Serdica

Mathematical Journal

Сердика

Математическо списание

Serdica Math. J. **38** (2012), 463–472

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

GROWTH FUNCTIONS OF F_r-SETS

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Communicated by V. Drensky

Abstract. In this paper we consider an open problem from [1], regarding the description of the growth functions of the free group acts. Using the language of graphs, we solve this problem by providing the necessary and sufficient conditions for a function to be a growth function for a free group act.

1. Introduction.

1.1. Definitions. Let G be a monoid.

Definition 1. *A G-act is a set S equipped with an operation $\circ : S \times G \to S$, satisfying the axioms of identity and associativity:*

(1) *For all $s \in S, s \circ 1 = s$, where 1 is the identity element in G.*

(2) *For all $g_1, g_2 \in G, s \in S, (s \circ g_1) \circ g_2 = s \circ (g_1 g_2)$.*

2010 *Mathematics Subject Classification:* 05C30, 20E08, 20F65.
Key words: Group action, growth function.

If G is a group, S is commonly called a G-set.

Definition 2. *A (directed, labelled) graph is a collection* $\Gamma = \{V, E, Y, \iota, \tau, \mathrm{Lab}\}$ *consisting of three sets and three functions* $\iota : E \to V$, $\tau : E \to V$ *and* $\mathrm{Lab} : E \to Y$. *The elements of* $V = V(\Gamma)$ *are called the vertices of* Γ, *the elements of* $E = E(\Gamma)$ *are called the edges of* Γ, $\iota(e)$ *is the initial vertex of the edge* e, $\tau(e)$ *is the terminal point of* e, $\mathrm{Lab}(e)$ *is the label of* e. *A graph* $\Gamma' = \{V', E', Y, \iota', \tau', \mathrm{Lab}'\}$ *is a subgraph of* Γ *if* $V' \subset V$, $E' \subset E$ *and the restrictions of* $\iota, \tau, \mathrm{Lab}$ *to* E' *are equal to* $\iota', \tau', \mathrm{Lab}'$.

A path in Γ is a sequence of edges, written as $p = e_1 e_2 \dots e_n$, such that $\iota(e_{i+1}) = \tau(e_i)$, for every $i = 1, 2, \dots, n-1$. We write $\iota(p) = \iota(e_1)$, $\tau(p) = \tau(e_n)$, and call these the initial and the terminal points of p, respectively. If $Y \subset G$, where G is a monoid, we define the label of p as $\mathrm{Lab}(p) = \mathrm{Lab}(e_1)\mathrm{Lab}(e_2) \cdots \mathrm{Lab}(e_n)$. If e is an edge with label y, and y has an inverse $y^{-1} \in Y$, we can define an inverse edge e^{-1}, as an edge such that

(1) $\iota(e^{-1}) = \tau(e)$,

(2) $\tau(e^{-1}) = \iota(e)$,

(3) $\mathrm{Lab}(e^{-1}) = \mathrm{Lab}(e)^{-1}$.

We similarly define an inverse of a path.

Now let $X = \{x_1, \dots, x_r\}$ be an alphabet on r letters, F_r be the free group of rank r with free generating set X and $Y = X \cup X^{-1}$. Suppose that Γ is a graph whose edges are assigned labels in Y, where every edge has a (unique) inverse, and where every vertex is the initial point of exactly one edge with every label in Y. Then we can define the action of F_r on $S = V(\Gamma)$ such that, given $y \in Y$ and $v_1, v_2 \in V$, $v_1 \circ y = v_2$ if and only if there exists an edge e with label y, $\iota(e) = v_1, \tau(e) = v_2$. Conversely, given any F_r-set S, we can construct a graph such that $V(\Gamma) = S$, and where for any $v_1, v_2 \in V$, $y \in Y$, there exists an edge e, $\iota(e) = v_1$, $\tau(e) = v_2$, $\mathrm{Lab}(e) = y$ if and only if $v_1 \circ y = v_2$. Using this idea, the study of F_r-sets becomes equivalent to the study of certain kinds of graphs.

Definition 3. *Let* S *be an* F_r*-set. We say that a directed, labelled graph* Γ *is the graph of* S *if* $V(\Gamma) = S$, *and for every* $v_1, v_2 \in V(\Gamma)$ *there exists an edge* e *such that* $\iota(e) = v_1, \tau(e) = v_2$ *if and only if* $v_1 \circ \mathrm{Lab}(e) = v_2$.

We will often call Γ the graph of the action of F_r on S, and refer to it as an F_r-graph.

1.2. Outline. This paper is concerned with a question posed in [1]. Given a set of elements A in a G-set S, and a filtration $G_0 \subset G_1 \subset \cdots$ in G, we can define the growth function of S with respect to A as $g(A, n) = |A \circ G_n|$. In particular, we may consider actions of F_r, the free group of r generators, with the filtration defined by the reduced length of words in F_r with respect to the alphabet Y. This immediately motivates the question:

Question. *What are the necessary and sufficient conditions for a function g to be the growth function of an F_r-set?*

The answer to this question will be of a constructive nature. We will first prove the necessity of certain conditions, and then show how, given a growth function meeting these conditions, an F_r-set can be constructed having this growth function.

2. Growth functions of F_r-sets.

2.1. Definitions. Let F_r be the free group generated by $X = \{x_1, \ldots, x_r\}$. The set $Y = X \cup X^{-1}$ is called the symmetric basis of F_r. Any element $g \in F_r$ can be written as a product of the elements in Y. The product of the shortest length equal to g is unique and called the reduced form of g. Defining $F_r(0) = \{1\}$ and $F_r(n)$ as the set of elements of F_r which can be written as the product of at most n elements of Y, we obtain a filtration of F_r, called standard. Now let Γ be an F_r-graph and $A \subset V(\Gamma)$. We set $B(A, 0) = A$, $B(A, n) = A \circ F_r(n)$. We call $B(A, n)$ the "ball of radius n around A". Clearly, each such ball is exactly the set of vertices which are connected to A by a path of length less than or equal to n. In a similar way, we can define the "sphere" of radius n around A, as $S(A, n) = B(A, n) \setminus B(A, n-1)$, and this becomes the set of vertices which are connected to A by a path of length n, but not by any shorter path. We say that A generates Γ if $V(\Gamma) = \bigcup_{i=0}^{\infty} B(A, n)$. It is obvious that if Γ is connected, any non-empty set of vertices is a generating set for the whole graph.

Definition 4. *The growth function of an F_r-graph Γ with respect to a set of vertices A is the function $g : \mathbb{N} \cup \{0\} \to \mathbb{N}$ defined by $g(A, n) = |B(A, n)|$.*

It will also be important for us occasionally to consider the "difference function" of a growth function g: $d_g(A, n) = |S(A, n)| = g(A, n) - g(A, n-1)$, where we set $g(A, -1) = 0$ for convenience. Now recall the following:

Definition 5. *For any graph Γ, and for any $v \in V(\Gamma)$,* $\mathrm{star}(v)$ *is the set of all edges $e \in E(\Gamma)$ such that $\iota(e) = v$.*

If the edges of Γ are given labels from Y, the symmetric basis of F_r, we say that $\mathrm{star}(v)$ is *standard* if it contains exactly one edge with every label in Y. Similarly, we say that $\mathrm{star}(v)$ is *regular* if it contains at most one edge with every possible label (note that $\mathrm{star}(v)$ is standard if and only if $\mathrm{star}(v)$ is regular and has $2r$ elements). The condition that the star of any vertex v is standard is equivalent to the condition that the action of the basis of F_r on v is well defined, and hence Γ is an F_r-graph if and only if the star of every vertex is standard and every edge has an inverse. So it will be quite useful for us to define the *deficit* of a vertex v, which is the number of edges that are "missing" from $\mathrm{star}(v)$. Recall one more definition from [1].

Definition 6. *Let Γ be an F_r-graph. For every vertex $v \in V(\Gamma)$, the deficit of v is denoted* $\mathrm{def}(v)$, *and* $\mathrm{def}(v) = 2r - |\mathrm{star}(v)|$.

In the following sections, we will often consider vertices as sitting inside multiple graphs at the same time. It will help to introduce some notation to avoid any confusion arising from this situation. Here, $\mathrm{star}_\Omega(v)$ will denote the star of v as a vertex of the graph Ω. If Γ is an F_r-graph, and $\Omega \subset \Gamma$, then $\mathrm{star}_\Omega(v) = \mathrm{star}_\Gamma(v) \cap E(\Omega)$. Similarly, we define $\mathrm{def}_\Omega(v) = 2r - |\mathrm{star}_\Gamma(v) \cap E(\Omega)|$.

2.2. Properties of subgraphs of F_r-sets. At the end of this section, we will use graphs to provide an answer to the question posed in [1]:

Question. *What are the necessary and sufficient conditions for a function g to be the growth function of an F_r-set?*

This will be much easier if we first prove some basic facts about F_r-graphs. We call a directed graph labelled by the symmetric basis $Y = X \cup X^{-1}$, of the free group F_r freely generated by $X = \{x_1, \ldots, x_r\}$, *regular* if the star of every vertex in Ω is regular and every edge has a (unique) inverse.

Proposition 1. *Any regular graph Ω can be embedded in an F_r-graph.*

Proof. We must find a graph labelled by Y, which contains Ω, where every edge has an inverse, and where the star of every vertex is standard. We will do this by attaching infinite trees to the vertices of Ω. For any letter $y \in Y$, let T_y be the unique connected tree labelled by Y, with a distinguished vertex $o(T_y)$ such that every edge has an inverse, the stars of all vertices $v \in V(T_y)$, $v \neq o(T_y)$ are standard, while the star of $o(T_y)$ consists of a single edge with label y.

Then for every vertex $v \in V(\Omega)$ such that $\mathrm{star}(v)$ is not standard, for every $y \in Y$ which is not represented in $\mathrm{star}(v)$, we attach T_y to our graph Ω

by letting $o(T_y) = v$. In this way we construct a new graph $\overline{\Omega}$, which contains Ω as a subgraph, and where every star is standard. So Ω is embedded into an F_r-graph. $\square$

In actuality we have done more than prove the fact; we have shown how to embed any graph Ω into an F_r-graph in a canonical way. Given Ω, we can now unambiguously refer to an F_r graph $\overline{\Omega}$ containing Ω. We call this embedding *canonical.*

Lemma 1. *Let Ω be a finite regular graph, $\Omega \subset \overline{\Omega}$ the canonical embedding and z a reduced word in F_r. Then there are as many paths with label z "leaving" Ω (initial point in Ω and terminal point outside Ω), as there are paths with label z "entering" Ω (initial point outside Ω and terminal point inside Ω).*

Proof. The action of $\langle z \rangle$, the cyclic group generated by z, on the set of vertices in Ω splits into finitely many orbits, and so it suffices to show that the number of paths with label z associated with a given orbit entering Ω is equal to the number of such paths which leave Ω. Consider any orbit. Suppose the orbit is finite, with n vertices, and choose some $v \in \Omega$ which is in this orbit. Then, consider the set

$$K = \{k \in \mathbb{Z}, 0 < k < n \ \text{ such that } \ v \circ z^{k-1} \in \Omega, \ \text{ but } \ v \circ z^k \notin \Omega\}.$$

Label the numbers in this set $k_1 < k_2 < \cdots < k_t$. Let

$$M = \{m \in \mathbb{Z}, 0 < m < n \ \text{ such that } \ v \circ z^{m-1} \in \Omega, \ \text{ but } \ v \circ z^m \notin \Omega\}.$$

Note that it is possible for the orbit to "leave" and "enter" Ω in this way several times, as the action of z on $v \circ z^k$ may travel down the tree on which $v \circ z^k$ sits, and back into Ω. It is obvious that between k_i and k_{i+1} there is exactly one integer m_i such that $v \circ z^{m_i}$ "enters" Ω, for the orbit cannot leave Ω and then leave again without first entering. Similarly there is exactly one such number $m_t, k_t < m_t \leq n$, and no such number $0 < m_0 < k_1$. Since there are n elements of the orbit, all such numbers correspond to different paths, and so there are t paths with label z entering Ω, and t paths leaving.

Similarly, suppose the orbit is infinite, and choose some $v \in \Omega$ which is in this orbit. Construct the set

$$K_+ = \{k > 0 \ \text{ such that } \ v \circ z^k \ \text{ "leaves" } \ \Omega\}.$$

That is, $v \circ z^k$ is not an element of Ω, but $v \circ z^{k-1}$ is in Ω. Then construct the set

$$M_+ = \{m > 0 \text{ such that } v \circ z^m \text{ “enters” } \Omega\}.$$

These sets are finite because Ω is finite, and so we can number these elements $m_1 < m_2 < \cdots < m_t$, $k_1 < k_2 < \cdots < k_s$. As in the finite case, between any two k_i, k_{i+1} we have exactly one m_i. And as Ω is finite, the orbit must eventually leave the graph forever, and so we can write $k_1 < m_1 < \cdots < k_t < m_t < k_{t+1}$. So, we count one more path with label z leaving Ω as we do paths with label z entering Ω. But we can construct analogous sets K_-, M_-, using positive powers of z^{-1} instead of positive powers of z. In the same way, we will count one more path with label z^{-1} leaving Ω as we do paths with label z^{-1} entering Ω. But paths with label z^{-1} leaving Ω can be considered paths with label z entering Ω, and vice versa. Hence, when we consider all powers of z, positive and negative, we count the same number of paths with label z entering Ω as leaving Ω. $\square$

It follows that there is some bijection f_z from the set of paths with label z leaving Ω to the set of paths with label z^{-1} leaving Ω. We will often make use of the fact that such a bijection exists without reference to what it actually is, and so will denote $f_z(p) = p'$, with the understanding that $p'' = p$. The key point being that we can split paths leaving Ω into pairs (p, p'), where $\mathrm{Lab}(p')$ is the inverse of $\mathrm{Lab}(p)$. For now, it is important only to consider an obvious corollary.

Corollary 1. *Let Ω be a finite regular graph, $\Omega \subset \overline{\Omega}$ the canonical embedding and $y \in Y$. Then the number of edges with label y “leaving” Ω is equal to the number of edges with label y “entering”.*

Whence it follows immediately that for any finite regular graph Ω, the number of edges “leaving” Ω in the canonical embedding is even. Equivalently, $\sum\limits_{v \in V(\Omega)} \mathrm{def}_\Omega(v) = 2C$ for some $C \geq 0$.

Lemma 2. *Let Ω be a finite regular graph and k a natural number satisfying $2k \leq \sum\limits_{v \in V(\Omega)} \mathrm{def}_\Omega(v)$. Then Ω can be embedded as a subgraph of a finite regular graph Ω' such that $V(\Omega') = V(\Omega)$ and*

$$\sum_{v \in V(\Omega')} \mathrm{def}_{\Omega'}(v) = \sum_{v \in V(\Omega)} \mathrm{def}_\Omega(v) - 2k.$$

P r o o f. Let v be any vertex of $\Omega \subset \overline{\Omega}$ such that $\mathrm{star}_\Omega(v)$ is not standard. We know that $\mathrm{star}_{\overline{\Omega}}(v)$ is standard, and so there must be at least one edge which is in $\mathrm{star}_{\overline{\Omega}}(v) \setminus \mathrm{star}_\Omega(v)$. Let e be any such edge. Then there exists e', $\mathrm{Lab}(e') = \mathrm{Lab}(e)^{-1}$ such that for some $v' \in V(\Omega)$, $e' \in \mathrm{star}_{\overline{\Omega}}(v') \setminus \mathrm{star}_\Omega(v')$. So, for every $y \in Y$ and $v \in V(\Omega)$ such that $\mathrm{star}_\Omega(v)$ contains no edge of label y, there corresponds a vertex v' such that $\mathrm{star}_\Omega(v')$ contains no edge of label y^{-1}. It follows that we can add a pair of edges e_1, e_1^{-1} to Ω, $\mathrm{Lab}(e_1) = y, \iota(e_1) = v, \tau(e_1) = v'$, while leaving the stars of all vertices in Ω regular, So Ω' is the graph constructed by adding k pairs of edges in this way. Since Ω' has $2k$ more edges than Ω, $\sum\limits_{v \in V(\Omega')} \mathrm{def}_{\Omega'}(v) = \sum\limits_{v \in V(\Omega)} \mathrm{def}_\Omega(v) - 2k$. $\square$

2.3. Constructing F_r-sets with particular growth functions. In [1], it was mentioned that any growth function g of a F_r-set satisfies certain necessary conditions (expressed in terms of the difference function of g):

1. $d(1) \leq 2rd(0)$,
2. $d(n) \leq (2r-1)d(n-1)$ for all $n > 1$.

It was not known whether these conditions were sufficient. Using the results of the previous section, we can solve this problem by proving the following theorem:

Theorem 1. *Let g be a function from $\mathbb{N}$ to $\mathbb{N} \cup \{0\}$. Define $d(0) = g(0)$, and $d(i) = g(i) - g(i-1)$ for $i > 0$. Then g is the growth function of some F_r act if and only if*

(i) $d(1) \leq 2rd(0)$,

(ii) $d(n) \leq (2r-1)d(n-1)$ *for all* $n > 1$,

(iii) $d(n) \leq (2r-1)d(n-1) - 1$ *if* $d(n-1)$ *is odd, for all* $n > 1$.

Before we begin the main proof, we note that the necessity of these conditions is very easy to prove. The first two conditions are already known to be necessary, as mentioned above. The third condition is necessary by Corollary 1. For suppose there is an F_r-graph Γ such that for some set of vertices A, $|S(A,n)| = 2k+1$. The vertices of $S(A,n)$ are exactly those vertices which are the terminal points of edges leaving $B(A,n-1)$. By Corollary 1, there is an even number of edges leaving $B(A,n-1)$. It follows that at least one vertex in $S(A,n)$ is connected to $B(A,n-1)$ by at least two edges. Now consider $S(A,n+1)$, which is the set of those vertices which are terminal points of edges leaving $B(A,n)$, or equivalently those vertices which are the terminal points of edges leaving $S(A,n)$ and which are not in $B(A,n-1)$. Suppose $|S(A,n+1)| = (2r-1)|S(A,n)|$. Then there are at least $(2r-1)|S(A,n)|$ edges leaving $S(A,n)$ and terminating outside of $B(A,n)$. But we have also said that there are at least $|S(A,n)|+1$ edges leaving $S(A,n)$ and terminating inside $B(A,n-1)$. Hence, there are at least $2r|S(A,n)|+1$ edges with initial point in $S(A,n)$. By the pigeonhole principle, some vertex in $S(A,n)$ has more than $2r$ edges in its star, and hence Γ is not an F_r-graph. Therefore, the third condition is necessary. We will now show that they are sufficient.

P r o o f. Given a function $g(n)$ meeting the conditions of Theorem 1, we will construct a graph Γ so that the star of every vertex in $V(\Gamma)$ is standard, every edge in $E(\Gamma)$ has an inverse, and for a set of vertices A, $|B(A,n)| = g(n)$ for all $n \geq 0$. To do this, we will construct an ascending chain $\Gamma_0 \subset \Gamma_1 \subset \ldots$ so that:

(1) for all $i \geq 0$, $V(\Gamma_i) = B(V(\Gamma_0), i)$,

(2) for all $i \geq 0$, the only vertices in Γ_i with non-standard stars are those in $S(V(\Gamma_0), i)$,

(3) for all $i \geq 0$, every edge in Γ_i has an inverse, and

(4) for all $i \geq 0$, $\sum_{v\in\Gamma_i} \mathrm{def}_{\Gamma_i}(v) \geq d(i+1)$.

Then, $\bigcup_{i=0}^{\infty} \Gamma_i = \Gamma$ will be an F_r-graph with growth function $g(n)$.

We create Γ_0 by letting $V(\Gamma_0)$ be a set containing $d(0)$ elements, and $E(\Gamma_0) = \varnothing$. If, for convenience, we define Γ_{-1} to be the empty graph, then conditions 1 through 3 are trivially satisfied. Since there are no edges in Γ_0, $\sum_{v\in\Gamma_0} \mathrm{def}_{\Gamma_0}(v) = 2rd(0) \geq d(1)$, and Γ_0 satisfies all conditions listed above.

Given Γ_i, we must show how to create Γ_{i+1}. By construction,

$$\sum_{v\in\Gamma_i} \mathrm{def}_{\Gamma_i}(v) \geq d(i+1).$$

We first apply Lemma 2 to Γ_i, adding pairs of mutually inverse edges to Γ_i, so that the deficit of the resulting graph (call it Ω_1) will satisfy $\sum_{v\in\Omega_1} \mathrm{def}_{\Omega_1}(v) = d(i+1)$ or $\sum_{v\in\Omega_1} \mathrm{def}_{\Omega_1}(v) = d(i+1)+1$, depending on whether $d(i+1)$ is even or odd, respectively. Note that by induction the only vertices which have non-standard stars in Γ_i are those in $S(V(\Gamma_0), i)$, and so Lemma 2 necessarily creates edges only between vertices in $S(V(\Gamma_0), i)$. Vertices in Γ_i with standard stars therefore remain untouched, and so Ω_1 continues to satisfy properties (1)–(4).

We then add $d(i+1)$ new vertices to Ω_1. We can create a set of ordered pairs $L = \{(v, y)\}$, where $(v, y) \in L$ if $v \in \Gamma_i$ and there is no edge with label y in $\text{star}_{\Omega_1}(v)$. $|L| = \sum\limits_{v \in \Gamma_i} \text{def}_{\Omega_1}(v)$, and so if $d(i+1)$ is even, then $d(i+1) = |L|$. In this case, $|L|$ is exactly equal to the number of vertices in Ω_1 which have empty stars. So, for every pair $(v, y) \in L$, we can create an edge (and its inverse) with label y from v to a vertex with an otherwise empty star. The resulting graph is Γ_{i+1}. Indeed, it has $g(i)+d(i+1) = g(i+1)$ vertices which form $B(V(\Gamma_0), i+1)$, and the star of every vertex which is also in Γ_i is standard. Also, there are $d(i+1)$ vertices which have only one attached edge, so that $\sum\limits_{v \in \Gamma_{i+1}} \text{def}_{\Gamma_{i+1}}(v)) = (2r-1)d(i+1)$, which is the upper bound for $d(i+2)$ (and therefore Γ_{i+1} satisfies properties (1)–(4)).

If $d(i+1)$ is odd, then $|L| = d(i+1) + 1$. We define $L' = L \setminus \{(u, x)\}$ for some $(u, x) \in L$. Then, for every $(v, y) \in L'$, we attach an edge with label y beginning at v and terminating at one of the $d(i+1)$ new vertices which do not yet have an attached edge, as we did in the even case. We must then add an edge whose initial point is u with label x. This attachment is always possible, by the existence of the bijection shown in Lemma 2. Note that u is the only vertex of Γ_i whose star has not yet been made standard, and x is the only edge which is missing from this star. So by necessity, the edge will terminate in one of the $d(i+1)$ vertices of $\Omega_1 \setminus \Gamma_i$. The resulting graph will be Γ_{i+1}, as it has $g(i) + d(i+1) = g(i+1)$ vertices which form $B(V(\Gamma_0), i+1)$, the star of every vertex which was in Γ_i is standard, and $\sum\limits_{v \in \Gamma_{i+1}} \text{def}_{\Gamma_{i+1}}(v) = (2r-1)d(i+1) - 1$, which is the upper bound for $d(i+2)$ (and therefore Γ_{i+1} satisfies properties (1)–(4). Additionally, because on each step we add a finite number of vertices and edges, the graphs in our ascending chain will all be finite. This allows us to apply Lemma 2 throughout the induction.

And so we are done, $\Gamma = \bigcup\limits_{i=0}^{\infty} \Gamma_i$ is an F_r-graph with the desired growth function. $\square$

Note that the proof can be modified (and is much simpler) in the case of the free monoid W_r, to show a known fact (see [1]) that a function f is the growth function of a W_r action if and only if $d_f(n) \leq r d_f(n-1)$ for all n.

Acknowledgements. I would like to thank Professor Yuri Bahturin for his guidance and supervision during the research which lead to this paper, as well as his invaluable help in writing and editing the paper itself. The research leading to this paper was funded by an NSERC USRA award.

REFERENCES

[1] Yu. Bahturin, A. Olshanskii. Actions of maximal growth. *Proc. London Math. Soc.* (*3*) **101**, *1* (2010), 27–72.

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Received March 20, 2012