Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **38** (2012), 395–416

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE GIBSON BOUNDS OVER FINITE FIELDS*

Mikhail V. Budrevich, Alexander E. Guterman

Communicated by M. Kochetov

Dedicated to Professor Yuri Bahturin on the occasion of his sixty fifth birthday

Abstract. We investigate the Pólya problem on the sign conversion between the permanent and the determinant over finite fields. The main attention is given to the sufficient conditions which guarantee non-existence of sing-conversion. In addition we show that $\mathbb{F}_3$ is the only field with the property that any matrix with the entries from the field is convertible. As a result we obtain that over finite fields there are no analogs of the upper Gibson barrier for the conversion and establish the lower convertibility barrier.

1. Introduction.

In this paper $\mathbb{F}_q$ denotes a finite field of q elements, its characteristic $\operatorname{char}\mathbb{F}_q = p > 2$, $\mathbb{F}$ denotes an arbitrary field, $M_{mn}(\mathbb{F}_q)$ is the set of $m \times n$ matrices with the entries from $\mathbb{F}_q$, $M_n(\mathbb{F}_q) = M_{nn}(\mathbb{F}_q)$. $\mathbb{F}_q^n$ denotes

2010 *Mathematics Subject Classification:* 15A15, 15A04.

Key words: permanent, determinant, finite fields, Pólya problem.

*The work on this project is partially supported by the grants MD-2502.2012.1 and RFBR 12-01-00140.

the vector space of n-vectors over $\mathbb{F}_q$. Let $X \circ A$ be the Hadamard (entrywise) product of the matrices X and A. By $A(i|j)$ we denote the matrix obtained from A by deleting the i-th row and the j-th column.

The permanent and determinant functions are well-known:

$$\operatorname{per}(A) = \sum_{\sigma \in S_n} a_{1\sigma(1)} \cdots a_{n\sigma(n)} \text{ and } \det(A) = \sum_{\sigma \in S_n} \operatorname{sgn}(\sigma) \mathrm{a}_{1\sigma(1)} \cdots \mathrm{a}_{\mathrm{n}\sigma(\mathrm{n})},$$

where A is an arbitrary square matrix, and S_n is the symmetric group of degree n, and $\operatorname{sgn}(\cdot)$ denotes the sign function.

The permanent and determinant functions look very similar, however, these two functions have considerably different behavior. For example, the difference between these functions becomes visible when we consider the complexity of their computation. The determinant function can be computed easily by the Gauss elimination algorithm in polynomial time. However, it is still an open problem if there exists a polynomial algorithm to compute the permanent. Moreover, Valiant [23] has shown that even computing the permanent of a (0,1)-matrix is a $\sharp P$-complete problem, i.e., this problem is an arithmetic analogue of Cook's hypothesis $P \neq NP$, see [6, 9, 10, 14] for details.

In 1913 Pólya [21] asked if there exists a possibility to compute the permanent function using the determinant function. The idea is based on the following observation: Consider the map $T\colon M_2(\mathbb{F}) \to M_2(\mathbb{F})$ given by the formula:

$$T : \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \to \begin{pmatrix} a_{11} & -a_{12} \\ a_{21} & a_{22} \end{pmatrix} \tag{1.1}$$

Direct computations show that

$$\operatorname{per}(A) = \det(T(A)) \tag{1.2}$$

for any $A \in M_2(\mathbb{F})$. The existence question for such T which satisfies (1.2), where T is nothing but the multiplication of some matrix elements by -1 is called Pólya's permanent problem, see [18].

Pólya's permanent problem for the field of complex numbers was solved negatively by Szegö [22], namely he proved that for $n \geq 3$ there is no generalization of the formula (1.1). Then several authors investigated possible types of apriori convertible or apriori non-convertible matrices. The first result showing that there exists a quantitative barrier for the sign conversion for $(0,1)$-matrices is due to Gibson [12]. Namely, Gibson proved that if $v(A) > \Omega_n = (n^2 + 3n - 2)/2$ then A is not convertible, where $v(A)$ is the number of nonzero entries of A, i.e., he provided an upper barrier Ω_n for the conversion over a field $\mathbb{F}$ of zero characteristic. In the extremal case, when $v(A) = \Omega_n$, the convertible matrix is unique up to the permutation equivalence. Later, see for example the paper [3] by Brualdi, Shader and the references therein, the attention was given to convertible $(0,1)$-matrices with less than Ω_n units such that after changing any of its zeros to a unit, the matrix becomes non-convertible. In the work by Little [16] the low barrier $\omega_n = n + 5$ for convertibility in $M_n(\mathbb{F})$, $\operatorname{char}\mathbb{F} = 0$, was obtained by the methods of graph theory. Namely he proved that any A with $v(A) \leq n+5$ is sign convertible. In [7] a short purely matricial proof of this result was established and it was shown that for any value v between ω_n and Ω_n both sign-convertible and sign non-convertible $(0,1)$-matrices with v units do exist.

In parallel over finite fields there were attempts to investigate the transformations converting the determinant to the permanent, see [5, 8], where it is proved that there are no bijective determinant-permanent converters.

This work is devoted to the investigation of sign-convertibility between the determinant and the permanent over finite fields.

Our paper is organized as follows. Section 1 is introductory. In Section 2 we show that over the field of 3 elements any matrix is sign-convertible. Section 3 is devoted to the proof that over any other field there are matrices which are not sign-convertible. In Section 4 we investigate Gibson convertibility barriers over finite fields in terms of the so called essential elements of matrices.

2. Pólya's problem over the field $\mathbb{F}_3$.

Definition 2.1. *A matrix $A \in M_n(\mathbb{F}_q)$ is called sign-convertible if there exists $X \in M_n(\{\pm 1\})$ such that* $\operatorname{per}(A) = \det(A \circ X)$*.*

In this section we are going to prove that over the field of 3 elements sign-convertibility is always possible. In order to do this we need to prove the following auxiliary lemmas.

Lemma 2.2. *Let $A \in M_n(\mathbb{F}_3)$ and* $\operatorname{per}(A) = 1$*. Then there exists a matrix $X \in M_n(\{\pm 1\})$ such that* $\det(A \circ X) = 1$*.*

P r o o f. We proceed by induction.

The base of induction is the case $n = 2$. Let

$$A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}.$$

Then consider the following matrix

$$X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}.$$

It is straightforward to check that $\text{per}(A) = \det(X \circ A)$.

The inductive step. Assume the statement is true for the matrices of size less than or equal to $n-1$. Consider $A \in M_n(\mathbb{F}_3)$.

If $\det(A) = 1$ then we consider $X = (x_{ij})$, where $x_{ij} = 1$ for all $1 \le i, j \le n$.

If $\det(A) = -1$ then we consider $X = (x_{ij})$, where $x_{1j} = -1$ and $x_{ij} = 1$ for all $1 \le j \le n$, $2 \le i \le n$.

Let us consider the case $\det(A) = 0$.

By the Laplace decomposition for $\det(A)$ along the first row we have

$$\det(A) = a_{11}\det(A(1|1)) - \cdots + (-1)^{n+1} a_{1n} \det(A(1|n)).$$

1. Assume there exists j such that $a_{1j}\det(A(1|j)) \neq 0$. There are two possible cases:

A. $(-1)^{1+j} a_{1j} \det(A(1|j)) = 1$. We consider $X = (x_{kl})$, where $x_{kl} = 1$ for all $1 \le k, l \le n$, with $(k,l) \neq (1,j)$, and $x_{1j} = -1$. Then $\det(X \circ A) = \det(A) - (-1)^{1+j} a_{1j} \det(A(1|j)) + (-1)^{1+j} x_{1j} a_{1j} \det(A(1|j)) = \det(A) - 2 = 1$. We have obtained the required matrix X.

B. $(-1)^{1+j} a_{1j} \det(A(1|j)) = -1$. We consider $X = (x_{kl})$, where $x_{kl} = 1$ for all $2 \le k \le n$, $1 \le l \le n$, and $x_{1j} = 1$, $x_{1,l} = -1$ for all $1 \le l \le n$ with $l \neq j$. Then $\det(X \circ A) = -(\det(A) - (-1)^{1+j} a_{1j} \det(A(1|j)) + (-1)^{1+j} x_{1j} a_{1j} \det(A(1|j))) = -(\det(A) + 2) = 1$.

2. Assume now that for any j we have $a_{1j}\det(A(1|j)) = 0$. Since $\text{per}(A) = 1$ then there exists a number k such that $a_{1k}\text{per}(A(1|k)) \neq 0$. Let us apply the inductive hypothesis for $A(1|k)$: there exists $Y' \in M_{n-1}(\{\pm 1\})$ such that $\det(A(1|k) \circ Y') \neq 0$. Consider $Y \in M_n(\{\pm 1\})$ such that $Y(1|k) = Y'$ and the other entries of Y are 1.

Now for the matrix $Y \circ A$ there exists a nonzero element in the Laplace decomposition of $\det(Y \circ A)$ along the first row. By Case 1 we can construct a matrix Z such that $\det(Z \circ (Y \circ A)) = 1$. Consider now $X = Z \circ Y$. $\square$

Lemma 2.3. *Let Σ be the sum of two or more elements equal to 1 in $\mathbb{F}_3$. Then it is possible to change the signs of the elements in Σ in such a way that the obtained sum $\Sigma' = 0$.*

Proof. Consider all possible cases for the value of Σ.

1. Let $\Sigma = 0$. Then $\Sigma' = \Sigma = 0$ (without any changes of the signs).

2. Let $\Sigma = 1$. Then Σ contains $3k+1$ elements. Thus Σ contains at least four elements, each of them equals to 1. Let us change the signs of two of them. Then for the new sum $\Sigma' = \Sigma - 2 - 2 = 0$.

3. Let $\Sigma = -1$. Let us change one sign. Thus we have $\Sigma' = \Sigma - 1 - 1 = -1 - 2 = 0$. $\square$

Corollary 2.4. *Let Σ be the sum of certain elements in $\mathbb{F}_3$. Assume that among them there are two or more non-zero elements. Then it is possible to change the signs of the elements in Σ in such a way that the obtained sum $\Sigma' = 0$.*

P r o o f. Indeed we change the signs of all elements equal to -1 to obtain $+1$ and then apply Lemma 2.3. $\square$

Lemma 2.5. *Let $A \in M_n(\mathbb{F}_3)$. Assume that* $\text{per}(A) = 0$. *Then there exists a matrix $X \in M_n(\{\pm 1\})$ such that* $\det(A \circ X) = 0$.

P r o o f. In the case $\det(A) = 0$ it is sufficient to take the matrix $X = (x_{ij})$ with $x_{ij} = 1$ for all $i, j = 1, \ldots, n$.

Thus further we assume that $\det(A) \neq 0$. We shall show that there exists $X \in M_n(\mathbb{F}_3)$ such that the columns of $X \circ A$ are linearly dependent. We now start the proof by induction.

For $n = 2$ we choose $X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$.

Assume the statement is true for all matrices of size less than or equal to $n - 1$. We would like to show that it is also true for the matrices of size n. Consider the number of non-zero elements in the rows of A.

Case 1. Assume that in each row there exist at least two nonzero elements. This implies that in the sum $\sum\limits_{j=1}^{n} a_{ij}x_{ij}$ there are at least two nonzero elements for any fixed $i = 1, \ldots, n$. Therefore we can apply Corollary 2.4 to this sum. We set $x_{ij} = -1$ if the application of Corollary 2.4 assumes change of the sign of a_{ij} and $x_{ij} = 1$ otherwise. Therefore we obtain the matrix $X = (x_{ij})$ satisfying $\sum\limits_{j=1}^{n} a_{ij}x_{ij} = 0$ for any i. Thus the rows of $X \circ A$ are linearly dependent over $\mathbb{F}_3$ with the coefficients $\lambda_1 = \cdots = \lambda_n = 1$. Therefore the required matrix X is constructed.

Case 2. It is straightforward to see that since $\det(A) \neq 0$, the matrix A does not contain a zero row. Thus it remains to consider only the following case.

Case 3. Assume that there exists a row of A with unique nonzero element. Permuting the rows and the columns of A we can move this nonzero element to position $(1,1)$. We obtain

$$PAQ = A^{(1)} = \begin{pmatrix} \pm 1 & 0 & \cdots & 0 \\ a_{21}^{(1)} & a_{22}^{(1)} & \cdots & a_{2n}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1}^{(1)} & a_{n2}^{(1)} & \cdots & a_{nn}^{(1)} \end{pmatrix},$$

where P, Q are permutation matrices. The following equalities are true:

$$0 = \operatorname{per}(A) = \operatorname{per}(A^{(1)}) = a_{11}^{(1)} \operatorname{per}(A^{(1)}(1|1)).$$

In addition $a_{11}^{(1)} \neq 0$ and therefore $\operatorname{per}(A^{(1)}(1|1)) = 0$. Thus by the inductive hypothesis for $A^{(1)}(1|1)$ we can find $Y \in M_n(\{\pm 1\})$ such that $\det(A^{(1)} \circ Y) = a_{11} y_{11} \det(A^{(1)}(1|1) \circ Y(1|1)) = 0$. We choose $X = P^{-1} Y Q^{-1}$. Then

$$\det(A \circ X) = \det\Big((P^{-1} A^{(1)} Q^{-1}) \circ (P^{-1} Y Q^{-1})\Big) = \det\Big(P^{-1} (A^{(1)} \circ Y) Q^{-1}\Big)$$

since P and Q are permutation matrices. Hence, by the multiplicativity of the determinant we have that

$$\det(A \circ X) = \det(P^{-1} Q^{-1}) \det\Big(A^{(1)} \circ Y\Big) = 0.$$

Thus the result follows. □

Theorem 2.6. *If $A \in M_n(\mathbb{F}_3)$, then there exists $X \in M_n(\{\pm 1\})$ such that* $\operatorname{per}(A) = \det(A \circ X)$.

Proof.

1. Let $\operatorname{per}(A) = 0$. By Lemma 2.5 we can choose a matrix $X \in M_n(\{\pm 1\})$ such that $\det(A \circ X) = 0$.

2. Let $\operatorname{per}(A) = \pm 1$. It is sufficient to consider the case when $\operatorname{per}(A) = 1$ (otherwise we multiply the first row of X by -1). By Lemma 2.2 we derive the required statement. □

3. Nonconvertible matrices over fields with more than 3 elements. Now we prove that $\mathbb{F}_3$ is the only field for which Pólya's sign-convertibility between the permanent and the determinant is always possible.

3.1. Fields of characteristic 3. In this section we use the presentation of the field $\mathbb{F}_{3^k}$ as a quotient ring $\mathbb{F}_{3^k} \simeq \mathbb{F}_3[x]/(f(x))$, where $\mathbb{F}_3[x]$ is a polynomial ring over $\mathbb{F}_3$ and $f(x)$ is an irreducible polynomial of degree k. The details related to this presentation can be found for example in [25, Chapter 9.5].

Theorem 3.1 [25, Theorem 9.5.7]**.** *For any prime p and $n \in \mathbb{N}$ there exists a field with p^n elements. This field is unique up to isomorphism.*

Theorem 3.2. [25, Corollary on p. 383]**.** *For any $n > 1$ there exists an irreducible polynomial $f(x) \in \mathbb{F}_p[x]$ of degree n and $\mathbb{F}_{p^n} \simeq \mathbb{F}_p[x]/(f(x))$.*

Proposition 3.3. *Let $\mathbb{F}_q$ be a finite field of characteristic 3, $q = 3^k$, $k > 1$. Then there exists a non-convertible (3×3)-matrix over $\mathbb{F}_q$.*

P r o o f. Let us choose an irreducible polynomial $f(x) \in \mathbb{F}_3[x]$, $^\circ f = k$, and consider $\mathbb{F}_{3^k} \simeq \mathbb{F}_3[x]/(f(x))$. Note that since $^\circ f = k > 1$ the element $x+1$ is defined correctly in this quotient ring. Let

$$A = \begin{pmatrix} x+1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in M_3(\mathbb{F}_{3^k}).$$

Let us show that the matrix A is not convertible. In the beginning we compute its permanent

$$\text{per} \begin{pmatrix} x+1 & -1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} = 2(x+1) \tag{3.1}$$

Suppose that A is convertible. Let $Y = (y_{ij}) \in M_3(\{\pm 1\})$ and $\det(Y \circ A) = 2(x+1)$. By the Laplace decomposition we obtain

$$\det(Y \circ A) = ax + b$$

where $a = y_{11}(y_{22}y_{33} - y_{23}y_{32})$ and

$$b = y_{11}(y_{22}y_{33} - y_{23}y_{32}) + y_{12}(y_{21}y_{33} - y_{23}y_{31}) + y_{13}(y_{21}y_{32} - y_{22}y_{31}).$$

Since Y is a converter we obtain

$$ax + b = 2(x+1). \tag{3.2}$$

By assumption $k > 1$, and therefore x and 1 are linearly independent independent over $\mathbb{F}_3$. It follows from the equality (3.2) that $a = 2$ and $b = 2$. Thus

$$2 = y_{11}(y_{22}y_{33} - y_{23}y_{32}). \tag{3.3}$$

Without loss of generality we assume that $y_{11} = 1$, otherwise we may multiply the first and the second rows of Y by -1 to obtain a new converter satisfying the required conditions. Then by the equality (3.3) we have that $y_{22}y_{33} - y_{23}y_{32} = 2$. Since $y_{ij} \in \{1, -1\}$ we have one of the following four possibilities for the matrix Y:

$$\begin{pmatrix} 1 & y_{12} & y_{13} \\ y_{21} & 1 & 1 \\ y_{31} & -1 & 1 \end{pmatrix}, \quad \begin{pmatrix} 1 & y_{12} & y_{13} \\ y_{21} & 1 & -1 \\ y_{31} & 1 & 1 \end{pmatrix},$$

$$\begin{pmatrix} 1 & y_{12} & y_{13} \\ y_{21} & -1 & 1 \\ y_{31} & -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} 1 & y_{12} & y_{13} \\ y_{21} & -1 & -1 \\ y_{31} & 1 & -1 \end{pmatrix}. \tag{3.4}$$

Note that all these matrices have the same determinant, and we can transform any of them to another one multiplying by -1 two of the rows or a row and a column. Hence, either all these four matrices listed in (3.4) define converters for A or none of them. Therefore, we can assume that Y has the following form:

$$Y = \begin{pmatrix} 1 & y_{12} & y_{13} \\ y_{21} & 1 & 1 \\ y_{31} & -1 & 1 \end{pmatrix}.$$

By the Laplace decomposition formula for $\det(Y \circ A)$ we have

$$\det(Y \circ A) = \det \begin{pmatrix} (x+1) & -y_{12} & y_{13} \\ y_{21} & 1 & 1 \\ y_{31} & -1 & 1 \end{pmatrix}$$

$$= 2(x+1) + y_{12} \det \begin{pmatrix} y_{21} & 1 \\ y_{31} & 1 \end{pmatrix} + y_{13} \det \begin{pmatrix} y_{21} & 1 \\ y_{31} & -1 \end{pmatrix} = \operatorname{per}(A) + \Delta,$$

where

$$\Delta = \det \begin{pmatrix} y_{21} & y_{12} + y_{13} \\ y_{31} & y_{12} - y_{13} \end{pmatrix}. \tag{3.5}$$

Let us show that $\Delta \neq 0$. Observe that $Y \in M_3(\{\pm 1\})$. Therefore, we have to consider the following two cases:

1. $y_{12} = -y_{13}$. Then (3.5) has the form

$$\Delta = \det \begin{pmatrix} y_{21} & 0 \\ y_{31} & -y_{12} \end{pmatrix} = -y_{21}y_{12} \neq 0$$

since $y_{ij} \in \{1, -1\}$.

2. $y_{12} = y_{13}$. Then (3.5) has the form

$$\Delta = \det \begin{pmatrix} y_{21} & -y_{12} \\ y_{31} & 0 \end{pmatrix} = y_{31}y_{12} \neq 0$$

The last inequality is also true since by the definition $y_{ij} \neq 0$.

Hence, $\Delta \neq 0$, and thus the required Y does not exists. □

3.2. Fields of characteristic $p > 3$.

Lemma 3.4. *Let $B \in M_3(\{\pm 1\}) \subseteq M_3(\mathbb{F}_q)$ and $\mathbb{F}_q$ be a field of characteristic $p > 3$. Then* $\det(B) \neq 2$.

P r o o f.

$$\det(B) = b_{11}b_{22}b_{33} + b_{12}b_{23}b_{31} + b_{21}b_{32}b_{13} - b_{13}b_{22}b_{31} - b_{12}b_{21}b_{33} - b_{11}b_{23}b_{32}.$$

Each of the above summands is either 1 or -1 since $B \in M_3(\{\pm 1\})$. Each entry b_{ij} is a factor of exactly two different summands. Hence, the change of the sign of b_{ij} does not reflect to the parity of the number of negative summands. In this way both the number of 1 and the number of -1 among these summands are always odd. Hence $\det(B) \in \{0, 4, -4\}$. Thus $\det B \neq 2$ in $\mathbb{F}_p$. □

Theorem 3.5. *Let $n \geq 3$ and $\mathbb{F}_q$ be any field with* $\operatorname{char} \mathbb{F}_q > 3$. *Then there exists a nonconvertible matrix $A_n \in M_n(\mathbb{F}_q)$.*

P r o o f. Consider

$$A_3 = \begin{pmatrix} -1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} \in M_3(\mathbb{F}_q)$$

Direct computations show that $\operatorname{per}(A_3) = 2$. By Lemma 3.4 for any $X \in M_3(\{\pm 1\})$ we have $\det(B \circ X) \neq 2$ and therefore A_3 is not convertible.

Now for arbitrary $n > 3$ we choose the matrix $A_n = I_{n-3} \oplus A_3 \in M_n(\mathbb{F}_q)$ which concludes the proof. □

Remark 3.6. Theorem 2.6, Proposition 3.3 and Theorem 3.5 provide examples of matrices which are convertible over the field $\mathbb{F}_3$ and are not convertible over some other fields.

4. Convertibility barriers for matrices over finite fields.

In the sequel we need the following notion of essential rank of a matrix.

Definition 4.1. *An* essential rank (*see* [20], [2]) $\mathrm{ess}\,(A)$ *of* $A \in M_{n,k}(\mathbb{F}_q)$ *is the maximal* r *such that any* r *columns in* A *are linearly independent.*

In Lemmas 4.2 and 4.3 below we construct two nonsingular matrices with nonzero elements that will be used in our further considerations.

Lemma 4.2. *Let* $\mathrm{char}(\mathbb{F}_q) = p > 3$*,* n *be not divisible by* p*, and* $G = (g_{ij}) \in M_n(\mathbb{F}_q)$ *be the following matrix:*

$$G = \begin{pmatrix} 2 & 2 & 2 & \dots & 2 & 3+\delta \\ 1 & 2 & 2 & \dots & 2 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & 1 & 2 & 1 \\ 1 & 1 & \dots & \dots & 1 & 1 \end{pmatrix}, \tag{4.1}$$

where

$$g_{ii} = \begin{cases} 1 & \text{if } i = n, \\ 3 & \text{if } 2n - i \text{ is divisible by } p, \\ 2 & \text{otherwise,} \end{cases}$$

$$g_{ij} = \begin{cases} 1 & \text{if } i > j \text{ or } j = n \text{ and } i \neq 1, \\ 2 & \text{if } i < j < n, \\ 3+\delta & \text{if } (i,j) = (1,n), \end{cases}$$

and

$$\delta = \begin{cases} 1 & \text{if } 2n+1 \text{ is divisible by } p, \\ 0 & \text{otherwise.} \end{cases}$$

Then $\mathrm{rk}\,(M) = n$*. Moreover, the sum of all entries in any row of* M *is not equal to* 0.

Proof. Apply elementary transformations to transfer the matrix into some simpler form. Namely,

1. Let us subtract the first column from the others. We obtain the matrix

$$G_1 = \begin{pmatrix} 2 & 0 & 0 & \dots & 0 & 1+\delta \\ 1 & 1 & 1 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & 0 & 1 & 0 \\ 1 & 0 & \dots & \dots & 0 & 0 \end{pmatrix}.$$

Note that the last row of the obtained matrix contains a unique nonzero element.

2. Then we subsequently subtract the last row from the rows with indexes $2, \dots, n-1$ with multiplicity 1 and from the first row with multiplicity 2, and obtain the matrix

$$G_2 = \begin{pmatrix} 0 & 0 & 0 & \dots & 0 & 1+\delta \\ 0 & 1 & 1 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & 0 & 1 & 0 \\ 1 & 0 & \dots & \dots & 0 & 0 \end{pmatrix}$$

which differs from G_1 only in the first column.

3. Now we use Laplace decomposition for the determinant of G_2 along the first row and then along the last row. This produces the factor $-(1+\delta)$, since the sign is $(-1)^{n+1+n} = -1$, and we receive:

$$\det G_2 = -(1+\delta) \cdot \det \begin{pmatrix} 1 & 1 & 1 & \dots & 1 \\ 0 & 1 & \dots & \dots & 1 \\ \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & \dots & 0 & 1 \end{pmatrix}. \tag{4.2}$$

In the right-hand side of the equality (4.2) there is an upper triangular $(n-2) \times (n-2)$ matrix with nonzero elements on the main diagonal. Hence, its determinant is nonzero. Since $\delta \in \{0, 1\}$, it follows that $(1+\delta) \neq 0$. Therefore, the right-hand side is nonzero, hence, $\operatorname{rk}(G) = n$.

Now let us check that the sum of elements in each row is different from 0. For the first row we have $2n + 1 + \delta$. By the definition of our matrix δ was chosen in such a way that $2n + 1 + \delta \not\equiv 0 \pmod{p}$.

Let $1 < k < n$. If $p \nmid (2n - k)$ then $g_{kk} = 2$. Thus we have that the sum of the elements in the k-th row is equal to $(k-1) + 2(n-k) + 1 = 2n - k \not\equiv 0 \pmod{p}$. If $p \mid (2n-k)$ then $g_{kk} = 3$. Thus we have that the sum of the elements in the k-th row is equal to $(k-1) + 2(n-k-1) + 3 + 1 = 2n - k + 1 \not\equiv 0 (\operatorname{mod} p)$. For the last row the sum is n which is not divisible by p by the assumption. □

Lemma 4.3. *Let* $\operatorname{char}(\mathbb{F}_q) = p > 3$*,* n *be divisible by* p*, and* $H = (h_{ij}) \in M_n(\mathbb{F}_q)$ *be the following matrix:*

$$(4.3) \qquad H = \begin{pmatrix} 2 & 2 & 2 & \dots & 2 & 1 \\ 1 & 2 & 2 & \dots & 2 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & 1 & 2 & 1 \\ 1 & 1 & \dots & \dots & 1 & 2 \end{pmatrix},$$

where

$$h_{ij} = \begin{cases} 1 & \text{if } i > j \text{ or } j = n, \\ 2 & \text{if } i < j < n, \end{cases} \quad \text{and} \quad h_{ii} = \begin{cases} 3 & \text{if } 2n - i \text{ is divisible by } p, \\ 2 & \text{otherwise.} \end{cases}$$

Then $\operatorname{rk}(H) = n$*. Moreover each row sum of the matrix* H *is different from* 0*.*

Proof. We are going to show that H is nonsingular. In order to do this we transform it to some simpler form by elementary transformations.

1. We subtract the first column from the others and then obtain the following matrix:
$$H_1 = \begin{pmatrix} 2 & 0 & 0 & \dots & 0 & -1 \\ 1 & 1 & 1 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & 0 & 1 & 0 \\ 1 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

2. There are only 2 nonzero elements in the first row, namely h_{11}, h_{1n}, and only two nonzero elements in the last row, namely h_{n1}, h_{nn}. If we add the last row to the first one, then we obtain the following matrix:
$$H_2 = \begin{pmatrix} 3 & 0 & 0 & \dots & 0 & 0 \\ 1 & 1 & 1 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 1 & \dots & \dots & 0 & 1 & 0 \\ 1 & 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

3. We use the Laplace decomposition of the determinant of H_2 along the first

row. Then the following equality holds:

$$\det H_2 = 3 \det \begin{pmatrix} 1 & 1 & \dots & 1 & 0 \\ 0 & 1 & \dots & 1 & 0 \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & 0 & 1 & 0 \\ 0 & \dots & \dots & 0 & 1 \end{pmatrix}.$$

4. In the right-hand side of the last equality there is an upper triangular matrix with nonzero elements on the main diagonal. Hence, its determinant is nonzero. Since $\operatorname{char} \mathbb{F}_q \neq 3$, it follows that $\det H_2 \neq 0$, hence $\operatorname{rk}(H) = n$.

Let us show that the sum of elements in each row is different from 0. In the first row we have $2n - 1 = -1$ in $\mathbb{F}_q$ since n is divisible by p.

For the k-th row $1 < k < n$ we have that if $p \not| (2n - k)$ then $h_{kk} = 2$, and the sum is $(k-1) + 2(n-k) + 1 = 2n - k \not\equiv 0 (\mathrm{mod} p)$. However, if k is such that $2n - k$ is divisible by p, then $h_{kk} = 3$, and hence the sum of the elements in the kth row is equal to $2n - k + 1 \not\equiv 0 (\mathrm{mod}\, p)$. The corresponding sum for the last row is equal to $n + 1 = 1$ in $\mathbb{F}_q$ since n is divisible by p. □

Now we construct an $n \times (n+1)$ matrix with all elements different from zero and essential rank equal n.

Lemma 4.4. *Let $\mathbb{F}_q$ be a field of characteristic $p > 3$. Let $C \in M_{n,n+1}(\mathbb{F}_q)$ be such that its first n columns coincide with the first n columns of G from Lemma* 4.2 *and the $(n+1)$st column is the sum of all columns of G. Let $D \in M_{n,n+1}(\mathbb{F}_q)$ be such that its first n columns coincide with the first n columns of H from Lemma* 4.3 *and the $(n+1)$st column is the sum of all columns of H. Then the following conditions are satisfied:*

1. *All elements of C and D are nonzero.*

2. $\operatorname{ess}(C) = \operatorname{ess}(D) = n$.

P r o o f. By their definitions matrices G and H do not contain zero elements. Therefore it is sufficient to show that the last columns of C and D also do not contain zero elements. By Lemmas 4.2 and 4.3 the sum of the elements in each row of G and H is not equal to 0. Therefore the vectors obtained as sums of column vectors also do not have zero elements and all elements of C and D are different from 0.

Now we are going to prove that $\operatorname{ess}(C) = \operatorname{ess}(D) = n$. Assume the opposite, namely that the essential rank $\operatorname{ess}(C) < n$ or $\operatorname{ess}(D) < n$. This implies that there exists a singular square submatrix of C or D of order n. By Lemmas 4.2 and 4.3 the submatrices $C(|n+1)$ and $D(|n+1)$ are nonsingular. Consider any submatrix $C(|k), k < n+1$. We subtract all the columns of $C(|k)$ from the last one. By the choice of C, the last column of the obtained matrix is equal to the k-th column of C. This implies that we have obtained a matrix which differs from G only by a permutation of columns. Therefore $C(|k)$ is nonsingular. This is the contradiction and thus the essential rank of C is equal to n. Similar considerations with D and H provide the same result. □

Now we are going to prove several technical lemmas about ± combinations of elements in a finite field.

Lemma 4.5. *Let us consider the field $\mathbb{F}_p$, $|\mathbb{F}_p| = p > 2$ be a prime number. Let $a_1, \ldots, a_{p-1} \in \mathbb{F}_p \setminus \{0\}$, possibly equal. Then for any $a \in \mathbb{F}_p$ there exists a $(p-1)$-tuple $(\delta_1, \ldots, \delta_{p-1})$, where $\delta_i \in \{\pm 1\}$, such that $\sum\limits_{i=1}^{p-1} \delta_i a_i = a$.*

Proof. It is sufficient to show that

$$\left|\left\{\sum_{i=1}^{k} \delta_i a_i \mid \delta_i \in \{\pm 1\}\right\}\right| \geq k+1 \tag{4.4}$$

for each $k = 1, \ldots, p-1$.

Note that $a_1 \neq -a_1$ since $\operatorname{char} \mathbb{F}_p \neq 2$. Hence $|\{a_1, -a_1\}| = 2$, and the condition (4.4) is satisfied for $k = 1$.

We proceed by induction on k. Note that if on some step we have obtained all the elements from $\mathbb{F}$, then the condition (4.4) is satisfied for all k and we are done. So, by induction we further assume that for any $l < k$ it holds that $N_l = \left|\left\{\sum\limits_{i=1}^{l} \delta_i a_i \mid \delta_i \in \{\pm 1\}\right\}\right| \geq l+1$ and $N_l < p$. To prove the lemma we need to show that after we add the element $a_k \in \mathbb{F}_p$ to the family $a_1, \ldots, a_{k-1}$ the number of different elements that can be obtained as ± combinations of $a_1, \ldots, a_k$ is greater than the number of such combinations for $a_1, \ldots, a_{k-1}$ at least by 1.

Let us denote the sets of all possible different ± combinations

$$\{b_1, \ldots, b_N\} := \left\{\sum_{i=1}^{k-1} \delta_i a_i \mid \delta_i \in \{\pm 1\}\right\}, \quad M := \left\{\sum_{i=1}^{k} \delta_i a_i \mid \delta_i \in \{\pm 1\}\right\}$$

and consider two induced sets:

$$M_1 = \{b_1 + a_k, \ldots, b_N + a_k\} \text{ and } M_2 = \{b_1 - a_k, \ldots, b_N - a_k\} \subseteq M.$$

Note that $|M_1| = |M_2| = N$ because all elements $b_1, \ldots, b_N$ are different. Hence

$$|M_1 \cup M_2| \geq N. \tag{4.5}$$

Suppose there is an equality in (4.5). Comparing this assumption with the fact that $|M_1| = |M_2| = N$ we obtain that $M_1 = M_2$. Since these sets are equal we have that the sums of all elements of these sets are equal, i.e.,

$$\sum_{i=1}^{N}(b_i + a_k) = \sum_{i=1}^{N}(b_i - a_k).$$

After simplification we obtain

$$2Na_k = 0.$$

This is a contradiction since no one of these factors is zero. Hence the inequality in formula (4.5) is strict. Thus $|M| \geq |M_1 \cup M_2| > N$. By the inductive hypothesis $N \geq (k-1)+1 = k$. This concludes the proof. □

Corollary 4.6. *Let us consider the field $\mathbb{F}_p$, where $|\mathbb{F}_p| = p$ is a prime number. Let $m \geq p$ and $a_1, \ldots, a_m$ be a tuple of fixed non-zero elements of $\mathbb{F}_p$, we admit that possibly some of them may coincide. Then for any $a \in \mathbb{F}_p$ there exists an m-tuple $(\delta_1, \ldots, \delta_m)$, where $\delta_i \in \{\pm 1\}$, $i = 1, \ldots, m$, such that $a = \sum_{i=1}^{m} \delta_i a_i$.*

Proof. Let us consider $x = a_p + \ldots + a_m$. By Lemma 4.5 there is a tuple $(\delta_1, \ldots, \delta_{p-1})$, where $\delta_i \in \{\pm 1\}$, such that $a - x = \sum_{i=1}^{p-1} \delta_i a_i$. Therefore

$$a = \sum_{i=1}^{p-1} \delta_i a_i + \sum_{i=p}^{m} a_i = \sum_{i=1}^{m} \delta_i a_i, \quad \text{where } \delta_i \in \{\pm 1\}, i = 1, \ldots, m. \qquad \square$$

The next example shows that the result of Lemma 4.5 cannot be further improved for $m < p - 1$.

Example 4.7. Let us consider $p-2$ non-zero elements $a_1 = a_2 = \ldots = a_{p-2} = 1$. Then not all elements of $\mathbb{F}_p$ can be represented as $\pm$ combinations of $a_1, \ldots, a_{p-2}$. Namely, 0 cannot be equal to a sum of all these elements with $\pm$ signs since $p-2 < p$ and $(p-2)$ is odd.

Now, we are going to prove that all elements of $\mathbb{F}_p \setminus \{0\}$ can be represented as $\pm$ combinations of any non-zero $p-2$ elements of $\mathbb{F}$.

Corollary 4.8. *Let $\mathbb{F}_p$ be a field, $|\mathbb{F}_p| = p > 2$, p be a prime number. Assume that $a_1, \ldots, a_m \in \mathbb{F}_p \setminus \{0\}$, $m \geq p-2$, are non-zero elements of $\mathbb{F}_p$, possibly some of them are equal. Then for any $a \in \mathbb{F}_p \setminus \{0\}$ there exists an m-tuple $(\delta_1, \ldots, \delta_m)$, where $\delta_i \in \{\pm 1\}$, $i = 1, \ldots, m$, such that $a = \sum\limits_{i=1}^{m} \delta_i a_i$.*

Proof. If $m \geq p-1$ then the result follows from Lemma 4.5 and Corollary 4.6. Let us show that the result holds if $m = p-2$.

For $p = 3$ we have $m = 1$ and $\{a_1, -a_1\} = \{1, 2\} = \mathbb{F}_3 \setminus \{0\}$.

Assume now that $p > 3$. The rest of the proof splits into the following two cases:

Case 1. Let us assume that among $a_1, \ldots, a_{p-2}$ there are at least two elements a, b such that $a \neq \pm b$. Without loss of generality $a_1 \neq \pm a_2$. Then the four elements $\pm a_1 \pm a_2$ are distinct. Indeed, assume in the contrary that there are equal elements among them. Then one of the following equalities is true:

$$2(a_1 \pm a_2) = 0;\ 2a_i = 0,\ i = 1, 2.$$

Each of these equalities contradicts either with $a_1 \neq \pm a_2$, or with $a_i \neq 0$, or with $p = \operatorname{char} \mathbb{F}_p > 2$.

Now we repeat the arguments from the proof of Lemma 4.5 showing that adjoining a_k adds at least one element to the set $\left\{\sum\limits_{i=1}^{k-1} \delta_i a_i \mid \delta_i \in \{\pm 1\}\right\}$. Since at the second step there are 4 different elements, then at $(p-2)$-nd step there are p different elements, i.e., $\left\{\sum\limits_{i=1}^{p-2} \delta_i a_i \mid \delta_i \in \{\pm 1\}\right\} = \mathbb{F}_p$.

Case 2. Let us assume now that all a_i are equal up to the signs. By choosing appropriate δ_i we may consider only the case $a_1 = \ldots = a_{p-2}$. Then

$$\sum_{i=1}^{p-2} \delta_i a_i = a_1 \left(\sum_{i=1}^{p-2} \delta_i \right),\ \delta_i \in \{\pm 1\}.$$

Multiplication by a non-zero a_1 is a bijection in the field $\mathbb{F}_p$. Thus it is sufficient to consider only the set $M = \left\{\sum\limits_{i=1}^{p-2} \delta_i \mid \delta_i \in \{\pm 1\}\right\}$. It is straightforward to see that $M = \mathbb{F} \setminus \{0\}$. Indeed, for any odd $l \in \{1, \ldots, p-1\}$ we have the representation

$$l = \sum_{i=1}^{l} 1 + \sum_{i=1}^{\frac{p-2-l}{2}} 1 + \sum_{i=1}^{\frac{p-2-l}{2}} (-1)$$

and for any even $l \in \{1, \dots, p-1\}$ it holds that

$$l = \sum_{i=1}^{p-l}(-1) + \sum_{i=1}^{\frac{l-2}{2}} 1 + \sum_{i=1}^{\frac{l-2}{2}}(-1).$$

This concludes the proof. $\square$

The following lemma is a key tool in the proof of the main results.

Lemma 4.9. *Let the matrix* $A = (a_{ij}) \in M_n(\mathbb{F}_q)$ *have no zero elements. Then there exists a matrix* $X \in M_n(\{\pm 1\})$ *such that* $\mathrm{per}(A \circ X) \neq 0$.

Proof. In case $\mathrm{per}(A) \neq 0$ we can take the matrix X with all elements equal to one.

Assume now that $\mathrm{per}(A) = 0$. Let us prove the existence of X by induction.

For $n = 2$ if $\mathrm{per}(A) = a_{11}a_{22} + a_{12}a_{21} = 0$ then for the matrix

$$X = \begin{pmatrix} 1 & 1 \\ -1 & 1 \end{pmatrix}$$

we have that $\mathrm{per}(A \circ X) \neq 0$. Indeed if for both matrices A and $A \circ X$ the permanent is zero then

$$0 = \mathrm{per}(A) + \mathrm{per}(A \circ X) = a_{11}a_{22} + a_{12}a_{21} + a_{11}a_{22} - a_{12}a_{21} = 2a_{11}a_{22} \neq 0$$

since $p \neq 2$ and $a_{11}, a_{22} \neq 0$.

Now we assume that for all matrices of size less than n the lemma is true. Let $A = (a_{ij}) \in M_n(\mathbb{F}_q)$, $\mathrm{per}(A) = 0$, $a_{ij} \neq 0$. Consider the Laplace decomposition of A along the first row

$$\mathrm{per}(A) = \sum_{i=1}^{n} a_{1i}\mathrm{per}(A(1|i)). \tag{4.6}$$

There are two possibilities:

Case 1. There exists a nonzero summand in the decomposition (4.6). Let $a_{1l}\mathrm{per}(A(1|l)) \neq 0$. We consider $X = (x_{ij})$, $x_{1l} = -1$, $x_{ij} = 1$ for all other (i, j). Then $\mathrm{per}(A \circ X) \neq 0$ since otherwise

$$0 = \mathrm{per}(A) - \mathrm{per}(A \circ X) = 2a_{1l}\mathrm{per}(A(1|l)) \neq 0.$$

Case 2. In the decomposition (4.6) all summands are equal to zero. Then by the inductive hypothesis applied to the matrix $A(1|1)$ we can choose $Y \in$

$M_n(\{\pm 1\})$ such that $\mathrm{per}(A(1|1) \circ Y(1|1)) \neq 0$. Then the Laplace decomposition of

$$\mathrm{per}(A \circ Y) = \sum_{i=1}^{n} y_{1i} a_{1i} \mathrm{per}(A(1|i) \circ Y(1|i))$$

has a non-zero summand. Thus arguing as in Case 1 with the matrix $A \circ Y$ we find $Z \in M_n(\{\pm 1\})$ such that $\mathrm{per}(A \circ Y \circ Z) \neq 0$. It remains to set $X = Y \circ Z$. □

Remark 4.10. In the paper [12] it was proved that if $\mathbb{F}$ is a field of zero characteristic, $A \in M_n(\mathbb{F})$ is a convertible matrix with $\mathrm{per}A > 0$, then A contains no more than $(n^2 + 3n - 2)/2$ non-zero elements, so-called upper Gibson barrier.

The following theorem demonstrates non-existence of similar barrier for the conversion over finite fields.

Theorem 4.11. *Let p be a prime number. Then for any $n \geq p - 2$ there exists a convertible matrix $A = (a_{ij}) \in M_n(\mathbb{F}_p)$ with nonzero permanent and without zero entries.*

Proof. Let us take $B \in M_n(\mathbb{F}_p)$ constructed as follows: $b_{11} = \ldots = b_{1n} = 1$ and for the submatrix $B(1|)$ located in the rows $2, \ldots, n$ we take the matrix C constructed in Lemma 4.4 if $p \nmid (n-1)$ and the matrix D constructed in Lemma 4.4 if $p \mid (n-1)$.

By Lemma 4.9 there exists $Y \in M_n(\{\pm 1\})$ such that $\alpha := \mathrm{per}(B \circ Y) \neq 0$. Let us consider $A = B \circ Y$ and prove that it is convertible.

We decompose the determinant of the matrix B by the first row

$$\det(B) = \sum_{i=1}^{n} (-1)^{i+1} b_{1i} \det(B(1|i)). \tag{4.7}$$

Since by Lemma 4.4 $\mathrm{ess}\,(B(1|)) = n-1$ the sum (4.7) consists of n nonzero summands. By Corollary 4.8 there are constants $\delta_1, \ldots, \delta_n \in \{\pm 1\}$ such that

$$\sum_{i=1}^{n} \delta_i (-1)^{i+1} c_{1i} \det(B(1|i)) = \alpha.$$

Consider now the matrix $Z = (z_{ij}) \in M_n(\{\pm 1\})$ defined by

$$z_{ij} = \begin{cases} \delta_j, & \text{if } i = 1,\ j = 1, \ldots, n, \\ 1, & \text{if } i \geq 2,\ j = 1, \ldots, n. \end{cases}$$

Hence,

$$\det(B \circ Z) = \alpha.$$

Now take $X = Y \circ Z$. Therefore,

$$\mathrm{per}(A) = \mathrm{per}(B \circ Y) = \alpha = \det(B \circ Z) = \det(B \circ Y \circ Y \circ Z) = \det(A \circ X).$$

Thus for any $n \geq p - 2$ we have constructed the convertible matrix with nonzero permanent. □

Low convertibility barriers for matrices over fields of zero characteristic were investigated in the papers [7, 16]. It was proved that $(0,1)$-matrices with less than $n + 6$ non-zero elements are always convertible. Below we provide the analog of this result for the matrices over finite fields.

Definition 4.12. *Let $A \in M_n(\mathbb{F}_q)$. A generalized diagonal of A is a set of n entries of A, no two of which lie in the same row or column.*

Definition 4.13. *Let $A = (a_{ij}) \in M_n(\mathbb{F}_q)$. An element a_{ij} is called an* essential *element if it belongs to a generalized diagonal consisting of nonzero elements.*

Lemma 4.14. *Let $A = (a_{ij}) \in M_n(\mathbb{F}_q)$ and a_{ij} be a non-essential element of A. Then A is convertible if and only if the matrix A' in which the element a_{ij} is equal to 0 is convertible.*

P r o o f. In the case a_{ij} is not essential we have that by its definition there is no $\sigma \in S_n$ such that $\prod_{k=1}^{n} a_{k\sigma(k)} \neq 0$ where $\sigma(i) = j$. We write both determinant and permanent of A as formal sums removing possible zero summands. Then the obtained expressions do not depend on a_{ij} and therefore we can substitute zero element instead of it. To prove the opposite implication we observe that this substitution do not effect the set of zero summands. □

Lemma 4.15. *Let $A \in M_n(\{0,1\})$ be a convertible $(0,1)$-matrix over the field of zero characteristic. Then any matrix with the same pattern of zeros is convertible over any field.*

P r o o f. The permanent of a $(0,1)$-matrix is equal to the number of generalized diagonals which does not contain any zero element. If a $(0,1)$-matrix A is convertible then there exists $X \in M_n(\{\pm 1\})$ such that all nonzero summands in the determinant of $A \circ X$ are positive, i.e. the formal expressions for the permanent of A and the determinant of $A \circ X$ coincide. Thus substituting arbitrary elements instead of ones we derive that the permanent of A and the determinant of $A \circ X$ are still equal as formal expressions and therefore they coincide. In particular any matrix with the pattern of zeros as in A is convertible by means of the matrix X. □

Below we provide the low barrier for the convertibility. This bound appears to be the same as the one established in [16] and [7] for $(0,1)$-matrices over fields of zero characteristic and is equal to $\omega_n = n+5$.

Theorem 4.16. *Let $A \in M_n(\mathbb{F}_q)$ and the number of the essential elements be not greater than $\omega_n = n+5$. Then A is convertible.*

P r o o f. By Lemma 4.14 we can substitute nonessential elements by zeros. The obtained matrix contains less than or equal to ω_n nonzero elements. Let us substitute these elements by 1 and consider the obtained matrix as (0,1)-matrix over the field of zero characteristic. By the theorem on the lower bound for the conversion [7] the obtained (0,1)-matrix is convertible. Let $X \in M_n(1,-1)$ be the corresponding converter. Then by Lemma 4.15 X is the converter for the matrix A. □

Remark 4.17. This bound is exact for the fields with more than 3 elements and of characteristic different from 2 since in Proposition 3.3 and Theorem 3.5 nonconvertible matrices which contain $n+6$ essential elements are constructed.

REFERENCES

[1] E. Artin. Geometric Algebra. New York, Interscience Publishers, Inc., 1957.

[2] R. E. Blahut. Theory and Practice of Error Control Codes. Reading, Massachusetts etc., Addison Wesley, 1983.

[3] R. A. Brualdi, B. L. Shader. On sing-nonsingular matrices and the conversion of the permanent into the determinant. Applied Geometry and Discrete Mathematics, Festschr. 65th Birthday Victor Klee, DIMACS, *Ser. Discret. Math. Theor. Comput. Sci.* **4** (1991), 117–134.

[4] R. A. Brualdi, B. L. Shader. Matrices of Sing-Solvable Linear Systems. Cambridge Tracts in Mathematics, vol. **116**. Cambridge, Cambridge Univ. Press, 1995.

[5] M. Budrevich, A. Guterman. Permanent has less zeros than determinant over finite fields. *Contemp. Math.* to appear.

[6] S. A. Cook. The complexity of theorem proving procedures. In: Proc. 3rd Ann. ACM Sympos. Theory Computing, Shaker Heights, Ohio 1971, 151–158.

[7] G. Dolinar, A. Guterman, B. Kuzma. Gibson Barriers for the Pólya's Problem. *Fundam. Prikl. Mat.* **16**, *8* (2010), 73–86 (in Russian); English translation in: *Fundamental and Applied Mathematics*, **16**, *8* (2010), 73–86.

[8] G. Dolinar, A. Guterman, B. Kuzma, M. Orel. On the Pólya's permanent problem over finite fields. *Eur. J. Comb.* **32**, *1* (2011), 116–132.

[9] M. R. Garey, D. S. Johnson. Computers and Intractability. A Guide to the Theory of NP-completeness. A Series of Books in the Mathematical Sciences. San Francisco, W. H. Freeman and Company, 1979.

[10] J. von zur Gathen. Permanent and determinant. *Linear Algebra Appl.* **96** (1987), 87–100.

[11] P. M. Gibson. An identity between permanents and determinants. *Amer. Math. Month.* **76** (1969), 270–271.

[12] P. M. Gibson. Conversion of the Permanent into the Determinant. *Proc. Amer. Math. Soc.* **27** (1971), 471–476.

[13] A. Guterman, G. Dolinar, B. Kuzma. Pólya's convertibility problem for symmetric matrices. *Mat. Zametki* **92**, *3* (Sept. 2012) to appear.

[14] R. M. Karp. Reducibility among combinatorial problems. In: Complexity of Computer Computations. New-York, Plenum Press, 1972, 85–104.

[15] V. Klee, R. Ladner, R. Manber. Sing-solvabiliy revisited. *Linear Algebra Appl.* **59** (1984), 132–157.

[16] C. H. C. Little. A characterization of convertible (0,1)-matrices. *J. Combin. Theory*, Ser. B **18** (1975), 187–208.

[17] M. Marcus, H. Minc. On the relation between the determinant and the permanent. *Illinois J. Math.* **5** (1961), 376–381.

[18] W. McCuaig. Pólya's permanent problem. *The Electronic Journal of Combinatorics*, **11** (2004), R79.

[19] H. Minc. Permanents. Encyclopedia of Mathematics and its Applications, vol. **6**. Reading, MA: Addison-Wesley Publishing Company, 1978.

[20] A. A. Nechaev, V. L. Kurakin. Linear Codes and Polylinear Recurrents. Moscow, 1997 (in Russian).

[21] G. PÓLYA. Aufgabe 424. *Arch. Math. Phys. Ser. 3*, **20** (1913), 271.

[22] G. SZEGÖ. Lösung zu Aufgabe 424. *Arch. Math. Phys. Ser. 3* **21** (1913), 291–292.

[23] L. G. VALIANT. The complexity of computing the permanent. *Theoret. Comput. Sci.* **8** (1979), 189–201.

[24] V. V. VAZIRANI, M. YANNAKAKIS. Pfaffian orientations, 0-1 permanents, and even cycles in directed graphs. *Discrete Applied Mathematics* **25** (1989), 179–190.

[25] *E. B. Vinberg.* Algebra. Moscow, Factorial, 2001(in Russian).

Mikhail Budrevich
Alexander E. Guterman
Department of Algebra
Faculty of Mathematics and Mechanics
Moscow State University
GSP-1, 119991 Moscow, Russia
e-mail: mbudrevich@yandex.ru
guterman@list.ru

Received February 27, 2012