Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **38** (2012), 417–432

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON ORDINARY AND $\mathbb{Z}_2$-GRADED POLYNOMIAL IDENTITIES OF THE GRASSMANN ALGEBRA

Viviane Ribeiro Tomaz da Silva*

Communicated by P. Koshlukov

Abstract. The main purpose of this paper is to provide a survey of results concerning the ordinary and $\mathbb{Z}_2$-graded polynomial identities of the infinite dimensional Grassmann algebra over a field of characteristic zero, as well as of its sequences of ordinary and $\mathbb{Z}_2$-graded codimensions and cocharacters. We also intend to describe briefly the techniques used by the authors in order to illustrate some important methods used in PI-theory.

1. Introduction. The infinite dimensional Grassmann algebra over a field of characteristic zero plays an important role in the PI-theory (the theory of algebras satisfying a polynomial identity). Furthermore, the description of its

2010 *Mathematics Subject Classification:* Primary: 16R10, Secondary: 16W55.

Key words: Ordinary polynomial identities, graded polynomial identities, cocharacters, codimensions, Grassmann algebra.

*Partially supported by Conselho Nacional de Desenvolvimento Científico e Tecnológico – CNPq – Brasil and "Para mulheres na Ciência" (L'Oréal/UNESCO/Academia Brasileira de Ciências).

ordinary and $\mathbb{Z}_2$-graded identities involves relevant ideas used in PI-theory. The main goal of this paper is to present and discuss briefly these results and their proofs. In the first part, we will deal with the ordinary identities of the Grassmann algebra and its codimension and cocharacter sequences, by exploring some methods used in their proofs. By proceeding similarly, the $\mathbb{Z}_2$-graded identities of the Grassmann algebra as well as its $\mathbb{Z}_2$-graded codimension and cocharacter sequences are presented in the second part of this paper.

Let us give the precise definitions. Let A be an associative algebra over a field F of characteristic zero, and $X = \{x_1, x_2, \dots\}$ be a countable infinite set of indeterminates. We say that a polynomial $f(x_1, \dots, x_n)$ in the free associative algebra $F\langle X\rangle$ is an *ordinary polynomial identity* of A, if $f(a_1, \dots, a_n) = 0$ for all $a_1, \dots, a_n \in A$. For instance, the Lie commutator $[x_1, x_2] := x_1x_2 - x_2x_1$ is an ordinary polynomial identity for any commutative algebra; moreover the commutator of length three $[x_1, x_2, x_3] := [[x_1, x_2], x_3]$ is an ordinary polynomial identity for the Grassmann algebra. If A satisfies a non-trivial ordinary identity, then we say that A is a *PI-algebra*. The set $\mathrm{Id}(A)$ of all ordinary identities of A is a *T-ideal* of $F\langle X\rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle X\rangle$.

On the other hand, if $X = Y \cup Z$ is a disjoint union of two countable sets of indeterminates $Y = \{y_1, y_2, \dots\}$ and $Z = \{z_1, z_2, \dots\}$, then the free associative algebra $F\langle X\rangle = F\langle Y, Z\rangle$ over X has a natural $\mathbb{Z}_2$-grading $\mathbf{F}^{(0)} \oplus \mathbf{F}^{(1)}$, where $\mathbf{F}^{(0)}$ (respectively $\mathbf{F}^{(1)}$) is the subspace of $F\langle X\rangle$ spanned by all monomials in the variables X having an even (respectively odd) number of variables of Z.

Recall that an F-algebra A is said to be a $\mathbb{Z}_2$-*graded algebra* over F (or a *superalgebra*) if there exist two subspaces $A^{(0)}, A^{(1)}$ such that $A = A^{(0)} \oplus A^{(1)}$ and the following relations are satisfied:

$$A^{(0)}A^{(0)} + A^{(1)}A^{(1)} \subseteq A^{(0)} \quad \text{and} \quad A^{(0)}A^{(1)} + A^{(1)}A^{(0)} \subseteq A^{(1)}.$$

We call $A^{(i)}$ the i-th homogeneous component of A and we say that a homogeneous element $a \in A^{(i)}$ has homogeneous degree i. A subspace $W \subseteq A$ is called *homogeneous* if and only if $W = (W \cap A^{(0)}) \oplus (W \cap A^{(1)})$.

On the other hand, due to the duality between $\mathbb{Z}_2$-gradings and $\mathbb{Z}_2$-actions on A, we can associate each $\mathbb{Z}_2$-grading of A to an automorphism φ of A of order 2 and vice versa. Still, if φ is an automorphism of A of order 2, then $\langle\varphi\rangle \cong \mathbb{Z}_2$ induces a natural $\mathbb{Z}_2$-grading $A = A^{(0)} \oplus A^{(1)}$, where

$$A^{(0)} = \{a \in A \mid \varphi(a) = a\} \qquad \text{and} \qquad A^{(1)} = \{a \in A \mid \varphi(a) = -a\}.$$

A polynomial $f(y_1, \dots, y_r, z_1, \dots, z_m) \in F\langle Y, Z\rangle$ is a $\mathbb{Z}_2$-*graded identity* of the superalgebra A if $f(a_1, \dots, a_r, b_1, \dots, b_m) = 0$ for all $a_1, \dots, a_r \in A^{(0)}$ and

$b_1, \ldots, b_m \in A^{(1)}$. The set $\mathrm{Id}^{\mathrm{gr}}(A)$ of all $\mathbb{Z}_2$-graded identities of A is a T_2-*ideal* of $F\langle Y, Z\rangle$, i.e., an ideal invariant under all endomorphisms of $F\langle Y, Z\rangle$ which preserve the $\mathbb{Z}_2$-grading.

Since the characteristic of the ground field F is zero, it is well known that the study of $\mathrm{Id}(A)$ as well of $\mathrm{Id}^{\mathrm{gr}}(A)$ is determined by their multilinear parts. More precisely, in the ordinary (respectively $\mathbb{Z}_2$-graded) case, if

$$P_n = \mathrm{span}_F\{x_{\sigma(1)} \cdots x_{\sigma(n)} \mid \sigma \in S_n\}$$

$$\left(\text{respectively } P_n^{\mathrm{gr}} = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(n)} \mid \sigma \in S_n,\, w_i \in \{y_i, z_i\},\, 1 \leqslant i \leqslant n\}\right),$$

then the study of $\mathrm{Id}(A)$ (respectively $\mathrm{Id}^{\mathrm{gr}}(A)$) is equivalent to the study of $P_n \cap \mathrm{Id}(A)$ (respectively $P_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(A)$), for all $n \geqslant 1$. Still, if we denote by $P_n(A)$ (respectively $P_n^{\mathrm{gr}}(A)$) the quotient space

$$P_n(A) := \frac{P_n}{P_n \cap \mathrm{Id}(A)} \quad \left(\text{respectively } P_n^{\mathrm{gr}}(A) := \frac{P_n^{\mathrm{gr}}}{P_n^{\mathrm{gr}} \cap \mathrm{Id}^{\mathrm{gr}}(A)}\right),$$

then its dimension

$$c_n(A) = \dim_F P_n(A) \quad (\text{respectively } c_n^{\mathrm{gr}}(A) = \dim_F P_n^{\mathrm{gr}}(A))$$

is called the *n-th ordinary (respectively $\mathbb{Z}_2$-graded) codimension* of A and carries relevant information about the ordinary (respectively $\mathbb{Z}_2$-graded) identities of A.

Moreover, in the $\mathbb{Z}_2$-graded case, we can consider also the multilinear spaces

$$\begin{aligned} P_{r,m} = \mathrm{span}_F\{w_{\sigma(1)} \cdots w_{\sigma(r+m)} \mid \sigma \in S_{r+m},\ & w_i = y_i, \text{ if } 1 \leqslant i \leqslant r, \\ & w_i = z_{i-r}, \text{ if } r+1 \leqslant i \leqslant r+m\} \end{aligned}$$

and

$$P_{r,m}(A) := \frac{P_{r,m}}{P_{r,m} \cap \mathrm{Id}^{\mathrm{gr}}(A)}.$$

The dimension $c_{r,m}(A) = \dim_F P_{r,m}(A)$ is called the *(r, m)-th graded codimension* of A. The $\mathbb{Z}_2$-graded codimensions $c_n^{\mathrm{gr}}(A)$ and $c_{r,m}(A)$ are related by

$$c_n^{\mathrm{gr}}(A) = \sum_{r=0}^{n} \binom{n}{r} c_{r,n-r}(A).$$

If we consider the natural action of the group S_n (respectively $\mathbb{Z}_2 \wr S_n$ and $S_r \times S_m$) on the space $P_n(A)$ (respectively $P_n^{\mathrm{gr}}(A)$ and $P_{r,m}(A)$), then we can investigate its structure as an S_n-module (respectively a $\mathbb{Z}_2 \wr S_n$-module and an $S_r \times S_m$-module) by describing its S_n-character (respectively $\mathbb{Z}_2 \wr S_n$-character and $S_r \times S_m$-character) called *ordinary cocharacter* (respectively *$\mathbb{Z}_2 \wr S_n$-cocharacter* and *$r \times m$-th graded cocharacter*).

While, in the ordinary case, it is well known that there is a one-to-one correspondence between irreducible S_n-characters and partitions $\lambda \vdash n$ (see Theorem 12.2.7 of [6]); for the $\mathbb{Z}_2$-graded case we have that there is a one-to-one correspondence between irreducible $S_r \times S_m$-characters (respectively $\mathbb{Z}_2 \wr S_n$-characters) and pairs of partitions (λ, μ), where $\lambda \vdash r$, $\mu \vdash m$ (respectively $|\lambda| + |\mu| = n$). Hence we have

$$\chi(A) = \sum_{\lambda \vdash n} m_\lambda \chi_\lambda, \quad \chi_{r,m}(A) = \sum_{\substack{\lambda \vdash r \\ \mu \vdash m}} m_{\lambda,\mu}\, \chi_\lambda \otimes \chi_\mu \quad \text{and} \quad \chi_n^{\mathrm{gr}}(A) = \sum_{|\lambda| + |\mu| = n} m'_{\lambda,\mu}\, \chi_{\lambda,\mu},$$

where m_λ (respectively $m_{\lambda,\mu}$ and $m'_{\lambda,\mu}$) denotes the multiplicity of the irreducible character χ_λ (respectively $\chi_\lambda \otimes \chi_\mu$ and $\chi_{\lambda,\mu}$) associated with the partition λ (respectively with the pair of partitions (λ, μ)). Moreover, in the $\mathbb{Z}_2$-graded case, we have $m_{\lambda,\mu} = m'_{\lambda,\mu}$ for all $\lambda \vdash r$, $\mu \vdash n - r$, with $r = 0, \ldots, n$ (see Theorem 10.4.5 of [8]), and thus here we will focus on $\chi_{r,m}(A)$ only. Furthermore, when convenient, we will identify the irreducible character χ_λ with the Young diagram corresponding to the partition λ.

Let E be the infinite-dimensional Grassmann algebra with presentation

$$E = F\langle 1, e_1, e_2, \ldots \mid e_i e_j = -e_j e_i \rangle.$$

Note that the set

$$B_E := \{e_{i_1} \cdots e_{i_t} \mid 1 \leqslant i_1 < \cdots < i_t, t \geqslant 0\}$$

is a basis of E over F. For a basic element $a = e_{i_1} \cdots e_{i_t} \in B_E$, the length $|a|$ is given by t, the quantity of generators in a. Moreover, we can decompose E as a direct sum of the subspaces

$$\mathbf{E}^{(\mathbf{0})} = \operatorname{span}_F\{a \in B_E \mid |a| \equiv 0 \pmod 2\}$$

and

$$\mathbf{E}^{(\mathbf{1})} = \operatorname{span}_F\{a \in B_E \mid |a| \equiv 1 \pmod 2\},$$

which are the center and the anti-commutative part of E, respectively. Clearly the subspaces $\mathbf{E}^{(\mathbf{0})}$ and $\mathbf{E}^{(\mathbf{1})}$ satisfy the necessary conditions to turn E a superalgebra, and one calls $\mathbf{E} = \mathbf{E}^{(\mathbf{0})} \oplus \mathbf{E}^{(\mathbf{1})}$ the *canonical grading* of E. Still, given a superalgebra $A = A^{(0)} \oplus A^{(1)}$, we can build, from the canonical grading of E, a new superalgebra called the *Grassmann envelope* of A and given by

$$G(A) := (A^{(0)} \otimes \mathbf{E^{(0)}}) \oplus (A^{(1)} \otimes \mathbf{E^{(1)}}).$$

It is worth to say that the importance of the Grassmann algebra for the PI-theory can be explicitated, for instance, through the celebrated Kemer's theory. In his works, Kemer proved [9, 10] that E appears in the classification of the so-called verbally prime algebras, and any associative PI-algebra over a field of characteristic zero is PI-equivalent to the Grassmann envelope of a finite dimensional associative superalgebra.

Here we will deal with the $\mathbb{Z}_2$-gradings of E by considering the automorphisms φ of E of order 2 associated with them. Thus, when we are working with the $\mathbb{Z}_2$-grading of E induced by the automorphism φ of E of order 2, we will refer to E as E_φ, and we will use the notations $\mathrm{Id}(E_\varphi)$, $P_{r,m}(E_\varphi)$, $c_n(E_\varphi)$ and $\chi_{r,m}(E_\varphi)$ instead of $\mathrm{Id}^{\mathrm{gr}}(E)$, $P_{r,m}(E)$, $c_n^{\mathrm{gr}}(E)$ and $\chi_{r,m}(E)$, respectively.

Note, for instance, that the canonical grading of E is induced by the automorphism φ such that $\varphi(e_i) = -e_i$, for all $i \geqslant 1$. In particular, the vector space $L := \mathrm{span}_F\{e_1, e_2, \dots\}$ is a homogeneous subspace of φ in this case.

In general, if φ is an automorphism of order 2 of the Grassmann algebra, then for each $i = 1, 2, \dots$ we have

$$\varphi(e_i) = \sum_{j=1}^{\infty} \alpha_{ji} e_j + \sum_{a_k \in B_E, |a_k| \geq 2} \beta_{ki} a_k,$$

in which only finite many scalars α_{ji}, β_{ki} are nonzero.

Consider then the linear part φ_l of φ defined by

$$\varphi_l(e_i) = \sum_{j=1}^{\infty} \alpha_{ji} e_j, \qquad i = 1, 2, \dots,$$

and extend φ_l to E as a homomorphism. We clearly obtain that the automorphism φ_l is a linear operator on the vector space $L = \mathrm{span}_F\{e_1, e_2, \dots\}$, that is, L is a homogeneous subspace of φ_l. Note that L can be presented as a direct sum of its subspaces, which are eigenspaces of φ_l. More precisely, $L = L_1 \oplus L_{-1}$, where

$$L_1 = \{v \in L \mid \varphi_l(v) = v\} \qquad \text{and} \qquad L_{-1} = \{v \in L \mid \varphi_l(v) = -v\}.$$

Observe that we have three possibilities for $\dim_F L_1$ and $\dim_F L_{-1}$:

Case 1: $\dim_F L_1 = \infty$ and $\dim_F L_{-1} = \infty$.

Case 2: $\dim_F L_1 = \ell < \infty$ and $\dim_F L_{-1} = \infty$.

Case 3: $\dim_F L_1 = \infty$ and $\dim_F L_{-1} = \ell < \infty$.

Moreover, one example of each case is given by the automorphisms $\varphi_l^{(\infty)}$, $\varphi_l^{(\ell)}$ and $\varphi_l^{(\ell^*)}$ defined as follows:

$$\varphi_l^{(\infty)}(e_i) = \begin{cases} e_i, & i \text{ even} \\ -e_i, & i \text{ odd} \end{cases} \qquad \varphi_l^{(\ell)}(e_i) = \begin{cases} e_i, & i = 1, \ldots, \ell \\ -e_i, & \text{otherwise} \end{cases}$$

$$\varphi_l^{(\ell^*)}(e_i) = \begin{cases} -e_i, & i = 1, \ldots, \ell \\ e_i, & \text{otherwise.} \end{cases}$$

2. Ordinary identities, codimensions and cocharacters. In 1973, Krakowski and Regev [11] computed the T-ideal of ordinary polynomial identities of the Grassmann algebra by using the theory of codimensions. We will recall briefly the useful and interesting method developed by them in [11].

Given an element $\sigma \in S_n$ and a subset $I \subseteq \{1, \ldots, n\}$, consider the integer $f_I^{(n)}(\sigma) \in \{-1, 1\}$ defined by the equality

$$a_{\sigma(1)} \cdots a_{\sigma(n)} = f_I^{(n)}(\sigma)\, a_1 \cdots a_n,$$

where $a_1, \ldots, a_n$ are arbitrary elements of B_E such that $a_1 \cdots a_n \neq 0$ and the set of indices i for which a_i is of odd length coincides with the I.

Define the $2^n \times n!$ matrix

$$H^{(n)} = (f_I^{(n)}(\sigma)) \tag{1}$$

determined by ordering the 2^n subsets of $\{1, \ldots, n\}$ and using them as row indices, and ordering the $n!$ elements of S_n and using them as column indices.

Krakowski and Regev proved that $c_n(E) = \mathrm{rank}_F H^{(n)}$ and thus, by studying the rank of $H^{(n)}$, they obtained a lower bound to $c_n(E)$, by showing $\mathrm{rank}_F\, H^{(n)} \geqslant 2^{n-1}$ and thus concluding

$$c_n(E) \geqslant 2^{n-1}. \tag{2}$$

On the other hand, an upper bound to $c_n(E)$ was found by working with codimensions of the J_d*-type algebras*, that is, algebras satisfying an identity of the form

$$x_1 \cdots x_d = \sum_{\substack{\sigma \in S_d \\ \sigma(1) \neq 1}} \alpha_\sigma x_{\sigma(1)} \cdots x_{\sigma(d)}.$$

Note that the Grassmann algebra E is an algebra of type J_3, since E satisfies the identity $[x_1, x_2, x_3] = 0$, that is, the identity $x_1x_2x_3 = x_2x_1x_3 + x_3x_1x_2 - x_3x_2x_1$. In [11], the authors proved that for any algebra A of type J_d

$$c_n(A) \leqslant (d-1)^{n-1}. \tag{3}$$

Hence as a corollary they established that $c_n(E) \leqslant 2^{n-1}$. In this way Krakowski and Regev concluded the following:

Theorem 1. *For any positive integer n, the n-th ordinary codimension of the Grassmann algebra E is given by $c_n(E) = 2^{n-1}$.*

Given any T-ideal I, denote by $c_n(I)$ the dimension of the vector space

$$P_n(I) := \frac{P_n}{P_n \cap I}.$$

Note that as a consequence of (3), we also get $c_n(\langle [x_1, x_2, x_3] \rangle_T) \leqslant 2^{n-1}$ and thus, since clearly $\langle [x_1, x_2, x_3] \rangle_T \subseteq \mathrm{Id}(E)$, we have

$$2^{n-1} \leqslant c_n(E) \leqslant c_n(\langle [x_1, x_2, x_3] \rangle_T) \leqslant 2^{n-1}$$

and therefore Krakowski and Regev concluded that in general, for the Grassmann algebra E over a field F of characteristic $\neq 2$, we have $c_n(E) = c_n(\langle [x_1, x_2, x_3] \rangle_T)$ and $P_n \cap \mathrm{Id}(E) = P_n \cap \langle [x_1, x_2, x_3] \rangle_T$. In the case when the characteristic of F is zero, this implies the following:

Theorem 2. *The T-ideal of ordinary polynomial identities for the Grassmann algebra E is generated by $[x_1, x_2, x_3]$, that is, $\mathrm{Id}(E) = \langle [x_1, x_2, x_3] \rangle_T$.*

It is important to say that, in 1962, Latyshev [12] proved that the T-ideal $\langle [x_1, x_2, x_3] \rangle_T$ is Spechtian. Moreover, in 1991, Di Vincenzo [4] worked also with the finite dimensional Grassmann algebras and gave a different proof for Theorem 2.

On the other hand, the cocharacter sequence of the Grassmann algebra was described by Olsson and Regev [13] in 1976. In their proof, the authors showed that some irreducible characters appeared in the cocharacter $\chi_n(E)$ and, by computing their dimensions, they established (2) and concluded, by using Theorem 1, the following result.

Theorem 3. *For any positive integer n, the n-th ordinary cocharacter of the Grassmann algebra E is given by* $\chi_n(E) = \sum_{i=0}^{n-1} \begin{array}{|c|l}\hline \multicolumn{2}{|c|}{\leftarrow n-i \rightarrow} \\ \hline \uparrow & \\ i & \\ \downarrow & \\ \cline{1-1}\end{array}$.

Observe that, in particular, Olsson and Regev gave a new proof to (2). Hence, by finding a set of generators to $P_n(\langle[x_1, x_2, x_3]\rangle_T)$ as a vector space, one may prove by a different way Theorem 1 (and also Theorem 2). Note that this was done in the proof of Theorem 4.1.8 of [8].

3. $\mathbb{Z}_2$-graded identities, codimensions and cocharacters.

As a consequence of the study of the ordinary case, we clearly have a complete description of the $\mathbb{Z}_2$-graded identities as well of the $\mathbb{Z}_2$-graded codimensions and cocharacters for the trivial grading $E = E \oplus 0$, that is, for the superalgebra $E_{\varphi_l^{(0^*)}}$. On the other hand, Giambruno, Mishchenko and Zaicev [7] proved in 2001 the following result about the canonical grading $\mathbf{E} = \mathbf{E^{(0)}} \oplus \mathbf{E^{(1)}}$, that is, the superalgebra $E_{\varphi_l^{(0)}}$.

Theorem 4. *For the canonical grading $E_{\varphi_l^{(0)}}$ of the Grassmann algebra we have the following:*

(*i*) $\mathrm{Id}(E_{\varphi_l^{(0)}}) = \langle[y_1, y_2], [y_1, z_1], z_1z_2 + z_2z_1\rangle_{T_2}$;

(*ii*) $c_n(E_{\varphi_l^{(0)}}) = 2^n$, *for all* $n \geqslant 1$;

(*iii*) $\chi_{r,m}(E_{\varphi_l^{(0)}}) = \boxed{\leftarrow r \rightarrow} \otimes \boxed{\begin{matrix}\uparrow \\ m \\ \downarrow\end{matrix}}$, *for all* $r, m \geqslant 0$.

In general, if φ_l is an arbitrary automorphism of order 2 of E such that L is a homogeneous subspace, then the $\mathbb{Z}_2$-graded codimensions $c_n(E_{\varphi_l})$ were investigated by Anisimov [2] in 2001. He proved the following result.

Theorem 5. *For any positive integer n:*

(*i*) *If* $\dim_F L_1 = \infty$ *and* $\dim_F L_{-1} = \infty$, *then*

$$c_n(E_\varphi) = c_n(E_{\varphi_l}) = 4^{n-\frac{1}{2}};$$

(*ii*) *If* $\dim_F L_{-1} = \ell < \infty$, *then*

$$c_n(E_{\varphi_l}) = 2^{n-1} \sum_{k=0}^{\min\{\ell,n\}} \binom{n}{k};$$

(*iii*) *If* $\dim_F L_1 = \ell < \infty$, *then* $c_n(E_{\varphi_l}) = 4^{n-\frac{1}{2}}$ *for* $n \leqslant \ell$ *and*

$$2^{n-1} \sum_{k=0}^{\ell} \binom{n}{k} \leqslant c_n(E_{\varphi_l}) \leqslant 2^n \sum_{k=0}^{\ell} \binom{n}{k} \quad \text{for } n \geqslant \ell + 1.$$

It is interesting that, in order to establish the above result, Anisimov (see [1] and [2]) generalized the method developed by Krakowski and Regev for the computation of the codimension sequence of E (in the ordinary case) by using the matrices $H^{(n)}$ given in (1). In [15], by applying this generalization, da Silva obtained the exact value of $c_n(E_{\varphi_l})$ in the open case left by Anisimov.

Theorem 6. *If* $\dim_F L_1 = \ell < \infty$ *and* $n \geqslant \ell + 1$, *then*

$$c_n(E_{\varphi_l}) = \begin{cases} 2^{n-1} \displaystyle\sum_{k=0}^{\ell} \binom{n}{k} + 2^{n-1} \binom{n-1}{\ell}, & \text{if } \ell \text{ is even} \\ 2^{n-1} \displaystyle\sum_{k=0}^{\ell} \binom{n}{k} + (2^{n-1} - 1) \binom{n-1}{\ell}, & \text{if } \ell \text{ is odd.} \end{cases}$$

Also in [2], Anisimov described the $\mathbb{Z}_2$-graded codimensions $c_n(E_\varphi)$ as well as the T_2-ideal $\mathrm{Id}(E_\varphi)$ for any automorphism φ of order 2 of E such that $\dim_F L_1 = \dim_F L_{-1} = \infty$.

Theorem 7. *Let* φ *be an automorphism of order* 2 *of the Grassmann algebra such that* $\dim_F L_1 = \dim_F L_{-1} = \infty$. *Then:*

(*i*) $\mathrm{Id}(E_\varphi) = \mathrm{Id}(E_{\varphi_l}) = \langle [u_1, u_2, u_3] \mid u_i \in Y \cup Z \rangle$;

(*ii*) $c_n(E_\varphi) = c_n(E_{\varphi_l}) = 4^{n-\frac{1}{2}}$ *for any positive integer* n.

The deep connections between the T_2-ideals $\mathrm{Id}(E_\varphi)$ and $\mathrm{Id}(E_{\varphi_l})$ as well as the $\mathbb{Z}_2$-graded codimensions $c_n(E_\varphi)$ and $c_n(E_{\varphi_l})$, when one of the eigenspaces L_1 or L_{-1} is finite-dimensional, were pointed out by Anisimov in 2003. Given an automorphism φ of the order 2 of E we say that φ is a *graded automorphism* if

$\varphi(\mathbf{E^{(i)}}) = \mathbf{E^{(i)}}$, for $i = 0, 1$, where $\mathbf{E} = \mathbf{E^{(0)}} \oplus \mathbf{E^{(1)}}$ is the canonical grading of E. In [3], Anisimov proved the following result.

Theorem 8. *Let φ be a graded automorphism of order 2 of the Grassmann algebra. If φ satisfies one of the conditions:*

(*i*) $\dim_F L_{-1} = \ell < \infty$ *and* $\prod_{j=1}^{\ell+1}(\varphi(e_{i_j}) - e_{i_j}) = 0$ *for any* $\ell + 1$ *generators* $e_{i_1}, \ldots, e_{i_{\ell+1}}$,

(*ii*) $\dim_F L_1 = \ell < \infty$ *and* $\prod_{j=1}^{\ell+1}(\varphi(e_{i_j}) + e_{i_j}) = 0$ *for any* $\ell + 1$ *generators* $e_{i_1}, \ldots, e_{i_{\ell+1}}$,

then

- $\mathrm{Id}(E_\varphi) = \mathrm{Id}(E_{\varphi_l})$;
- $c_n(E_\varphi) = c_n(E_{\varphi_l})$ *for any positive integer* n.

Note that the conditions (*i*) and (*ii*) are natural for the graded automorphisms of order 2 of E. Indeed, in [3], Anisimov proved that the conditions (*i*) and (*ii*) hold for any such automorphism with $\dim L_{-1} = 1$ and $\dim L_1 = 1$, respectively. In other words, given any graded automorphism of order 2 of E we have

$$\mathrm{Id}(E_\varphi) = \mathrm{Id}(E_{\varphi_l}) \text{ and } c_n(E_\varphi) = c_n(E_{\varphi_l}), \text{ if } \dim L_{-1} = 1 \text{ or } \dim L_1 = 1. \tag{4}$$

In particular, the previous results show that in order to know $\mathrm{Id}(E_\varphi)$ and $c_n(E_\varphi)$ (as well as $\chi_{r,m}(E_\varphi)$), for any automorphism φ of order 2 of E in any of the cases mentioned above, it suffices to study $\mathrm{Id}(E_{\varphi_l})$ and $c_n(E_{\varphi_l})$ (and $\chi_{r,m}(E_{\varphi_l})$), respectively.

Di Vincenzo and da Silva described in [5] (see also [14]) the $\mathbb{Z}_2$-graded identities, codimensions and cocharacters of E_{φ_l}. Observe that a complete description of $\mathrm{Id}(E_{\varphi_l})$, $c_n(E_{\varphi_l})$ and $\chi_{r,m}(E_{\varphi_l})$ is given when we describe the $\mathbb{Z}_2$-graded polynomial identities, as well as its $\mathbb{Z}_2$-graded codimension and cocharacter sequences, for the superalgebras $E_{\varphi_l^{(\infty)}}$, $E_{\varphi_l^{(\ell)}}$ and $E_{\varphi_l^{(\ell^*)}}$ defined in the introduction.

In [5], the authors exploited the deep relations between the $S_r \times S_m$-modules $P_{r,m}(E_{\varphi_l^{(\infty)}})$, $P_{r,m}(E_{\varphi_l^{(\ell^*)}})$ and $P_{r+m}(E)$ established by the linear isomorphism $\psi_{r,m} : P_{r+m} \longrightarrow P_{r,m}$ induced by the map

$$x_i \mapsto \begin{cases} y_i & \text{for } i = 1, \ldots, r \\ z_{i-r} & \text{for } i = r+1, \ldots, r+m \end{cases}.$$

And thus, by using the knowledge of the ordinary case given by Theorems 1, 2 and 3, they obtained a new proof of items (i) and (ii) of Theorem 5, as well as the following results about the $\mathbb{Z}_2$-graded polynomial identities and cocharacter sequences for the superalgebras $E_{\varphi_l^{(\infty)}}$ and $E_{\varphi_l^{(\ell^*)}}$.

Theorem 9. *Let $U = Y \cup Z$. Then*

(i) $\mathrm{Id}(E_{\varphi_l^{(\infty)}}) = \langle [u_1, u_2, u_3] \mid u_i \in U \rangle_{T_2}$

(ii) $\mathrm{Id}(E_{\varphi_l^{(\ell^*)}}) = \langle [u_1, u_2, u_3],\, z_1 z_2 \cdots z_{\ell+1} \mid u_i \in U \rangle_{T_2}.$

Theorem 10. *If $d = \infty, \ell^*$ with $\ell \geqslant 0$, then, for all $r, m \geqslant 1$, we have*

$$(i)\ \ \chi_{0,m}(E_{\varphi_l^{(d)}}) = \begin{cases} \displaystyle\sum_{j=0}^{m-1} \emptyset \otimes \begin{array}{|c|c|}\hline \uparrow & \leftarrow j \rightarrow \\ \cline{2-2} m-j & \multicolumn{1}{c}{} \\ \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, & \text{if either } d = \infty \\ & \text{or } d = \ell^* \text{ with } m \leqslant \ell \\ 0, & \text{if } d = \ell^* \text{ with } m \geqslant \ell + 1 \end{cases}$$

$$(ii)\ \ \chi_{r,m}(E_{\varphi_l^{(d)}}) = \begin{cases} \displaystyle 2\sum_{i=0}^{r-1}\sum_{j=0}^{m-1} \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{\leftarrow r-i \rightarrow} \\ \hline \uparrow & \multicolumn{1}{c}{} \\ i & \multicolumn{1}{c}{} \\ \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \otimes \begin{array}{|c|c|}\hline \uparrow & \leftarrow j \rightarrow \\ \cline{2-2} m-j & \multicolumn{1}{c}{} \\ \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array}, & \text{if either } d = \infty \\ & \text{or } d = \ell^* \text{ with } m \leqslant \ell \\ 0, & \text{if } d = \ell^* \text{ with } m \geqslant \ell + 1 \end{cases}$$

$$(iii)\ \ \chi_{r,0}(E_{\varphi_l^{(d)}}) = \sum_{i=0}^{r-1} \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{\leftarrow r-i \rightarrow} \\ \hline \uparrow & \multicolumn{1}{c}{} \\ i & \multicolumn{1}{c}{} \\ \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1}\end{array} \otimes\ \emptyset, \qquad \text{for } d = \infty, \ell^*.$$

While the restriction on the quantity of generators e_i with $\mathbb{Z}_2$-degree 1 leads us to a nilpotent condition expressed by the $\mathbb{Z}_2$-graded identity $z_1 z_2 \cdots z_{\ell+1}$ of $E_{\varphi_l^{(\ell^*)}}$, we have that the restriction on the quantity of generators e_i with $\mathbb{Z}_2$-degree 0 has deeper consequences. The study of the superalgebras $E_{\varphi_l^{(\ell)}}$ was done by Di Vincenzo and da Silva in [5] by using the so-called *Y-proper polynomials*, that is, the polynomials $f \in F\langle Y, Z \rangle$ such that all the variables $y_i \in Y$ occurring in f appear in commutators only.

Here, we denote by $B(Y)$ the set of all Y-proper polynomials and we recall (see [6]) that, if A is a unitary superalgebra over a field F of characteristic zero, then its T_2-ideal $\mathrm{Id}^{\mathrm{gr}}(A)$ is determined by its multilinear Y-proper polynomials, that is, by the elements of the spaces $\Gamma_{r,m} := P_{r,m} \cap B(Y)$ which are $\mathbb{Z}_2$-graded identities for A. Furthermore, we can consider the quotient space

$$\Gamma_{r,m}(A) := \frac{\Gamma_{r,m}}{\Gamma_{r,m} \cap \mathrm{Id}^{\mathrm{gr}}(A)}$$

and define in an obvious way the *Y-proper (r,m)-th codimension* and *cocharacter* of A, which are denoted by $\gamma_{r,m}(A)$ and $\xi_{r,m}(A)$, respectively. Due to the deep relations between the $S_r \times S_m$-structure (and dimension) of the spaces $P_{r,m}(A)$ and $\Gamma_{r,m}(A)$, one may obtain a complete description of the $\mathbb{Z}_2$-graded identities, codimensions and cocharacters of a superalgebra A, by studying the Y-proper spaces. It is exactly this strategy which was used in [5] in the investigation of $E_{\varphi_l^{(\ell)}}$.

Furthermore, by considering

$$I := \langle [u_1, u_2, u_3] \mid u_i \in Y \cup Z \rangle$$

and that, for any $\sigma \in S_n$, we have

$$[u_{\sigma(1)}, u_{\sigma(2)}] \cdots [u_{\sigma(n-1)}, u_{\sigma(n)}] = (-1)^{\sigma} [u_1, u_2] \cdots [u_{n-1}, u_n] \pmod{I},$$

where $(-1)^{\sigma}$ is the sign of the permutation σ, the authors easily established the crucial lemmas:

Lemma 11. *If $r \equiv 0 \pmod 2$ and $m \geqslant 0$, then for each $f \in \Gamma_{r,m}$ there exists $g \in \Gamma_{0,m}$ such that*

$$f(y_1, \ldots, y_r, z_1, \ldots, z_m) \equiv g(z_1, \ldots, z_m)[y_1, y_2] \cdots [y_{r-1}, y_r] \pmod{I}.$$

Moreover we have

(*i*) *If $r \geqslant \ell + 1$, then $f \in \mathrm{Id}(E_{\varphi_l^{(\ell)}})$;*

(*ii*) *If $r \leqslant \ell$, then $f \in \mathrm{Id}(E_{\varphi_l^{(\ell)}})$ if and only if $g \in \mathrm{Id}(E_{\varphi_l^{(\ell-r)}})$.*

Lemma 12. *If $r \equiv 1 \pmod 2$ and $m \geqslant 1$, then for each $f \in \Gamma_{r,m}$ there exists $g \in \Gamma_{1,m}$ such that*

$$f(y_1, \ldots, y_r, z_1, \ldots, z_m) \equiv g(z_1, \ldots, z_m, y_1)[y_2, y_3] \cdots [y_{r-1}, y_r] \pmod{I}.$$

Moreover we have

(*i*) *If $r \geqslant \ell + 1$, then $f \in \mathrm{Id}(E_{\varphi_l^{(\ell)}})$;*

(*ii*) *If $r \leqslant \ell$, then $f \in \mathrm{Id}(E_{\varphi_l^{(\ell)}})$ if and only if $g \in \mathrm{Id}(E_{\varphi_l^{(\ell-r+1)}})$.*

Note that these lemmas have the relevant rule to show where the investigation about $E_{\varphi_l^{(\ell)}}$ should focus, namely on the study of the spaces $\Gamma_{0,m} \cap \mathrm{Id}(E_{\varphi_l^{(h)}})$ and $\Gamma_{1,m} \cap \mathrm{Id}(E_{\varphi_l^{(h)}})$, for all h. The next step consisted on to find an appropriate set of generators for $\Gamma_{0,m}$ modulo I:

Definition 13. *Given a subset $T = \{j_1, \dots, j_t\} \subseteq \{1, \dots, m\}$ such that $j_1 < j_2 < \cdots < j_t$ and $t = |T|$ is even, we consider the elements $i_1 < i_2 < \cdots < i_r$ of the complementary set $\{1, \dots, m\} - T$. We define the polynomial f_T in $\Gamma_{0,m}$ by:*

$$f_T = z_{i_1} \cdots z_{i_r} [z_{j_1}, z_{j_2}] \cdots [z_{j_{t-1}}, z_{j_t}].$$

A specific linear combination of these polynomials f_T has a crucial role.

Definition 14. *For $m \geqslant 2$ let*

$$g_m(z_1, \dots, z_m) = \sum_{\substack{T \\ |T| \text{ even}}} (-2)^{-\frac{|T|}{2}} f_T,$$

moreover, define

$$g_1(z_1) = z_1.$$

Indeed, concerning the spaces $\Gamma_{0,m}$ and $\Gamma_{1,m}$, we have:

Proposition 15. *The polynomials*

$$g_{\ell+2}(z_1, \dots, z_{\ell+2}), \quad [g_{\ell+1}(z_1, \dots, z_{\ell+1}), y] \quad \text{and} \quad g_{\ell+1}(z_1, \dots, z_{\ell+1})[z_{\ell+2}, y]$$

are $\mathbb{Z}_2$-graded polynomial identities for $E_{\varphi_l^{(\ell)}}$.

The main results of [5] are the following:

Theorem 16. *The T_2-ideal $\mathrm{Id}(E_{\varphi_l^{(\ell)}})$ is generated by the set of the following polynomials:*

(*i*) $[u_1, u_2, u_3], \quad u_i \in Y \cup Z,$

(*ii*) $[y_1, y_2] \cdots [y_{\ell-1}, y_\ell][y_{\ell+1}, u_{\ell+2}], \quad u_{\ell+2} \in Y \cup Z \qquad$ (*if ℓ is even*),

(iii) $[y_1, y_2] \cdots [y_{\ell-2}, y_{\ell-1}][y_\ell, y_{\ell+1}]$ *(if ℓ is odd),*

(iv) $g_{\ell-r+2}(z_1, \ldots, z_{\ell-r+2})[y_1, y_2] \cdots [y_{r-1}, y_r]$ *(for all $r \leqslant \ell$, r even),*

(v) $[g_{\ell-r+2}(z_1, \ldots, z_{\ell-r+2}), y_1][y_2, y_3] \cdots [y_{r-1}, y_r]$ *(for all $r \leqslant \ell$, r odd),*

(vi) $g_{\ell-r+2}(z_1, \ldots, z_{\ell-r+2})[z_{\ell-r+3}, y_1][y_2, y_3] \cdots [y_{r-1}, y_r]$ *(for all $r \leqslant \ell$, r odd).*

Theorem 17. *Given $\ell \geqslant 0$, we have, for all $r, m \geqslant 1$:*

(i) $$\chi_{0,m}(E_{\varphi_l^{(\ell)}}) = \sum_{j=0}^{\min\{m-1,\ell\}} \emptyset \otimes \begin{array}{|c|c|}\hline \uparrow & \leftarrow j \rightarrow \\ \cline{2-2} m-j & \multicolumn{1}{c}{} \\ \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array}$$

(ii) $$\chi_{r,0}(E_{\varphi_l^{(\ell)}}) = \begin{cases} \displaystyle\sum_{i=0}^{\min\{r-1,\ell\}} \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{\leftarrow r-i \rightarrow} \\ \hline \uparrow & \multicolumn{1}{c}{} \\ i & \multicolumn{1}{c}{} \\ \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \otimes \emptyset, & \textit{if } \ell \textit{ is even} \\ \displaystyle\sum_{i=0}^{\min\{r-1,\ell-1\}} \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{\leftarrow r-i \rightarrow} \\ \hline \uparrow & \multicolumn{1}{c}{} \\ i & \multicolumn{1}{c}{} \\ \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \otimes \emptyset, & \textit{if } \ell \textit{ is odd} \end{cases}$$

(iii) $$\chi_{r,m}(E_{\varphi_l^{(\ell)}}) = \sum_{i=0}^{r-1} \sum_{j=0}^{m-1} m_{i,j} \begin{array}{|c|c|}\hline \multicolumn{2}{|c|}{\leftarrow r-i \rightarrow} \\ \hline \uparrow & \multicolumn{1}{c}{} \\ i & \multicolumn{1}{c}{} \\ \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array} \otimes \begin{array}{|c|c|}\hline \uparrow & \leftarrow j \rightarrow \\ \cline{2-2} m-j & \multicolumn{1}{c}{} \\ \downarrow & \multicolumn{1}{c}{} \\ \cline{1-1} \end{array},$$

$$\textit{where } m_{i,j} = \begin{cases} 2, & \textit{if } i + j \leqslant \ell - 1 \\ 1, & \textit{if } i + j = \ell \\ 0, & \textit{otherwise.} \end{cases}$$

The interested reader can find other proofs of these results in [14]. In particular, the $\mathbb{Z}_2$-graded cocharacters are described directly by da Silva, without using the Y-proper polynomials.

The following corollary is a consequence of the following results of this section:

Corollary 18.

(*i*) *If* $\dim_F L_1 = \infty$ *and* $\dim_F L_{-1} = \infty$, *then*
$\mathrm{Id}(E_\varphi) = \mathrm{Id}(E_{\varphi_l^{(\infty)}}) = \langle [u_1, u_2, u_3] \mid u_i \in Y \cup Z \rangle_{T_2}$,
$c_n(E_\varphi) = c_n(E_{\varphi_l^{(\infty)}}) = 4^{n-\frac{1}{2}}$ *and* $\chi_{r,m}(E_\varphi) = \chi_{r,m}(E_{\varphi_l^{(\infty)}})$
(*see Theorem* 10).

(*ii*) *If* φ *is a graded automorphism and* ℓ *is a positive integer then:*

	$\mathrm{Id}(E_\varphi)$	$c_n(E_\varphi)$	$\chi_{r,m}(E_\varphi)$
$\dim_F L_1 = 1$	$\mathrm{Id}(E_{\varphi_l^{(1)}})$ *see Theorem* 16	$c_n(E_{\varphi_l^{(1)}})$ *see Theorems* 5 *and* 6	$\chi_{r,m}(E_{\varphi_l^{(1)}})$ *see Theorem* 17
$\dim_F L_1 = \ell > 1$ *and* $\prod_{j=1}^{\ell+1}(\varphi(e_{i_j}) + e_{i_j}) = 0$ *for any* $\ell+1$ *generators* $e_{i_1}, \ldots, e_{i_{\ell+1}}$	$\mathrm{Id}(E_{\varphi_l^{(\ell)}})$ *see Theorem* 16	$c_n(E_{\varphi_l^{(\ell)}})$ *see Theorems* 5 *and* 6	$\chi_{r,m}(E_{\varphi_l^{(\ell)}})$ *see Theorem* 17
$\dim_F L_{-1} = 1$	$\mathrm{Id}(E_{\varphi_l^{(1^*)}})$ *see Theorem* 9	$c_n(E_{\varphi_l^{(1^*)}})$ *see Theorem* 5	$\chi_{r,m}(E_{\varphi_l^{(1^*)}})$ *see Theorem* 10
$\dim_F L_{-1} = \ell > 1$ *and* $\prod_{j=1}^{\ell+1}(\varphi(e_{i_j}) - e_{i_j}) = 0$ *for any* $\ell+1$ *generators* $e_{i_1}, \ldots, e_{i_{\ell+1}}$	$\mathrm{Id}(E_{\varphi_l^{(\ell^*)}})$ *see Theorem* 9	$c_n(E_{\varphi_l^{(\ell^*)}})$ *see Theorem* 5	$\chi_{r,m}(E_{\varphi_l^{(\ell^*)}})$ *see Theorem* 10

It is worth to say that the above results remain valid if 1 does not belong to E (see [2, 3, 14]).

REFERENCES

[1] N. Yu. Anisimov. Involution codimensions of Grassmann algebra. *Vestn. Mosk. Univ., Ser. I 3* (2001), 25–29 (in Russian); English translation in *Mosc. Univ. Math. Bull.* **56**, *3* (2001), 25–29.

[2] N. Anisimov. $\mathbb{Z}_p$-codimensions of $\mathbb{Z}_p$-identities of Grassmann Algebra. *Commun. Algebra* **29**, *9* (2001), 4211–4230.

[3] N. Anisimov. Linearization method of computing $\mathbb{Z}_2$-codimensions of identities of the Grassmann Algebra. In: Polynomial Identities and Combinatorial

Methods (Eds. A. Giambruno et al.), Pantelleria, Italy. Lect. Notes Pure Appl. Math. vol. **235**, 2003, 1–26, New York, Marcel Dekker.

[4] O. M. Di Vincenzo. A note on the identities of the Grassmann algebras. *Boll. Unione Mat. Ital., VII. Ser.* **A 5**, *3* (1991), 307–315.

[5] O. M. Di Vincenzo; V. R. T. da Silva. On $\mathbb{Z}_2$-graded polynomial identities of the Grassmann algebra. *Linear Algebra Appl.* **431**, *1–2* (2009), 56–72.

[6] V. Drensky. Free Algebras and PI-Algebras. Graduate Course in Algebra. Singapore, Springer, 2000.

[7] A. Giambruno, S. Mishchenko, M. Zaicev. Polynomial identities on superalgebras and almost polynomial growth. *Commun. Algebra* **29**, *9* (2001), 3787–3800.

[8] A. Giambruno, M. Zaicev. Polynomial Identities and Asymptotic Methods. Mathematical Surveys and Monographs, vol. **122**, Providence, RI, Amer. Math. Soc., 2005.

[9] A. R. Kemer. Varieties and $\mathbb{Z}_2$-graded algebras. *Izv. Akad. Nauk SSSR, Ser. Mat.* **48**, *5* (1984), 1042–1059 (in Russian); English translation in *Math. USSR, Izv.* **25** (1985), 359–374.

[10] A. R. Kemer. Ideals of Identities of Associative Algebras. Translations of Mathematical Monographs vol. **87**, Providence, RI, American Mathematical Society, 1991.

[11] D. Krakowski, A. Regev. The polynomial identities of the Grassmann algebra. *Trans. Amer. Math. Soc.* **181** (1973), 429–438.

[12] V. N. Latyshev. On the choice of basis in a T-ideal. *Sib. Mat. Zh.* **4**, *5* (1963), 1122–1127 (in Russian).

[13] J. B. Olsson, A. Regev. Colength sequence of some T-ideals. *J. Algebra* **38** (1976), 100–111.

[14] V. R. T. da Silva. Codimensões, Cocaracteres, Identidades e Polinômios Centrais $\mathbb{Z}_2$-Graduados da Álgebra de Grassmann. Tese de Doutorado, Departamento de Matemática, Universidade Federal de Minas Gerais, 2008.

[15] V. R. T. da Silva. $\mathbb{Z}_2$-Codimensions of the Grassmann algebra. *Commun. Algebra* **37**, *9* (2009), 3342–3359.

Departamento de Matemática
Instituto de Ciências Exatas
Universidade Federal de Minas Gerais
30161-970 – Belo Horizonte – Brasil
e-mail: `viviane@mat.ufmg.br`

Received March 4, 2012