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UNIVERSAL ENVELOPING ALGEBRAS OF NONASSOCIATIVE STRUCTURES

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Dedicated to Yuri Bahturin on his 65th birthday

Abstract. This is a survey paper to summarize the latest results on the universal enveloping algebras of Malcev algebras, triple systems and Leibniz n-ary algebras.

1. Introduction. Nonassociative algebras are very rich in algebraic structure. They have important applications not only in many branches of mathematics but also in physics. Perhaps, a Lie algebra is the most known example of a nonassociative algebra. In the classical theory of Lie algebras, the famous Poincaré-Birkhoff-Witt (PBW) theorem (Poincaré (1900), G. D. Birkhoff (1937),

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Witt (1937))[40, 4, 53] establishes a fundamental connection between Lie and associative algebras by relating the universal associative enveloping algebra to any Lie algebra L.

Let L be a Lie algebra over a field $\mathbb{F}$. Then the *universal associative enveloping* algebra of L is a pair $(U(L), \varepsilon)$ such that

1. $U(L)$ is a unital associative algebra over $\mathbb{F}$.

2. $\varepsilon : L \to U(L)^-$ is a Lie algebra homomorphism.

3. If (A, ψ) is another such pair, then there exists a unique homomorphism $\theta : U(L) \to A$ such that $\theta \circ \varepsilon = \psi$.

The Poincaré-Birkhoff-Witt theorem describes the structure of a linear basis of $U(L)$:

Theorem 1. *Let L be a Lie algebra over a field $\mathbb{F}$, and $(U(L), \varepsilon)$ its universal enveloping algebra. If $\{x_\alpha \mid \alpha \in I\}$ is a basis of L over $\mathbb{F}$ where I is an index set with a total order $\leq$, then monomials of the form*

$$\varepsilon(x_{\alpha_1})^{k_1} \cdots \varepsilon(x_{\alpha_r})^{k_r}$$

where $\alpha_1 < \cdots < \alpha_r$ and the integers $k_i \geq 0$ are arbitrary and form a basis of $U(L)$ over $\mathbb{F}$.

In [26] Lazard constructed a universal enveloping algebra for a Lie algebra over a principal ideal domain. In general, not every Lie ring can be isomorphically imbedded into the commutator Lie ring A^- of an associative ring A. The corresponding counter-example was given by Shirshov [49].

For other nonassociative structures analogous results have been also discovered which allowed to describe their representations by means of universal enveloping algebras and to make calculations in a similar way as in a ring of polynomials. In particular, for color Lie algebras (superalgebras) introduced by V. Rittenberg and D. Wyler [41] and extensively studied by M. Scheunert [43] the generalized PBW and Ado theorems hold true which appeared to be a powerful technique for studying representations.

2. Malcev algebras.

2.1. PBW-theorem. Moufang-Lie algebras (now called Malcev algebras) were introduced by Malcev [32] as the tangent algebras of analytic Moufang loops.

Definition 1. *A* Malcev algebra *is a vector space M over a field $\mathbb{F}$ with a bilinear product $[a, b]$ satisfying anticommutativity and the Malcev identity for* *all* $a, b, c \in M$*:*

$$[a, a] = 0, \qquad [J(a, b, c), a] = J(a, b, [a, c]).$$

where

$$J(a, b, c) = [[a, b], c] + [[b, c], a] + [[c, a], b].$$

Clearly, every Lie algebra is a Malcev algebra, and so Malcev algebras generalize the class of Lie algebras.

All of the known examples of finite-dimensional Malcev algebras arise from the following construction.

Example 1. Let A be an alternative algebra, that is, its associator $(a, b, c) = (ab)c - a(bc)$ satisfies

$$(a, b, c) = -(b, a, c),$$
$$(a, b, c) = -(a, c, b),$$

for any $a, b, c \in A$. If we introduce a new product by

$$[x, y] = xy - yx,$$

then we obtain a new algebra denoted by A^- which satisfies both anticommutativity and Malcev identity. Hence, A^- is a Malcev algebra. Obviously, any subspace M of A closed under $[\cdot, \cdot]$ is also an example of a Malcev algebra.

In [24] Kuzmin developed the complete theory of Malcev algebras of finite-dimensions. In particular, he showed that every central simple finite-dimensional Malcev algebra over a field of characteristic different from 2 and 3 is either a Lie algebra or the algebra of the form $\mathbb{O}^-/\mathbb{F}1$, where $\mathbb{O}$ is the octonion algebra over $\mathbb{F}$ and 1 denotes the identity of $\mathbb{O}$.

A Malcev algebra is said to be *special* if it can be realized as a subalgebra of A^-, where A is an alternative algebra. One of the open questions in the area of Malcev algebras is the well known *speciality problem*: is every Malcev algebra special? This problem has been formulated by Kuzmin in the Dniester Notebook (see also [14, 44, 50]). It is clear that $\mathbb{O}^-/\mathbb{F}1$ is a special Malcev algebra. Moreover, all examples of finite-dimensional Malcev algebras we know are special. Some of the most important recent work related to the speciality problem for Malcev algebras is in a series of papers by Shestakov and Zhukavets [46, 47, 48]. In particular, they showed that a free Malcev superalgebra on one odd generator is special. As a consequence, the Malcev Grassmann algebra is also special.

Let us now discuss the results of Shestakov and Zhukavets in more details.

Definition 2. *A* superalgebra $M = M_0 \oplus M_1$ *is a* $\mathbb{Z}_2$*-graded vector space such that*

$$M_0M_0 \subseteq M_0,\ M_1M_1 \subseteq M_0,\ M_1M_0 \subseteq M_1,\ M_0M_1 \subseteq M_1.$$

The elements from M_0 and M_1 are called *homogeneous* of degrees 0 and 1, respectively. Then:

Definition 3. *A superalgebra M is called a* Malcev superalgebra *if the following super-identities hold:*

$$\begin{aligned}
&xy = -(-1)^{\overline{xy}}yx,\\
&((xy)z)t - x((yz)t) - (-1)^{\overline{y}(\overline{z}+\overline{t})}(x(zt))y - (-1)^{\overline{t}(\overline{y}+\overline{z})}((xt)y)z\\
&\ = (-1)^{\overline{yz}}(xz)(yt),
\end{aligned}$$

where $\overline{x}, \overline{y}, \overline{z}, \overline{t}$ denote the degrees of the homogeneous elements x, y, z, and t, respectively.

Let $\mathrm{Malc}[x]$ be a free Malcev superalgebra generated by one odd element x. As proved in [45], $\mathrm{Malc}[x]$ admits the following linear basis:

$$\{x^k,\ x^{4k}x^2,\ x^{4k+1}x^2 \mid k > 0\}.$$

Using this fact, the authors showed in [48] that there is an injective homomorphism from $\mathrm{Malc}[x]$ into A^-/I where $A = \mathrm{Alt}[x]$ is a free alternative superalgebra on one odd generator x and $I = (A^2A^2)A + A(A^2A^2)$. In other words, $\mathrm{Malc}[x]$ is a special Malcev algebra.

There is another way of constructing Malcev algebras which is based on the concept of generalized alternative nucleus of A where A is an arbitrary algebra. Namely, for any algebra A its *generalized alternative nucleus* is given by

$$\mathrm{N_{alt}}(A) = \{a \in A \mid (a, x, y) = -(x, a, y) = (x, y, a),\ x, y \in A\},$$

where $(x, y, z) = (xy)z - x(yz)$. This is not necessarily a subalgebra of A but it is a subalgebra of A^-, that is, it is closed under $[x, y] = xy - yx$. In fact, it can be proven that $\mathrm{N_{alt}}(A)$ is a Malcev algebra relative to $[\,\cdot\,, \cdot\,]$.

In 2004, Pérez-Izquierdo and Shestakov [38] extended the PBW theorem from Lie algebras to Malcev algebras. For any Malcev algebra M, over a field of characteristic different fron 2 and 3, they constructed a universal *nonassociative* enveloping algebra $U(M)$ which shares many properties of the universal associative enveloping algebras of Lie algebras: $U(M)$ is linearly isomorphic to the polynomial algebra $P(M)$ and has a natural (nonassociative) Hopf algebra structure. We note that the PBW theorem does not answer the speciality problem since $U(M)$ is not necessarily alternative.

Theorem 2 (Pérez-Izquierdo and Shestakov [38])**.** *For every Malcev algebra M over a field $\mathbb{F}$ of characteristic $\neq 2,3$ there exists a nonassociative algebra $U(M)$ and an injective algebra morphism $\iota\colon M \to U(M)^-$ such that $\iota(M) \subseteq \mathrm{N}_{\mathrm{alt}}(U(M))$. Furthermore, $U(M)$ is a universal object with respect to such morphisms.*

We next recall the construction of $U(M)$ given in [38]. Let $\{a_i \mid i \in \Lambda\}$ be a linear basis of M over $\mathbb{F}$ where Λ is an index set with some fixed total order $\leq$. Consider $F(M)$, the unital free nonassociative algebra on the set of generators a_i, $i \in \Lambda$. Clearly, M can be viewed as a subset of $F(M)$. Then we generate the ideal $I(M)$ of $F(M)$ by the following elements:

$$ab - ba - [a,b], \qquad (a,x,y) + (x,a,y), \qquad (x,a,y) + (x,y,a),$$

for all $a,b \in M$ and all $x,y \in F(M)$. Let us set $U(M) = F(M)/I(M)$. As follows immediately from the definition of $U(M)$, the mapping $\iota : M \to U(M)^-$ given by $a \mapsto \iota(a) = \overline{a} = a + I(M)$ is a Malcev homomorphism such that $\iota(M) \subseteq N_{\mathrm{alt}}(U(M))$. As shown in [38] a linear basis of $U(M)$ consists of 1 and the following left-normed monomials

$$\overline{a}_{i_1}(\overline{a}_{i_2}(\cdots(\overline{a}_{i_{n-1}}\overline{a}_{i_n})\cdots)),$$

where $i_1 \leq \ldots \leq i_n$.

In [5, 8, 52] the structure constants for the universal nonassociative enveloping algebras $U(M)$ for the 4-dimensional solvable Malcev algebra, the 5-dimensional nilpotent Malcev and the one-parameter family of 5-dimensional solvable (non-nilpotent) Malcev algebras have been determined by means of constructing a representation of $U(M)$ by differential operators on the polynomial algebra $P(M)$.

Since $U(M)$ is not always alternative, it seems reasonable to consider its alternator ideal

$$I(U(M)) = \langle (x,x,y),\ (y,x,x) \mid x,y \in U(M) \rangle$$

and its maximal alternative quotient $A(M) = U(M)/I(U(M))$, which is the universal alternative enveloping algebra of M. The map $\tau : M \to A(M)^-$ defined by $\tau(a) = a + I(U(M))$ is a universal homomorphism. It remains an open problem whether τ is always an injective homomorphism which is equivalent to the speciality problem.

This construction produces new examples of infinite dimensional alternative algebras.

Example 2 (Tvalavadze, Bremner [52]). Let $\mathbb{M}_\gamma = \mathrm{Span}\{a, b, c, d, e\}$ be a 5-dimensional Malcev algebra with the following multiplication table:

$$[b,c] = 2d, \quad [a,b] = -b, \quad [a,c] = -c, \quad [a,d] = d, \quad [a,e] = \gamma e \, (\gamma \neq 0).$$

Then the universal alternative enveloping algebra $A(\mathbb{M}_\gamma)$ of $\mathbb{M}_\gamma$ has the following basis:

$$\{a^i d, \quad a^i b^j c^k e^m \mid i, j, k, m \geq 0\}$$

with the following multiplication table:

$$\begin{aligned}
a^i d \cdot a^r d &= 0, \\
a^i d \cdot a^r b^n c^p e^s &= \delta_{0n}\delta_{0p}\delta_{0s} a^i (a-1)^r d, \\
a^i b^j c^k e^m \cdot a^r d &= \delta_{j0}\delta_{k0}\delta_{m0} a^{i+r} d, \\
a^i b^j c^k e^m \cdot a^r b^n c^p e^s &= a^i (a+j+k-\gamma m)^r b^{j+n} c^{k+p} e^{m+s} + \delta_{m,0}\delta_{s,0}\delta_{j+n,1}\delta_{k+p,1} T^{ir}_{jk},
\end{aligned}$$

where

$$T^{ir}_{jk} = \begin{cases} 0 & \text{if } (j,k) = (0,0), \\ (a-1)^{i+r} d - a^i (a+1)^r d & \text{if } (j,k) = (1,0), \\ -(a-1)^{i+r} d - a^i (a+1)^r d & \text{if } (j,k) = (0,1), \\ a^i (a-1)^r d - a^i (a+2)^r d & \text{if } (j,k) = (1,1). \end{cases}$$

In particular this implies that every Malcev algebra $\mathbb{M}_\gamma$ in the one-parameter family of solvable 5-dimensional Malcev algebras is special.

It is worth mentioning that $A(M)$ is not always infinite-dimensional as in the following lemma (the proof of this lemma can be found in [6]):

Lemma 1 (Shestakov). *The universal alternative enveloping algebra $U(\mathbb{M})$ of the 7-dimensional simple Malcev algebra $\mathbb{M}$ over $\mathbb{R}$ is isomorphic to the division algebra $\mathbb{O}$ of real octonions: $U(\mathbb{M}) \cong \mathbb{O}$.*

2.2. Chevalley's and Ado-Iwasawa theorems. In the theory of universal enveloping algebras of Lie algebras classical Chevalley's theorem asserts that the center of the universal enveloping algebra $U(L)$ of a semi-simple Lie algebra L is isomorphic to a polynomial ring in n variables where n is the dimension of the Cartan subalgebra of L. Moreover, by Kostant's theorem $U(L)$ is in fact a free module over its center. An analogous result holds for Malcev algebras.

Theorem 3 (Zhelyabin, Shestakov [54])**.** *The center of the universal nonassociative envelope for a finite-dimensional semisimple Malcev algebra over a field of characteristic zero is a ring of polynomials in a finite number of variables equal to the dimension of its Cartan subalgebra, and the universal nonassociative enveloping algebra is a free module over its center.*

If a Malcev algebra is not semisimple, then the center of the universal nonassociative envelope can have a different structure [52].

Theorem 4. *Let* $\mathbb{M} = \operatorname{span}\{a, b, c, d, e\}$ *belong to the one-parameter family of solvable 5-dimensional Malcev algebras over a field* $\mathbb{F}$*. Then*

$$Z(U) = \begin{cases} \mathbb{F}[\, d^{\gamma m} e^m \,] & \text{if } \gamma = l/m \text{ with } l, m \in \mathbb{Z},\ (l, m) = 1,\ m > 0; \\ \mathbb{F} & \text{if } \gamma \notin \mathbb{Q}. \end{cases}$$

In the rest of this section we discuss the extension of the Ado-Iwasawa theorem to Malcev algebras. In the case of Lie algebras, Ado's theorem states that every finite-dimensional Lie algebra over a field of characteristic zero has a faithful finite-dimensional representation. Another way of stating this result is that every finite-dimensional Lie algebra L is a subalgebra of $\operatorname{End} V^-$ for some finite-dimensional vector space V, so can be viewed as an algebra of matrices. In [15] Filippov constructed an example of a nilpotent finite-dimensional Malcev algebra over an associative commutative ring Φ ($\frac{1}{6} \in \Phi$) of index 8 on a set of 6 generators which has no faithful representation.

Definition 4. *A linear map* $\rho : A \to \operatorname{End} V$*, where* A *is a Malcev algebra and* V *is a vector space, is called a* representation *if the sum* $A \oplus V$ *is a Malcev algebra under the following operation:*

$$(x_1 + v_1) \cdot (x_2 + v_2) = x_1 x_2 + \rho(x_2)(v_1) - \rho(x_1)(v_2).$$

Moreover, ρ *is said to be* faithful *if*

$$\operatorname{Ker}(\rho) = \{x \in A \mid \rho(x) = 0\} = \{0\}.$$

As follows from Theorem 2 any Malcev algebra M can be realized as a subalgebra of $\mathrm{N}_{\mathrm{alt}}(A)$ for some algebra A. In [38] the authors proved that if M is finite-dimensional, then A can be taken finite-dimensional, too. Namely, the following generalized version of Ado's theorem for Malcev algebras holds.

Theorem 5. *Let M be a finite-dimensional Malcev algebra over a field of characteristic not equal to 2 and 3. Then, there exist a unital finite-dimensional algebra A and a monomorphism of Malcev algebras $\iota : M \to \mathrm{N}_{\mathrm{alt}}(A)$.*

3. Triple systems.

3.1. Lie and Jordan triple systems. A triple system over a field $\mathbb{F}$ is a linear space over $\mathbb{F}$ with a trilinear operator. Among the triple systems the most important examples are *Lie triple systems* (LTS) and *Jordan triple systems* (JTS). They were introduced and systematically studied by N. Jacobson [18].

Definition 5. *A triple system V with a trilinear operator $[\cdot, \cdot, \cdot]$ is said to be a Lie triple system if $[\cdot, \cdot, \cdot]$ satisfies the following identities:*

$$[x, y, z] = -[y, x, z], \tag{3.1}$$

$$[x, y, z] + [z, x, y] + [y, z, x] = 0, \tag{3.2}$$

$$[u, v, [x, y, z]] = [[u, v, x], y, z] + [x, [u, v, y], z] + [x, y, [u, v, z]] \tag{3.3}$$

for any $x, y, z, u, v \in V$.

By introducing the following ternary operation $[a, b, c] := [[a, b], c]$ any Lie algebra $\mathfrak{g}$ can be turned into a Lie triple system. Other examples of Lie triple systems naturally arise from sets of skew-symmetric elements of $\mathfrak{g}$ relative to involutary automorphisms. Namely, let $\varphi : \mathfrak{g} \to \mathfrak{g}$ be any automorphism of $\mathfrak{g}$ such that $\varphi^2 = \mathrm{id}$. Then $\mathfrak{g}_\varphi = \{x \in \mathfrak{g} \mid \varphi(x) = -x\}$ is closed under $[[\cdot, \cdot], \cdot]$. Hence, $\mathfrak{g}_\varphi$ is itself a Lie triple system.

Definition 6. *Let $\mathfrak{g}$ be a Lie algebra, and T a Lie triple system with a triple product $[\cdot, \cdot, \cdot]$. An* embedding *of T into $\mathfrak{g}$ is a linear transformation $\theta : T \to \mathfrak{g}$ such that*

(i) $\theta([x, y, z]) = [[\theta(x), \theta(y)], \theta(z)]$, *for any $x, y, z \in T$,*

(ii) $\theta(T)$ *generates $\mathfrak{g}$.*

It can be easily shown that if $\theta : T \to \mathfrak{g}$ is an embedding then

$$\mathfrak{g} = \theta(T) \oplus [\theta(T), \theta(T)].$$

In [18] Jacobson noticed that any Lie triple system T can be embedded into its *standard enveloping* Lie algebra denoted by $L_s(T)$. Let us briefly recall the construction of $L_s(T)$ [17].

Definition 7. *A linear transformation $D : T \to T$ is said to be a* derivation *if*

$$D([a, b, c]) = [D(a), b, c] + [a, D(b), c] + [a, b, D(c)]$$

for any $a, b, c \in T$.

Equation (3.3) (Definition 5) implies that for any fixed $a, b \in T$, $D_{a,b}(c) = [a, b, c]$ is a derivation which will be called *inner*. The set $\mathrm{InnDer}(T)$ of all inner derivations has a natural Lie algebra structure given by

$$[D_{a,b}, D_{c,d}] = D_{a,b}D_{c,d} - D_{c,d}D_{a,b}.$$

Note that the relation

$$[D_{a,b}, D_{c,d}] = D_{D_{a,b}(c),d} + D_{c,D_{a,b}(d)}$$

proves the closure of $\mathrm{InnDer}(T)$ under the Lie bracket. Let

$$L_s(T) = T \oplus \mathrm{InnDer}(T).$$

By setting

$$[(x, D_{a,b}), (y, D_{c,d})] = (D_{a,b}(y) - D_{c,d}(x), D_{x,y} + [D_{a,b}, D_{c,d}])$$

$L_s(T)$ can be transformed into a Lie algebra. Note that

$$T = \{(x, D) \in L_s(T) \mid \varphi((x, D)) = -(x, D)\}$$

where $\varphi : L_s(T) \to L_s(T)$ such that $\varphi((a, D)) = (-a, D)$ is an automorphism of order 2. Thus, every LTS arises as the set of skew-symmetric elements for a suitable automorphism of order 2.

With any Lie triple system we can also associate its *universal enveloping* Lie algebra $L_u(T)$ defined in the following natural way. Let $L(T)$ be a free Lie algebra based on T, and $\tau : T \to L(T)$ be a linear mapping given by $\tau(x) = x$ for every $x \in T$. Consider $I \lhd L(T)$ generated by all elements of the form

$$\tau([a,b,c]) - [[\tau(a),\tau(b)],\tau(c)]$$

where $a,b,c \in T$. Then $L_u(T) = L(T)/I$. Moreover, for any Lie algebra $\mathfrak{g}$ and $\theta : T \to \mathfrak{g}$ satisfying $\theta([a,b,c]) = [[\theta(a),\theta(b)],\theta(c)]$ there exists a unique homomorphism $\psi : L_u(T) \to \mathfrak{g}$ such that $\theta = \psi \circ \tau$.

In [28] Lister determined all simple finite-dimensional Lie triple systems over an algebraically closed field of zero characteristic. The main idea of his proof was to reduce the classification problem to the determination of similarity classes among the automorphisms of period 2 in the simple Lie algebras. Recall that the Lie algebras of types A_n, B_n, C_n, D_n together with the exceptional algebras G_2, F_4, E_6, E_7, E_8 form a complete list of simple finite-dimensional Lie algebras over an algebraically closed field of zero characteristic. A simple Lie triple system of a non-exceptional simple Lie algebra is isomorphic to one of the following:

(1) The set of matrices of trace zero skew-symmetric relative to an automorphism $X \to P_{n,r}^{-1} X P_{n,r}$, where $P_{n,r} = \operatorname{diag}\{\underbrace{1,\dots,1}_{r},\underbrace{-1,\dots,-1}_{n-r}\}$;

(2) The set of symmetric matrices of trace zero;

(3) The set of symplectic symmetric matrices of trace zero (the order of matrices must be even);

(4) The set of skew-symmetric matrices of order $2n+1$ that are also skew-symmetric relative to $X \to P_{2n+1,r}^{-1} X P_{2n+1,r}$;

(5) The set of skew-symmetric matrices of order $2n$ that are additionally symplectic symmetric;

Let $P'_{n,r} = \operatorname{diag}\{\underbrace{1,\dots,1}_{r},\underbrace{-1,\dots,-1}_{\frac{n}{2}-r},\underbrace{1,\dots,1}_{r},\underbrace{-1,\dots,-1}_{\frac{n}{2}-r}\}$ for even n.

(6) The set of symplectic skew-symmetric matrices which are skew-symmetric relative to $X \to P'^{-1}_{n,r} X P'_{n,r}$;

(7) The set of symplectic skew-symmetric matrices which are additionally symmetric.

Let L be a simple exceptional Lie algebra with a fixed Cartan subalgebra denoted by H. Choose the following canonical basis for L:

$$h_1,\dots,h_r,e_{\alpha_1},\dots,e_{\alpha_k}$$

where $h_1, \dots, h_r$ is a basis for H, and $\{e_{\alpha_1}, \dots, e_{\alpha_k}\}$ is a system of root vectors. If L has one of the following types: G_2, F_4, E_7 or E_8, then every Lie triple system of L is defined by an appropriate automorphism φ of the form:

$$\varphi(h) = h, \quad \varphi(e_\alpha) = \pm e_\alpha \tag{3.4}$$

where $h \in H$, and e_α is a root vector. The only case left is when L has type E_6. If ψ is an inner automorphism of E_6 of period 2, then it is equivalent to an automorphism of form (3.4). Choose in L a canonical basis $h_1, \dots, h_6, e_\alpha, \dots$. The roots of L are

$$\alpha_{pq}(h) = \lambda_p - \lambda_q,$$

$$\alpha_{p,q,s}(h) = \lambda_p + \lambda_q + \lambda_s,$$

$$-\alpha_{p,q,s}(h),\ \alpha_0(h) = \sum_{i=1}^{6} \lambda_i,\ -\alpha_0(h),$$

where $h = \sum_{i=1}^{6} \lambda_i h_i$, p, q, s are all distinct, and $p, q, s = 1, \dots, 6$. Let e_{pq}, e_{pqs}, e'_{pqs}, e_0, and e'_0 be the corresponding root vectors. It is known from [28] that all outer automorphisms of E_6 are equivalent to an automorphism χ given by:

$$\chi(e_{12}) = \pm e_{12},\ \chi(e_{23}) = \pm e_{41},\ \chi(e_{34}) = \pm e_{34},$$

$$\chi(e_{45}) = \pm e_{63},\ \chi(e_{56}) = \pm e_{56},\ \chi(e_{135}) = \pm e_{135}.$$

where $\{\alpha_{12}, \alpha_{23}, \alpha_{34}, \alpha_{45}, \alpha_{56}, \alpha_{135}\}$ forms a set of simple roots of L.

Jordan triple systems are generalizations of Jordan algebras (commutative algebras satisfying $(xy)x^2 = x(yx^2)$). The following definition was given by Meyberg in [33]:

Definition 8. *A triple system V with a trilinear operator $\{\cdot, \cdot, \cdot\}$ is said to be a Jordan triple system if $\{\cdot, \cdot, \cdot\}$ satisfies the following identities:*

$$\{x, y, z\} = \{x, z, y\}, \tag{3.5}$$

$$\{u, v, \{x, y, z\}\} = -\{\{v, u, x\}, y, z\} + \{x, \{u, v, y\}, z\} + \{x, y, \{u, v, z\}\} \tag{3.6}$$

for any $x, y, z, u, v \in V$.

Simple Jordan triple systems over an algebraically closed field of characteristic different from 2 have been classified by O. Loos in [30]. In [35, 36]

E. Neher obtained the complete description of the real finite-dimensional simple Jordan triple systems.

Let J be a Jordan algebra over a field of characteristic different from 2. Introducing the following triple product

$$\{x, y, z\} = (xy)z + (zy)x - y(xz)$$

J becomes a Jordan triple system. However, the converse is not true: not every JTS comes from some Jordan algebra in this way. For instance, the following two examples of JTS from [37] do not have Jordan algebras associated with them.

Example 3. Consider an associative algebra D with involution i. The set of $p \times q$ matrices $M_{p,q}(D)$ with entries from D forms a JTS under the triple product given by

$$\{X, Y, Z\} = X(i(Y))^t Z + Z(i(Y))^t X.$$

Example 4. Let $A_p(\mathbb{F})$ be the set of all skew-symmetric matrices of order p over a base field $\mathbb{F}$. Then $A_p(\mathbb{F})$ becomes a JTS under

$$\{X, Y, Z\} = XYZ + ZYX.$$

Meyberg showed [34] that every Jordan triple system $(T, \{\dot{}, \dot{}, \dot{}\})$ defines a Lie triple system denoted by T^- with the following triple product

$$[x, y, z] = \{x, y, z\} - \{y, x, z\}.$$

In most cases, this construction carries central simple Jordan triple systems into central simple Lie triple system.

Jordan triple systems are closely related to the so-called Jordan pairs [31] defined as follows:

Definition 9. *Let $V = (V_+, V_-)$ be a pair of vector spaces equipped with the following trilinear maps: $V_\varepsilon \times V_{-\varepsilon} \times V_\varepsilon \to V_\varepsilon$ where $\varepsilon = \pm$ satisfying*

$$\{x_\varepsilon, y_{-\varepsilon}, z_\varepsilon\} = \{z_\varepsilon, y_{-\varepsilon}, x_\varepsilon\},$$

$$\{x_\varepsilon, y_{-\varepsilon}, \{u_\varepsilon, v_{-\varepsilon}, w_\varepsilon\}\} - \{u_\varepsilon, v_{-\varepsilon}, \{x_\varepsilon, y_{-\varepsilon}, z_\varepsilon\}\} =$$

$$\{\{x_\varepsilon, y_{-\varepsilon}, u_\varepsilon\}, v_{-\varepsilon}, w_\varepsilon\} - \{u_\varepsilon, \{y_{-\varepsilon}, x_\varepsilon, v_{-\varepsilon}\}, w_\varepsilon\}$$

With any Jordan triple system V we can associate a Jordan pair (V, V) equipped with $\{x_\varepsilon, y_{-\varepsilon}, z_\varepsilon\} = \{x, y, z\}$.

We next will be concerned with the universal associative enveloping algebras of Lie (Jordan) triple systems (see [18] for more details).

Let $F\langle X\rangle$ be a free associative algebra generated by $X = \{x_\alpha\}$. Let V be a Lie triple system with basis y_α, $\alpha \in I$, and the following multiplication table:

$$[y_\alpha, y_\beta, y_\gamma] = \sum_\delta \mu^\delta_{\alpha\beta\gamma} y_\delta,$$

where $\alpha, \beta, \gamma \in I$ and $\mu^\delta_{\alpha\beta\gamma} \in \mathbb{F}$.

The *universal associative envelope* of V is defined in the following way. Set $U(V) = F\langle X\rangle/J$ where J is the ideal of $F\langle X\rangle$ generated by the elements:

$$[[x_\alpha, x_\beta], x_\gamma] - \sum_\delta \mu^\delta_{\alpha\beta\gamma} x_\delta.$$

There is a natural linear map $\varepsilon : V \mapsto U(V)$ defined by $\varepsilon(y_\alpha) = x_\alpha$, $\alpha \in I$. Moreover, for any unital associative algebra A and a linear map $\sigma : V \mapsto A$ such that

$$\sigma([a, b, c]) = [[\sigma(a), \sigma(b)], \sigma(c)]$$

there exists a unique homomorphism of unital associative algebras $\psi : U(V) \mapsto A$ such that $\psi \circ \varepsilon = \sigma$.

Since any Lie triple system T has a standard imbedding into $L_s(T)$ defined above and any Lie algebra is imbedded into its universal associative envelope we have that the universal homomorphism ε must be injective.

In a similar manner we can define the universal associative envelope for a Jordan triple system.

Some interesting recent work towards generalization of Lie triple systems has been done in [7]. The authors have introduced the concept of Leibniz triple systems in such a way that any Lie triple system automatically satisfies the defining identities of Leibniz triple systems. Besides, they have constructed universal Leibniz envelopes for Leibniz triple systems and obtained some results regarding polynomial identities in Leibniz algebras.

3.2. Anti-Lie and anti-Jordan triple systems. We start this section with the notion of an (ε, δ)-Freudenthal-Kantor triple system (FKTS) which generalizes the concept of a generalized Jordan triple system (GJTS) of the second order.

Let V be a vector space over a field $\mathbb{F}$ of characteristic different from 2 and 3 with a trilinear product $(\,\cdot\,,\,\cdot\,,\,\cdot\,) : V \times V \times V \to V$.

Definition 10. *Let $\varepsilon, \delta = \pm 1$. A pair $(V, (\,\cdot\,,\,\cdot\,,\,\cdot\,))$ is called an (ε, δ)-Freudenthal-Kantor triple system if for any $x, y, z, a, b, c \in V$ its triple product has the following properties:*

$$(a, b, (x, y, z)) = ((a, b, x), y, z) + \varepsilon(x, (b, a, y), z) + (x, y, (a, b, z)),$$

$$K(K(a, b)x, y) - L(y, x)K(a, b) + \varepsilon K(a, b)L(x, y) = 0,$$

where

$$L(a, b)c = (a, b, c), \quad K(a, b)c = (a, c, b) - \delta(b, c, a).$$

The concept of a $(-1, 1)$-FKTS coincides with that of a GJTS of the second order. For $\varepsilon = 1$ and $\delta = -1$ we obtain the standard definition of an anti-Jordan triple system (AJTS), namely:

Definition 11. *Let V be a vector space that possesses a triple product $\langle\,\cdot\,,\,\cdot\,,\,\cdot\,\rangle$ satisfying*

$$\langle x, y, x\rangle = 0, \tag{3.7}$$

$$\langle x, y, \langle u, v, w\rangle\rangle - \langle u, v, \langle x, y, w\rangle\rangle = \langle\langle x, y, u\rangle, v, w\rangle + \langle u, \langle y, x, v\rangle, w\rangle \tag{3.8}$$

for all $x, y, u, v, w \in V$. Then $(V, \langle\,\cdot\,,\,\cdot\,,\,\cdot\,\rangle)$ defines an anti-Jordan triple system.

The next concept is closely related to anti-Jordan triple systems and useful to obtain constructions of Lie superalgebras.

Definition 12. *Let V be a vector space over a field $\mathbb{F}$ of characteristic differernt from 2. Suppose that a triple product $[\,\cdot\,,\,\cdot\,,\,\cdot\,]$ in V satisfies* (3.2)*,* (3.3) *and the following equation*

$$[x, y, z] = [y, x, z] \tag{3.9}$$

for all $x, y, z \in V$. Then $(V, [\,\cdot\,,\,\cdot\,,\,\cdot\,])$ defines an anti-Lie triple system.

Let $(V, \langle\,\cdot\,,\,\cdot\,,\,\cdot\,\rangle)$ be an AJTS (or, equivalently, a $(1, -1)$-FKTS). Introducing the second triple product by

$$[x, y, z] = \langle x, y, z\rangle + \langle y, x, z\rangle \tag{3.10}$$

we obtain an anti-Lie triple system.

As was shown in [1, 19, 22] all simple Lie (super)algebras over a field $\mathbb{F}$ of characteristic different from 2 and 3 can be constructed from suitable Freudenthal-Kantor triple systems by means of the standard embedding method. Let us briefly recall this construction from [20]. Consider an (ε, δ)-FKTS denoted by U. Then the direct sum $T = U \oplus U$ can be turned into a LTS (if $\delta = 1$) or an anti-LTS (if $\delta = -1$) by introducing the following triple product: let $x = x_1 \oplus x_2$, $y = y_1 \oplus y_2$ and $z = z_1 \oplus z_2$ be arbitrary elements from T. Then, by definition,

$$[x, y, z] := ((L(x_1, y_2) - \delta L(y_1, x_2))z_1 + \delta K(x_1, y_1)z_2)$$

$$\oplus (-\varepsilon K(x_2, y_2)z_1 + \varepsilon(L(y_2, x_1) - \delta L(x_2, y_1))z_2),$$

where the operators $L(\cdot, \cdot)$ and $K(\cdot, \cdot)$ are as in Definition 10. Denote the set of all left-multiplications $L(\cdot, \cdot)$ (inner derivations) of $(T, [\cdot, \cdot, \cdot])$ by $\mathrm{InnDer}(T)$. It is a Lie algebra under the standard Lie bracket on the set of linear transformations of T. Then for $\delta = 1$ the direct sum

$$L = \mathrm{InnDer}(T) \oplus T$$

becomes a Lie algebra under the following Lie bracket:

$$[u, v] = L(u, v),$$

$$[L(u, v), z] = L(u, v)(z),$$

$$[L(u, v), L(x, y)] = L(u, v)L(x, y) - L(x, y)L(u, v),$$

for any $u, v, x, y \in T$. For $\delta = -1$, $(L, [\cdot, \cdot])$ is a Lie superalgebra with even component $L_{\overline{0}} = \mathrm{InnDer}(T)$ and the odd component $L_{\overline{1}} = T$.

We can now give a few examples of finite-dimensional anti-Jordan triple systems over a field $\mathbb{F}$.

Example 5. Let $X, Y, Z \in M_{m,n}(\mathbb{F})$, $m = 2r$, and $A = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$ where I_r is the $r \times r$ identity matrix. Then $M_{m,n}(\mathbb{F})$ is an anti-Jordan triple system relative to the triple product

$$\langle X, Y, Z \rangle = XY^tAZ - ZY^tAX.$$

Example 6. Let $X, Y, Z \in M_{m,n}(\mathbb{F})$, $n = 2r$, and $B = \begin{pmatrix} 0 & I_r \\ -I_r & 0 \end{pmatrix}$ where I_r is the $r \times r$ identity matrix. Then $M_{m,n}(\mathbb{F})$ is an anti-Jordan triple system under

$$\langle X, Y, Z \rangle = XBY^tZ - ZBY^tX.$$

Example 7. Let $X, Y, Z \in M_{n,n}(\mathbb{F})$. Then it becomes an anti-Jordan triple system via

$$\langle X, Y, Z\rangle = XYZ - ZYX.$$

Example 8. Let T be a vector space over $\mathbb{F}$, and $B : T \times T \to \mathbb{F}$ be an alternating nondegenerate bilinear form. Consider an invertible endomorphism f of T satisfying

$$\alpha B(f(x), f(y)) = B(x, y), \quad f^2 = \beta \cdot \mathrm{id}$$

for nonzero scalars $\alpha, \beta \in \mathbb{F}$, and $x, y \in T$. Then T is an anti-Jordan triple system with triple product:

$$\langle x, y, z\rangle = B(x, f(y))z + B(f(y), z)x + B(x, z)f(y),$$

where $x, y, z \in T$.

In all of the above examples the resulting anti-Jordan triple system turns out to be *simple*. Based on Faulkner's and Ferrar's [16] classification of the finite-dimensional simple anti-Jordan pairs S. Bashir [2] obtained the complete description of finite-dimensional simple anti-Jordan triple systems over an algebraically closed field of characteristic zero. In order to state the classification theorem we recall the definition of an anti-Jordan pair:

Definition 13. *Let $V = (V^+, V^-)$ be a pair of vector spaces over a field $\mathbb{F}$ with trilinear maps $\{\cdot, \cdot, \cdot\}^\varepsilon : V^\varepsilon \times V^{-\varepsilon} \times V^\varepsilon \to V^\varepsilon$ such that $\{x, y, z\}^\varepsilon := D^\varepsilon(x, y)z$, for $\varepsilon = \pm$. Then V is called an* anti-Jordan pair *if for all $x, u \in V^\varepsilon$, $y, v \in V^{-\varepsilon}$ and $\varepsilon = \pm$*

$$\{x, y, x\}^\varepsilon = 0,$$

$$[D^\varepsilon(x, y), D^\varepsilon(u, v)] = D^\varepsilon(\{x, y, u\}^\varepsilon, v) + D^\varepsilon(u, \{y, x, v\}^{-\varepsilon}).$$

If $V = (V^+, V^-)$ is an anti-Jordan pair with a trilinear product $\langle \cdot, \cdot, \cdot\rangle$, then $T(V) = V^+ \oplus V^-$ under

$$\langle x^+ \oplus x^-, y^+ \oplus y^-, z^+ \oplus z^-\rangle = \langle x^+, y^-, z^+\rangle \oplus \langle x^-, y^+, z^-\rangle$$

is an anti-Jordan triple system *associated with V*.

Theorem 6 (Bashir [2])**.** *T is a finite-dimensional simple anti-Jordan triple system over an algebraically closed field $\mathbb{F}$ of characteristic zero if and only if T is isomorphic to one of the following:*

(*a*) *The triple system associated with one of the three simple anti-Jordan pairs* $\mathfrak{M}_{mn}(\mathbb{F})$, $\mathfrak{ss}_n(\mathbb{F})$ *and the symplectic anti-Jordan pair* (*see [16]*).

(*b*) *The anti-Jordan triple systems of Examples* 5, 6, 7, 8.

In what follows we focus on the universal envelope of anti-Jordan triple systems. There has not been much research done on enveloping algebras of anti-Jordan triple systems. It is still an open problem whether every AJTS (or at least a finite-dimensional AJTS) can be isomorphically embedded into an associative algebra.

Definition 14. *We call* $(\mathbb{T}, \langle\cdot\rangle)$ *a* symplectic anti-Jordan triple system *if there exists an alternating non-degenerate bilinear form* B *on* $\mathbb{T}$ *such that*

$$\langle x, y, z\rangle = B(x, y)z + B(y, z)x + B(x, z)y \tag{3.11}$$

for all $x, y, z \in \mathbb{T}$.

Let T be a finite-dimensional anti-Jordan triple system with basis $\{x_1, \ldots, x_s\}$ and the multiplication table as follows

$$\langle x_i, x_j, x_k\rangle = \sum_{m=1}^{s} c_{ijk}^{m} x_m. \tag{3.12}$$

We note that any associative algebra A becomes an anti-Jordan triple system with respect to the triple product defined by

$$(a, b, c) = abc - cba.$$

In a similar way as for Lie triple systems we can construct the universal associative envelope of T. Namely, we let $U(T)$ be the unital free associative algebra on generators $x_1, \ldots, x_s$ modulo the ideal J generated by the elements

$$x_i x_j x_k - x_k x_j x_i - \sum_{m=1}^{s} c_{ijk}^{m} x_m. \tag{3.13}$$

Clearly, there exists a universal homomorphism $\tau : T \to U(T)$. However, it is not known whether this homomorphism is generally injective.

Using noncommutative Gröbner-Shirshov bases in [51] the author studied the universal associative envelope of symplectic T and established a generalization of the PBW theorem for a finite-dimensional simple symplectic anti-Jordan triple system.

From now on we assume that $\mathbb{T}$ is a simple symplectic anti-Jordan triple system. Since $\mathbb{T}$ is simple, its alternating bilinear form B is non-degenerate and, therefore, $\dim \mathbb{T} = 2n$. It is well known that there exists a basis $e_1, e_2, \ldots, e_{2n-1}$, e_{2n} of $\mathbb{T}$ such that $B(e_{2i-1}, e_{2i}) = 1$, $i = 1, \ldots, n$, and the remaining $B(e_k, e_j) = 0$, $k < j$.

Definition 15. *We say that indices i, j form a* pair *if the corresponding $B(e_i, e_j)$ is nonzero.*

Lemma 2. *Let $A = F[e_1, \ldots, e_{2n}]$ be a unital free associative algebra. Then $U(T) = A/J$ where J is generated by the following elements*

$$e_k^2 e_{k+1} - e_{k+1} e_k^2 - 2e_k, \quad e_{k+1}^2 e_k - e_k e_{k+1}^2 + 2e_{k+1}, \tag{3.14}$$

$$e_k e_{k+1} e_i - e_i e_{k+1} e_k - e_i, \quad e_k e_i e_{k+1} - e_{k+1} e_i e_k - e_i, \tag{3.15}$$

$$e_i e_k e_{k+1} - e_{k+1} e_k e_i - e_i, \quad e_r e_s e_m - e_m e_s e_r, \tag{3.16}$$

where $i, k, r, s, m = 1, 2, \ldots, 2n$ are such that $i \notin \{k, k+1\}$ and $k, k+1$ is a pair but none of r, s, m form a pair.

For $n = 1$ we consider $\widetilde{A} = F[e_1, e_2, t]$ which is a free associative algebra on the free generators e_1, e_2, t. Consider the set of the following elements in $\widetilde{A}$:

$$e_1 t + t e_1 - 2e_1, \quad e_2 t + t e_2 - 2e_2, \tag{3.17}$$

$$e_1 e_2 - e_2 e_1 - t. \tag{3.18}$$

We denote by $\widetilde{J}$ the ideal of $\widetilde{A}$ generated by these elements. Let us now set $\widetilde{U(\mathbb{T})} = \widetilde{A}/\widetilde{J}$.

For $n > 1$ we consider $\widetilde{A} = F[e_1, \ldots, e_{2n}, t]$ which is a free associative algebra on the free generators $e_1, \ldots, e_{2n}, t$. Consider the following elements in $\widetilde{A}$:

$$e_i t - e_i, \quad t e_i - e_i, \tag{3.19}$$

$$t^2 - t, \quad e_r e_s - e_s e_r, \tag{3.20}$$

$$e_{q+1} e_q - e_q e_{q+1} + t, \tag{3.21}$$

where $i = 1, \ldots, 2n$, $q = 1, 3, \ldots, 2n-1$, and (r, s) is not a pair. Let $\widetilde{J}$ be the ideal generated by these elements. Set

$$\widetilde{U(\mathbb{T})} = F[e_1, \ldots, e_{2n}, t]/\widetilde{J}.$$

Lemma 3. *$U(T)$ and $\widetilde{U(T)}$ are isomorphic as associative algebras.*

Theorem 7 ([51])**.** *Let T be a simple symplectic anti-Jordan triple system with basis e_1, e_2. Then the imbedding of T into $\widetilde{U(T)}$ is injective. Moreover, the following products*

$$t^k e_1^i e_2^j, \tag{3.22}$$

where $k, i, j \in \mathbb{N} \cup \{0\}$ form a basis of the vector space $U(T)$.

Theorem 8 ([51])**.** *Let T be a simple symplectic anti-Jordan triple system with basis $e_1, \ldots, e_{2n}$, $n \geq 2$. Then the imbedding of T into $\widetilde{U(T)}$ is injective. Moreover, the following products*

$$e_1^{k_1} \ldots e_{2n}^{k_n} \text{ and } t,$$

where $k_1, \ldots, k_n \in \mathbb{N} \cup \{0\}$ form a basis of the vector space $U(T)$.

Using the explicit structure of the PBW-basis for $U(T)$ it is possible to compute its center denoted by $Z(U)$. Namely, the following holds true.

Theorem 9 ([51])**.** *Let $\mathbb{T}$ be a finite-dimensional simple symplectic anti-Jordan triple system over a field $\mathbb{F}$. Then*

$$Z(U) = \begin{cases} \mathbb{F}1 & \text{if } \dim \mathbb{T} = 2, \\ \operatorname{span}\{1, t\}_{\mathbb{F}} & \text{if } \dim \mathbb{T} > 2, \end{cases}$$

where $Z(U)$ is the center of the universal enveloping algebra.

4. Leibniz n-ary algebras. Recall that n-Lie algebras (sometimes referred to as Filippov algebras or Nambu-Lie algebras) were introduced by Filippov in 1985 [13]. For $n = 2$, the class of n-Lie algebras coincides with the class of Lie algebras.

Definition 16. *An n-Lie algebra over a field $\mathbb{F}$ is a vector space L over $\mathbb{F}$ with an n-ary operation $[-, -, \ldots, -] : L \times \ldots \times L \to L$ satisfying the anticommutative identity and the generalized Jacobi identity:*

$$[x_1, \ldots, x_i, \ldots, x_j, \ldots, x_n] = -[x_1, \ldots, x_j, \ldots, x_i, \ldots, x_n], \tag{4.1}$$

$$[[x_1, \ldots, x_n], y_2, \ldots, y_n] = \sum_{i=1}^{n} [x_1, \ldots, [x_i, y_2, \ldots, y_n], \ldots, x_n]. \tag{4.2}$$

Example 9. Let V_n be an $(n+1)$-dimensional vector space with the basis

$$\{e_1, e_2, \ldots, e_{n+1}\}.$$

Then we define

$$[x_1, x_2, \ldots, x_n] = \det A$$

where A is the following matrix

$$\begin{pmatrix} e_1 & e_2 & \ldots & e_{n+1} \\ x_{11} & x_{21} & \ldots & x_{n+1,1} \\ x_{12} & x_{22} & \ldots & x_{n+1,2} \\ \vdots & \vdots & \ddots & \vdots \\ x_{1n} & x_{2n} & \ldots & x_{n+1,n} \end{pmatrix},$$

and $x_i = x_{1i}e_1 + x_{2i}e_2 + \cdots + x_{n+1,i}e_{n+1}$. Then V_n equipped with this n-bracket is an n-Lie algebra.

In [29] Ling showed that any simple finite-dimensional n-Lie algebra over an algebraically closed field $\mathbb{F}$ of zero characteristic is isomorphic to V_n defined above. In other words, for a fixed $n \geq 3$ there exists a unique (up to an isomorphism) simple finite-dimensional n-Lie algebra. A natural question about the classification of irreducible representations arises for simple finite-dimensional n-Lie algebras. In a light of Ling's results, only representations of V_n need to be studied.

Recall that to any n-Lie algebra $\mathfrak{g}$ we can assign the Lie algebra $L(\mathfrak{g}) = \bigwedge^{n-1} \mathfrak{g}$ with the bracket:

$$[a_1 \wedge \cdots \wedge a_{n-1}, b_1 \wedge \cdots \wedge b_{n-1}] = \sum_{i=1}^{n-1} (-1)^{i+n} [a_1, \ldots, a_{n-1}, b_i] \wedge b_1 \wedge \cdots \wedge \hat{b}_i \wedge \cdots \wedge b_{n-1}$$

where $a_1, \ldots, a_{n-1}, b_1, \ldots, b_{n-1} \in \mathfrak{g}$. Filippov proved that $L(V_n) \cong so_{n+1}$.

Definition 17. *A representation of an n-Lie algebra $\mathfrak{g}$ in a vector space M is a linear map $\rho : \bigwedge^{n-1} \mathfrak{g} \to End(M)$ defined by $\rho(a_1 \wedge \cdots \wedge a_{n-1})(m) = [a_1, \ldots, a_{n-1}, m]$ such that*

$$\rho([a_1, \ldots, a_n] \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2})$$

$$= \sum_{i=1}^{n} (-1)^{i+n} \rho(a_1 \wedge \cdots \wedge \hat{a}_i \wedge \cdots \wedge a_n) \rho(a_i \wedge a_{n+1} \wedge \cdots \wedge a_{2n-2}),$$

where $a_1, \dots, a_{2n-2} \in \mathfrak{g}$.

Note that the above condition implies that M is an ordinary Lie module over $L(\mathfrak{g})$.

Dzhumadil'daev in [10] proved that any finite-dimensional n-Lie representation of V_n, $n \geq 2$, is completely reducible. Moreover, any finite-dimensional irreducible n-Lie V_n-module can be extended to $L(V_n)$-module (so_{n+1}-module) with the highest weight $t\pi_1$ (π_1 is a fundamental weight of so_{n+1}) for a nonnegative integer t.

Let A be an associative algebra. It becomes an n-ary algebra under

$$(x_1, \dots, x_n) = \sum_{\sigma \in S_n} (-1)^\sigma x_{\sigma(1)} \dots x_{\sigma(n)}.$$

It will be denoted by A^- and called the *commutator algebra* of A. By analogy with other nonassociative structures, it seems reasonable to study embeddings of an n-Lie algebra into A^- where A is an associative algebra. Pojidaev in [39] showed that such an enveloping algebra does not exist for a semisimple finite-dimensional ternary Lie algebra (3-Lie algebra). He also conjectured the existence of the universal associative enveloping algebras for simple n-Lie algebras. This problem was partially solved in [12] where for even n and any $(n+1)$-dimensional n-Lie algebra L the authors constructed a universal associative enveloping algebra $U(L)$ and showed that the natural map $L \to U(L)$ is injective.

In 2002 Casas, Insua, and Ladra [9] introduced the generalized version of n-Lie algebras called Leibniz n-algebras.

Definition 18. *A Leibniz n-algebra is a vector space with n-linear map* $[-, -, \dots, -]$ *satisfying the* Leibniz n-identity

$$\begin{aligned}
&[[x_1, \dots, x_n], y_1, \dots, y_{n-1}] \\
&= \sum_{i=1}^{n} [x_1, \dots, x_{i-1}, [x_i, y_1, \dots, y_{n-1}], x_{i+1}, \dots, x_n]
\end{aligned}$$

When $n = 2$ one recovers Leibniz algebras. Clearly, any n-Lie algebra is a Leibniz n-ary algebra which additionally satisfies anticommutativity identity. A Lie triple system is a particular example of a Leibniz 3-algebra that also satisfies the following conditions:

$$[x, y, z] + [y, z, x] + [z, x, y] = 0,$$

$$[x, y, y] = 0.$$

In what follows let $K = L^{\otimes(n-1)}$ denote the tensor product of $n-1$ copies of L. Moreover, K can always be turned into a Leibniz algebra by introducing the following binary product:

$$[a_1 \otimes \cdots \otimes a_{n-1}, b_1 \otimes \cdots \otimes b_{n-1}] = \sum_{i=1}^{n-1} a_1 \otimes \cdots \otimes [a_i, b_1, \ldots, b_{n-1}] \otimes \cdots \otimes a_{n-1}.$$

Definition 19. *A* representation *of a Leibniz n-algebra L in a vector space V is defined by n multilinear mappings*

$$\rho_i : K \to \mathrm{End}(V),$$

$0 \leq i \leq n-1$, *satisfying the following axioms:*

$$\rho_k([l_1, \ldots, l_n], l_{n+1}, \ldots, l_{2n-2})$$
$$= \sum_{i=1}^{n} \rho_i(l_1, \ldots, \hat{l}_i, \ldots, l_n) \cdot \rho_k(l_i, l_{n+1}, \ldots, l_{2n-2}),\ 2 \leq k \leq n,$$

$$[\rho_1(l_n, \ldots, l_{2n-2}), \rho_k(l_1, \ldots, l_{n-1})]$$
$$= \sum_{i=1}^{n-1} \rho_k(l_1, \ldots, l_{i-1}, [l_i, l_n, \ldots, l_{2n-2}], l_{i+1}, \ldots, l_{n-1}),\ 1 \leq k \leq n.$$

A particular example of representation is the adjoint representation when $V = L$ and all ρ_i are given by

$$\mathrm{ad}_i(l_1, \ldots, l_{n-1})(l) = [l_1, \ldots, l_{i-1}, l, l_i, \ldots, l_{n-1}].$$

In order to construct the universal associative envelope for a Leibniz n-algebra we introduce the following notation. Consider n isomorphic copies of K: one *left* copy, $(n-2)$ *middle* copies and one *right* copy denoted by K_l, K_{m_i}, $1 \leq i \leq n-2$, K_r, respectively. We next consider the tensor algebra of the direct product of all isomorphic copies of K, that is,

$$T(K_l \oplus K_{m_1} \oplus \ldots \oplus K_{m_{n-2}} \oplus K_r).$$

Let $x_1 \otimes \cdots \otimes x_{n-1}$ be an element from K. Then we denote by $l_1 \otimes \cdots \otimes l_{n-1}$, $m_1^i \otimes \cdots \otimes m_{n-1}^i$, $r_1 \otimes \cdots \otimes r_{n-1}$ the corresponding elements from K_l, K_{m_i}, K_r. Setting

$$\rho_1(x_1, \ldots, x_{n-1}) = l_1 \otimes \cdots \otimes l_{n-1},$$

$$\rho_i(x_1,\dots,x_{n-1}) = m_1^{i-1}\otimes\cdots\otimes m_{n-1}^{i-1},$$
$$\rho_n(x_1,\dots,x_{n-1}) = r_1\otimes\cdots\otimes r_{n-1},$$

where $x_1,\dots,x_{n-1}\in L$ we can rewrite the defining relations (see Definition 19) as follows

(1) If $2\le k\le n-1$,

$$\begin{aligned}&[m_1^{k-1},\dots,m_n^{k-1}]\otimes m_{n+1}^{k-1}\otimes\dots\otimes m_{2n-2}^{k-1}\\ &= (l_2\otimes\dots\otimes l_n)\cdot(m_1^{k-1}\otimes m_{n+1}^{k-1}\otimes\dots\otimes m_{2n-2}^{k-1})\\ &+\sum_{i=2}^{n-1}(m_1^{i-1}\otimes\dots\otimes\widehat{m}_i^{i-1}\otimes\dots\otimes m_n^{i-1})\cdot(m_i^{k-1}\otimes m_{n+1}^{k-1}\otimes\dots\otimes m_{2n-2}^{k-1})\\ &+(r_1\otimes\dots\otimes r_{n-1})\cdot(m_n^{k-1}\otimes m_{n+1}^{k-1}\otimes\dots\otimes m_{2n-2}^{k-1}).\end{aligned}$$

(2) If $k=n$,

$$\begin{aligned}&[r_1,\dots,r_n]\otimes r_{n+1}\otimes\dots\otimes r_{2n-2} = (l_2\otimes\dots\otimes l_n)\cdot(r_1\otimes r_{n+1}\otimes\dots\otimes r_{2n-2})\\ &+\sum_{i=2}^{n-1}(m_1^{i-1}\otimes\dots\otimes\widehat{m}_i^{i-1}\otimes\dots m_n^{i-1})\cdot(r_i\otimes r_{n+1}\otimes\dots\otimes r_{2n-2})\\ &+(r_1\otimes\dots\otimes r_{n-1})\cdot(r_n\otimes r_{n+1}\otimes\dots\otimes r_{2n-2}).\end{aligned}$$

(3) If $k=1$,

$$\begin{aligned}&(l_n\otimes\dots\otimes l_{2n-2})\cdot(l_1\otimes\dots\otimes l_{n-1})-(l_1\otimes\dots\otimes l_{n-1})\cdot(l_n\otimes\dots\otimes l_{2n-2})\\ &=[l_1\otimes\dots\otimes l_{n-1}, l_n\otimes\dots\otimes l_{2n-2}].\end{aligned}$$

(4) If $2\le k\le n-1$,

$$\begin{aligned}&(l_n\otimes\dots\otimes l_{2n-2})\cdot(m_1^{k-1}\otimes\dots\otimes m_{n-1}^{k-1})-(m_1^{k-1}\otimes\dots\otimes m_{n-1}^{k-1})\cdot(l_n\otimes\dots\otimes l_{2n-2})\\ &=[m_1^{k-1}\otimes\dots\otimes m_{n-1}^{k-1}, m_n^{k-1}\otimes\dots\otimes m_{2n-2}^{k-1}].\end{aligned}$$

(5) If $k=n$,

$$\begin{aligned}&(l_n\otimes\dots\otimes l_{2n-2})\cdot(r_1\otimes\dots\otimes r_{n-1})-(r_1\otimes\dots\otimes r_{n-1})\cdot(l_n\otimes\dots\otimes l_{2n-2})\\ &=[r_1\otimes\dots\otimes r_{n-1}, r_n\otimes\dots\otimes r_{2n-2}]\end{aligned}$$

Definition 20. *The universal enveloping algebra of the Leibniz n-algebra L is the unitary associative algebra*

$$U_n(L) = T(K_l \oplus K_{m_1} \oplus \ldots \oplus K_{m_{n-2}} \oplus K_r)/I,$$

where I is the n-sided ideal corresponding to the relations (1)–(5).

The following result was obtained in [9]:

Theorem 10. *The category of representations of the Leibniz n-algebra L is equivalent to the category of right modules over $U_n(L)$.*

Our next goal is to construct an explicit basis of PBW-type for L. Let L be a finite-dimensional Leibniz n-algebra with a basis $\{e_1, \ldots, e_d\}$, and let

$$F[x_{s_1,\ldots,s_{n-1}}, y^k_{s_1,\ldots,s_{n-1}}, z_{s_1,\ldots,s_{n-1}}]$$

be a free associative algebra on the generators $x_{s_1,\ldots,s_{n-1}}, y^k_{s_1,\ldots,s_{n-1}}, z_{s_1,\ldots,s_{n-1}}$ where $s_1, \ldots, s_{n-1} \in \{1, \ldots, d\}$, $1 \le k \le n-2$. Let $e_{s_1} \otimes \cdots \otimes e_{s_{n-1}}$ be a basis element from K. Then we denote by $l_{s_1} \otimes \cdots \otimes l_{s_{n-1}}$, $m^i_{s_1} \otimes \cdots \otimes m^i_{s_{n-1}}$, $r_{s_1} \otimes \cdots \otimes r_{s_{n-1}}$ the corresponding elements from K_l, K_{m_i}, K_r.

Then we can define the following morphism

$$\varepsilon : T(K_l \oplus K_{m_1} \oplus \cdots \oplus K_{m_{n-2}} \oplus K_r) \to F[x_{s_1,\ldots,s_{n-1}}, y^k_{s_1,\ldots,s_{n-1}}, z_{s_1,\ldots,s_{n-1}}]$$

by

$$\varepsilon(l_{s_1} \otimes \cdots \otimes l_{s_{n-1}}) = x_{s_1,\ldots,s_{n-1}},$$
$$\varepsilon(m^i_{s_1} \otimes \cdots \otimes m^i_{s_{n-1}}) = y^i_{s_1,\ldots,s_{n-1}},$$
$$\varepsilon(r_1 \otimes \cdots \otimes r_{s_{n-1}}) = z_{s_1,\ldots,s_{n-1}}.$$

The relations (1)–(5) are translated into

(R1)

$$\begin{aligned}
&\varepsilon([m^{k-1}_{s_1}, \ldots, m^{k-1}_{s_n}] \otimes m^{k-1}_{s_{n+1}} \otimes \ldots \otimes m^{k-1}_{s_{2n-2}}) \\
&= x_{s_2,\ldots,s_n} \cdot y^{k-1}_{s_1,s_{n+1},\ldots,s_{2n-2}} + \sum_{i=2}^{n-1} y^{i-1}_{s_1,\ldots,\widehat{s}_i,\ldots,s_n} \cdot y^{k-1}_{s_i,s_{n+1},\ldots,s_{2n-2}} \\
&+ z_{s_1,\ldots,s_{n-1}} \cdot y^{k-1}_{s_n,s_{n+1},\ldots,s_{2n-2}},
\end{aligned}$$

(R2)

$$\varepsilon([r_{s_1}, \ldots, r_{s_n}] \otimes r_{s_{n+1}} \otimes \cdots \otimes r_{s_{2n-2}})$$

$$= x_{s_2,\dots,s_n} \cdot z_{s_1,s_{n+1},\dots,s_{2n-2}} + \sum_{i=2}^{n-1} y^{i-1}_{s_1,\dots,\widehat{s}_i,\dots,s_n} \cdot z_{s_i,s_{n+1},\dots,s_{2n-2}}$$
$$+ z_{s_1,\dots,s_{n-1}} \cdot z_{s_n,s_{n+1},\dots,s_{2n-2}},$$

(R3)

$$\varepsilon([l_{s_1} \otimes \cdots \otimes l_{s_{n-1}}, l_{s_n} \otimes \cdots \otimes l_{s_{2n-2}}])$$
$$= x_{s_n,s_{n+1},\dots,s_{2n-2}} \cdot x_{s_1,\dots,s_{n-1}} - x_{s_1,\dots,s_{n-1}} \cdot x_{s_n,s_{n+1},\dots,s_{2n-2}},$$

(R4)

$$\varepsilon([m^{k-1}_{s_1} \otimes \cdots \otimes m^{k-1}_{s_{n-1}}, m^{k-1}_{s_n} \otimes \cdots \otimes m^{k-1}_{s_{2n-2}}])$$
$$= x_{s_n,s_{n+1},\dots,s_{2n-2}} \cdot y^{k-1}_{s_1,\dots,s_{n-1}} - y^{k-1}_{s_1,\dots,s_{n-1}} \cdot x_{s_n,s_{n+1},\dots,s_{2n-2}},$$

(R5)

$$\varepsilon([r_{s_1} \otimes \cdots \otimes r_{s_{n-1}}, r_{s_n} \otimes \cdots \otimes r_{s_{2n-2}}])$$
$$= x_{s_n,s_{n+1},\dots,s_{2n-2}} \cdot z_{s_1,\dots,s_{n-1}} - z_{s_1,\dots,s_{n-1}} \cdot x_{s_n,s_{n+1},\dots,s_{2n-2}}.$$

Calculating a Gröbner-Shirshov basis for the ideal

$$\varepsilon(I) \subset F[x_{s_1,\dots,s_{n-1}}, y^k_{s_1,\dots,s_{n-1}}, z_{s_1,\dots,s_{n-1}}]$$

generated by (R1)-(R5), Casas, Insua, and Ladra proved a PBW theorem for Leibniz n-algebras. Namely, the following holds true.

Theorem 11. *Let L be a Leibniz n-algebra of dimension d. Then a linear basis of the universal enveloping algebra $U_n(L)$ is formed by the monomials of the type*

$$x^{a_{11\dots1}}_{11\dots1} \dots \widehat{x}^{a_{\alpha_1\dots\alpha_{n-1}}}_{\alpha_1\dots\alpha_{n-1}} h(y^1_{11\dots1}, \dots, y^{n-2}_{dd\dots d}) \cdot z^e_{s_1,\dots,s_{n-1}},$$

where

$$h(y^1_{11\dots1}, \dots, y^{n-2}_{dd\dots d})$$

is a monic monomial and $e = 0, 1$.

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