{X_{j_4}][X_l}, X_p] \end{aligned}$$

by (4), (5) and (3), we have

$$f = v(X)[X_{j_4}, \underbrace{X_p][X_m}, X_l] + v(X)[X_p, \underbrace{X_m][X_{j_4}}, X_l] - v(X)[X_m, \underbrace{X_{j_4}][X_l}, X_p].$$

Here the first term can be ignored by Claim 2, whereas the second and third terms are vw-polynomials of the second type.

Thus, $l \notin \{j_2, j_4\}$. If $j_4 < j_2$ then f is a vw-polynomial of the third type. So we are left to prove the following:

Claim 7. The lemma is true if $j_2 < j_4$.

Using (7), we get

$$\begin{aligned} v(X)\,[X_m, \underbrace{X_{j_2}][X_p}, X_{j_4}] &= -v(X)\,[X_m, \underbrace{X_{j_4}][X_p}, X_{j_2}] - v(X)\,[X_{j_2}, \underbrace{X_m][X_p}, X_{j_4}] \\ &\quad -v(X)\,[X_{j_4}, \underbrace{X_m][X_p}, X_{j_2}] - v(X)\,[X_{j_2}, \underbrace{X_{j_4}][X_p}, X_m] - v(X)\,[X_{j_4}, \underbrace{X_{j_2}][X_p}, X_m]. \end{aligned} \tag{10}$$

The first term is a vw-polynomial of the third type. The other terms can be ignored by Claim 2. (As $l \notin \{j_2, j_4\}$, we can assume that $l = i_{2q}$, and by (7) we may reorder l with j_2 or j_4, returning to the cases already considered.)

This completes the proof. □

Fix $n \in \mathbb{N}$. To every canonical vw-polynomial f of degree $|I|$ in indeterminates X_i for $i \in I \subseteq \mathbb{N}(n)$ we associate an operator $\xi_n(f) : K[X] \Longrightarrow K[X;Y]$. The fashion of defining ξ_n depends on the type of the vw-polynomial (defined prior Lemma 5.5):

Definition 5.6. *Let f be a canonical vw-polynomial in the indeterminates X_i, $i \in I$, $|I| = 2q+4$. Let $k < l < m < p$ be the largest numbers in I (that is, $i < k$ for every $i \in I \setminus \{k,l,m,p\}$). Set $J = \{k,l,m,p\}$.*

(1) *Suppose that f is of the first type, and let*

$$f = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}] \cdot w_J^{(\nu)} \qquad (\nu \in \{1,2,3,4,5\}). \tag{11}$$

For every ν define j_1, j_2 as follows:

ν	1	2	3	4	5
j_1	3	1	1	1	4
j_2	4	3	4	2	2

and then define $\xi_n(f) : K[X] \Longrightarrow K[X;Y]$ by

$$\xi_n(f) = \varepsilon(I \setminus j_2; \phi), \quad \textit{where} \quad \phi(i) = 3^{ni} \quad \textit{for all} \quad i \in I \setminus \{j_1, j_2\}, \quad \phi(j_1) = 2^{j_1}.$$

(2) *Let f be of the second type:*

$$f = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}]\, [X_{j_2}, X_{j_1}][X_{j_3}, X_{j_4}], \tag{12}$$

where $j_1 < j_2$, $\{j_1, j_2, j_4\} = \{l, m, p\}$ and $j_3 \neq k$.

Then we set

$$\xi_n(f) = \varepsilon(I \setminus j_4; \phi), \quad \textit{where} \quad \phi(i) = 3^{ni} \quad \textit{for all} \quad i \in I \setminus \{j_3, j_4\}, \quad \phi(j_3) = 2^{j_3}.$$

(3) *Suppose that f is of the third type:*

$$f = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}]\, [X_m, \underbrace{X_p][X_{j_2}}, X_{j_1}], \ (j_1 < j_2).$$

Then we set

$$\xi_n(f) = \varepsilon(I \setminus j_2; \phi), \quad \textit{where} \quad \phi(i) = 3^{ni} \quad \textit{for all} \quad i \in I \setminus \{j_1, j_2\} \quad \textit{and} \quad \phi(j_1) = 2^{j_1}.$$

Next we define $\xi_n(f)$ for canonical v-polynomials f and for canonical multilinear commutators.

Definition 5.7. (1) *Let f be a canonical v-polynomial:*

$$f = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}],$$

where $i_1 < \cdots < i_{2q}$, $I = \{i_1, \ldots, i_{2q}\} \subset \mathbb{N}(n)$.
Then we set $\xi_n(f) = \varepsilon(I; \phi)$, where $\phi(i) = 3^{ni}$ for all $i \in I$.

(2) *Let f be a canonical multilinear commutator of degree greater than 2:*

$$f = [X_{i_1}, \ldots, X_{i_q}], \tag{13}$$

where $q \geq 3$, $i_1 < \cdots < i_{q-2}$, $i_\nu < i_q$ for all $\nu \in \{1, \ldots, q-1\}$, and $\{i_1, \ldots, i_q\} \in \mathbb{N}(n)$. Then we set

$$\xi_n(f) = \varepsilon(I \setminus i_{q-1}; \phi), \quad \textit{where} \quad \phi(i) = 2^i \quad \textit{for all} \quad i \in I \setminus \{i_{q-1}\}.$$

We call the elements of the form (13) **canonical u-polynomials**. If f is a multilinear polynomial of the form $a(X)b(X)$, where $a(X)$ is a canonical v-polynomial and $b(X)$ is a canonical u-polynomial then f is called a **canonical vu-polynomial**. In this case we set $\xi_n(f) = \xi_n(a) \cdot \xi_n(b)$.

A multilinear polynomial f is called **canonical of rank** l if f is of the form

$$f = a_1(X)a_2(X) \cdots a_{l-1}(X)a_l(X), \tag{14}$$

where every polynomial $a_i(X)$ for $i = 1, \ldots, l-1$ is either canonical vw-polynomial or canonical vu-polynomial, and $a_l(X)$ is a canonical v-polynomial. For a canonical polynomial f as in (14) we set

$$\xi_n(f) = \xi_n(a_1) * \xi_n(a_2) * \cdots * \xi_n(a_l) \tag{15}$$

and

$$\overrightarrow{f} = ({}^\circ(a_1(X), \ldots, {}^\circ(a_{l-1}(X), {}^\circ(a_l(X)) \in \mathbb{N}^l\,. \tag{16}$$

Definition 5.8. *Let $\leq$ be a lexicographic ordering on $\mathbb{N}^l$. If f is a canonical polynomial and $\xi_n(f) = \varepsilon(I, \phi)$ then we denote by $\alpha(f)$ the number of $i \in I$ such that $\phi(i)$ is even, that is, $\phi(i) = 2^i$.*

In order to understand the principle of constructing the operators $\xi_n(f)$, one has to keep in mind the following remarks:

Remark 2. The functions ϕ in the definition of the operators $\xi_n(f)$ always take values in the set $M = \{2, 2^2, \dots, 2^n, 3^n, 3^{2n}, \dots, 3^{n^2}\}$. This set has been selected so that the following condition would hold: if $A, B \subset M$ and $\sum_{i\in A} i = \sum_{j\in B} j$ then $A = B$. Define a mapping E from the set of all numbers of the form $\sum_{i\in A} i$, where $A \subseteq M$, to the set of all subsets of M as follows: $E\left(\sum_{i\in A} i\right) = A$. Thus E is well defined.

Remark 3. The operators $\xi_n(f)$ are constructed so that they would have an extreme property with respect to the function α introduced in Definition 5.8: it follows from Lemmas 5.1 and 5.2 that, if f is a canonical polynomial (a vw-polynomial or a vu-polynomial) and $\xi_n(f) = \varepsilon(I, \phi)$ then $\varepsilon(I; \phi) \cdot f \neq 0$, whereas $\varepsilon(J; \psi) \cdot f = 0$ for a pair (J, ψ) such that either $|J| > |I|$ or $|J| = |I|$ and $|\{j \in J : \psi(j) \text{ is even}\}| > \alpha(f)$. In particular, this implies the following lemma:

Lemma 5.9. *If $f, g \in P_n$ are canonical polynomials of rank l and $\alpha(f) > \alpha(g)$ then $\xi_n(f) \cdot g(X) \neq 0$.*

Let $T^{(m)}$ be the T-ideal of $K[X]$ generated by the polynomial t_m. Recall that $t_m = [X_1, X_2, X_3] \cdots [X_{3m-2}, X_{3m-1}, X_{3m}]$. Obviously, $T^{(m)}$ is stable under the derivatives $\partial/\partial X_i$ ($i \in \mathbb{N}$), and hence is a T-ideal in $\overline{K[X]}$.

Theorem 5.10. *Let T be a T-ideal in $\overline{K[X]}$. Then the following conditions are equivalent:*

(1) *$T \subseteq T^{(m)}$;*

(2) *for every KS_n-module $M \subseteq T_n$ one has $h'(M) \geq m$.*

P r o o f. Denote by $\overline{B}^{(m)}$ the set of all polynomials $f \in \overline{K[X]}$ such that f has no consequences in $K[X; Y]$ of the form $g(X_1, \dots, X_{m-1}, Y_1, \dots, Y_n)$ that are linear in $X_1, \dots, X_{m-1}$ and alternating in $Y_1, \dots, Y_n$ ($n \in \mathbb{N}$). It is easy to observe that $\overline{B}^{(m)}$ is a T-ideal in $\overline{K[X]}$. In order to prove Theorem 1.2 it suffices to show that $\overline{B}^{(m)} = T^{(m)}$ (see Proposition 3.4).

It follows from Lemma 5.1 (2) that $t_m(X) \in \overline{B}^{(m)}$. Therefore, $T^{(m)} \subseteq \overline{B}^{(m)}$. So it suffices to show that if $f \in \overline{B}^{(m)}$ then $f \in T^{(m)}$.

We use induction on m. If $m = 1$, this is equivalent to the following well known fact: if f does not imply any standard polynomial then $f \in T^{(1)}$ (see [5, 7.1.2]).

Let $m = l$ and $f \in \overline{B}_n^{(l)}$ $(n \in \mathbb{N})$. As $\overline{B}^{(l)} \subset \overline{B}^{(l-1)}$, the inductive assumption implies that $f \in T^{(l-1)}$. The space $T_n^{(l-1)}$ is spanned by polynomials of the form $a_1(X) \cdots a_k(X)$, where every polynomial $a_1(X), \dots, a_k(X)$ is either a canonical commutator or w-polynomial, and the total number of the w-polynomials and u-polynomials (that is, canonical commutators of degree greater than 2) in this product is at least $l - 1$. Then it follows from Lemma 5.5 that

$$f \equiv \sum_{i=1}^{d} c_i f_i \,(\operatorname{mod} T^{(l)}), \tag{17}$$

where $c_1, \dots, c_d \in K$ and $f_1, \dots, f_d$ are canonical polynomials of rank l.

We shall prove that $c_i = 0$ for all $i = 1, \dots, d$. Indeed, suppose the contrary. Then there is $0 < i \le d$ such that

(*i*) $c_i \neq 0$;

(*ii*) if $\alpha(f_j) > \alpha(f_i)$ for $0 < j \le d$ then $c_j = 0$, where α is as in Definition 5.8;

(*iii*) if $\overrightarrow{f_j} > \overrightarrow{f_i}$ and $\alpha(f_j) = \alpha(f_i)$ then $c_j = 0$.

Suppose that $f_i = a_1(X) \cdots a_{l-1}(X) a_l(X)$, where every polynomial $a_j(X)$ for $1 \le j \le l-1$ is either a canonical vw-polynomial or a canonical vu-polynomial, and $a_l(X)$ is a canonical v-polynomial. Apply the operator $\xi_n(f_i)$ to f.

First consider $\xi_n(f_i) \cdot f_i$ (keeping in mind (15)). By the definition of $\xi_n(f_i)$, for every $\mu = 1, \dots, l - 1$ we have:

$$\xi_n(f_i) \cdot a_\mu(X) = \xi_n(a_\mu) \cdot a_\mu(X) = 2^{\lambda_\mu} \cdot Y^{(r_\mu)} X_{t_\mu} + \sum_{\nu=1}^{r_\mu} \gamma_\nu Y^{(r_\mu - \nu)} X_{t_\mu} Y^{(\nu)},$$

where $\lambda_\mu \ge 0$. In addition, $\xi_n(f_i) \cdot a_l(X) = 2^{\lambda_l} Y^{(r_l)}$ for some integer $\lambda_l \ge 0$, see Lemma 5.1. Express $\xi_n(f_i) \cdot f_i$ as a linear combination of distinct Y-words (in what follows we shall call such an expression a Y-expansion of the polynomial). Then the Y-word

$$\left(\prod_{\mu=1}^{l} 2^{\lambda_\mu} \right) \cdot Y^{(r_1)} X_{t_1} Y^{(r_2)} \cdots X_{t_{l-2}} Y^{(r_{l-1})} X_{t_{l-1}} Y^{(r_l)} \tag{18}$$

occurs in the Y-expansion of $\xi_n(f_i) \cdot f_i$.

Suppose that we have already proved that the Y-word (18) for $j \neq i$ $(1 \le j \le d)$ cannot occur with a non-zero coefficient in the Y-expansion of

$\xi_n(f_i) \cdot f_j$. Then the Y-word (18) occurs in the Y-expansion of the polynomial $g(X;Y) = \xi_n(f_i) \cdot (\sum_{i=1}^{d} c_i f_i)$ with coefficient c_i. Therefore, $g(X;Y) \neq 0$. On the other hand, $g(X;Y)$ is alternating in all indeterminates except $X_{t_1}, \ldots, X_{t_{l-1}}$, which contradicts the assumption that $f \in T^{(l)}$ and (17). This contradiction shows that $c_i = 0$ in (17) for all $i = 1, \ldots, d$, and hence $f \in T^{(l)}$, as required.

We are left to show that the Y-word (18) does not occur with a non-zero coefficient in the Y-expansion of $\xi_n(f_j)$. Suppose the contrary. Let $f_j = b_1(X) \cdots b_{l-1}(X) b_l(X)$, where $b_\mu(X)$ for $\mu = 1, \ldots, l-1$ are canonical vu-polynomials or vw-polynomials and $b_l(X)$ is a canonical v-polynomial.

Note that we can assume that

(*iv*) $c_j \neq 0$.

As $\xi_n(f_i) \cdot f_j \neq 0$, it follows from Lemma 5.9 that $\alpha(f_j) \geq \alpha(f_i)$, which means, in view of (*ii*) and (*iv*), that

(*v*) $\alpha(f_j) = \alpha(f_i)$.

Now it follows from (*iv*), (*v*) and (*iii*) that $\overrightarrow{f}_j \leq \overrightarrow{f}_i$. If $\overrightarrow{f}_j < \overrightarrow{f}_i$ then it follows from Remark 2 that the Y-word (18) occurs with a non-zero coefficient in the Y-expansion of $\xi_n(f_i) \cdot f_j$. Therefore, we can assume that

(*vi*) $\overrightarrow{f}_j = \overrightarrow{f}_i$.

Let $\xi_n(f_i) = \varepsilon(I, \phi)$ and $I_0 = \{\nu : \phi(\nu) = 2^\nu\}$, $I_1 = I \setminus I_0$. Set $J = \mathbb{N}(n) \setminus I$. Clearly, in (18) we have $t_1, \ldots, t_{l-1} \in J$. In view of Remark 3 and (*v*) one easily observes that $\xi_n(f_i) \cdot f_j \neq 0$ implies that $b_\mu(X)$ for every $\mu \in \{1, \ldots, l-1\}$ is either (*a*) a canonical vw-polynomial or (*b*) a canonical vu-polynomial.

(*a*) In this case

$$b_\mu(X) = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}] \cdot [X_{j_1}, \underbrace{X_{j_2}][X_{j_3}}, X_{j_4}],$$

and then the following conditions hold (see Definition 5.6):

$$|J \cap \{j_1, j_2, j_3, j_4\}| = 1;$$

$$|I_0 \cap \{j_1, j_2, j_3, j_4\}| = 1;$$

$$\{i_1, \ldots, i_{2q}\} \subseteq I_1.$$

(b) In this case

$$b_\mu(X) = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}] \cdot [X_{j_1}, \dots, X_{j_k}],$$

and then the following conditions hold:

$$|J \cap \{j_1, \dots, j_k\}| = 1;$$

$$|I_0 \cap \{j_1, \dots, j_k\}| = k - 1;$$

$$\{i_1, \dots, i_{2q}\} \subseteq I_1.$$

Suppose now that the Y-word (18) occurs with a non-zero coefficient in the Y-expansion of $\xi_n(f_i) \cdot f_j$. In view of (vi) and Remark 2, for every $\mu = 1, \dots, l-1$ the Y-word $Y^{(r_\mu)} X_{t_\mu}$ (which occurs with a non-zero coefficient in the Y-expansion of the polynomial $\xi_n(f_i) \cdot a_\mu(X) = \xi_n(a_\mu) \cdot a_\mu(X)$) also occurs in the Y-expansion of the polynomial $\xi_n(f_i) \cdot b_\mu(X) = \xi_n(a_\mu) \cdot b_\mu(X)$. We shall show that in this case $a_\mu(X) = b_\mu(X)$ for all $\mu = 1, \dots, l-1$, and then automatically we have $a_l(X) = b_l(X)$. It follows from this that $f_i = f_j$, which contradicts the assumption.

So we are proving that $a_\mu(X) = b_\mu(X)$ for all $\mu = 1, \dots, l-1$. Note that $\xi_n(a_\mu)$ determines the mapping $\phi : I \Longrightarrow \mathbb{N}$ and $r_\mu = \sum_{i \in I} \phi(i)$. Define a mapping E as in Remark 2. There are the following possibilities:

(1) $a_\mu(X)$ is a canonical vw-polynomial and $b_\mu(X)$ is a canonical vu-polynomial. Then $E(r_\mu)$ contains exactly one even number (by the construction of $\xi_n(f_i)$). But this contradicts (b) which requires $|I_0 \cap \{j_1, \dots, j_k\}| = k - 1 \geq 2$.

(2) $a_\mu(X)$ is a canonical vu-polynomial and $b_\mu(X)$ is a canonical vw-polynomial. Then $E(r_\mu)$ contains exactly $t-1$ even numbers, where t is the degree of the canonical u-polynomial that occurs in $a_\mu(X)$. As $t \geq 3$, this contradicts (a), which requires $|I_0 \cap \{j_1, j_2, j_3, j_4\}| = 1$.

(3) $a_\mu(X)$ and $b_\mu(X)$ are canonical vu-polynomials. In this case the equality $a_\mu(X) = b_\mu(X)$ follows straightforwardly from our assumption and Remark 2. Indeed, both of them are of the form $v(X)[X_{j_1}, \dots, X_{j_{k-2}}, X_{t_\mu}, X_{j_{k-1}}]$, where $v(X)$ is a canonical v-word in the indeterminates $X_{k_1}, \dots, X_{k_r}$, and $\{2^{j_1}, \dots, 2^{j_k - 1}\}$ (respectively, $\{3^{nk_1}, \dots, 3^{nk_r}\}$) is the set of all even (respectively, odd) numbers in $E(r_\mu)$.

(4) $a_\mu(X)$ and $b_\mu(X)$ are canonical vw-polynomials. Our reasoning here splits in three parts depending on the type of $a_\mu(X)$.

(4-1) $a_\mu(X)$ is a vw-polynomial of the first type. Suppose that $a_\mu(X)$ is a polynomial defined by (11) in Definition 5.6. Let $k < l < m < p$ be the largest numbers in the set $\{i_1, \dots, i_{2q}, j_1, j_2, j_3, j_4\}$. By the construction of $\xi_n(a_\mu)$ (see Definition 5.6 (1), $E(r_\mu)$ contains a unique even number 2^e, where $e \in \{k, l, m, p\}$ and $i_\mu \in \{k, l, m, p\}$. With this in mind, one observes that $b_\mu(X)$ cannot be a vw-polynomial of the second or of the third type. Therefore, $b_\mu(X) = [X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}] w_{J'}^{(\lambda)}$, where $J' = \{k, l, m, p\}$ and $1 \leq \lambda \leq 5$. One can check straightforwardly (using Definition 5.6 (1)) that if in the Y-expansions of $\xi_n(a_\mu) \cdot a_\mu(X)$ and $\xi_n(a_\mu) \cdot b_\mu(X)$ the terms of the form $Y^{(\alpha)} X_\beta$ coincide, then $a_\mu(X) = b_\mu(X)$.

(4-2) $a_\mu(X)$ is a vw-polynomial of the second type, and $k < l < m < p$ are as above. So

$$[X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}] \cdot [X_{j_2}, \underbrace{X_{j_1}][X_{j_3}]}, X_{j_4}],$$

where $j_1 < j_2$, $\{j_1, j_2, j_3, j_4\} = \{k, l, m, p\}$ and $j_3 \neq k$. Then we have $r_\mu = 3^{nj_1} + 3^{nj_2} + 2^{j_3} + \sum_{i=1}^{2q} 3^{in}$, where $t_\mu = j_4$. As $j_3 \notin \{k, l, m, p\}$ and $\xi_n(a_\mu) \cdot b_\mu \neq 0$, one observes that b_μ cannot be a vw-polynomial of the first type in view of Remark 2.

Suppose that b_μ is a vw-polynomial of the second type. One can check straightforwardly (using item (2) of Definition 5.6) that if the terms of the form $Y^{(\alpha)} X_\beta$ are the same in the Y-expansions of $\xi_n(a_\mu) \cdot a_\mu(X)$ and $\xi_n(a_\mu) \cdot b_\mu(X)$, then $a_\mu(X) = b_\mu(X)$.

Finally, $b_\mu(X)$ cannot be a vw-polynomial of the third type. Indeed, $t_\mu = j_4 \in \{l, m, p\}$, and one checks straightforwardly (using item (3) of Definition 5.6) that if the Y-expansion of $\xi_n(a_\mu) \cdot b_\mu(X) \neq 0$ contains the term $Y^{(\alpha)} X_\beta$ then $\beta \in \{l, m, p\}$.

(4-3) $a_\mu(X)$ is a vw-polynomial of the third type. Then

$$[X_{i_1}, X_{i_2}] \cdots [X_{i_{2q-1}}, X_{i_{2q}}] \cdot [X_m, \underbrace{X_p][X_{j_2}}, X_{j_1}], \quad \text{where } j_1 < j_2.$$

It follows from item (3) of Definition 5.6 that $t_\mu = j_2 \notin \{l, m, p\}$ and $E(r_\mu)$ contains a unique even number 2^e for some $e \in \{l, m, p\}$. If $b_\mu(X)$ is of the first or of the second type and the Y-expansion of $\xi_n(a_\mu) \cdot b_\mu(X)$ contains the term $Y^{(\alpha)}X_\beta$ then either $E(\alpha)$ contains 2^i for $i \in \{l, m, p\}$ or $\beta \in \{l, m, p\}$. Therefore, $b_\mu(X)$ is a vw-polynomial of the third type. But then one checks straightforwardly (using item (3) of Definition 5.6) that if the Y-expansions of $\xi_n(a_\mu) \cdot a_\mu(X)$ and $\xi_n(a_\mu) \cdot b_\mu(X)$ contain the term $Y^{(\alpha)}X_\beta$ then $a_\mu(X) = b_\mu(X)$. This completes the proof of the theorem. □

Fix an arbitrary sequence $I = (i_1, \dots, i_{m-1})$ of pairwise distinct numbers in $\mathbb{N}(n)$ and a sequence $\Pi = (P_1, \dots, P_m)$ of m pairwise disjoint sets P_i such that $\cup_{i=1}^m P_i = \mathbb{N}(n) \setminus M(I)$. Denote by $S(I, \Pi)$ the subset of S_n formed by all $\sigma \in S_n$ such that $\sigma^{-1}(i_1) < \cdots < \sigma^{-1}(i_{m-1})$, and in the permutation

$$\sigma(1), \sigma(2), \dots, i_1, \dots, i_2, , \dots, i_{m-1}, , \dots, \sigma(n-1), \sigma(n)$$

the set of symbols occurring prior i_1 coincides with P_1 and of those between $i_{\nu-1}$ and i_ν coincides with P_ν. (In particular, i_1 is located at the $(|P_1|+1)$-th position.) Thus, $S(I, \Pi)$ depends on the ordering of the elements $\{i_1, \dots, i_{m-1}\}$ and on the choice of the subsets $P_1, \dots, P_m$.

Set $J = \{i_1, \dots, i_{m-1}\}$. If I' is a reordering of $i_1, \dots, i_{m-1}$ and Π' is a similar sequence for I' then $S(I, \Pi) \cap S(I', \Pi')$ is empty. It is easy to observe that for every fixed J we have $S_n = \cup_{I,\Pi} S(I, \Pi)$, where I ranges over the ordering of the elements of J, and Π are as above. In other words, $S_n = \cup_{I,\Pi} S(I, \Pi)$ is a partition of S_n.

This partition defines an equivalence relation on S_n, which is denoted by $R_J^{(n)}$. So $\sigma \equiv \sigma' \pmod{R_J^{(n)}}$ means $\sigma, \sigma' \in S(I, \Pi)$ for some I, Π as above.

Let $M \subseteq \mathbb{N}(n)$ and $L = \mathbb{N}(n) \setminus M = \{l_1, \dots, l_k\}$, where $l_1 < \cdots < l_k$. If $\sigma \in S_n$ then we denote by σ_L the permutation of the set L such that $D_M(X_{\sigma(1)} \cdots X_{\sigma(n)}) = X_{\sigma_i(l_1)} \cdots X_{\sigma_i(l_k)}$, where D_M is the differential operator in X_i ($i \in M$). For a sequence I it is convenient to denote by $M(I)$ the set $\{i_1, \dots, i_{m-1}\}$, that is, $M(I) = J$. In this notation state the following proposition.

Proposition 5.11. *Let $f = \sum_{\sigma \in S_n} \alpha_\sigma X_{\sigma(1)}, \dots, X_{\sigma(n)} \in P_n$, where $\alpha_\sigma \in K$. Then the following conditions are equivalent:*

(1) *There is no polynomial $g \neq 0$ following from f which is linear in $X_1, \dots, X_{m-1}$ and alternating in indeterminates from Y.*

(2) *For every set $S(I, \Pi) \subset S_n$ with $M(I) = J$ and for arbitrary $L \subset \mathbb{N}(n) \setminus M(I)$ we have*

$$\sum_{\sigma \in S(I,\Pi)} (-1)^{\sigma_L} \alpha_\sigma = 0. \tag{19}$$

P r o o f. (1) $\Longrightarrow$ (2). Let J be an arbitrary subset of cardinality $m-1$ in $\mathbb{N}(n)$. We shall prove that (19) holds for every set $S(I,\Pi)$ with $M(I)=J$.

Let $L\subset\mathbb{N}(n)\setminus J$. Consider the operator $\xi:=\varepsilon(L,\phi)$, where $\phi(i)=2^i$ for all $i\in L$ and $\phi(i)=3^{n_i}$ otherwise. It follows from Remark 2 that $\xi(X_{\sigma(1)}\cdots X_{\sigma(n)})=\pm\xi(X_{\tau(1)}\cdots X_{\tau(n)})$ if and only if $\sigma\equiv\tau(\operatorname{mod}R_J^{(n)})$. Taking into account Lemmas 5.1 and 5.2 we observe that in the Y-expansion of the polynomial $\xi(f)$ the coefficient of the Y-word that corresponds to $S(I,\Pi)$ with $M(I)=J$ is equal, up to the sign, to $2^{|L|}\sum_{\sigma\in S(I,\Pi)}(-1)^{\sigma_L}\alpha_\sigma$. Note that the polynomial $\xi(f)$ is alternating in the indeterminates in Y and linear in X_i ($i\in J$) and is a consequence of f. Therefore, $\sum_{\sigma\in S(I,\Pi)}(-1)^{\sigma_L}\alpha_\sigma=0$.

(2) $\Longrightarrow$ (1). Suppose the contrary. Then there exists an operator $\xi=\varepsilon(\mathbb{N}(n)\setminus J;\phi)$ such that $|J|=m-1$ and $\xi(f)\neq 0$. The elements $\xi(X_{\sigma(1)}\cdots X_{\sigma(n)})$ for $\sigma\in S_n$ are Y-words (with coefficients ±1). In addition, if $\sigma=\tau\,(\operatorname{mod}R_J^{(n)})$ then the corresponding Y-words coincide. Let $M=\{i\in L:\phi(i)\text{ is even}\}$. Using Lemmas 5.1 and 5.2, we observe that the coefficient of the Y-word $\xi(X_{\sigma(1)}\cdots X_{\sigma(n)})$ equals $\pm2^{|M|}\sum_{\tau\in S(I,\Pi)}(-1)^{\tau_M}\alpha_\tau$, where $M(I)=J$ and $\sigma\in S(I,\Pi)$. By (2), this number equals zero. This is a contradiction, and the proposition follows. $\square$

Lemma 5.12. *Let $B^{(m)}$ be the T-ideal of $K[X]$ generated by the polynomials f that have no consequences of the form $g(X_1,\ldots,X_{m-1};Y)$, where g is linear in $X_1,\ldots,X_{m-1}$ and alternating in the indeterminates in Y. Then $B^{(m)}$ is stable under the derivations $\partial/\partial X_i$ for all $i\in\mathbb{N}$.*

P r o o f. It suffices to check that $\dfrac{\partial}{\partial X_i}T_n^{(m)}\subseteq T_{n-1}^{(m)}$ for $i\in\mathbb{N}(n)$ and $n\geq m$. For every $\sigma\in S_n$ denote by $\sigma'\in S_{n-1}$ the permutation that satisfies $\dfrac{\partial}{\partial X_n}X_{\sigma(1)}\cdots X_{\sigma(n)}=X_{\sigma'(1)}\cdots X_{\sigma'(n-1)}$. Let $J\subseteq\mathbb{N}(n-1)$, where $|J|=m-1$. It is easy to see that if $\sigma\equiv\tau\pmod{R_J^{(n)}}$ then $\sigma'\equiv\tau'\pmod{R_J^{(n-1)}}$. Therefore, if (19) holds for $f=\sum\limits_{\sigma\in S_n}X_{\sigma(1)}\cdots X_{\sigma(m-1)}$ then (19) remains valid for $\dfrac{\partial f(X)}{\partial X_n}=\sum\limits_{\sigma'\in S_{n-1}}X_{\sigma'(1)}\cdots X_{\sigma'(m-1)}$, and the lemma follows. $\square$

It follows from this that $B^{(m)}=\overline{B}^{(m)}$, where $\overline{B}^{(m)}$ is the T-ideal in $\overline{K[X]}$ discussed in the proof of Theorem 5.10.

The proof of Theorem 1.4 follows from Theorem 5.10 and Lemma 5.12. This also implies Theorem 1.2.

REFERENCES

[1] A. Z. Ananin, A. R. Kemer. Varieties of associative algebras, whose subvariety lattices are distributive. *Sib. Mat. Zh.* **17** (1976), 723–730. (in Russian); English translation in *Sib. Math. J.* **17** (1976), 549–554.

[2] C. W. Curtis, I. Reiner. Representation Theory of Finite Groups and Associative Algebras. Interscience, John Willey and Sons, New York-London, 1962.

[3] G. K. Genov. The Spechtness of certain varieties of associative algebras over a field of zero characteristic. *C. R. Acad. Bulgare Sci.* **29** (1976), 939–941 (in Russian).

[4] G. K. Genov. Some Specht varieties of associative algebras. *Pliska Stud. Math. Bulgar.* **2** (1981), 30–40 (in Russian).

[5] A. Giambruno, M. Zaicev. Polynomial Identities and Asymptotic Methods. Mathematical Surveys and Monographs vol. **122**, Providence, RI, American Mathematical Society, 2005.

[6] G. D. James. The Representation Theory of the Symmetric Groups. Lecture Notes in Mathematics vol. **682**, Berlin–Heidelberg–New York, Springer-Verlag, 1978.

[7] V. N. Latyshev. On certain varieties of associative algebras. *Izv. Akad. Nauk SSSR, Ser. Mat.* **37** (1973), 1010–1037 (in Russian); English translation in *Math. USSR, Izv.* **7** (1973), 1011–1038.

[8] V. N. Latyshev. Partially ordered sets and nonmatrix identities of associative algebras. *Algebra i Logika* **15** (1976), 53–70 (in Russian); English translation in *Algebra Logic* **15** (1976), 34–45.

[9] V. N. Latyshev. Complexity of nonmatrix varieties of associative algebras. I, II. *Algebra i Logika* **16** (1977), 149–183, 184–199 (in Russian); English translation in *Algebra Logic* **16** (1977), 98–122, 122–133.

[10] V. N. Latyshev. Finite basis property of identities of certain rings. *Usp. Mat. Nauk* **32**, *4* (1977), 259–260 (in Russian).

[11] F. D. Murnaghan. On the analysis of the Kroneker product of irreducible representations of S_n. *Proc. Nat. Acad. Sci. U.S.A.* **41** (1955), 515–518.

[12] A. P. Popov. On the Specht property of some varieties of associative algebras. *Pliska Stud. Math. Bulgar.* **2** (1981), 41–53 (in Russian).

[13] W. Specht. Gezetze in Ringen, I. *Math. Z.* **52** (1950), 557–589.

A. E. Zalesskii
Institut Matematiki Nats. Acad. Navuk Belarusi
11, Vulitsa Surganova Str.
Minsk, Belarus
and
Dipartimento di Matematica e Applicazioni
Università degli Studi di Milano-Bicocca
via R. Cozzi 53, Milano, 20126, Italy
e-mail: `alexandre.zalesski@gmail.com`

Received January 30, 2012