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GENERALIZED HOMOGENEOUS BESOV SPACES AND THEIR APPLICATIONS

Hatem Mejjaoli

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Abstract. In this paper we define the homogeneous Besov spaces associated with the Dunkl operators on $\mathbb{R}^d$, and we give a complete analysis on these spaces and same applications.

1. Introduction.

Dunkl operators T_j $(j = 1, \dots, d)$ introduced by Dunkl in [13] are parameterized differential-difference operators on $\mathbb{R}^d$ that are related to finite reflection groups. Over the last years, much attention has been paid to these operators in various mathematical (and even physical) directions. In this prospect, Dunkl operators are naturally connected with certain Schrödinger operators for Calogero-Sutherland-type quantum many-body systems [3, 12, 17]. Moreover, Dunkl operators allow generalizations of several analytic structures, such as Laplace operator, Fourier transform, heat semigroup, wave equations, and Schrödinger equations ([11, 15, 21, 22, 23, 24]).

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In the present paper, we intend to continue our study of generalized spaces of type Sobolev associated with Dunkl operators started in [20]. Indeed, in [20] we provided a general theory for the Littlewood-Paley associated with the Dunkl operators. Furthermore, we study some functions spaces associated with Dunkl operators: generalized Sobolev spaces, generalized Hölder spaces and BMO associated with the Dunkl operators.

Next, in second paper [18] we have continue our investigation of function spaces; generalized Bessel potential spaces, nonhomogeneous Besov spaces and Triebel-Lizorkin spaces associated with Dunkl operators. We obtain their basic properties and apply them to estimate the solutions of the Dunkl-Schrödinger and the Dunkl heat equations.

The main subject of this paper is the study of the homogeneous Dunkl-Besov spaces, establish refined Sobolev inequalities between the homogeneous Dunkl-Besov spaces and many spaces as the homogeneous Dunkl-Riesz spaces and the generalized Lorentz spaces. Generalize the Gagliardo-Nirenberg inequality in the context of Dunkl theory. We shall also consider a few applications of these results to the generalized heat equations and generalized Schrödinger equations.

The contents of the paper is as follows. In §2 we recall some basic results about the harmonic analysis associated with the Dunkl operators. In §3 we introduce the homogeneous Littlewood-Paley decomposition associated with the Dunkl operators. We shall obtain Bernstein's inequalities. §4 is devoted to study the Dunkl-Riesz potential spaces, the homogeneous Dunkl-Besov spaces. According to a standard process in the Euclidean case (cf. [28]), we shall consider equivalent norms, lifting properties, interpolations and dualities of these spaces. In §5 we summarize some results on embeddings and paraproduct operators, which depend on the index γ associated to the multiplicity function of the root system. We consider also some applications of the homogeneous Dunkl-Besov spaces to differential-difference equations. We shall obtain Strichartz type estimates of the solutions of the Dunkl-Schrödinger equation, a space-time estimate of the solutions of the Dunkl heat equation. We give also as applications a Sobolev inequalities in generalized Lorentz spaces.

2. Preliminaries. In order to confirm the basic and standard notations we briefly overview the theory of Dunkl operators and related harmonic analysis. Main references are [11, 13, 14, 15, 24, 25, 26, 29].

2.1. Root system, reflection group and multiplicity function. Let $\mathbb{R}^d$ be the Euclidean space equipped with a scalar product $\langle , \rangle$ and let $\|x\| = \sqrt{\langle x, x\rangle}$. For α in $\mathbb{R}^d\backslash\{0\}$, σ_α denotes the reflection in the hyperplane $H_\alpha \subset \mathbb{R}^d$

perpendicular to α, i.e., for $x \in \mathbb{R}^d$, $\sigma_\alpha(x) = x - 2\|\alpha\|^{-2}\langle \alpha, x\rangle \alpha$. A finite set $R \subset \mathbb{R}^d\backslash\{0\}$ is called a root system if $R \cap \mathbb{R}\alpha = \{\pm\alpha\}$ and $\sigma_\alpha R = R$ for all $\alpha \in R$. We normalize each $\alpha \in R$ as $\langle \alpha, \alpha\rangle = 2$. We fix a $\beta \in \mathbb{R}^d\backslash \cup_{\alpha\in R} H_\alpha$ and define a positive root system R_+ of R as $R_+ = \{\alpha \in R \mid \langle \alpha, \beta\rangle > 0\}$. The reflections $\sigma_\alpha, \alpha \in R$, generate a finite group $W \subset O(d)$, called the reflection group. A function $k : R \to \mathbb{C}$ on R is called a multiplicity function if it is invariant under the action of W. We introduce the index γ as

$$\gamma = \gamma(k) = \sum_{\alpha\in R_+} k(\alpha).$$

Throughout this paper, we will assume that $k(\alpha) \geq 0$ for all $\alpha \in R$. We denote by ω_k the weight function on $\mathbb{R}^d$ given by

$$\omega_k(x) = \prod_{\alpha\in R_+} |\langle \alpha, x\rangle|^{2k(\alpha)},$$

which is invariant and homogeneous of degree 2γ. In the case that the reflection group W is the group $\mathbb{Z}_2^d$ of sign changes, the weight function ω_k is a product function of the form $\prod_{j=1}^d |x_j|^{k_j}$, $k_j \geq 0$. We denote by c_k the Mehta-type constant defined by

$$c_k = \int_{\mathbb{R}^d} e^{\frac{-\|x\|^2}{2}} \omega_k(x)dx.$$

In the following we denote by

$C(\mathbb{R}^d)$	the space of continuous functions on $\mathbb{R}^d$.
$C^p(\mathbb{R}^d)$	the space of functions of class C^p on $\mathbb{R}^d$.
$\mathcal{E}(\mathbb{R}^d)$	the space of C^∞-functions on $\mathbb{R}^d$.
$\mathcal{S}(\mathbb{R}^d)$	the Schwartz space of rapidly decreasing functions on $\mathbb{R}^d$.
$D(\mathbb{R}^d)$	the space of C^∞-functions on $\mathbb{R}^d$ which are of compact support.
$\mathcal{S}'(\mathbb{R}^d)$	the space of temperate distributions on $\mathbb{R}^d$.

2.2. The Dunkl operators. Let $k : R \to \mathbb{C}$ be a multiplicity function on R and R_+ a fixed positive root system of R. Then the Dunkl operators T_j, $1 \leq j \leq d$, are defined on $C^1(\mathbb{R}^d)$ by

$$T_j f(x) = \frac{\partial}{\partial x_j} f(x) + \sum_{\alpha\in R_+} k(\alpha)\alpha_j \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x\rangle},$$

where $\alpha = (\alpha_1, \alpha_2, \cdots, \alpha_d)$. Similarly as ordinary derivatives, each T_j satisfies for all f, g in $C^1(\mathbb{R}^d)$ and at least one of them is W-invariant,

$$T_j(fg) = (T_j f)g + f(T_j g)$$

and for all f in $C_b^1(\mathbb{R}^d)$ and g in $\mathcal{S}(\mathbb{R}^d)$,

$$\int_{\mathbb{R}^d} T_j f(x) g(x) \omega_k(x) dx = -\int_{\mathbb{R}^d} f(x) T_j g(x) \omega_k(x) dx.$$

Furthermore, according to [13, 14], the Dunkl operators T_j, $1 \leq j \leq d$ commute and there exists the so-called Dunkl's intertwining operator V_k such that $T_j V_k = V_k(\partial/\partial x_j)$ for $1 \leq j \leq d$ and $V_k(1) = 1$. We define the Dunkl-Laplace operator $\triangle_k$ on $\mathbb{R}^d$ by

$$\begin{aligned}\triangle_k f(x) &= \sum_{j=1}^{d} T_j^2 f(x) \\ &= \triangle f(x) + 2 \sum_{\alpha \in R^+} k(\alpha) \left(\frac{\langle \nabla f(x), \alpha \rangle}{\langle \alpha, x \rangle} - \frac{f(x) - f(\sigma_\alpha(x))}{\langle \alpha, x \rangle^2} \right),\end{aligned}$$

where $\triangle$ and ∇ are the usual Euclidean Laplacian and nabla operators on $\mathbb{R}^d$ respectively. Since the Dunkl operators commute, their joint eigenvalue problem is significant, and for each $y \in \mathbb{R}^d$, the system

$$T_j u(x, y) = y_j u(x, y), \; j = 1, \ldots, d, \quad \text{and} \quad u(0, y) = 1$$

admits a unique analytic solution $K(x, y)$, $x \in \mathbb{R}^d$, called the Dunkl kernel, which has a holomorphic extension to $\mathbb{C}^d \times \mathbb{C}^d$. For $x, y \in \mathbb{C}^d$, the kernel satisfies

(a) $K(x, y) = K(y, x)$,
(b) $K(\lambda x, y) = K(x, \lambda y)$ for $\lambda \in \mathbb{C}$,
(c) $K(wx, wy) = K(x, y)$ for $w \in W$.

2.3. The Dunkl transform. For functions f on $\mathbb{R}^d$ we define L^p-norms of f with respect to $\omega_k(x)dx$ as

$$\|f\|_{L_k^p(\mathbb{R}^d)} = \left(\int_{\mathbb{R}^d} |f(x)|^p \omega_k(x) dx \right)^{\frac{1}{p}},$$

if $1 \leq p < \infty$ and $\|f\|_{L_k^\infty(\mathbb{R}^d)} = \text{ess } \sup_{x \in \mathbb{R}^d} |f(x)|$. We denote by $L_k^p(\mathbb{R}^d)$ the space of all measurable functions f on $\mathbb{R}^d$ with finite L_k^p-norm.

The Dunkl transform $\mathcal{F}_D$ on $L_k^1(\mathbb{R}^d)$ is given by

$$\mathcal{F}_D(f)(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)K(x,-iy)\omega_k(x)dx.$$

Some basic properties are the following (cf. [15] and [11]): For all $f \in L_k^1(\mathbb{R}^d)$,

(a) $\|\mathcal{F}_D(f)\|_{L_k^\infty(\mathbb{R}^d)} \le c_k^{-1}\|f\|_{L_k^1(\mathbb{R}^d)}$,

(b) $\mathcal{F}_D(f(\cdot/\lambda)))(y) = \lambda^{2\gamma+d}F_D(f)(\lambda y)$ for $\lambda > 0$,

(c) if $\mathcal{F}_D(f)$ belongs to $L_k^1(\mathbb{R}^d)$, then

$$\mathcal{F}_D(f)(y) = \frac{1}{c_k}\int_{\mathbb{R}^d} f(x)K(x,-iy)\omega_k(x)dx \quad a.e.$$

and moreover, for all $f \in \mathcal{S}(\mathbb{R}^d)$,

(d) $\mathcal{F}_D(T_jf)(y) = iy_j\mathcal{F}_D(f)(y)$,

(e) if we define $\overline{\mathcal{F}_D}(f)(y) = \mathcal{F}_D(f)(-y)$, then

$$\mathcal{F}_D\overline{\mathcal{F}_D} = \overline{\mathcal{F}_D}\mathcal{F}_D = Id.$$

Proposition 1. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}(\mathbb{R}^d)$ onto itself and for all f in $\mathcal{S}(\mathbb{R}^d)$,*

$$\int_{\mathbb{R}^d} |f(x)|^2\omega_k(x)dx = \int_{\mathbb{R}^d} |\mathcal{F}_D(f)(\xi)|^2\omega_k(\xi)d\xi.$$

In particular, the Dunkl transform $f \to \mathcal{F}_D(f)$ can be uniquely extended to an isometric isomorphism on $L_k^2(\mathbb{R}^d)$.

We define the tempered distribution $\mathcal{T}_f$ associated with $f \in L_k^p(\mathbb{R}^d)$ by

$$\langle \mathcal{T}_f, \phi\rangle = \int_{\mathbb{R}^d} f(x)\phi(x)\omega_k(x)dx \tag{1}$$

for $\phi \in \mathcal{S}(\mathbb{R}^d)$ and denote by $\langle f, \phi\rangle_k$ the integral in the righthand side.

Definition 1. *The Dunkl transform $\mathcal{F}_D(\tau)$ of a distribution $\tau \in \mathcal{S}'(\mathbb{R}^d)$ is defined by*

$$\langle \mathcal{F}_D(\tau), \phi\rangle = \langle \tau, \mathcal{F}_D(\phi)\rangle$$

for $\phi \in \mathcal{S}(\mathbb{R}^d)$.

In particular, for $f \in L_k^p(\mathbb{R}^d)$, it follows that for $\phi \in \mathcal{S}(\mathbb{R}^d)$,

$$\langle \mathcal{F}_D(f), \phi\rangle = \langle \mathcal{F}_D(\mathcal{T}_f), \phi\rangle = \langle \mathcal{T}_f, \mathcal{F}_D(\phi)\rangle = \langle f, \mathcal{F}_D(\phi)\rangle_k.$$

Theorem 1. *The Dunkl transform $\mathcal{F}_D$ is a topological isomorphism from $\mathcal{S}'(\mathbb{R}^d)$ onto itself.*

2.4. The Dunkl convolution.

Definition 2. *Let y be in $\mathbb{R}^d$. The Dunkl translation operator $f \mapsto \tau_y f$ is defined on $\mathcal{S}(\mathbb{R}^d)$ by*

$$\mathcal{F}_D(\tau_y f)(x) = K(ix, y)\mathcal{F}_D(f)(x), \quad \text{for all } x \in \mathbb{R}^d. \tag{2}$$

Proposition 2. *If $f(x) = F(\|x\|)$ in $\mathcal{E}(\mathbb{R}^d)$, then we have*

$$\tau_y f(x) = V_k\Big[F(\sqrt{\|x\|^2 + \|y\|^2 + 2\langle x, .\rangle})\Big](x), \quad \text{for all } x \in \mathbb{R}^d$$

where V_k is the Dunkl intertwining operator. (*Cf.* [*25*]).

Using the Dunkl translation operator, we define the Dunkl convolution product of functions as follows (cf. [29]).

Definition 3. *The Dunkl convolution product of f and g in $\mathcal{S}(\mathbb{R}^d)$ is the function $f *_D g$ defined by*

$$f *_D g(x) = \int_{\mathbb{R}^d} \tau_x f(-y) g(y) \omega_k(y) dy, \quad \text{for all } x \in \mathbb{R}^d. \tag{3}$$

This convolution is commutative and associative and satisfies the following properties. (Cf. [26]).

Proposition 3. i) *For f and g in $\mathcal{S}(\mathbb{R}^d)$ the function $f *_D g$ belongs to $\mathcal{S}(\mathbb{R}^d)$ and we have*

$$\mathcal{F}_D(f *_D g)(y) = \mathcal{F}_D(f)(y)\mathcal{F}_D(g)(y), \quad \text{for all } y \in \mathbb{R}^d. \tag{4}$$

ii) *Let $1 \leq p, q, r \leq \infty$, such that $\frac{1}{p} + \frac{1}{q} - \frac{1}{r} = 1$. If f is in $L^p_k(\mathbb{R}^d)$ and g is a radial element of $L^q_k(\mathbb{R}^d)$, then $f *_D g \in L^r_k(\mathbb{R}^d)$ and we have*

$$\|f *_D g\|_{L^r_k(\mathbb{R}^d)} \leq \|f\|_{L^p_k(\mathbb{R}^d)} \|g\|_{L^q_k(\mathbb{R}^d)} \, . \tag{5}$$

iii) *Let $W = \mathbb{Z}_2^d$. We have the same result for all f belongs to $L^p_k(\mathbb{R}^d)$ and $g \in L^q_k(\mathbb{R}^d)$.*

Definition 4. *The Dunkl convolution product of a distribution S in $\mathcal{S}'(\mathbb{R}^d)$ and a function ϕ in $\mathcal{S}(\mathbb{R}^d)$ is the function $S *_D \phi$ defined by*

$$S *_D \phi(x) = \langle S_y, \tau_{-y}\phi(x)\rangle.$$

Proposition 4. *Let f be in $L_k^p(\mathbb{R}^d)$, $1 \le p \le \infty$, and ϕ in $\mathcal{S}(\mathbb{R}^d)$. Then the distribution $\mathcal{T}_f *_D \phi$ is given by the function $f *_D \phi$. If we assume that ϕ is arbitrary for $d = 1$ and radial for $d \ge 2$, then $\mathcal{T}_f *_D \phi$ belongs to $L_k^p(\mathbb{R}^d)$. Moreover, for all $\psi \in \mathcal{S}(\mathbb{R}^d)$,*

$$\langle \mathcal{T}_f *_D \phi, \psi \rangle = \langle \check{f}, \phi *_D \check{\psi} \rangle_k, \tag{6}$$

where $\check{\psi}(x) = \psi(-x)$, and

$$\mathcal{F}_D(\mathcal{T}_f *_D \phi) = \mathcal{F}_D(\mathcal{T}_f)\mathcal{F}_D(\phi). \tag{7}$$

For each $u \in \mathcal{S}'(\mathbb{R}^d)$, we define the distributions $T_j u$, $1 \le j \le d$, by

$$\langle T_j u, \psi \rangle = -\langle u, T_j \psi \rangle$$

for all $\psi \in \mathcal{S}(\mathbb{R}^d)$. Then $\langle \triangle_k u, \psi \rangle = \langle u, \triangle_k \psi \rangle$ and these distributions satisfy the following properties (see 2.3 (d)):

$$\begin{aligned} \mathcal{F}_D(T_j u) &= i y_j \mathcal{F}_D(u), \\ \mathcal{F}_D(\triangle_k u) &= -\|y\|^2 \mathcal{F}_D(u). \end{aligned} \tag{8}$$

In the following we denote $\mathcal{T}_f$ given by (1) by f for simplicity.

3. Homogeneous Dunkl-Littlewood-Paley decomposition.

One of the main tools in this paper is the homogeneous Littlewood-Paley decompositions of distributions associated with the Dunkl operators into dyadic blocs of frequencies.

Lemma 1. *Let us define by $\mathcal{C}$ the ring of center 0, of small radius $\frac{1}{2}$ and great radius 2. It exist two radial functions ψ and φ the values of which are in the interval $[0,1]$ belonging to $D(\mathbb{R}^d)$ such that*

$$\begin{aligned} &\operatorname{supp}\psi \subset B(0,1), \quad \operatorname{supp}\varphi \subset \mathcal{C} \\ &\forall \xi \in \mathbb{R}^d, \quad \psi(\xi) + \sum_{j \ge 0} \varphi(2^{-j}\xi) = 1 \\ &\forall \xi \in \mathcal{C}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1 \\ &|n-m| \ge 2 \Rightarrow \operatorname{supp}\varphi(2^{-n}.) \cap \operatorname{supp}\varphi(2^{-m}.) = \emptyset \\ &j \ge 1 \Rightarrow \operatorname{supp}\psi \cap \operatorname{supp}\varphi(2^{-j}.) = \emptyset. \end{aligned}$$

Notation. We denote by

$$(9)\qquad \forall\, j\in\mathbb{Z},\quad \Delta_j f=\mathcal{F}_D^{-1}\left(\varphi\left(\frac{\xi}{2^j}\right)\mathcal{F}_D(f)\right),\qquad S_j f=\sum_{n\leq j-1}\Delta_n f.$$

The distribution $\Delta_j f$ is called the j-th dyadic block of the homogeneous Littlewood-Paley decomposition of f associated with the Dunkl operators.

Throughout this paper we define ϕ and χ by $\phi=\mathcal{F}_D^{-1}(\varphi)$ and $\chi=\mathcal{F}_D^{-1}(\psi)$.

When dealing with the Littlewood-Paley decomposition, it is convenient to introduce the functions $\widetilde{\psi}$ and $\widetilde{\varphi}$ belonging to $D(\mathbb{R}^d)$ such that $\widetilde{\psi}\equiv 1$ on $\operatorname{supp}\psi$ and $\widetilde{\varphi}\equiv 1$ on $\operatorname{supp}\varphi$ as well the operators $\widetilde{S}_j$ and $\widetilde{\Delta}_j$ defined by

$$\mathcal{F}_D(\widetilde{\Delta}_j f)=\widetilde{\varphi}\left(\frac{\xi}{2^j}\right)\mathcal{F}_D(f),\quad \mathcal{F}_D(\widetilde{S}_j f)=\widetilde{\psi}\left(\frac{\xi}{2^j}\right)\mathcal{F}_D(f).$$

We remark that

$$\mathcal{F}_D(S_j f)(\xi)=\widetilde{\psi}\left(\frac{\xi}{2^j}\right)\mathcal{F}_D(S_j f)(\xi)\text{ and }\mathcal{F}_D(\Delta_j f)(\xi)=\widetilde{\varphi}\left(\frac{\xi}{2^j}\right)\mathcal{F}_D(\Delta_j f)(\xi).$$

We put

$$\widetilde{\phi}=\mathcal{F}_D^{-1}(\widetilde{\varphi}),\text{ and }\widetilde{\chi}=\mathcal{F}_D^{-1}(\widetilde{\psi}).$$

Definition 5. *Let us denote by $\mathcal{S}'_{h,k}(\mathbb{R}^d)$ the space of tempered distribution such that*

$$\lim_{j\to-\infty}S_j u=0\quad in\quad \mathcal{S}'(\mathbb{R}^d).$$

Remark 1. i) If a tempered distribution u is such that its Dunkl transform $\mathcal{F}_D(u)$ is locally integrable near 0, then u belongs to $\mathcal{S}'_{h,k}(\mathbb{R}^d)$.

ii) A non zero constant function u does not belongs to $\mathcal{S}'_{h,k}(\mathbb{R}^d)$.

iii) The space $\mathcal{S}'_{h,k}(\mathbb{R}^d)$ is exactly the space of tempered distributions for which we may write

$$u=\sum_{j\in\mathbb{Z}}\Delta_j u.$$

Proposition 5 (Bernstein inequalities)**.** *For all $\alpha\in\mathbb{N}^d$ and $\sigma\in\mathbb{R}$, for all $j\in\mathbb{Z}$, for all $1\leq p,q\leq\infty$ and for all $f\in\mathcal{S}'(\mathbb{R}^d)$, we have*

i) $\|\Delta_j f\|_{L^q_k(\mathbb{R}^d)}\leq\|\widetilde{\phi}\|_{L^r_k(\mathbb{R}^d)}\|\Delta_j f\|_{L^p_k(\mathbb{R}^d)}2^{j(d+2\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)}$, *with* $\dfrac{1}{q}=\dfrac{1}{p}+\dfrac{1}{r}-1.$

ii) $\|S_j f\|_{L^q_k(\mathbb{R}^d)} \leq \|\widetilde{\chi}\|_{L^r_k(\mathbb{R}^d)} \|S_j f\|_{L^p_k(\mathbb{R}^d)} 2^{j(d+2\gamma)\left(\frac{1}{p}-\frac{1}{q}\right)}$, *with* $\dfrac{1}{q} = \dfrac{1}{p} + \dfrac{1}{r} - 1$.

iii) $\|(\sqrt{-\triangle_k})^\sigma \Delta_j f\|_{L^p_k(\mathbb{R}^d)} \leq \|\mathcal{F}_D^{-1}(\|\xi\|^\sigma \widetilde{\varphi})\|_{L^1_k(\mathbb{R}^d)} \|\Delta_j f\|_{L^p_k(\mathbb{R}^d)} 2^{j\sigma}$.

Moreover if $W = \mathbb{Z}_2^d$, *we have*

iv) $\|T^\alpha \Delta_j f\|_{L^p_k(\mathbb{R}^d)} \leq 2^{\frac{d}{2}} \|T^\alpha \widetilde{\phi}\|_{L^1_k(\mathbb{R}^d)} \|\Delta_j f\|_{L^p_k(\mathbb{R}^d)} 2^{j|\alpha|}$.

v) $\|T^\alpha S_j f\|_{L^p_k(\mathbb{R}^d)} \leq 2^{\frac{d}{2}} \|T^\alpha \widetilde{\chi}\|_{L^1_k(\mathbb{R}^d)} \|S_j f\|_{L^p_k(\mathbb{R}^d)} 2^{j|\alpha|}$.

Proof. The proof is similar to the nonhomogeneous case (cf. [18]). □

Definition 6. *For* $s \in \mathbb{R}$, *the operator* $\mathcal{R}^s_k$ *from* $\mathcal{S}'_{h,k}(\mathbb{R}^d)$ *to* $\mathcal{S}'_{h,k}(\mathbb{R}^d)$ *is defined by*

$$\mathcal{R}^s_k(f) = \mathcal{F}_D^{-1}(\|\cdot\|^s \mathcal{F}_D f).$$

The operator $\mathcal{R}^{-s}_k$ *is called Dunkl-Riesz potentials.*

4. $\dot{\mathcal{B}}^{s,k}_{p,q}$, $\dot{\mathcal{H}}^s_{p,k}$ **spaces and basic properties.** In this section we define analogues of the homogeneous Besov and Riesz potential spaces associated with the Dunkl operators on $\mathbb{R}^d$ and obtain their basic properties. In particular, we use the homogeneous Dunkl-Littlewood-Paley decomposition of f in $\mathcal{S}'_{h,k}(\mathbb{R}^d)$, obtained in the previous section, and apply the standard process used in the Euclidean case.

4.1. Definitions. From now, we make the convention that for all nonnegative sequence $\{a_q\}_{q\in\mathbb{Z}}$, the notation $(\sum_q a_q^r)^{\frac{1}{r}}$ stands for $\sup_q a_q$ in the case $r = \infty$. Let $s \in \mathbb{R}$ and $1 \leq p \leq \infty$. For a sequence $\{u_j\}_{j\in\mathbb{Z}}$ of functions on $\mathbb{R}^d$, we define

$$\|\{u_j\}\|_{l^s_q(L^p_k(\mathbb{R}^d))} = \Big(\sum_{j\in\mathbb{Z}} (2^{js}\|u_j\|_{L^p_k(\mathbb{R}^d)})^q\Big)^{\frac{1}{q}}.$$

Definition 7. *Let* $s \in \mathbb{R}$ *and* $p, q \in [1, \infty]$. *The homogeneous Dunkl-Besov spaces* $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ *is the space of distribution in* $\mathcal{S}'_{h,k}(\mathbb{R}^d)$ *such that*

$$\|f\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} := \Big(\sum_{j\in\mathbb{Z}} (2^{sj}\|\Delta_j f\|_{L^p_k(\mathbb{R}^d)})^q\Big)^{\frac{1}{q}} < \infty.$$

Proposition 6 ([1]). *Let $s \in \mathbb{R}$, p and q two elements of $[1,\infty]$, the space $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ is the set of $f \in \mathcal{S}'_{h,k}(\mathbb{R}^d)$ verifying*

$$\|\dot{f}\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} = \Big(\int_0^\infty (t^{-s}\|f *_D \phi_t\|_{L^p_k(\mathbb{R}^d)})^q \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty.$$

Definition 8. *For $s \in \mathbb{R}$ and $1 \le p \le \infty$, the homogeneous Dunkl-Riesz potential space $\dot{\mathcal{H}}^s_{p,k}(\mathbb{R}^d)$ is defined as the space $\mathcal{R}_k^{-s}(L^p_k(\mathbb{R}^d))$, equipped with the norm $\|f\|_{\dot{\mathcal{H}}^s_{p,k}(\mathbb{R}^d)} = \|\mathcal{R}^s_k(f)\|_{L^p_k(\mathbb{R}^d)}$.*

Proposition 7. *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$ satisfy*

$$s < \frac{d+2\gamma}{p}, \quad or \quad s = \frac{d+2\gamma}{p} \quad and\ q = 1.$$

Let $\{u_j\}_{j\in\mathbb{Z}}$ be a sequence of functions such that $\|\{u_j\}\|_{l^s_q(L^p_k(\mathbb{R}^d))} < \infty$.

(1) *If $\operatorname{supp} \mathcal{F}_D(u_j) \subset 2^j R$ for some annulus R centered at the origin, then $f = \sum_{j\in\mathbb{Z}} u_j$ belongs to $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ and there exists a positive constant $C(s)$ such that $\|f\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} \le C(s)\|\{u_j\}\|_{l^s_q(L^p_k(\mathbb{R}^d))}$.*

(2) *If $s > 0$ and $\operatorname{supp} \mathcal{F}_D(u_j) \subset 2^j B$ for some ball B centered at the origin.*

Then $f = \sum_{j\in\mathbb{Z}} u_j$ belongs to $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ and there exists a positive constant $C(s)$ such that

$$\|f\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} \le C(s)\|\{u_j\}\|_{l^s_q(L^p_k(\mathbb{R}^d))}.$$

Proof. We obtain these results by a similar ideas used in the nonhomogeneous case. (cf. [18]). □

Corollary 1. *Let p, q be as above. The definitions of the space $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ do not depend on the choice of the couple (φ, ψ) defining the homogeneous Dunkl-Littlewood-Paley decomposition.*

Proposition 8. *Let $s \in \mathbb{R}$ and $1 \le p, q \le \infty$.*

i) The operator $\triangle_k$ is a linear continuous operator from $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ into $\dot{\mathcal{B}}^{s-2,k}_{p,q}(\mathbb{R}^d)$ and from $\dot{\mathcal{H}}^s_{p,k}(\mathbb{R}^d)$ into $\dot{\mathcal{H}}^{s-2}_{p,q}(\mathbb{R}^d)$.

ii) When $W = \mathbb{Z}_2^d$, the operators T_j, $j = 1, \ldots, d$ are a linear continuous operators from $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ into $\dot{\mathcal{B}}^{s-1,k}_{p,q}(\mathbb{R}^d)$ and from $\dot{\mathcal{H}}^{s}_{p,k}(\mathbb{R}^d)$ into $\dot{\mathcal{H}}^{s-1}_{p,q}(\mathbb{R}^d)$.

Proof. We obtain these results by a similar ideas used in the nonhomogeneous case. (cf. [18]). □

Theorem 2. *Let $s, t \in \mathbb{R}$ and $1 \le p, q \le \infty$. The operator $\mathcal{R}^t_k$ is a linear continuous injective operator from $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ onto $\dot{\mathcal{B}}^{s-t,k}_{p,q}(\mathbb{R}^d)$, and from $\dot{\mathcal{H}}^{s}_{p,k}(\mathbb{R}^d)$ onto $\dot{\mathcal{H}}^{s-t}_{p,k}(\mathbb{R}^d)$.*

Proof. Since $\mathcal{F}_D$ satisfies (4), we can apply the same arguments used in the proof of Theorem 5.1.1 in [28]. □

4.2. Embeddings. As in the Euclidean case (cf. [28]), the monotone character of l_q-spaces and Minkowski's inequality yield the following.

Proposition 9. *If $1 \le q_1 < q_2 \le \infty$ we have*

$$\dot{\mathcal{B}}^{s,k}_{p,q_1}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{B}}^{s,k}_{p,q_2}(\mathbb{R}^d), \quad (1 \le p \le \infty,\ s \in \mathbb{R}). \tag{10}$$

Moreover

$$\dot{\mathcal{B}}^{s,k}_{p,1}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{H}}^{s}_{p,k}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{B}}^{s,k}_{p,\infty}(\mathbb{R}^d), \quad (1 \le p \le \infty,\ s \in \mathbb{R}). \tag{11}$$

If $s_0 \ne s_1$ we also have

$$(\dot{\mathcal{H}}^{s_0}_{p,k}(\mathbb{R}^d), \dot{\mathcal{H}}^{s_1}_{p,k}(\mathbb{R}^d))_{\theta,q} = \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d) \quad (1 \le p, q \le \infty,\ \theta \in (0,1)), \tag{12}$$

where $s = (1-\theta)s_0 + \theta s_1$.

Proof. We obtain these results by a similar ideas used in the nonhomogeneous case. (cf. [18]). □

Proposition 10. *We assume that $s - \dfrac{d+2\gamma}{p} = s_1 - \dfrac{d+2\gamma}{p_1}$. Then the following inclusion hold*

$$\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{B}}^{s_1,k}_{p_1,q_1}(\mathbb{R}^d), \quad (1 \le p \le p_1 \le \infty,\ 1 \le q \le q_1 \le \infty,\ s,\ s_1 \in \mathbb{R}).$$

Proof. In order to prove the inclusion, we use the estimate.

$$\Delta_j f = \widetilde{\phi}_j *_D \Delta_j f.$$

The Proposition 5 i) give that

$$\begin{aligned}\|\Delta_j f\|_{L_k^{p_1}(\mathbb{R}^d)} &= \|\widetilde{\phi}_j *_D \Delta_j f\|_{L_k^{p_1}(\mathbb{R}^d)}\\ &\leq C2^{j(d+2\gamma)(\frac{1}{p}-\frac{1}{p_1})}\|\Delta_j f\|_{L_k^{p}(\mathbb{R}^d)}.\end{aligned}$$

By definition of the homogeneous Dunkl-Besov spaces, we therefore infer

$$\begin{aligned}\|f\|_{\dot{\mathcal{B}}^{s_1,k}_{p_1,q_1}(\mathbb{R}^d)} &= \left(\sum_{j=-\infty}^{\infty}(2^{js_1}\|\Delta_j f\|_{L_k^{p_1}(\mathbb{R}^d)})^{q_1}\right)^{\frac{1}{q_1}}\\ &\leq C\left(\sum_{j\in\mathbb{Z}}(2^{js_1}2^{j(d+2\gamma)(\frac{1}{p}-\frac{1}{p_1})}\|\Delta_j f\|_{L_k^{p}(\mathbb{R}^d)})^{q_1}\right)^{\frac{1}{q_1}}\\ &\leq C\left(\sum_{j\in\mathbb{Z}}(2^{js}\|\Delta_j f\|_{L_k^{p}(\mathbb{R}^d)})^{q_1}\right)^{\frac{1}{q_1}}\\ &\leq C\|f\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)},\end{aligned}$$

since $q \leq q_1$. This gives the inclusion. $\square$

As a consequence of real and complex interpolations, we can deduce multiplicative inequalities, which will be needed in the theory of differential-difference operators.

Theorem 3. (1) *If u belongs to $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)\cap\dot{\mathcal{B}}^{t,k}_{p,q}(\mathbb{R}^d)$, then u belongs to $\dot{\mathcal{B}}^{\theta s+(1-\theta)t,k}_{p,q}(\mathbb{R}^d)$ for all $\theta\in[0,1]$ and*

$$\|u\|_{\dot{\mathcal{B}}^{\theta s+(1-\theta)t,k}_{p,q}(\mathbb{R}^d)} \leq \|u\|^{\theta}_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}\|u\|^{1-\theta}_{\dot{\mathcal{B}}^{t,k}_{p,q}(\mathbb{R}^d)}.$$

(2) *If u belongs to $\dot{\mathcal{B}}^{s,k}_{p,\infty}(\mathbb{R}^d)\cap\dot{\mathcal{B}}^{t,k}_{p,\infty}(\mathbb{R}^d)$ and $s<t$, then $u\in\dot{\mathcal{B}}^{\theta s+(1-\theta)t,k}_{p,1}(\mathbb{R}^d)$ for all $\theta\in(0,1)$ and there exists a positive constant $C(t,s)$ such that*

$$\|u\|_{\dot{\mathcal{B}}^{\theta s+(1-\theta)t,k}_{p,1}(\mathbb{R}^d)} \leq C(t,s)\|u\|^{\theta}_{\dot{\mathcal{B}}^{s,k}_{p,\infty}(\mathbb{R}^d)}\|u\|^{1-\theta}_{\dot{\mathcal{B}}^{t,k}_{p,\infty}(\mathbb{R}^d)}.$$

(3) *If u belongs to $\dot{\mathcal{B}}^{s,k}_{p,\infty}(\mathbb{R}^d)\cap\dot{\mathcal{B}}^{s+\varepsilon,k}_{p,\infty}(\mathbb{R}^d)$ and $\varepsilon>0$, then u belongs to $\dot{\mathcal{B}}^{s,k}_{p,1}(\mathbb{R}^d)$ and there exists a positive constant C such that*

$$\|u\|_{\dot{\mathcal{B}}^{s,k}_{p,1}(\mathbb{R}^d)} \leq \frac{C}{\varepsilon}\|u\|_{\dot{\mathcal{B}}^{s,k}_{p,\infty}(\mathbb{R}^d)}\log_2\left(e+\frac{\|u\|_{\dot{\mathcal{B}}^{s+\varepsilon,k}_{p,\infty}(\mathbb{R}^d)}}{\|u\|_{\dot{\mathcal{B}}^{s,k}_{p,\infty}(\mathbb{R}^d)}}\right).$$

Proof. The proof is the same as in the nonhomogeneous frame work, and thus omitted. □

4.3. Subspace dense. We proceed as in the Euclidean case (cf. [27] and [28]), we prove.

Lemma 2. *Let a and b two numbers real such that $0 < a < b$ and $(u_t)_{t>0}$ a family of distributions such that*

i) $\operatorname{supp}\mathcal{F}_D(u_t) \subset \{\xi \in \mathbb{R}^d : \frac{a}{t} \le \|\xi\| \le \frac{b}{t}\}$,

ii) $\left(\int_0^\infty (t^{-s}\|u_t\|_{L^p_k(\mathbb{R}^d)})^q \frac{dt}{t}\right)^{\frac{1}{q}} < \infty$.

Then

$$\left\|\int_0^\infty u_t \frac{dt}{t}\right\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} \le C(a,b,s)\left(\int_0^\infty (t^{-s}\|u_t\|_{L^p_k(\mathbb{R}^d)})^q \frac{dt}{t}\right)^{\frac{1}{q}}.$$

If $s > 0$, we have the same conclusion when we replace i) by

iii) $\operatorname{supp}\mathcal{F}_D(u_t) \subset \{\xi \in \mathbb{R}^d : \|\xi\| \le \frac{b}{t}\}$.

Proposition 11. *For $q < \infty$, the subspace*

$$\left\{u \in \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d) : \quad \text{support}(\mathcal{F}_D(u)) \quad \textit{is} \quad \textit{compact}\right\}$$

is dense in $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$.

Proof. Let $f \in \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ and $f_\varepsilon = \displaystyle\int_\varepsilon^{\frac{1}{\varepsilon}} f *_D \phi_t \frac{dt}{t}$. From the previous lemma we deduce that f_ε belongs to $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$, on the other hand since $q < \infty$ we see that f_ε tends to f in norm $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$. □

Proposition 12. *If q is a real number greater than 1 and (s,p) is couple of real numbers so that $s < \dfrac{d+2\gamma}{p}$ and p greater than 1, then the space $\displaystyle\bigcap_{s\in\mathbb{R}} \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ is a dense subspace of $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$.*

Proof. The proof of this consists in writing that

$$\begin{aligned}\|u - S_j u\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} &\le \left\|\sum_{i\ge j}\Delta_i u\right\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}\\ &\le C\Big(\sum_{i\ge j} 2^{qis}\|\Delta_i u\|^q_{L^p_k(\mathbb{R}^d)}\Big)^{\frac{1}{q}}.\end{aligned}$$

So, the proposition is proved because the remainder term of a convergent series tends to 0. $\square$

Proposition 13. *For all $s \in \mathbb{R}$ and $1 \leq p, q \leq \infty$, the couple $(\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d), \|\cdot\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)})$ is a normed space. If besides q is finite then $D(\mathbb{R}^d) \cap \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ is densely embedded in $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$.*

Proof. It is obvious that $\|\cdot\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}$ is a semi-norm. Let us assume that $\|u\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} = 0$ for some u in $\mathcal{S}'_{h,k}(\mathbb{R}^d)$. This implies that Supp $\mathcal{F}_D(u) \subset \{0\}$ and thus that for any $j \in \mathbb{Z}$ we have $S_j u = u$. As u belongs to $\mathcal{S}'_{h,k}(\mathbb{R}^d)$, we must have $\lim\limits_{j\to-\infty} S_j u = 0$ so that we can conclude that $u = 0$. Now, if q is finite and $u \in \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$, it is obvious that the sequence of general term $\sum\limits_{|m|\leq n} \Delta_m u$ belongs to $\mathcal{E}(\mathbb{R}^d) \cap \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ and tends to u in $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$. Arguing like as the nonhomogeneous case (cf. [18]), it is then easy to exhibit a sequence of functions of $D(\mathbb{R}^d) \cap \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ which tends to u in $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$. $\square$

Theorem 4. *If $s < \dfrac{d+2\gamma}{p}$, then $(\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d), \|\cdot\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)})$ is a Banach space. For any p, the space $\dot{\mathcal{B}}^{\frac{d+2\gamma}{p},k}_{p,1}(\mathbb{R}^d)$ is also a Banach space.*

Proof. Let us first prove that $(\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d), \|\cdot\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)})$ is continuously embedded in $\mathcal{S}'(\mathbb{R}^d)$. The case $q = 1$ and $s = \dfrac{d+2\gamma}{p}$ is easy because the series $\sum\limits_j \Delta_j u$ is convergent in $L^\infty_k(\mathbb{R}^d)$. As $u = \sum\limits_j \Delta_j u$, this implies that u belongs to $L^\infty_k(\mathbb{R}^d)$. Besides, we have

$$\dot{\mathcal{B}}^{\frac{d+2\gamma}{p},k}_{p,1}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{B}}^{0,k}_{\infty,1}(\mathbb{R}^d) \hookrightarrow L^\infty_k(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d). \tag{13}$$

Let us now assume that $s < \dfrac{d+2\gamma}{p}$. Using that

$$\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{B}}^{s-\frac{d+2\gamma}{p},k}_{\infty,\infty}(\mathbb{R}^d).$$

By a simple calculation we prove one can find a large integer M such that for all nonnegative j

$$|\langle \Delta_j u, \chi\rangle| \leq C2^{j(\frac{d+2\gamma}{p}-s)}\|u\|_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{p},k}_{\infty,\infty}(\mathbb{R}^d)}\|\chi\|_{L^1_k(\mathbb{R}^d)}$$
$$\leq C2^{j(\frac{d+2\gamma}{p}-s)}\|u\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}\|\chi\|_{M,\mathcal{S}}$$

where

$$\|\chi\|_{M,\mathcal{S}} := \sup_{x\in\mathbb{R}^d,\ |\nu|\leq M}(1+\|x\|)^M|T^\nu u(x)|.$$

Because u belongs to ${\mathcal{S}'}_{h,k}(\mathbb{R}^d)$, we have $\langle u,\chi\rangle = \sum_j \langle \Delta_j u,\chi\rangle$. Therefore, for large enough M

$$|\langle u,\chi\rangle| \leq C\|u\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}\|\chi\|_{M,\mathcal{S}} \tag{14}$$

and we can conclude that $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$.

We still have to prove that for all triplet (s,p,q) satisfying the hypothesis of the theorem, the set $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ is a Banach space. So let us consider a Cauchy sequence $(u_n)_n$ in $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$. Using (13) or (14), this implies that a temperate distribution u exists such that the sequence $(u_n)_n$ converges to u in $\mathcal{S}'(\mathbb{R}^d)$. We now have to state that u belongs to ${\mathcal{S}'}_{h,k}(\mathbb{R}^d)$. Let us first assume that $s < \dfrac{d+2\gamma}{p}$. Since u_n belongs to ${\mathcal{S}'}_{h,k}(\mathbb{R}^d)$, we have, thanks to (14),

$$\forall\, j\in\mathbb{Z},\ \forall\, n\in\mathbb{N},\quad |\langle S_j u_n,\chi\rangle| \leq C2^{j(\frac{d+2\gamma}{p}-s)}\|u_n\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}\|\chi\|_{M,\mathcal{S}}.$$

As the sequence $(u_n)_n$ tends to u in $\mathcal{S}'(\mathbb{R}^d)$, we have

$$\forall\, j\in\mathbb{Z},\ \forall\, n\in\mathbb{N},\quad |\langle S_j u,\chi\rangle| \leq C2^{j(\frac{d+2\gamma}{p}-s)}\sup_{n\in\mathbb{N}}\|u_n\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}\|\chi\|_{M,\mathcal{S}}.$$

Thus u belongs to ${\mathcal{S}'}_{h,k}(\mathbb{R}^d)$.

The case when u belongs to $\dot{\mathcal{B}}^{\frac{d+2\gamma}{p},k}_{p,1}(\mathbb{R}^d)$ is a little bit different. Let $\varepsilon>0$. As $(u_n)_n$ is a Cauchy sequence in $\dot{\mathcal{B}}^{\frac{d+2\gamma}{p},k}_{p,1}(\mathbb{R}^d) \hookrightarrow \dot{\mathcal{B}}^{0,k}_{\infty,1}(\mathbb{R}^d)$, there exists an integer N such that

$$\forall\, j\in\mathbb{Z},\ \forall\, n\geq N,\quad \sum_{m\leq n}\|\Delta_m u_n\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{\varepsilon}{2} + \sum_{m\leq n}\|\Delta_m u_N\|_{L^\infty_k(\mathbb{R}^d)}.$$

Let us choose J small enough so that

$$\sum_{m\le J}\|\Delta_m u_N\|_{L^\infty_k(\mathbb{R}^d)} \le \frac{\varepsilon}{2}.$$

As u_n belongs to ${\mathcal{S}'}_{h,k}(\mathbb{R}^d)$, we have

$$\forall\, j \le J,\ \forall\, n \ge N, \quad \|S_j u_n\|_{L^\infty_k(\mathbb{R}^d)} \le \varepsilon.$$

As sequence $(u_n)_n$ tends to u in $L^\infty_k(\mathbb{R}^d)$, this implies that

$$\forall\, j \le J, \quad \|S_j u\|_{L^\infty_k(\mathbb{R}^d)} \le \varepsilon.$$

This proves that u belongs to ${\mathcal{S}'}_{h,k}(\mathbb{R}^d)$. Next, arguing like in the nonhomogeneous case completes the proof. □

4.4. Comparison with nonhomogeneous spaces. We recall the definition of nonhomogeneous Besov spaces associated with the Dunkl operators (cf. [18]).

Definition 9. *For $s \in \mathbb{R}$ and $p, q \in [1, \infty]$, we write*

$$\|f\|_{B^{s,k}_{p,q}(\mathbb{R}^d)} = \|S_0 f\|_{L^p_k(\mathbb{R}^d)} + \Big(\sum_{j\ge 1}(2^{sj}\|\Delta_j f\|_{L^p_k(\mathbb{R}^d)})^q\Big)^{\frac{1}{q}}.$$

The Besov space $B^{s,k}_{p,q}(\mathbb{R}^d)$ associated with the Dunkl operators is defined by

$$B^{s,k}_{p,q}(\mathbb{R}^d) = \Big\{f \in \mathcal{S}'(\mathbb{R}^d):\ \|f\|_{B^{s,k}_{p,q}(\mathbb{R}^d)} < \infty\Big\}.$$

We give now another definition equivalently for the Besov space $B^{s,k}_{p,q}(\mathbb{R}^d)$.

Proposition 14. *Let $s \in \mathbb{R}$, p and q two elements of $[1, \infty]$, the space $B^{s,k}_{p,q}(\mathbb{R}^d)$ is the set of $f \in \mathcal{S}'(\mathbb{R}^d)$ verifying*

$$\|\dot{f}\|_{B^{s,k}_{p,q}(\mathbb{R}^d)} = \|f *_D \psi\|_{L^p_k(\mathbb{R}^d)} + \Big(\int_0^1 (t^{-s}\|f *_D \phi_t\|_{L^p_k(\mathbb{R}^d)})^q \frac{dt}{t}\Big)^{\frac{1}{q}} < \infty.$$

Theorem 5. i) *We assume that $f \in \mathcal{S}'(\mathbb{R}^d)$ and $0 \notin \operatorname{supp}\mathcal{F}_D(f)$. Then for all s in $\mathbb{R}$ and $1 \le p, q \le \infty$ we have*

$$f \in \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d) \Longleftrightarrow f \in B^{s,k}_{p,q}(\mathbb{R}^d).$$

ii) *For all $s > 0$ and $1 \leq p, q \leq \infty$, we have*

$$B^{s,k}_{p,q}(\mathbb{R}^d) = \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d) \cap L^p_k(\mathbb{R}^d). \tag{15}$$

On the other hand the norm of f in $B^{s,k}_{p,q}(\mathbb{R}^d)$ is equivalently to $\|f\|_{L^p_k(\mathbb{R}^d)} + \|f\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}$.

iii) *For all $s < 0$ and $1 \leq p, q \leq \infty$, we have*

$$B^{s,k}_{p,q}(\mathbb{R}^d) = \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d) + L^p_k(\mathbb{R}^d). \tag{16}$$

Moreover the norm of f in $B^{s,k}_{p,q}(\mathbb{R}^d)$ is equivalently to

$$\inf\Big\{\|f_1\|_{L^p_k(\mathbb{R}^d)} + \|f_2\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} \,:\; f = f_1 + f_2\Big\}.$$

P r o o f. i) If $\mathcal{F}_D(f)(\xi) = 0$ in a neighborhood of $\xi = 0$ and if $f \in \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$, then $S_0 f$ being a finite sum of the form $\sum_j S_0(\Delta_j f)$ belongs to $L^p_k(\mathbb{R}^d)$. Thus $f \in B^{s,k}_{p,q}(\mathbb{R}^d)$. Conversely, if $f \in B^{s,k}_{p,q}(\mathbb{R}^d)$ then $\Delta_j f = \Delta_j(S_0 f)$ if $j < 0$. Thus $\Delta_j f \in L^p_k(\mathbb{R}^d)$ for all j and since $\Big(\sum_{j<0}(2^{js}\|\Delta_j f\|_{L^p_k(\mathbb{R}^d)})^q\Big)^{\frac{1}{q}}$, is a finite sum, $f \in \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$.

ii) In order to prove (15), we first note that it is obvious that for all $s \in \mathbb{R}$

$$\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d) \cap L^p_k(\mathbb{R}^d) \subset B^{s,k}_{p,q}(\mathbb{R}^d).$$

Conversely, if $f \in B^{s,k}_{p,q}(\mathbb{R}^d)$, then $\|\Delta_j f\|_{L^p_k(\mathbb{R}^d)} \leq C\|S_0 f\|_{L^p_k(\mathbb{R}^d)}$ for $j < 0$. Thus if $s > 0$

$$\left(\sum_{j<0}(2^{js}\|\Delta_j f\|_{L^p_k(\mathbb{R}^d)})^q\right)^{\frac{1}{q}} \leq C\|S_0 f\|_{L^p_k(\mathbb{R}^d)}$$

then $f \in \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$.

iii) We assume that $s < 0$. If f belongs to $B^{s,k}_{p,q}(\mathbb{R}^d)$, we put

$$f_1 = f *_D \chi,\; f_2 = \int_0^1 f *_D \phi_t \frac{dt}{t}.$$

This gives that

$$\|f_1\|_{L^p_k(\mathbb{R}^d)} = \|f *_D \chi\|_{L^p_k(\mathbb{R}^d)} \leq C\|f\|_{B^{s,k}_{p,q}(\mathbb{R}^d)}.$$

Moreover Lemma 2 implies that

$$\|f_2\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} \leq C\bigg(\int_0^1 \Big(t^{-s}\|f *_D \phi_t\|_{L^p_k(\mathbb{R}^d)}\Big)^q \frac{dt}{t}\bigg)^{\frac{1}{q}} \leq C\|f\|_{B^{s,k}_{p,q}(\mathbb{R}^d)}.$$

Conversely, let $f = f_1 + f_2$, where $f_1 \in L^p_k(\mathbb{R}^d)$ and $f_2 \in \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$. By relation (5) we obtain

$$\|f_1\|_{B^{s,k}_{p,q}(\mathbb{R}^d)} \leq C\|f_1\|_{L^p_k(\mathbb{R}^d)}.$$

On the other hand there exists $c > 0$ such that $\chi *_D \phi_t = 0$ for $t \leq c$, this gives that

$$f_2 *_D \chi = \int_c^\infty (f_2 *_D \phi_t *_D \chi)\frac{dt}{t}.$$

Hence

$$\begin{aligned}
\|f_2 *_D \chi\|_{L^p_k(\mathbb{R}^d)} &\leq C\|\chi\|_{L^1_k(\mathbb{R}^d)} \int_c^\infty \|f_2 *_D \phi_t\|_{L^p_k(\mathbb{R}^d)}\frac{dt}{t} \\
&\leq C\|\chi\|_{L^1_k(\mathbb{R}^d)} \int_c^\infty \Big(t^{-s}\|f_2 *_D \phi_t\|_{L^p_k(\mathbb{R}^d)}\Big)\, t^s\frac{dt}{t} \\
&\leq C\|\chi\|_{L^1_k(\mathbb{R}^d)} \bigg(\int_c^\infty t^{-sq'}\frac{dt}{t}\bigg)^{\frac{1}{q'}} \|f_2\|_{B^{s,k}_{p,q}(\mathbb{R}^d)}.
\end{aligned}$$

The inequalities

$$\Big(\int_0^1 (t^{-s}\|f_2 *_D \phi_t\|_{L^p_k(\mathbb{R}^d)})^q\frac{dt}{t}\Big)^{\frac{1}{q}} \leq C\|f_2\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}$$

is immediately. We obtain then

$$\|f_2\|_{B^{s,k}_{p,q}(\mathbb{R}^d)} \leq C\|f_2\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)}.$$

This gives then the result. $\square$

5. Application.

5.1. Paraproduit algorithm associated with Dunkl operators. In this section, we are going to study how the product acts on homogeneous Besov spaces associated with the Dunkl operators. This is could be well useful in non-linear partial differential-difference equations. Let us consider two temperate distributions u and v in ${\mathcal{S}'}_{h,k}(\mathbb{R}^d)$, and we write

$$u = \sum_{p\in\mathbb{Z}} \Delta_p u \quad \text{and} \quad v = \sum_{q\in\mathbb{Z}} \Delta_q v.$$

Formally, the product can be written as

$$uv = \sum_{p,q\in\mathbb{Z}} \Delta_p u \Delta_q v.$$

Now we introduce the paraproduct operator associated with the Dunkl operators.

Definition 10. *We define the homogeneous paraproduct operator* Π_a: ${\mathcal{S}'}_{h,k}(\mathbb{R}^d) \to {\mathcal{S}'}_{h,k}(\mathbb{R}^d)$ *by*

$$\Pi_a u = \sum_{q\geq 1} (S_{q-2}\, a)\Delta_q u,$$

where $u \in {\mathcal{S}'}_{h,k}(\mathbb{R}^d)$; $\{\Delta_q a\}$ *and* $\{\Delta_q u\}$ *are the homogeneous Littlewood-Paley decompositions and* $S_q a = \sum_{p\leq q-1} \Delta_p a$.

Let R indicate the following bilinear symmetric operator defined by

$$R(u,v) = \sum_{|p-q|\leq 1} \Delta_p u \Delta_q v, \text{ for all } u, v \in {\mathcal{S}'}_{h,k}(\mathbb{R}^d).$$

Obviously from Definition 10 it is clear that

$$uv = \Pi_u v + \Pi_v u + R(u,v).$$

The following theorems describe the action of the homogeneous paraproduct and remainder on the homogeneous Besov spaces associated with the Dunkl operators.

Theorem 6. *Let* $1 \leq p, r \leq \infty$ *and* $s \in \mathbb{R}$.

1) *If* $s > 0$ *then* Π *is a bilinear continuous from* $L^\infty_k(\mathbb{R}^d) \times \dot{\mathcal{B}}^{s,k}_{p,r}(\mathbb{R}^d)$ *to* $\dot{\mathcal{B}}^{s,k}_{p,r}(\mathbb{R}^d)$ *and there exists a positive constant* C *such that*

$$\|\Pi\|_{\mathcal{L}(L^\infty_k(\mathbb{R}^d)\times\dot{\mathcal{B}}^{s,k}_{p,r}(\mathbb{R}^d),\dot{\mathcal{B}}^{s,k}_{p,r}(\mathbb{R}^d))} \leq C^{s+1}.$$

2) *If* $t<0$ *and* $1\leq r,r_1,r_2\leq\infty$ *are such that* $\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}$ *and* $0<s+t<\frac{d+2\gamma}{p}$, *then* Π *is a bilinear continuous from* $\dot{\mathcal{B}}^{t,k}_{\infty,r_1}(\mathbb{R}^d)\times\dot{\mathcal{B}}^{s,k}_{p,r_2}(\mathbb{R}^d)$ *to* $\dot{\mathcal{B}}^{s+t,k}_{p,r}(\mathbb{R}^d)$ *and there exists a positive constant* C *such that*

$$\|\Pi\|_{\mathcal{L}(\dot{\mathcal{B}}^{t,k}_{\infty,r_1}(\mathbb{R}^d)\times\dot{\mathcal{B}}^{s,k}_{p,r_2}(\mathbb{R}^d),\dot{\mathcal{B}}^{s+t,k}_{p,r}(\mathbb{R}^d))}\leq\frac{C^{s+t}}{-t}.$$

Theorem 7. *Let* $(s_1,s_2)\in\mathbb{R}^2$ *and* $1\leq p,p_1,p_2,r,r_1,r_2\leq\infty$. *Assume that*

$$\frac{1}{p}\leq\frac{1}{p_1}+\frac{1}{p_2},\quad\frac{1}{r}=\frac{1}{r_1}+\frac{1}{r_2}\leq 1\quad\text{and}\quad s_1+s_2>(d+2\gamma)\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right).$$

Then the remainder R *maps* $\dot{\mathcal{B}}^{s_1,k}_{p_1,r_1}(\mathbb{R}^d)\times\dot{\mathcal{B}}^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)$ *in* $\dot{\mathcal{B}}^{s_{1,2},k}_{p,r}(\mathbb{R}^d)$ *and there exists a positive constant* C *such that*

$$\|R\|_{\mathcal{L}(\dot{\mathcal{B}}^{s_1,k}_{p_1,r_1}(\mathbb{R}^d)\times,\dot{\mathcal{B}}^{s_2,k}_{p_2,r_2}(\mathbb{R}^d),\dot{\mathcal{B}}^{s_{1,2},k}_{p,r}(\mathbb{R}^d))}\leq\frac{C^{s_1+s_2+1}}{s_1+s_2},$$

with $s_{1,2}=s_1+s_2-(d+2\gamma)\left(\frac{1}{p_1}+\frac{1}{p_2}-\frac{1}{p}\right)$.

Corollary 2. 1) *Let* $s>0$ *and* $p,r\in[1,\infty]$. *Then* $\dot{\mathcal{B}}^{s,k}_{p,r}(\mathbb{R}^d)\cap L^\infty_k(\mathbb{R}^d)$ *is an algebra and there exists a positive constant* C *such that*

$$\|uv\|_{\dot{\mathcal{B}}^{s,k}_{p,r}(\mathbb{R}^d)}\leq C[\|u\|_{L^\infty_k(\mathbb{R}^d)}\|v\|_{\dot{\mathcal{B}}^{s,k}_{p,r}(\mathbb{R}^d)}+\|v\|_{L^\infty_k(\mathbb{R}^d)}\|u\|_{\dot{\mathcal{B}}^{s,k}_{p,r}(\mathbb{R}^d)}].$$

2) *Moreover, for any* (s_1,s_2), *any* p_2 *and any* r_2 *such that* $s_1+s_2>\frac{d+2\gamma}{p_1}$ *and* $s_1<\frac{d+2\gamma}{p_1}$, *we have*

$$\|uv\|_{\dot{\mathcal{B}}^{s,k}_{p_2,r_2}(\mathbb{R}^d)}\leq C\Big[\|u\|_{\dot{\mathcal{B}}^{s_1,k}_{p_1,\infty}(\mathbb{R}^d)}\|v\|_{\dot{\mathcal{B}}^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)}+\|u\|_{\dot{\mathcal{B}}^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)}\|v\|_{\dot{\mathcal{B}}^{s_1,k}_{p_1,\infty}(\mathbb{R}^d)}\Big],$$

where $s=s_1+s_2-\frac{d+2\gamma}{p_1}$.

3) *Moreover, for any* (s_1,s_2), *any* p_2 *and any* (r_1,r_2) *such that* $s_1+s_2>\frac{d+2\gamma}{p_1}$, $s_1<\frac{d+2\gamma}{p_1}$, $\frac{1}{r_1}+\frac{1}{r_2}=1$, *we have*

$$\|uv\|_{\dot{\mathcal{B}}^{s,k}_{p,\infty}(\mathbb{R}^d)}\leq C\Big[\|u\|_{\dot{\mathcal{B}}^{s_1,k}_{p_1,r_1}(\mathbb{R}^d)}\|v\|_{\dot{\mathcal{B}}^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)}+\|u\|_{\dot{\mathcal{B}}^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)}\|v\|_{\dot{\mathcal{B}}^{s_1,k}_{p_1,r_1}(\mathbb{R}^d)}\Big].$$

4) *Moreover, for any* (s_1, s_2), *any* (p_1, p_2, p) *and any* (r_1, r_2) *such that* $s_j < \dfrac{d+2\gamma}{p_j}$, $s_1 + s_2 > (d+2\gamma)\left(\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p}\right)$ *and* $p \geq \max(p_1, p_2)$, *we have*

$$\|uv\|_{\dot{\mathcal{B}}^{s_{1,2},k}_{p,r}(\mathbb{R}^d)} \leq C\|u\|_{\dot{\mathcal{B}}^{s_1,k}_{p_1,r_1}(\mathbb{R}^d)}\|v\|_{\dot{\mathcal{B}}^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)}.$$

with $s_{1,2} = s_1 + s_2 - (d+2\gamma)\left(\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p}\right)$ *and* $r = \max(r_1, r_2)$.

5) *Moreover, for any* (s_1, s_2), *any* (p_1, p_2, p) *and any* (r_1, r_2) *such that* $s_j < \dfrac{d+2\gamma}{p_j}$, $s_1+s_2 > (d+2\gamma)\left(\dfrac{1}{p_1} + \dfrac{1}{p_2} - \dfrac{1}{p}\right)$, $p \geq \max(p_1, p_2)$, *and* $\dfrac{1}{r_1}+\dfrac{1}{r_2} = 1$, *we have*

$$\|uv\|_{\dot{\mathcal{B}}^{s_{1,2},k}_{p,\infty}(\mathbb{R}^d)} \leq C\|u\|_{\dot{\mathcal{B}}^{s_1,k}_{p_1,r_1}(\mathbb{R}^d)}\|v\|_{\dot{\mathcal{B}}^{s_2,k}_{p_2,r_2}(\mathbb{R}^d)}.$$

Proof. The prove of these results used the same method as in [20]. □

Remark 2. In the classical case, a similar result can be found in [5, 8, 9], where the authors used another methods that we can not adapt at the moment.

5.2. The slowly hypoellipticity. In this subsection we treat differential difference equations, given by replacing the Laplacian Δ in a differential equation with the Dunkl-Laplacian Δ_k, and consider some basic properties of the solutions in homogeneous Dunkl-Besov spaces. Though the process is a standard way, we sketch their proofs to understand the essential parts.

We consider the linear equation

$$-\triangle_k u + \sum_{1\leq i,j\leq d} c_{i,j} T_i u T_j u + cu = 0 \tag{17}$$

with $c_{i,j} \in \mathbb{R}$ and $c \geq 0$.

Theorem 8. *We assume that* $W = \mathbb{Z}_2^d$. *If* u *is a solution of* (17) *such that* u *belongs to* $\dot{\mathcal{B}}^{2,k}_{1,2}(\mathbb{R}^d) \cap \dot{W}^{1,\infty}_k(\mathbb{R}^d)$, *where* $\dot{W}^{1,\infty}_k(\mathbb{R}^d) := \Big\{f \in D'(\mathbb{R}^d) :\ T_j f \in L^\infty_k(\mathbb{R}^d),\ j = 1, \dots, d\Big\}$, *then* u *belongs to* $\dot{\mathcal{B}}^{n,k}_{1,2}(\mathbb{R}^d) \cap L^\infty_k(\mathbb{R}^d)$ *for all* $n \in \mathbb{N}$ *and in particular,* $u \in \mathcal{E}(\mathbb{R}^d)$.

Proof. If u in $\dot{\mathcal{B}}^{2,k}_{1,2}(\mathbb{R}^d)$, then each $T_i u \in \dot{\mathcal{B}}^{1,k}_{1,2}(\mathbb{R}^d)$. Therefore, it follows from Corollary 2, (1) that $c_{i,j} T_i u T_j u \in \dot{\mathcal{B}}^{1,k}_{1,2}(\mathbb{R}^d) \cap L^\infty_k(\mathbb{R}^d)$. Hence, we can deduce that

$$-\triangle_k u + cu \in \dot{\mathcal{B}}^{1,k}_{1,2}(\mathbb{R}^d).$$

Since the operator $-\triangle_k + cI$ is isomorphism from $\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)$ in $\dot{\mathcal{B}}^{s-2,k}_{p,q}(\mathbb{R}^d)$ for all $s \in \mathbb{R}$ and $(p, q \in [1,\infty]^2)$, it follows that $u \in \dot{\mathcal{B}}^{3,k}_{1,2}(\mathbb{R}^d)$. By iteration we deduce that $u \in \dot{\mathcal{B}}^{n+2,k}_{1,2}(\mathbb{R}^d)$ for all $n \in \mathbb{N}$. Then it follows from the Proposition 10 that $u \in \dot{\mathcal{B}}^{n-\frac{d+2\gamma}{2}+2,k}_{2,2}(\mathbb{R}^d)$. On the other hand, the Sobolev imbedding theorem (see [20], Theorem 4.3) yields that

$$\dot{\mathcal{H}}^s_{2,k}(\mathbb{R}^d) = \dot{\mathcal{B}}^{s,k}_{2,2}(\mathbb{R}^d) \hookrightarrow C^{s-\gamma-\frac{d}{2}}(\mathbb{R}^d) \quad \text{if } s > \gamma + \frac{d}{2}.$$

There by, the desired result follows. $\square$

5.3. Dunkl-Schrodinger equation.

Notations. We denote by:

p' conjugate of $p \in [1, \infty]$ given by $\dfrac{1}{p} + \dfrac{1}{p'} = 1$.

$\mathcal{I}_k(t)$ the group of isometries on $L^2_k(\mathbb{R}^d)$ generated by the skew-adjoint operator $i\triangle_k$ i.e. $\mathcal{I}_k(t) = e^{it\triangle_k}$.

For any interval I of $\mathbb{R}$ (bounded or unbounded) and a Banach space X, we define the mixed space-time $L^q(I, X)$ Banach space of (classes of) measurable functions $u : I \to X$ such that $\|u\|_{L^q(I,X)} < \infty$, with

$$\begin{aligned}
\|u\|_{L^q(I,X)} &= \Big(\int_I \|u(t,.)\|^q_X dt\Big)^{\frac{1}{q}}, \quad \text{if } 1 \le q < \infty, \\
\|u\|_{L^\infty(I,X)} &= \operatorname*{ess\,sup}_{t\in I} \|u(t,.)\|_X.
\end{aligned}$$

Similarly, we shall write $C(\overline{I}, X)$, for $1 \le r \le \infty$ the space of functions from $\overline{I}$ into X such that the map

$$t \mapsto \|u(t,.)\|_X$$

is continuous.

Proposition 15 ([22]). *If $p \in [2,\infty]$ and $t \neq 0$, then $\mathcal{I}_k(t)$ maps $L^{p'}_k(\mathbb{R}^d)$ continuously to $L^p_k(\mathbb{R}^d)$ and*

$$\|\mathcal{I}_k(t)g\|_{L^p_k(\mathbb{R}^d)} \le \frac{1}{(c_k^2|t|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{p})}} \|g\|_{L^{p'}_k(\mathbb{R}^d)}. \tag{18}$$

Corollary 3. *If* $t \neq 0$*, then*

$$\|\mathcal{I}_k(t)g\|_{\dot{\mathcal{H}}^s_{p,k}(\mathbb{R}^d)} \leq \frac{1}{(c_k^2|t|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{p})}}\|g\|_{\dot{\mathcal{H}}^s_{p,k}(\mathbb{R}^d)} \quad \textit{for all} \quad g \in \dot{\mathcal{H}}^s_{p,k}(\mathbb{R}^d),$$

where $s \in \mathbb{R}$ *and* $p \in [2, \infty]$*. Moreover*

$$\|\mathcal{I}_k(t)g\|_{\dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d)} \leq \frac{1}{(c_k^2|t|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{p})}}\|g\|_{\dot{\mathcal{B}}^{s,k}_{p',q}(\mathbb{R}^d)} \quad \textit{for all} \quad g \in \dot{\mathcal{B}}^{s,k}_{p,q}(\mathbb{R}^d),$$

where $s \in \mathbb{R}$ *and* $p \in [2, \infty]$*.*

P r o o f. Fix $t \neq 0$ and let $u(t,.) = \mathcal{I}_k(t)g$. Given $v \in \mathcal{S}(\mathbb{R}^d)$ it is easy to see that

$$\mathcal{F}_D^{-1}(v\mathcal{F}_D(u(t,.))) = \mathcal{I}_k(t)(\mathcal{F}_D^{-1}(v\mathcal{F}_D(g)). \tag{19}$$

In particular, it follows from (18) that

$$\|\mathcal{F}_D^{-1}(v\mathcal{F}_D(u(t,.)))\|_{L^p_k(\mathbb{R}^d)} \leq \frac{1}{(c_k^2|t|^{2\gamma+d})^{(\frac{1}{2}-\frac{1}{p})}}\|\mathcal{F}_D^{-1}(v\mathcal{F}_D(g)\|_{L^{p'}_k(\mathbb{R}^d)},$$

$$2 \leq p \leq \infty.$$

The result follows immediately from the above estimate and the definitions of the homogeneous Dunkl-Sobolev and Dunkl-Besov norms. □

Definition 11. *We say that the exponent pair* (q, r) *is* $\frac{d+2\gamma}{2}$*-admissible if* $q, r \geq 2$, $\left(q, r, \frac{d+2\gamma}{2}\right) \neq (2, \infty, 1)$ *and*

$$\frac{1}{q} + \frac{d+2\gamma}{2r} \leq \frac{d+2\gamma}{4}. \tag{20}$$

If equality holds in (20) *we say that* (q, r) *is sharp* $\frac{d+2\gamma}{2}$*-admissible, otherwise we say that* (q, r) *is nonsharp* $\frac{d+2\gamma}{2}$*-admissible. Note in particular that when* $d + 2\gamma > 2$ *the endpoint*

$$P = \left(2, \frac{2d+4\gamma}{d+2\gamma-2}\right)$$

is sharp $\frac{d+2\gamma}{2}$*-admissible.*

In the follows, we recall the result proved in [22].

Theorem 9 (Strichartz-type Schrödinger estimate)**.**
Suppose that $d \geq 1$ *and* (q,r) *and* (q_1,r_1) *are* $\dfrac{d+2\gamma}{2}$*-admissible pairs. If* u *is a solution to the problem*

$$\left\{\begin{array}{lcl} \partial_t u(t,x) - i\triangle_k u(t,x) & = & f(t,x),\ (t,x)\in I\times\mathbb{R}^d \\ u_{|t=0} & = & g \end{array}\right. \tag{21}$$

for some data, g*,* f *and an interval* I *of* $\mathbb{R}$ *(bounded or not), then*

$$\|u\|_{L^q(I,L^r_k(\mathbb{R}^d))} + \|u\|_{C(\overline{I},L^2_k(\mathbb{R}^d))} \leq C\left(\|g\|_{L^2_k(\mathbb{R}^d)} + \|f\|_{L^{q_1'}(I,L^{r_1'}_k(\mathbb{R}^d))}\right). \tag{22}$$

In practice, we use the integral formulation of (21)

$$u(t,x) = \mathcal{I}_k(t)g(x) + \int_0^t \mathcal{I}_k(t-s)f(s,x)ds, \tag{23}$$

which is essentially equivalent. In the follows, we note by Φ_k the operator defined by

$$\Phi_k(f)(t,x) := \int_0^t \mathcal{I}_k(t-s)f(s,x)ds. \tag{24}$$

The estimate of Theorem can be generalized to various spaces involving Dunkl operators and Dunkl transform.

Corollary 4. *Let* I *be an interval of* $\mathbb{R}$ *(bounded or not).*

(1) *If* (q,r) *and* (q_1,r_1) *are* $\frac{d+2\gamma}{2}$*-admissible pairs, then there exits a constant* C *such that*

$$\begin{array}{rcl} \|\mathcal{I}_k(.)g\|_{L^q(\mathbb{R},\dot{\mathcal{H}}^s_{r,k}(\mathbb{R}^d))} & \leq & C\|g\|_{\dot{\mathcal{H}}^s_{r,k}(\mathbb{R}^d)}, \\ \|\Phi_k(f)\|_{L^q(I,\dot{\mathcal{H}}^s_{r,k}(\mathbb{R}^d))} & \leq & C\|f\|_{L^{q_1'}(I,\dot{\mathcal{H}}^s_{r_1',k}(\mathbb{R}^d))}. \end{array}$$

(2) *If* (q,r) *and* (q_1,r_1) *are* $\dfrac{d+2\gamma}{2}$*-admissible pairs, then there exits a constant* C *independent of* I *such that*

$$\begin{array}{rcl} \|\mathcal{I}_k(\cdot)g\|_{L^q(\mathbb{R},\dot{\mathcal{B}}^{s,k}_{r,2}(\mathbb{R}^d))} & \leq & C\|g\|_{L^{q_1'}(I,\dot{\mathcal{B}}^{s,k}_{r_1',2}(\mathbb{R}^d))}, \\ \|\Phi_k(f)\|_{L^q(I,\dot{\mathcal{B}}^{s,k}_{r,2}(\mathbb{R}^d))} & \leq & C\|f\|_{L^{q_1'}(I,\dot{\mathcal{B}}^{s,k}_{r_1',2}(\mathbb{R}^d))}. \end{array}$$

P r o o f. (1) Using

$$\mathcal{I}_k(t)\Big(\mathcal{F}_D^{-1}(\|\xi\|^s)\mathcal{F}_D(g))\Big) = \mathcal{F}_D^{-1}\Big(\|\xi\|^s)\mathcal{F}_D(\mathcal{I}_k(t)g)\Big)$$

we deduce the result.

(2) Using the homogenous Littlewood-Paley decomposition associated with Dunkl operators, it is easy to establish similar estimates in homogeneous Dunkl-Besov spaces. □

5.4. Generalized heat equation. The generalized Dunkl heat equation reads

$$(25)\qquad \begin{cases} \partial_t u(t,x) - \triangle_k u(t,x) &= f(t,x),\ (t,x) \in [0,\infty) \times \mathbb{R}^d \\ u_{|t=0} &= g. \end{cases}$$

Rösler in [24] introduced the generalized heat semi-group $H_k(t)$ for the Dunkl-Laplace operator

$$H_k(t)f(x) := \begin{cases} \displaystyle\int_{\mathbb{R}^d} \Gamma_k(t,x,y)f(y)\omega_k(y)dy & \text{if} \quad t>0 \\ f(x) & \text{if} \quad t=0, \end{cases}$$

where Γ_k is the generalized heat kernel defined by

$$\Gamma_k(t,x,y) := \frac{c_k}{(4t)^{\gamma+\frac{d}{2}}} e^{-\frac{\|x\|^2+\|y\|^2}{4t}} K\left(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}\right); \quad x,y \in \mathbb{R}^d,\ t>0.$$

In practice, we use the integral formulation of (25)

$$(26)\qquad u(t,x) = H_k(t)g(x) + \int_0^t H_k(t-s)f(s,x)ds.$$

Theorem 10. *Let s be a positive real number and $(p,r) \in [1,\infty]^2$. A constant C exists which satisfies the following property. For $u \in \dot{\mathcal{B}}_{p,r}^{-2s,k}(\mathbb{R}^d)$, we have*

$$(27)\qquad C^{-1}\|u\|_{\dot{\mathcal{B}}_{p,r}^{-2s,k}(\mathbb{R}^d)} \le \Big\| \|t^s H_k(t)u\|_{L_k^p(\mathbb{R}^d)} \Big\|_{L^r(\mathbb{R}^+,\frac{dt}{t})} \le C\|u\|_{\dot{\mathcal{B}}_{p,r}^{-2s,k}(\mathbb{R}^d)}.$$

For proof this result we need the following lemma.

Lemma 3 ([23])**.** *There exist two positive constants κ and C depending only on φ such that for all $1 \leq p \leq \infty$, $\tau \geq 0$ and $j \in \mathbb{Z}$, we have*

$$\|\Delta_j(H_k(\tau)u)\|_{L^p_k(\mathbb{R}^d)} \leq Ce^{-\kappa 2^{2j}\tau}\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)}.$$

Proof of Theorem 10. Using Lemma 3, the fact that the operator Δ_j commutes with the operator $H_k(t)$ and the definition of the homogeneous Dunkl-Besov (semi) norm, we get

$$\|t^s H_k(t)u\|_{L^p_k(\mathbb{R}^d)} \leq C\|u\|_{\dot{\mathcal{B}}^{-2s,k}_{p,r}(\mathbb{R}^d)} \sum_{j\in\mathbb{Z}} t^s 2^{2js} e^{-\kappa t 2^{2j}} c_{r,j}$$

where $(c_{r,j})_{j\in\mathbb{Z}}$ denotes, as in all this proof, a generic element of the unit sphere of $l^r(\mathbb{Z})$. In the case when $r = \infty$, the required inequality comes immediately from the following easy result: for any positive s, we have

$$\sup_{t>0} \sum_{j\in\mathbb{Z}} t^s 2^{2js} e^{-\kappa t 2^{2j}} < \infty. \tag{28}$$

In the case $r < \infty$, using Hölder inequality and inequality (28) we obtain

$$\begin{aligned}
&\int_0^\infty t^{rs}\|H_k(t)u\|^r_{L^p_k(\mathbb{R}^d)}\frac{dt}{t} \\
&\leq C\|u\|^r_{\dot{\mathcal{B}}^{-2s,k}_{p,r}(\mathbb{R}^d)} \int_0^\infty \Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js} e^{-\kappa t 2^{2j}}\Big)^{r-1} \Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js} e^{-\kappa t 2^{2j}} c^r_{r,j}\Big)\frac{dt}{t} \\
&\leq C\|u\|^r_{\dot{\mathcal{B}}^{-2s,k}_{p,r}(\mathbb{R}^d)} \int_0^\infty \Big(\sum_{j\in\mathbb{Z}} t^s 2^{2js} e^{-\kappa t 2^{2j}} c^r_{r,j}\Big)\frac{dt}{t}.
\end{aligned}$$

This gives directly the result by Fubini's theorem.

In order to prove the other inequality, let us observe that for any s greater than -1, we have

$$\Delta_j u = \frac{1}{\Gamma(s+1)}\int_0^\infty t^s(-\triangle_k)^{s+1}H_k(t)\Delta_j u dt.$$

Then Lemma 3, Proposition 5 and the fact that the operator Δ_j commutes with the operator $H_k(t)$, leads to

$$\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)} \leq C\int_0^\infty t^s 2^{2j(s+1)} e^{-\kappa t 2^{2j}}\|H_k(t)\Delta_j u\|_{L^p_k(\mathbb{R}^d)}dt. \tag{29}$$

In the case $r = \infty$, we simple write

$$\begin{aligned}\|\Delta_j u\|_{L^p_k(\mathbb{R}^d)} &\leq C\Big(\sup_{t>0} t^s \|H_k(t)u\|_{L^p_k(\mathbb{R}^d)}\Big)\int_0^\infty 2^{2j(s+1)}e^{-\kappa t2^{2j}}\,dt \\ &\leq C2^{2js}\Big(\sup_{t>0} t^s \|H_k(t)u\|_{L^p_k(\mathbb{R}^d)}\Big).\end{aligned}$$

In the case $r < \infty$, Hölder's inequality with the weight $e^{-\kappa t2^{2j}}$ gives

$$\Big(\int_0^\infty t^s 2^{2j(s+1)}e^{-\kappa t2^{2j}}\|H_k(t)\Delta_j u\|_{L^p_k(\mathbb{R}^d)}dt\Big)^r$$

$$\leq C2^{-2j(r-1)}\int_0^\infty t^s 2^{2j(s+1)}e^{-\kappa t2^{2j}}\|H_k(t)\Delta_j u\|^r_{L^p_k(\mathbb{R}^d)}dt.$$

Thanks to (28) and Fubini's theorem, we infer from (29) that

$$\sum_{j\in\mathbb{Z}} 2^{-2jrs}\|\Delta_j u\|^r_{L^p_k(\mathbb{R}^d)} \leq C\int_0^\infty t^{rs}\|H_k(t)u\|^r_{L^p_k(\mathbb{R}^d)}\frac{dt}{t}.$$

The theorem is proved. □

5.5. Sobolev embedding Theorem. The main results of this subsection are in sprit of the classical case (cf. [2, 9, 10, 16, 19]).

Theorem 11. *Let $p \in [1,\infty]$ and let $s \in \mathbb{R}$ such that $0 < s < \dfrac{d+2\gamma}{r}$, then we have the continuous embedding*

$$\dot{\mathcal{B}}^{s,k}_{r,r}(\mathbb{R}^d) \hookrightarrow L^p_k(\mathbb{R}^d),$$

where $p = \dfrac{r(2\gamma+d)}{2\gamma+d-rs}$.

P r o o f. Let f be a Schwartz class, we have

$$|f(x)| \leq \sum_{j\in\mathbb{Z}} |\Delta_j f(x)|.$$

Fix $N \in \mathbb{Z}$. Then

$$\sum_{j\geq N} |\Delta_j f(x)| \leq \sum_{j\geq N} 2^{-sj}\sup_j(2^{sj}|\Delta_j f(x)|) \leq 2^{-sN}H(x),$$

where $H \in L^r_k(\mathbb{R}^d)$, as $f \in \dot{\mathcal{B}}^{s,k}_{r,r}(\mathbb{R}^d)$

$$\big\|\sup_j(2^{sj}|\Delta_j f(x)|)\big\|_{L^r_k(\mathbb{R}^d)} \leq \Big\|\sum_j (2^{sj}|\Delta_j f(x)|)^r\Big\|_{L^1_k(\mathbb{R}^d)}.$$

On the other hand,

$$\sum_{j<N}|\Delta_j f(x)| \leq \sum_{j<N} 2^{-(s-\frac{d+2\gamma}{r})N}\sup_j(2^{(s-\frac{d+2\gamma}{r})j}|\Delta_j f(x)|) \leq C2^{-(s-\frac{d+2\gamma}{r})N}G(x),$$

where $G \in L^\infty_k(\mathbb{R}^d)$, as $f \in \dot{\mathcal{B}}^{s-\frac{d+2\gamma}{r},k}_{\infty,\infty}(\mathbb{R}^d)$ by a Proposition 10. Finally,

$$|f(x)| \leq C\Big(2^{-(s-\frac{d+2\gamma}{r})N}G(x) + 2^{-sN}H(x)\Big).$$

We optimize on N, and

$$|f(x)| \leq CH^{\frac{r}{p}}(x)G^{1-\frac{r}{p}}(x),$$

so that

$$\|f\|^p_{L^p_k(\mathbb{R}^d)} \leq C\|H\|^r_{L^r_k(\mathbb{R}^d)}\|G\|^{p-r}_{L^\infty_k(\mathbb{R}^d)}. \qquad \square$$

We did slightly better the following

Theorem 12. *Let* $p \in [1,\infty]$ *and let* $s \in \mathbb{R}$ *such that* $0 < s < \dfrac{d+2\gamma}{r}$, *then we have*

$$\|f\|_{L^p_k(\mathbb{R}^d)} \leq C\|f\|^{1-\frac{r}{p}}_{\dot{\mathcal{B}}^{-(\frac{2\gamma+d}{r}-s),k}_{\infty,\infty}(\mathbb{R}^d)}\|f\|^{\frac{r}{p}}_{\dot{\mathcal{B}}^{s,k}_{r,r}(\mathbb{R}^d)}$$

where $p = \dfrac{r(2\gamma+d)}{2\gamma+d-rs}$.

Theorem 13. *Let* $1 < p < \infty$ *and* $0 < s < \dfrac{d+2\gamma}{p}$ *be given. There exists a positive constant* C *such that for all function* $f \in \dot{\mathcal{H}}^s_{p,k}(\mathbb{R}^d)$ *we have*

$$\|f\|_{L^q_k(\mathbb{R}^d)} \leq C\|f\|^{1-\theta}_{\dot{\mathcal{H}}^s_{p,k}(\mathbb{R}^d)}\|f\|^{\theta}_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{p},k}_{\infty,\infty}(\mathbb{R}^d)}, \tag{30}$$

where $\theta = \dfrac{sp}{d+2\gamma}$ *and* $q = \dfrac{p(d+2\gamma)}{d+2\gamma-ps}$.

Proof. By density we can suppose that f belongs to $\mathcal{S}(\mathbb{R}^d)$. It is easy to see that

$$f = \int_0^\infty H_k(t)\triangle_k f dt$$

and decompose the integral in two parts:

$$f = \int_0^A H_k(t)\triangle_k f dt + \int_A^\infty H_k(t)\triangle_k f dt,$$

where A is a constant to be fixed later.

On the other hand, by Theorem 10

$$\|H_k(t)\triangle_k f\|_{L^\infty_k(\mathbb{R}^d)} \leq \frac{C}{t^{1-\frac{1}{2}(s-\frac{d+2\gamma}{p})}} \|f\|_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{p},k}_{\infty,\infty}(\mathbb{R}^d)}.$$

Therefore after integrating we get

$$\int_A^\infty \|H_k(t)\triangle_k f\|_{L^\infty_k(\mathbb{R}^d)} dt \leq A^{\frac{1}{2}(s-\frac{d+2\gamma}{p})} \|f\|_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{p},k}_{\infty,\infty}(\mathbb{R}^d)}.$$

On the other hand, denoting $g = (-\triangle_k)^{\frac{s}{2}} f$, we have

$$H_k(t)\triangle_k f = \frac{1}{(-t)^{1-\frac{s}{2}}} H_k(t)(-t\triangle_k)^{1-\frac{s}{2}} g.$$

We proceed as in [20], we prove

$$|H_k(t)(-t\triangle_k)^{1-\frac{s}{2}} g(x)| \leq C(s) M_k(g)(x)$$

where $M_k(g)$ is a maximal function of g associated with the Dunkl operators (cf. [26]).

This leads to

$$\left| \int_0^A H_k(t)\triangle_k f(x) dt \right| \leq C A^{\frac{s}{2}} M_k(g)(x).$$

In conclusion, we get

$$\left| \int_0^\infty H_k(t)\triangle_k f(x) dt \right| \leq C\left(A^{\frac{s}{2}} M_k(g)(x) + A^{\frac{1}{2}(s-\frac{d+2\gamma}{p})} \|f\|_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{p},k}_{\infty,\infty}(\mathbb{R}^d)} \right),$$

and the choice of A such that

$$A^{\frac{d+2\gamma}{2p}} M_k(g)(x) = \|f\|_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{p},k}_{\infty,\infty}(\mathbb{R}^d)}$$

ensures that

$$\left|\int_0^\infty H_k(t)\triangle_k f(x)dt\right| \le C\Big(M_k(g)(x)\Big)^{1-\frac{ps}{d+2\gamma}} \|f\|^{\frac{ps}{d+2\gamma}}_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{p},k}_{\infty,\infty}(\mathbb{R}^d)}.$$

Finally taking the L^q_k norm with $q = \dfrac{p(d+2\gamma)}{d+2\gamma-ps}$, end the proof thanks to Theorem 6.1 of [26] i.e. (the maximal function M_k is bounded of $L^q_k(\mathbb{R}^d)$ into itself for $q > 1$). $\square$

Theorem 14 (Precised Sobolev inequality). *Let $1 \le p < q < \infty$. For all function f such that $\nabla_k f \in L^p_k(\mathbb{R}^d)$ and such that $f \in \dot{\mathcal{B}}^{-\beta,k}_{\infty,\infty}(\mathbb{R}^d)$ we have*

$$\|f\|_{L^q_k(\mathbb{R}^d)} \le C\|\nabla_k f\|^{\theta}_{L^p_k(\mathbb{R}^d)}\|f\|^{1-\theta}_{\dot{\mathcal{B}}^{-\beta,k}_{\infty,\infty}(\mathbb{R}^d)}, \tag{31}$$

where $\nabla_k f := (T_1 f, \dots, T_d f)$, $\theta = \dfrac{p}{q}$ and $\beta = \dfrac{\theta}{1-\theta}$.

Proof. Firstly we recall the following result

Lemma 4. *Let $(a_j)_{j\in\mathbb{Z}}$ a sequence and let $s = \theta s_1 + (1-\theta)s_2$ with $0 < \theta < 1$ and $s \ne s_1$. Then for all $r, r_1, r_2 \in [1,\infty]$ we have*

$$\|2^{js}a_j\|_{l^r} \le C\|2^{js_1}a_j\|^{\theta}_{l^{r_1}}\|2^{js_2}a_j\|^{1-\theta}_{l^{r_2}}. \tag{32}$$

We apply now this Lemma for the dyadic blocs $\Delta_j f$ with $s = 0$, $s_1 = 1$, $s_2 = -\beta$ and $r = r_1 = 2$ and $r_2 = \infty$. We obtain

$$\Big(\sum_{j\in\mathbb{Z}}|\Delta_j f(x)|^2\Big)^{\frac{1}{2}} \le C\Big(\sum_{j\in\mathbb{Z}}2^{2j}|\Delta_j f(x)|^2\Big)^{\frac{\theta}{2}}\big(\sup_{j\in\mathbb{Z}}2^{-\beta j}|\Delta_j f(x)|\big)^{1-\theta}. \tag{33}$$

Thus by Holder's inequality we deduce that

$$\begin{aligned}&\Big\|\Big(\sum_{j\in\mathbb{Z}}|\Delta_j f|^2\Big)^{\frac{1}{2}}\Big\|_{L^q_k(\mathbb{R}^d)}\\ &\qquad\le C\Big\|\Big(\sum_{j\in\mathbb{Z}}2^{2j}|\Delta_j f|^2\Big)^{\frac{1}{2}}\Big\|^{\theta}_{L^p_k(\mathbb{R}^d)}\big(\sup_{j\in\mathbb{Z}}2^{-\beta j}\|\Delta_j f\|_{L^\infty_k(\mathbb{R}^d)}\big)^{1-\theta}.\end{aligned}$$

Finally by using the characterization theorem of Lebesgue spaces via the Littlewood-Paley decomposition associated to the Dunkl operators we obtain the result. $\square$

Theorem 15. *Let $1 < p < q < \infty$. For all function f such that $f \in \dot{\mathcal{H}}^{s_1}_{p,k}(\mathbb{R}^d) \bigcap \dot{\mathcal{B}}^{-\beta,k}_{\infty,\infty}(\mathbb{R}^d)$ we have*

$$\|f\|_{\dot{\mathcal{H}}^{s}_{p,k}(\mathbb{R}^d)} \leq C \|f\|^{\theta}_{\dot{\mathcal{H}}^{s_1}_{p,k}(\mathbb{R}^d)} \|f\|^{1-\theta}_{\dot{\mathcal{B}}^{-\beta,k}_{\infty,\infty}(\mathbb{R}^d)}, \tag{34}$$

where $\theta = \dfrac{p}{q}$, $s = \theta s_1 - (1-\theta)\beta$ with $\beta > 0$, $-\beta < s < s_1$.

P r o o f. It suffices to prove that

$$\|(-\triangle_k)^{\frac{s-s_1}{2}} f\|_{L^q_k(\mathbb{R}^d)} \leq C \|f\|^{\theta}_{L^p_k(\mathbb{R}^d)} \|f\|^{1-\theta}_{\dot{\mathcal{B}}^{-\beta-s_1,k}_{\infty,\infty}(\mathbb{R}^d)}. \tag{35}$$

Indeed we use the following identity

$$(-\triangle_k)^{-\frac{\delta}{2}} f(x) = \frac{1}{\Gamma(\frac{\delta}{2})} \int_0^\infty t^{\frac{\delta}{2}-1} H_k(t) f(x) dt, \tag{36}$$

with $\delta = s_1 - s > 0$.

Let T be a parameter when will be choose later

$$(-\triangle_k)^{-\frac{\delta}{2}} f(x) = \frac{1}{\Gamma(\frac{\delta}{2})} \int_0^T t^{\frac{\delta}{2}-1} H_k(t) f(x) dt + \frac{1}{\Gamma(\frac{\delta}{2})} \int_T^\infty t^{\frac{\delta}{2}-1} H_k(t) f(x) dt. \tag{37}$$

We proceed as in [20] we obtain

$$|H_k(t) f(x)| \leq C M_k(f)(x).$$

On the other hand we use Theorem 10 and the fact that f belongs to $\dot{\mathcal{B}}^{-\beta-s_1,k}_{\infty,\infty}(\mathbb{R}^d)$ we deduce that

$$|H_k(t) f(x)| \leq C t^{\frac{-\beta-s_1}{2}} \|f\|_{\dot{\mathcal{B}}^{-\beta-s_1,k}_{\infty,\infty}(\mathbb{R}^d)}.$$

Thus by applying the preceding estimates on the right part of (37) we obtain

$$|(-\triangle_k)^{-\frac{\delta}{2}} f(x)| \leq \frac{C_1}{\Gamma(\frac{\delta}{2})} T^{\frac{\delta}{2}} M_k(f)(x) + \frac{C_2}{\Gamma(\frac{\delta}{2})} T^{\frac{\delta-\beta-s_1}{2}} \|f\|_{\dot{\mathcal{B}}^{-\beta-s_1,k}_{\infty,\infty}(\mathbb{R}^d)}. \tag{38}$$

We fix now

$$T = \left(\frac{\|f\|_{\dot{\mathcal{B}}^{-\beta-s_1,k}_{\infty,\infty}(\mathbb{R}^d)}}{M_k(f)(x)} \right)^{\frac{2}{\beta+s_1}},$$

we obtain

$$|(-\triangle_k)^{-\frac{\delta}{2}} f(x)| \leq \frac{C_1+C_2}{\Gamma(\frac{\delta}{2})}\Big(M_k(f)(x)\Big)^{\theta}\|f\|^{1-\theta}_{\dot{\mathcal{B}}^{-\beta-s_1,k}_{\infty,\infty}(\mathbb{R}^d)}.$$

Thus we deduce that

$$\|(-\triangle_k)^{-\frac{\delta}{2}} f\|_{L^q_k(\mathbb{R}^d)} \leq \frac{C_1+C_2}{\Gamma(\frac{\delta}{2})}\|M_k(f)\|^{\theta}_{L^p_k(\mathbb{R}^d)}\|f\|^{1-\theta}_{\dot{\mathcal{B}}^{-\beta-s_1,k}_{\infty,\infty}(\mathbb{R}^d)}.$$

For conclure we used the Theorem 6.1 of [26]. □

Corollary 5 (Precised Gagliardo-Nirenberg inequality). *Let* $1 \leq p < q$ *such that* $p\left(1+\dfrac{1}{d+2\gamma}\right) \leq q$. *Then, if* $\nabla_k f \in L^p_k(\mathbb{R}^d)$ *and* $f \in L^r_k(\mathbb{R}^d)$ *we have*

$$\|f\|_{L^q_k(\mathbb{R}^d)} \leq C\|f\|^{1-\frac{p}{q}}_{L^r_k(\mathbb{R}^d)}\|\nabla_k f\|^{\frac{p}{q}}_{L^p_k(\mathbb{R}^d)}, \tag{39}$$

where $r = \left(\dfrac{q}{p}-1\right)(d+2\gamma)$.

Proof. We apply Theorem 15 with $s = 0$ and $s_1 = 1$, then $\beta = \dfrac{p}{q-p} = \dfrac{d+2\gamma}{r}$ and we apply the Propositions 11 and 10, we have the following inclusions

$$L^r_k(\mathbb{R}^d) \subset \dot{\mathcal{B}}^{0,k}_{r,\infty}(\mathbb{R}^d) \subset \dot{\mathcal{B}}^{-\beta,k}_{\infty,\infty}(\mathbb{R}^d).$$

Hence we infer

$$\|f\|_{\dot{\mathcal{B}}^{-\beta,k}_{\infty,\infty}(\mathbb{R}^d)} \leq C\|f\|_{\dot{\mathcal{B}}^{0,k}_{r,\infty}(\mathbb{R}^d)} \leq C'\|f\|_{L^r_k(\mathbb{R}^d)}.$$

Thus the result is immediately. □

Theorem 16. *Let* $1 < q, r \leq \infty$. *For all function* f *belongs to* $\dot{\mathcal{H}}^{s_1}_{r,k}(\mathbb{R}^d)\bigcap L^q_k(\mathbb{R}^d)$ *we have*

$$\|f\|_{\dot{\mathcal{H}}^{s}_{p,k}(\mathbb{R}^d)} \leq C\|f\|^{1-\theta}_{\dot{\mathcal{H}}^{s_1}_{p,k}(\mathbb{R}^d)}\|f\|^{\theta}_{L^q_k(\mathbb{R}^d)}, \tag{40}$$

where $\dfrac{1}{p} = \dfrac{\theta}{q} + \dfrac{1-\theta}{r}$ *and* $\theta = 1 - \dfrac{s}{s_1}$.

P r o o f. We decompose f as in the follow

$$f = S_j f + (Id - S_j)f.$$

We proceed as in [20] p.21 we prove that

$$|S_j(-\triangle_k)^{\frac{s}{2}} f(x)| \leq C 2^{js} M_k(f)(x)$$

and

$$|(Id - S_j)(-\triangle_k)^{\frac{s}{2}} f(x)| \leq C 2^{-j(s_1-s)} M_k((-\triangle_k)^{\frac{s_1}{2}} f)(x).$$

Thus

$$|(-\triangle_k)^{\frac{s}{2}} f(x)| \leq C 2^{js} M_k(f)(x) + C 2^{-j(s_1-s)} M_k((-\triangle_k)^{\frac{s_1}{2}} f)(x).$$

Choosing j such that

$$2^j \asymp \left(\frac{M_k((-\triangle_k)^{\frac{s_1}{2}} f)(x)}{M_k(f)(x)} \right)$$

we infer that

$$|(-\triangle_k)^{\frac{s}{2}} f(x)| \leq C [M_k(f)(x)]^{1-\frac{s}{s_1}} [M_k((-\triangle_k)^{\frac{s_1}{2}} f)(x)]^{\frac{s}{s_1}}.$$

Applying Hölder inequality, we obtain

$$\|f\|_{\dot{\mathcal{H}}^s_{p,k}(\mathbb{R}^d)} \leq C \|M_k(f)\|^{1-\frac{s}{s_1}}_{L^q_k(\mathbb{R}^d)} \|M_k((-\triangle_k)^{\frac{s_1}{2}} f\|^{\frac{s}{s_1}}_{L^q_k(\mathbb{R}^d)}.$$

For conclure we used the Theorem 6.1 of [26]. □

For any measurable function f on $\mathbb{R}^d$, we define its distribution and rearrangement functions

$$d_{f,k}(\lambda) := m_k(\Big\{|f| \geq \lambda\Big\}), \quad f^*_k(s) := \inf \Big\{\lambda : \quad d_{f,k}(\lambda) \leq s\Big\}.$$

For $1 \leq p < \infty$ and $1 \leq q \leq \infty$, define

$$\|f\|_{L^{p,q}_k(\mathbb{R}^d)} = \begin{cases} \Big(\displaystyle\int_0^\infty (s^{\frac{1}{p}} f^*_k(s))^q \frac{ds}{s} \Big)^{\frac{1}{q}} & \text{if} \quad q < \infty \\ \displaystyle\sup_{s>0} s^{\frac{1}{p}} f^*_k(s) & \text{if} \quad q = \infty. \end{cases}$$

The generalized Lorentz spaces $L^{p,q}_k(\mathbb{R}^d)$ is defined as the set of all measurable functions f such that $\|f\|_{L^{p,q}_k(\mathbb{R}^d)} < \infty$.

It is easy to see that $L^{p,p}_k(\mathbb{R}^d) = L^p_k(\mathbb{R}^d)$ and that generalized Lorentz spaces can be derived from $L^p_k(\mathbb{R}^d)$ spaces by the real interpolation method. In particular, when $1 < p < \infty$ we have $L^{p,q}_k(\mathbb{R}^d) = [L^1_k(\mathbb{R}^d), L^\infty_k(\mathbb{R}^d)]_{\theta,q}$, with $\dfrac{1}{p} = 1 - \theta$.

Theorem 17. *Assume that $W = \mathbb{Z}_2^d$. Let $q \in [1,\infty]$ and let $s \in \mathbb{R}$ such that $0 < s < \dfrac{2\gamma + d}{q}$, then we have*

$$\|f\|_{L^{p,q}_k(\mathbb{R}^d)} \leq C\|f\|^{1-\frac{q}{p}}_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{q},k}_{\infty,q}(\mathbb{R}^d)} \|f\|^{\frac{q}{p}}_{\dot{\mathcal{B}}^{s,k}_{q,q}(\mathbb{R}^d)}, \tag{41}$$

where $p = \dfrac{q(2\gamma + d)}{2\gamma + d - qs}$.

Proof. Let f be in $\mathcal{S}(\mathbb{R}^d)$, we have

$$\|f\|^q_{L^{p,q}_k(\mathbb{R}^d)} = p\int_0^\infty \lambda^q \Big(d_{f,k}(\lambda)\Big)^{\frac{q}{p}} \frac{d\lambda}{\lambda}.$$

For $A > 0$, we put $f = f_{1,A} + f_{2,A}$ with $f_{1,A} = A^{d+2\gamma}\psi(A.) *_D f$, and $f_{2,A} = A^{d+2\gamma}\phi(A.) *_D f$. We proceed as [27, 28], we prove

$$\int_0^\infty A^{sq-d-2\gamma-1}\|f_{1,A}\|^q_{L^\infty_k(\mathbb{R}^d)} dA \leq C\|f\|^q_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{q},k}_{\infty,q}(\mathbb{R}^d)} \tag{42}$$

and

$$\int_0^\infty A^{sq-1}\|f_{2,A}\|^q_{L^\infty_k(\mathbb{R}^d)} dA \leq C\|f\|^q_{\dot{\mathcal{B}}^{s,k}_{q,q}(\mathbb{R}^d)}. \tag{43}$$

For all $\lambda > 0$, we have

$$\Big\{|f| \geq \lambda\Big\} \subset \Big\{|f_{1,A}| \geq \frac{\lambda}{2}\Big\} \bigcup \Big\{|f_{2,A}| \geq \frac{\lambda}{2}\Big\}.$$

We take now $\lambda = \lambda(A)$ such that

$$\|f_{1,A}\|_{L^\infty_k(\mathbb{R}^d)} = \frac{\lambda}{4}.$$

Then we deduce from the choice of λ, that then

$$d_{f,k}(\lambda) \leq d_{f_{2,A_\lambda},k}\left(\frac{\lambda}{2}\right).$$

By Bienaymene-Tchebytchev inequality, we have

$$d_{f_{2,A_\lambda},k}\left(\frac{\lambda}{2}\right) \leq 2^q \lambda^{-q} \|f_{2,A_\lambda}\|^q_{L^q_k(\mathbb{R}^d)}.$$

Moreover

$$\begin{aligned}\|f\|^q_{L^{p,q}_k(\mathbb{R}^d)} &= p\int_0^\infty \lambda^q \big(d_{f,k}(\lambda)\big)^{\frac{q}{p}} \frac{d\lambda}{\lambda} \\ &\leq p\int_0^\infty \lambda(A)^{q-1}\lambda'(A)\left(d_{f_{2,A_\lambda},k}\left(\frac{\lambda}{2}\right)\right)^{\frac{q}{p}} dA.\end{aligned}$$

From definitions of λ and f_{2,A_λ}, we deduce

$$\|f\|^q_{L^{p,q}_k(\mathbb{R}^d)} \leq C\bigg[\int_0^\infty A^{(d+2\gamma)q}\|\psi(A.)*_D f\|^q_{L^\infty_k(\mathbb{R}^d)}\left(d_{f_{2,A_\lambda},k}\left(\frac{\lambda}{2}\right)\right)^{\frac{q}{p}}\frac{dA}{A}$$

$$+\int_0^\infty A^{(d+2\gamma)(q-1)}\|\psi(A.)*_D f\|^{q-1}_{L^\infty_k(\mathbb{R}^d)}\|\Theta(A.)*_D f\|_{L^\infty_k(\mathbb{R}^d)}\left(d_{f_{2,A_\lambda},k}\left(\frac{\lambda}{2}\right)\right)^{\frac{q}{p}} dA\bigg]$$

$$= I_1 + I_2,$$

where

$$\Theta(Ax) = \langle \nabla\psi(Ax), x\rangle.$$

Applying Hölder inequality, we obtain

$$\begin{aligned}I_1 &\leq C\bigg(\int_0^\infty A^{qs-(d+2\gamma)}\|f_{1,A}\|^q_{L^\infty_k(\mathbb{R}^d)}\frac{dA}{A}\bigg)^{1-\frac{q}{p}}\bigg(\int_0^\infty A^{qs}\|f_{2,A}\|^q_{L^q_k(\mathbb{R}^d)}\frac{dA}{A}\bigg)^{\frac{q}{p}} \\ &\leq C\left(\|f\|^{\frac{q}{p}}_{\dot{\mathcal{B}}^{s,k}_{q,q}(\mathbb{R}^d)}\|f\|^{1-\frac{q}{p}}_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{q},k}_{\infty,q}(\mathbb{R}^d)}\right)^q.\end{aligned}$$

Proceeding in an exactly similar manner for I_1, we obtain

$$I_2 \leq C\bigg(\int_0^\infty A^{(d+2\gamma)(q-1)}\|\psi(A.)*_D f\|^{q-1}_{L^\infty_k(\mathbb{R}^d)}\|\Theta(A.)*_D f\|_{L^\infty_k(\mathbb{R}^d)}\frac{dA}{A}\bigg)^{\frac{q}{p}}$$

$$\times\bigg(\int_0^\infty A^{qs}\|f_{2,A}\|^q_{L^q_k(\mathbb{R}^d)}\frac{dA}{A}\bigg)^{\frac{q}{p}}.$$

By a simple calculations it is easy to obtain

$$I_2 \leq C\left(\|f\|^{\frac{q}{p}}_{\dot{\mathcal{B}}^{s,k}_{q,q}(\mathbb{R}^d)}\|f\|^{1-\frac{q}{p}}_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{q},k}_{\infty,q}(\mathbb{R}^d)}\right)^q.$$

Combining our estimates for I_1 and I_2 we have proved that

$$\|f\|^q_{L^{p,q}_k(\mathbb{R}^d)} \leq C\left(\|f\|^{\frac{q}{p}}_{\dot{\mathcal{B}}^{s,k}_{q,q}(\mathbb{R}^d)}\|f\|^{1-\frac{q}{p}}_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{q},k}_{\infty,q}(\mathbb{R}^d)}\right)^q,$$

which is the desired result. □

Corollary 6. *Assume that $W = \mathbb{Z}_2^d$. Let s be a real number in the interval $\left(0, \dfrac{d+2\gamma}{q}\right)$ and let q be a real number in $[1,\infty]$ There is a constant C such that, for any function $f \in \dot{\mathcal{B}}^{s,k}_{q,q}(\mathbb{R}^d)$, the following inequality holds:*

$$\left(\int_{\mathbb{R}^d}\frac{|f(x)|^q}{\|x\|^{sq}}\omega_k(x)dx\right)^{\frac{1}{q}} \leq C\|f\|^{\theta}_{\dot{\mathcal{B}}^{s,k}_{q,q}(\mathbb{R}^d)}\|f\|^{1-\theta}_{\dot{\mathcal{B}}^{s-\frac{d+2\gamma}{q},k}_{\infty,q}(\mathbb{R}^d)}, \tag{44}$$

where $\theta = 1 - \dfrac{qs}{d+2\gamma}$.

For proof this result we need the following lemma which we prove as the Euclidean case.

Lemma 5. *Let $1 \leq p_1, p_2, q_1, q_2 \leq \infty$. If $f \in L^{p_1,q_1}_k(\mathbb{R}^d)$ and $g \in L^{p_2,q_2}_k(\mathbb{R}^d)$, then*

$$\|fg\|_{L^{p,q}_k(\mathbb{R}^d)} \leq C\|f\|_{L^{p_1,q_1}_k(\mathbb{R}^d)}\|g\|_{L^{p_2,q_2}_k(\mathbb{R}^d)}, \tag{45}$$

where $\dfrac{1}{p} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$ and $\dfrac{1}{q} = \dfrac{1}{q_1} + \dfrac{1}{q_2}$.

P r o o f o f C o r o l l a r y 6. Let as in the previous theorem $1 < p < \infty$ and $s \in \left(0, \dfrac{d+2\gamma}{q}\right)$ with $\dfrac{1}{p} = \dfrac{1}{q} - \dfrac{s}{d+2\gamma}$. We take $g(x) = \dfrac{1}{\|x\|^s}$ and apply (45), in the specific form

$$\|fg\|_{L^{q,q}_k(\mathbb{R}^d)} \leq C\|f\|_{L^{p,q}_k(\mathbb{R}^d)}\|g\|_{L^{r,\infty}_k(\mathbb{R}^d)}$$

where $r = \dfrac{d+2\gamma}{s}$ and $p = \dfrac{q(d+2\gamma)}{d+2\gamma-qs}$. As $g \in L_k^{r,\infty}(\mathbb{R}^d)$, we have

$$\left(\int_{\mathbb{R}^d} \frac{|f(x)|^q}{\|x\|^{sq}} \omega_k(x)dx\right)^{\frac{1}{q}} \leq C\|f\|_{L_k^{p,q}(\mathbb{R}^d)}.$$

Combining this with (41), we obtain (44). $\square$

In the follow we prove a special case of Corollary 6 without assume that $W = \mathbb{Z}_2^d$.

Theorem 18. *Let $\dfrac{d+2\gamma}{4} < s < \dfrac{d+2\gamma}{2}$ be given. There exists a positive constant C such that for all function $u \in \dot{\mathcal{H}}^s_{2,k}(\mathbb{R}^d)$ we have*

$$\int_{\mathbb{R}^d} \frac{|u(x)|^2}{\|x\|^{2s}} \omega_k(x)dx \leq C\|u\|^2_{\dot{\mathcal{H}}^s_{2,k}(\mathbb{R}^d)}. \tag{46}$$

For proof this theorem we need the following lemma, which we obtain by a simple calculations.

Lemma 6. *Let s be a real number in the interval $(0, \gamma + \frac{d}{2})$. Then the function $x \mapsto \|x\|^{-2s}$ belongs to the Dunkl-Besov space $\dot{\mathcal{B}}^{d+2\gamma-2s,k}_{1,\infty}(\mathbb{R}^d)$.*

P r o o f o f T h e o r e m 18. Let us define

$$I_{s,k}(u) := \int_{\mathbb{R}^d} \frac{|u(x)|^2}{\|x\|^{2s}} \omega_k(x)dx = \langle \|\cdot\|^{-2s}, u^2 \rangle.$$

Using homogeneous Littlewood-Paley decomposition and the fact that u^2 belongs to ${\mathcal{S}'}_{h,k}(\mathbb{R}^d)$, we can write

$$\begin{aligned} I_{s,k}(u) &= \sum_{|n-m|\leq 2} \langle \Delta_n(\|\cdot\|^{-2s}), \Delta_m(u^2) \rangle \\ &\leq C \sum_{|n-m|\leq 2} \langle 2^{n(\frac{d+2\gamma}{2}-2s)} \Delta_n(\|\cdot\|^{-2s}), 2^{-m(\frac{d+2\gamma}{2}-2s)} \Delta_m(u^2) \rangle. \end{aligned}$$

Lemma 6 claims that $\|\cdot\|^{-2s}$ belongs to $\dot{\mathcal{B}}^{\frac{d+2\gamma}{2}-2s,k}_{2,\infty}(\mathbb{R}^d)$. Corollary 2 yields

$$\|u^2\|_{\dot{\mathcal{B}}^{2s-\frac{d+2\gamma}{2},k}_{2,1}(\mathbb{R}^d)} \leq C\|u\|^2_{\dot{\mathcal{H}}^s_{2,k}(\mathbb{R}^d)}.$$

Thus
$$I_{s,k}(u) \leq C\|u\|^2_{\dot{\mathcal{H}}^s_{2,k}(\mathbb{R}^d)}. \qquad \square$$

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Department of Mathematics
College of sciences
King Faisal University
P.O. 380 Ahsaa 31982
Kingdom of Saudi Arabia
e-mail: `hmejjaoli@kfu.edu.sa`
`hatem.mejjaoli@ipest.rnu.tn`

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