AN INTEGRAL EQUATION FOR PULSATIONS OF THE ZERO VELOCITY OVALS IN THE ASTEROIDAL ELLIPTICAL PROBLEM

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Summary:

The well known theory of stability according Hill in the restricted asteroidal circular problem, becomes a rather complicated question as soon as we pass to the more general asteroidal elliptical problem, in which the orbital excentricity of the disturbing planet differs from zero Hitherto all theories resulted always in asymptotical semiconvergent developments of the classics.
However this procedure renders every criterion of stability doubtful as the explicit time appears beyond the trigonometrical functions. The author uses a well known theorem on the homogeneous functions and succeeds in integrating completely the integral equation pertaining thereto. In this way the whole problem is reduced to quadratures and a very simple survey of the strongly pulsating zero velocity ovals is obtained.
Complete havoc is played just with the most interesting ovals, marking the transgression from a satellite to a planetary case.

In a previous paper two problems concerning the asteroidal movements have been formulated.

I. The so called circular, restricted problem of three bodies, with an asteroid of zero mass and a disturbing planet, going round the central body (Sun) in a circle (Hill-Poincaré).

II. The more general asteroidal elliptical problem, in which the disturbing planet revolves about the central body in a Keplerian ellipse, of finite, constant excentricity.

I pointed out that unfortunately the passage from the first (I) to the more general case (II) is rather difficult.

After having detected the cause of these difficulties in the existence of certain curves of small divisors, so called curves of resonance [1], [2], [3], I tried to study the problem in question more thoroughly.

The first step was the construction of certain secular solutions [4], [5].

Summing up I proved, that the curves of resonance can always be isolated and avoided, but then the movements of the more general elliptical problem (II) differ very greatly from the starting orbits of the so called restricted circular asteroidal problem (I).

Now the question arises whether the aforesaid changes, and possibly even secular changes, caused by the disturbing planet revolving in an Keplerian ellipse, do not effect the stability and even secular stability according Hill of the movements in question.

This appears more than probable, and a study of this problem is the main subject of the present paper. Previously such investigations were carried out by means of the so called equipotential surfaces, and we owe to the genius of the American astronomer G. W. Hill † the discovery of surfaces of zero velocity for the circular restricted problem I. [6].

Starting with the Jacobian integral of energy, that luckily enough in the case of the simplified circular problem, is always extant, and remembering that the square velocity for all real motions must remain positive, one is unabled to divide the whole plane (space) into regions of real and imaginary movements. In this manner G. W. Hill finds out certain limiting ovals, within which all movement, once started, must remain confined for ever.

There is no possibility to escape beyond such an impenetrable oval curtain — unless the whole movement became imaginary.

As to the thorough study of these ovals I must refer to the original paper of Hill [6], or in a more elementary way to text books such as [7].

There are easily obtained planetary ovals — satellite ovals and ovals with both possibilities of transgression from a planet case to a satellite movement: then limiting ovals preventing the asteroid not to approach any nearer to the central body or to the disturbing planet, or else obliging the small body to remain for ever within a certain maximum, utmost distance from the central body or from the disturbing planet. So G. W. Hill succeeded in ascertaining that our Moon can never recede from the Earth beyond the quadruple amount of its present distance-radius [6].

Now in my paper quoted above [1], I predicted that, as a consequence of the curves of resonance and the strong disturbing effect of the orbital excentricity of the planet, — the Hill zero velocity ovals and surfaces must start oscillating or changing secularly from their originally fixed position.

But at that time I understood it to be proved by the asymptotically divergent series of the classical theory of perturbations. And indeed at that time no other means appeared to be at our disposal. Hitherto there was no knowledge of any integral analogous to that of Jacobi for the circular restricted problem, which could have been taken advantage of for the more general elliptical asteroidal problem.

The efforts of the classics always resulted in asymptotically divergent developments, and these can never grant a stability according Hill for ever, because of the existence of explicit time [8], [9], [10].

But luckily, by the same method I succeed in integrating completely the integral equation, which replaces the Hill equation of the zero velocity ovals in the more general case of asteroidal elliptical problem, and thus defines the pulsations of the curves in question.

And indeed the generalised integral equation of the zero velocity ovals can easily be reduced to mere quadratures.

The amplitude of the pulsations proves to be of the order twice the orbital excentricity of the disturbing planet. Such a big amount, which appears in no way to be lessened or factored by the small mass of the disturbing planet, means very much.

Just in the case of the passages of the most interesting ovals of Hill from those, closed entirely round the central body (planetary ovals) to others, which start suddenly with the change of the constant of energy enclosing both central body and the disturbing planet, complete havoc is played with the connection of the originally satellite and planetary ovals. In this way many passages of an asteroid from a planetary body to a satellite type unforeseen by Hill become possible.

But the most interesting fact is that two neighbouring curves never intersect one another and therefore there can never be any talk of an envelope of Hill oscillating ovals. Moreover we are enabled to ascertain the whole character of stability by an extremely simple survey.

The results of Callandreau [9] as well as of Wilkens [8] being found by means of classical semi-convergent methods, with the possibility of explicit time seem to remain in our case entirely out of question — at least with the system of coordinates chosen hereafter. Still even these are testing in favour of our view.

In general it appears proved that the pulsating zero velocity ovals do not undergo any secular or long periodical changes, the period of their pulsations always being exactly the same as the short-periodic time of revolution of the disturbing planet.

On the contrary it remains more than not excluded that the asteroidal movements are subject to strong peculiar, secular variations, these all being caused by the mere effect of the orbital excentricity of the disturbing planet.

Let us start with the well known equations of the asteroidal elliptical problem. The fundamental system of rectangular coordinates is supposed to be fixed (not rotating). The origin is assumed to be in the Sun. Then denoting the relative coordinates to the Sun by resp. x, y, z for the asteroid of mass zero, $m=0$, so that the distance between the asteroid and the Sun of mass M is r. X, Y, Z are relative rectangular coordinates of Jupiter, the disturbing planet.

Let us denote farther m_1 the mass of Jupiter, k^2 the Gaussian constant of attraction. Then taking Δ_1 for the distance of the asteroid from Jupiter, whose prescribed Keplerian orbit is a fixed ellipse, for the relative coordinates of Jupiter with respect to the Sun we can write

$$X=R\cos v_1,\quad Y=R\sin v_1,\quad Z=0.$$

which means that we take the plane of the Jupiter ellipse for the fundamental plane XY and v_1 signifies the polar angle. The equations of motion are then

(1) $$m\ddot{x} = \frac{\partial V}{\partial x}, \text{ etc.}$$

(2) $$V = k^2 \frac{M+m}{r} m + mS,\ m = 0,\ S = k^2 m_1 \left(\frac{1}{\Delta} - \frac{xX+yY}{R^3}\right).$$

On multiplying the equations of motion (1) by respectively $\dot{x}, \dot{y}, \dot{z}$ adding and integrating the following integrodifferential equation is gained, whose first notions go back till to O. Callandreau (9) and the classics.

$$\sum_{x,y,z} \dot{x}\ddot{x} = \sum_{x,y,z} \frac{\partial U}{\partial x}\dot{x} + \frac{\partial U}{\partial t} - \frac{\partial U}{\partial t}$$

or else

(3) $$\dot{x}^2 + \dot{y}^2 + \dot{z}^2 = 2U - L - \int_{t_0}^{t} dt \frac{\partial 2U}{\partial t},\quad U = \frac{V}{m}.$$

It is to be expressly noted that in consequence of our assumption of, a prescribed Keplerian ellipse for the disturbing planet, the potential function U contains the independent variable (time) t, not only implicitly through the coordinates x, y, z, but also explicitly through the coordinates of Jupiter X, Y, Z 0.

Only the particular case of the restricted circular problem (I) admits of the well known simplification for which $\frac{\partial U}{\partial t} = 0$. Then and only then the right handside reduces to

(4) $$2U - L = 0$$

and defines the well known surfaces and ovals of zero velocity, which we transform as a rule into a rotating system.

Now our general integrodifferential equation (3) for the asteroidal elliptical case (II), replacing the classical integral of energy appears rather complicated. And indeed we must bear in mind above all things the compelling duty to perform the differentiation under the sign of the integral with respect to the explicit time, but to carry out the complete integration with respect to time both explicit and implicit. Moreover the structure of the function U is very circumstantial, especially in all the unavoidable cases of the trigonometrical developments. I refer in this respect to the classical papers of Tisserand [10], Callandreau [9], Newcomb, Le Verrier, Laplace etc.

The same circumstances still hold if we pass from the aforesaid integrodifferential equation (3) of energy to the simple integral equation defining the zero velocity ovals.

The reasoning inferred from (3) remains the same as in the Hill theory.

The square velocity figuring on the lefthand side of (3) must remain positive for all real movements of the asteroid, therefore the following inequality must for ever be acurately satisfied,

$$2U - L - \int_{t_0}^{t} dt\, 2\frac{\partial U}{\partial t} \geq 0, \tag{5}$$

which equation, in case of equality, supersedes the definition of the Hill zero velocity ovals of the more general asteroidal elliptical problem (II). As occurred previously, with the entire Jacobian Integral, even in our case of this integral equation (5), it seems to have hitherto been completely overlooked, that this equation, replacing the classical Hill-energy relation, is completely solvable, and thus the whole problem can always be reduced to mere quadratures.

To attain this end I take advantage of a most important property of the function U, which was well known to Gyldén and Theodor von Oppolzer [11].

And indeed the right hand side, despite all tremendously complicated trigonometrical developments of the classics nevertheless remains in its original form, a homogenous function of negative first dimension of Euler, with respect to both series of variables.

$$x,\ y,\ z,\quad X,\ Y,\ Z,$$

so that if we put

$$x = Rx, \text{ etc}$$

$$X = R\bar{X}, \tag{6}$$

then according to the well-known theorem of Euler, it will be

$$U(Rx, Ry, Rz, RX, RY, RZ) = R^{-1}U(x, y, z, X, Y, Z) = R^{-1}U. \tag{7}$$

This Eulerian homogeneity can be more easily surveyed if we pass from rectangular to polar coordinates, owing to the particular structure of the function U (2) (3). And indeed the transformation (6) means a periodic contraction and dilatation of the whole space, with the periodical oscillations of the prescribed Keplerian changes of R the radius vector of the disturbing planet. Thus we can transcribe the equations (2) in the form

$$V = k^2\frac{M+m}{r}m + k^2 m m_1\left(\frac{1}{\Delta} - \frac{r\cos\sigma}{R^2}\right), \tag{8}$$

and then recalling that, by the aforesaid oscillations, no angle can be affected, and still less one where both legs start from the same apex-point as R namely the Sun (as origin of coordinates)
we find immediately that $r = Rr$, $\Delta = R\Delta$, $U = R^{-1}U$,

$$U = R^{-1}\left\{k^2\frac{M}{r} + k^2 m_1\left(\frac{1}{\Delta} - \frac{r\cos\sigma}{a_1^2}\right)\right\} = R^{-1}U. \tag{9}$$

However it is to be expressly pointed out that the Eulerian homogeneity in question (and especially in the coordinates of the disturbing planet X, Y, Z) holds good even in the case when a rotating system of new axes of coordinates is introduced. But as was explained in another more general way in the previous paper, we must compel Jupiter to oscillate perpetually along the ξ axis, keeping to this line and never departing from it during its whole revolution round the Sun.

So if we wish to reach our aim viz. get rid of the explicit time, we must choose for the rotating speed of the new rectangular system, whose ξ axis always points to the revolving disturbing planet, just the speed of Jupiter. And in fact for this particular choice, η_1 as well as ξ_1 disappears and $\frac{r \cos \sigma}{R^2}$ is reduced to

$$\frac{xX+yY}{R^3} = \frac{\xi\xi_1+\eta\eta_1}{R^3} = \frac{\xi}{R}, \quad \xi = R\xi, \quad \xi_1 = R. \tag{10}$$

The main achievement thus arrived at, can easily be tested and at the same time made more comprehensible, if we choose to introduce the Eulerian transformation just from the start namely in the original equations of movement holding for the fixed system of rectangular coordinates (1).

As a direct computation would appear rather tedious through the carrying out of many second derivations, it appears advisable to apply the Lagrangean scheme and, only after the introduction, of the rotating, non uniform system to choose for Lagrangian variables resp. ξ, η, ζ.

Using the same notation as before, we find out for the expression of the Lagrangean kinetic Energy

$$(11) \quad 2T \quad m(\dot{x}^2+\dot{y}^2+\dot{z}^2) \quad m\{R^2(\dot{x}^2+ \quad)+\dot{R}^2(x^2+\ldots)+2R\dot{R}(\bar{x}\dot{\bar{x}}+\ldots)\}$$

After the introduction of the rotating system animated with the Jovian speed we find out for

$$2T = m\Big\{\dot{R}^2 r^2 + 2R\dot{R}\sum_{\xi,\eta,\zeta}\xi\dot{\bar{\xi}} + \dot{R}^2 \sum \dot{\xi}^2 + R^2\dot{v}_2^1(\xi^2+\eta^2) + \tag{12}$$
$$+2R^2\dot{v}_1(\bar{\xi}\eta \cdot \bar{\eta}\dot{\xi})\Big\}.$$

Then the Lagrangean scheme leads finally to the equations

$$R^2\xi - 2v_1 R^2\eta + 2R\dot{R}\xi - \eta RH_1 + \xi RH_2 = \frac{\partial U}{\partial \xi},$$

$$(13) \qquad R^2\ddot{\eta} + 2v_1 R^2\xi + 2R\dot{R}\eta + \xi RH_1 + \eta RH_2 = \frac{\partial U}{\partial \eta},$$

$$R\zeta + 2R\dot{R}\zeta + \zeta R^2\dot{v}^2 + \zeta RH_2 = \frac{\partial U}{\partial \zeta},$$

$$H_1 = R\ddot{v}_1 + 2\dot{R}\dot{v}_1, \quad H_2 = \ddot{R} - R\dot{v}_1^2.$$

If we wish to get rid of the terms $\bar{\eta}$, ξ, it is advisable to choose the hitherto not disposed of velocity of the rotating system in such a way that it coincides with the rotational Keplerian speed of the disturbing planet, and indeed the equation

$$H_1 \equiv v_1 R + 2\dot{R}v_1 = 0,$$

gives $\frac{\dot{v}_1}{v_1} = -2\frac{\dot{R}}{R}$, on integrating $\log v_1 = \log R^{-2} + \log c$ or else

$$R^2 v_1 = c = k\sqrt{a_1}\sqrt{1-e_1^2}\sqrt{M+m_1} \tag{14}$$

But this equality expresses just the Keplerian law of areas for the disturbing Jupiter, and simultaneously for this choice of the non uniform speed of the rotational system, the coefficient H_2 goes over into

$$H_2 = \ddot{R} - v_1^2 R = -\frac{k^2(M+m_1)}{R^2}, \tag{15}$$

this equation expressing equality of accelerations. In this way we succeed in transcribing the fundamental equations of motion (13) in the form

$$\begin{aligned} \frac{d(R^2\dot{\xi})}{dt} - 2v_1 R^2\dot{\eta} &= \frac{1}{R}\frac{\partial\Omega}{\partial\xi}, \\ \frac{d(R^2\dot{\eta})}{dt} + 2v_1 R^2\dot{\xi} &= \frac{1}{R}\frac{\partial\Omega}{\partial\eta}, \\ \frac{d(R^2\dot{\zeta})}{dt} + \frac{c^2\zeta}{R^2} &= \frac{1}{R}\frac{\partial\Omega}{\partial\zeta}, \end{aligned} \tag{16}$$

where we have made use of the sameness

$$R^2\ddot{\xi} + 2R\dot{R}\dot{\xi} = \frac{d}{dt}(R^2\dot{\xi}), \text{ etc.} \tag{17}$$

and the identities

$$\Omega = \frac{k^2(M+m_1)}{2a_1^3}r^2 + U = k^2M\left(\frac{r^2}{2a_1^3} + \frac{1}{r}\right) + k^2m_1\left(\frac{\Delta^2}{2a_1^3} + \frac{1}{\Delta}\right) \tag{18}$$

$$r\cos\sigma = \frac{r^2 + a_1^2 - \Delta^2}{2a_1}, \quad U = \frac{k^2M}{r} + k^2m_1\left(\frac{1}{\Delta} - \frac{r\cos\sigma}{a_1^2}\right), \quad \bar{R} = a_1.$$

After multiplying (16) resp. by the factors $2R^2\dot{\xi}$, $2R^2\dot{\eta}$, $2R^2\dot{\zeta}$ and summing up we get the result:

$$\frac{d}{dt}(R^2\dot{\xi})^2 + \frac{d}{dt}(R^2\dot{\eta})^2 + \frac{d}{dt}(R^2\dot{\zeta})^2 + \frac{dc^2\zeta^2}{dt} = 2\left\{\frac{d(R\Omega)}{dt} - \frac{\partial(R\Omega)}{\partial t}\right\}.$$

The integration is carried out, if we combine the square as

(19) $$R^4(\dot{\xi}^2+\dot{\eta}^2+\dot{\bar{\zeta}}^2)+c^2\zeta^2=2\bar{\Omega}R-L-\int_{t_0}^{t}dt2\bar{\Omega}\frac{\partial R}{\partial t}\geqq 0.$$

Let us limit ourselves this time to the questions of stability for $\bar{\zeta}=0$. All throughout every real movement, defined by the equations of motion the left-hand side of (19) must remain positive, representing a positive square of the velocity of the asteroid. Consequently the limiting surfaces (or ovals) — if they exist — are defined by the last relation. Now the integral equation can be written

(20) $$2\Omega R-L-\int_{t_0}^{t}dt2\Omega\frac{\partial R}{\partial t}=0.$$

And this equation is immediately solved by repeated substitutions. The resulting development is always absolutely and uniformly convergent, admitting the same dominating function as the exponential.

$$2\bar{\Omega}\quad\frac{L}{R}\left\{1+\int_{t_0}^{t}dt\frac{1}{R}\frac{\partial R}{\partial t}+\int_{t_0}^{t}dt\left(\frac{1}{R}\frac{\partial R}{\partial t}\int_{t_0}^{t}dt\frac{1}{R}\frac{\partial R}{\partial t}\right)\right.$$

$$\left.+\int_{t_0}^{t}dt\left[\frac{1}{R}\frac{\partial R}{\partial t}\int_{t_0}^{t}dt\left(\frac{1}{R}\frac{\partial R}{\partial t}\int_{t_0}^{t}dt\frac{1}{R}\frac{\partial R}{\partial t}\right)\right]+\cdots\right\}$$

(21) $$2\Omega\quad\frac{L}{R}\left\{1+\int_{a_1}^{R}d\log R+\frac{\left(\int_{a_1}^{R}d\log R\right)^2}{2!}+\frac{\left(\int_{a_1}^{R}d\log R\right)^3}{3!}+\ldots\right\}$$

Let us choose the starting radius vector $R=a_1$ which corresponds to the true anomaly $\cos v_0=e$, $t=t_0$. For this choice we obtain putting $a_1=1$

(22) $$2\bar{\Omega}\quad\frac{L}{R}\left\{1+\log R+\frac{(\log R)^2}{2!}+\frac{(\log R)^3}{3!}+\ldots\right\}.$$

The series arrived at is manifestly, uniformly, and absulutely convergent, as having the same dominating function as the exponential function.

The convergency holds good for all points of the complex Gaussian plane, with the exception of the point at infinity.

Moreover we can sum up the whole series in the form of an exponential thus finding the result

(23) $$2\Omega=\frac{L}{R}e^{\log R}=L.$$

The solving function is given by $\overline{\Omega}$, (18) — but by this it is not said that the original unknown in (5), (8), (9), (18) namely the function U remains unchanged — and indeed we must return to the original variables, which means to put

$$(24) \qquad \overline{\xi}=\xi R^{-1},\quad \eta=\eta R^{-1},\quad \zeta=\zeta R^{-1},\quad \xi_1=R,\ \eta_1=\zeta_1=0,$$

whereby the final solution of the general integral equation appears to be given by

$$-\frac{c^2\zeta^2}{R^3}+2\Omega\left(\frac{r}{R},\frac{1}{R}\right)=L,\ \Omega=\frac{\Omega\left(\frac{\xi}{R},\frac{\eta}{R},\frac{\zeta}{R}\right)}{R}$$

This important result gives the pulsations of the original Hill zero velocity oval.

Summing up we gather from the preceeding analysis the following theorem holding good for the case II of an asteroidal elliptical problem (meant in space and in a rotating system of coordinates).

The zero velocity surfaces (ovals) of the restricted circular problem I through the influence of the orbital excentricity of the disturbing planet Jupiter, *undergo by passage to the more general asteroidal elliptical problem strong oscillations with the period of revolution of the disturbing planet.*

The size of these oscillations is of the same order as the orbital excentricity of the planet being in no way lessened or factored by any disturbing small mass m_1.

On the contrary no secular changes appear to be ascertained. But this circumstance does not exclude a very probable possibility of secular changes of the orbits themselves. Another expression of the result, just obtained, could even be worded as follows:

The original zero velocity surfaces of the starting restricted circular problem undergo, by passage to the more general elliptical asteroidal problem in the XY plane for $\zeta=0$, merely such changes as may arise by the corresponding oscillations of the original constant of energy C, which goes over into $\frac{L}{R}$.

As the consequences of the theory explained herewith appear to be farreaching, it will be not out of place to try and prove the equation (23), etc. of the strongly pulsating ovals once more directly.

For that purpose let us start with the equations (13), multiply by the factor $\frac{R^2}{c^2}$ and use the integral of areas $R^2 v_1 = c$, (14)

$$c=k\sqrt{a_1}\sqrt{1-e_1^2}\sqrt{M+m_1}.$$

Let us now introduce the true anomaly v_1 of the disturbing planet instead of the time as independent variable. The original derivation with respect to the time will be replaced by resp.

$$\ddot{\xi} = \frac{d^2\xi}{dv_1^2}\frac{c_1^2}{R^4} - 2c_1\frac{\dot{R}}{R^3}\frac{d\xi}{dv_1}, \quad \dot{\xi} = \frac{c_1}{R^2}\frac{d\xi}{dv_1}, \quad \dot{\eta} = \frac{c_1}{R^2}\frac{d\eta}{dv_1} \tag{25}$$

After all computations have been carried out, we get the equations of Prof. Petr and Nechvile (13) in the form

$$\frac{d^2\xi}{dv_1^2} - 2\frac{d\eta}{dv_1} = \frac{R}{c^2}\frac{\partial\Omega}{\partial\xi},$$

(26)
$$\frac{d^2\eta}{dv_1^2} + 2\frac{d\xi}{dv_1} = \frac{R}{c^2}\frac{\partial\Omega}{\partial\eta},$$

$$\frac{d^2\zeta}{dv_1^2} + \zeta = \frac{R}{c^2}\frac{\partial\Omega}{\partial\zeta}, \qquad \varrho = \frac{c^2}{R}, \quad \Omega \text{ being given by (18)}$$

Hence by multiplying respectively by $\frac{d\xi}{dv_1}, \frac{d\eta}{dv_1}, \frac{d\zeta}{dv_1}$ adding and putting $\varrho = \frac{c^2}{R}$, we obtain:

$$\frac{d^2\xi}{dv_1^2}\frac{d\xi}{dv_1} + \frac{d^2\eta}{dv_1^2}\frac{d\eta}{dv_1} + \frac{d^2\zeta}{dv_1^2}\frac{d^2\zeta}{dv_1} + \zeta\frac{d\zeta}{dv_1} = \frac{d}{dv_1}\left(\frac{\Omega}{\varrho}\right) - \frac{\partial}{\partial v_1}\left(\frac{\Omega}{\varrho}\right),$$

after integration with respect to v_1

(27)
$$\left(\frac{d\xi}{dv_1}\right)^2 + \left(\frac{d\eta}{dv_1}\right)^2 + \left(\frac{d\zeta}{dv_1}\right)^2 + \zeta^2 - 2\frac{\Omega}{\varrho} - L - \int_{v_0}^{v_1} 2\,\Omega\,\frac{\partial\frac{1}{\varrho}}{dv_1}\,dv_1 = 0,$$

which integral equation in case of the extreme, namely the equation of pulsating curves for $\zeta = 0$ is solved by repeated substitution, as previously and leads to the result:

(28)
$$2\Omega = L\varrho + \varrho\int_{v_0}^{v_1} L\varrho\frac{\partial\frac{1}{\varrho}}{\partial v_1}\,dv_1 + L\varrho\int_{v_0}^{v_1} dv_1\left[\varrho\frac{\partial\frac{1}{\varrho}}{\partial v_1}\int_{v_0}^{v_1} dv_1\varrho\frac{\partial\frac{1}{\varrho}}{\partial v_1}\right] + \dots$$

If we change simply the necessary quadratures from

$$\frac{\partial\frac{1}{\varrho}}{\partial v_1}\,dv_1 = d\left(\frac{1}{\varrho}\right) = -\frac{1}{\varrho^2}\,d\varrho$$

and remember that

(29)
$$e^{\mp\log\varrho} = 1 \mp \log\varrho + \frac{(\log\varrho)^2}{2!} \mp \frac{(\log\varrho)^3}{3!} + \dots$$

the same method, as all previous ones gives the result

(30)
$$2\Omega(\xi, \eta) = 2\Omega(r, \Delta) = \varrho L e^{-\log\varrho} = L.$$

which leads to the resulting pulsating oval

(31)
$$2\overline{\Omega}\left(\frac{\xi}{R},\frac{\eta}{R}\right)=2\Omega\left(\frac{r}{R},\frac{\Delta}{R}\right)=L$$

or else

(32)
$$(r^2+m_1\Delta^2)\frac{1}{R^2}+2R\left(\frac{1}{r}+\frac{m_1}{\Delta}\right)-L=0, \text{ q. e. d.}$$

In the picture p. 68 the original Hill-Darwin ovals for $C = 3.491, 3.888, 4.018$ have been drawn by means of a Nomogram. The method chosen uses three parallel axes at the following distances: 0, 1, 11.

The unity of length on the middle axis was chosen ten times longer than that used on both the others, in order to avoid the detrimental condensation of the peculiar coordinates.

By means of this Nomogram the fixed ovals of the restricted circular asteroidal problem were obtained for $m_1 = \frac{1}{10}$, $e_1 = \frac{1}{10}$.

Then all pulsating increases and especially those belonging to a Maximum and a Minimum, can be achieved by merely shortening and lengthening to the extent of 10% every vector radius — all starting from the origin (the Sun).

And indeed, on beginning with the original Hill-Darwin oval of equation (30) (In the picture full line, while the extreme ovals are drawn in ---—),

and putting $r=\xi u$, $\Delta=\delta=\eta u$ i.e. $\xi=\frac{r}{u}$, $\eta=\frac{\delta}{u}$,

we easily get $\frac{1}{r}+\frac{m_1}{\Delta}=\frac{1}{u}$ or else $\frac{u}{r}+\frac{m_1 u}{\Delta}=1$,

which is $\frac{1}{\xi}+\frac{m_1}{\eta}=1$ a fixed hyperbola,

then
$$r^2+m_1\Delta^2+\frac{2}{u}=L$$

gives
$$(\xi^2+m_1\eta^2)u^2=L-\frac{2}{u},$$

$$\xi^2+m_1\eta^2=(Lu-2)\,u^{-3}$$

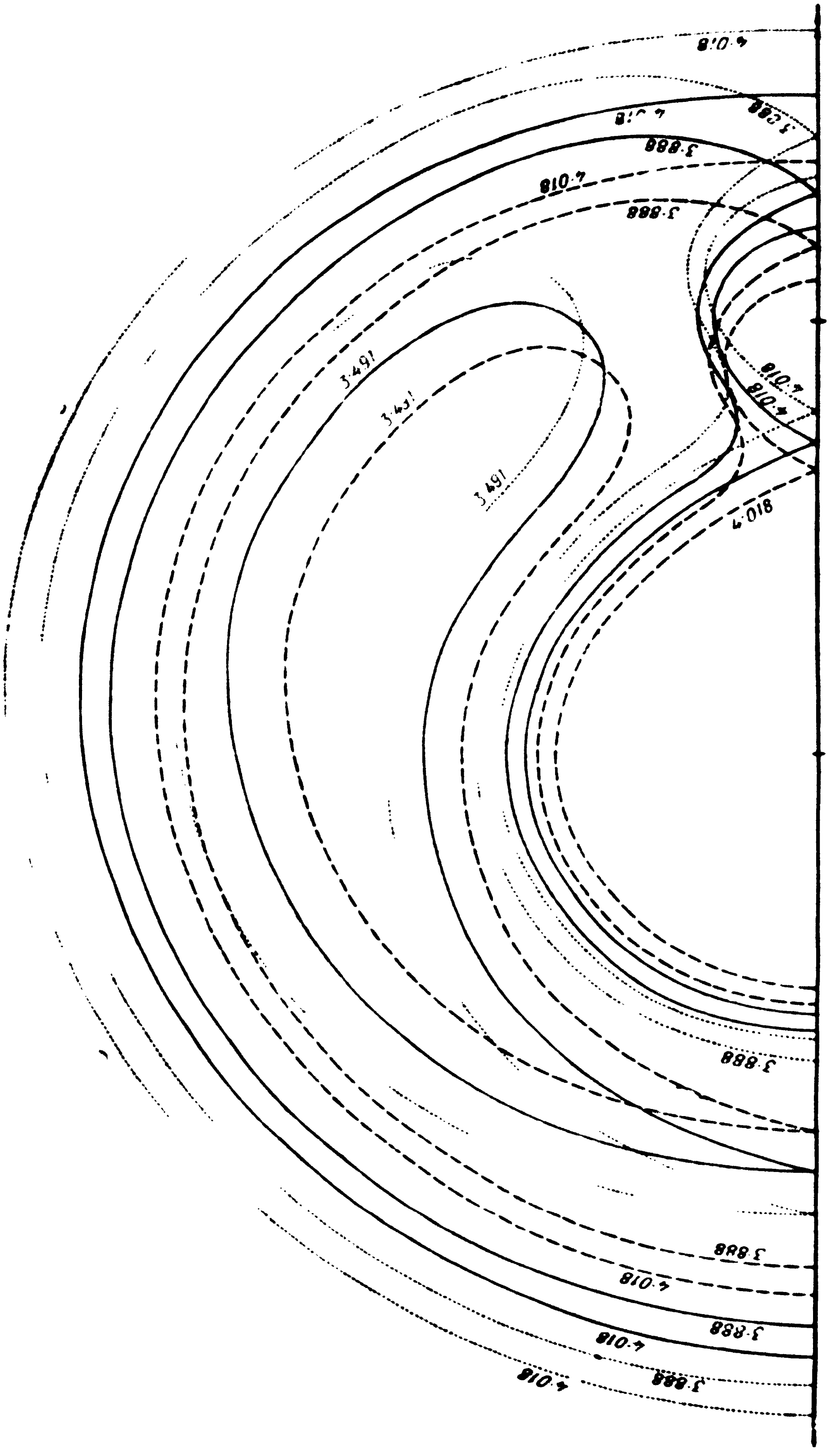
or else
$$\frac{\xi^2}{(u^{-3/2}\sqrt{Lu-2})^2}+\frac{\eta^2}{(m_1^{-1/2}u^{-3/2}\sqrt{Lu-2})^2}=1,$$

$$m_1^{-1/2}=\sqrt{10}=3{\cdot}1622777.$$

The axes of the changeable ellipse are resp.

$$a_\eta=3{\cdot}163\,u^{-3/2}\sqrt{Lu-2},\quad \beta_\xi=u^{-3/2}\sqrt{Lu-2},$$

where we put successively $u=1, 2, 3\ldots$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while the hyperbola $\frac{1}{\xi} + \frac{m_1}{\eta} = 1$ remains fixed for all. In the case of the general pulsating surface, oval (32), we easily find

$$(r^2 + m_1 \varDelta^2)\frac{1}{R^2} + 2R\left(\frac{1}{r} + \frac{m_1}{\varDelta}\right) = L, \quad R = \frac{a_1(1-e_1^2)}{1+e_1\cos v_1},$$

Putting again $\frac{1}{u} = R\left(\frac{1}{r} + \frac{m_1}{\varDelta}\right)$, which is $(r^2 + m_1 \varDelta^2)\frac{1}{R^2} = L - \frac{2}{u}$,

we replace this time, resp. by:

$$\frac{Ru}{r} = \frac{1}{\xi}, \quad \frac{Ru}{\varDelta} = \frac{1}{\eta} \quad \text{or else} \quad \xi = \frac{r}{Ru}, \quad \eta = \frac{\varDelta}{Ru}$$

thus obtaining $\frac{1}{\xi} + \frac{m_1}{\eta} = 1$ the fixed hyperbola

and a movable curve

$$(r^2 + m_1 \varDelta^2)\frac{1}{R^2} = L - \frac{2}{u},$$

when transcribed

$$\frac{\xi^2}{(u^{3/2}\sqrt{Lu-2})^2} + \frac{\eta^2}{(m_1^{1/2}u^{3/2}\sqrt{Lu-2})^2} = 1.$$

The semiaxes of this ellipse being fixed by

$$\alpha_\eta = 3{\cdot}162\sqrt{Lu-2}\,u^{-3/2}, \quad \beta_\xi = \sqrt{Lu-2}\,u^{-3/2}$$

$$r = \xi u \begin{cases} 1-e_1^2 \\ 1-e_1 \\ 1+e_1 \end{cases} \qquad \varDelta = \eta u \begin{cases} 1-e_1^2 \\ 1-e_1 \\ 1+e_1 \end{cases}$$

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ЕДНО ИНТЕГРАЛНО УРАВНЕНИЕ ЗА ПУЛСАЦИИТЕ НА ОВАЛИТЕ НА НУЛЕВАТА СКОРОСТ В АСТЕРОИДНАТА ЕЛИПТИЧНА ЗАДАЧА

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РЕЗЮМЕ

Добре известната теория на устойчивостта по Хил за специалната астероидна кръгова задача се усложнява, ако преминем към случая на по-общата елиптична астероидна задача, при която ексцентрицитетът на орбитата на смущаващата планета е различен от нула. Досегашните теории довеждаха винаги до класическите семиконвергентни развития. Тези методи правят обаче всеки критерий за устойчивост несигурен.

Авторът използува една добре известна теорема за хомогенните функции и успява да реши напълно интегралното уравнение, което играе роля в тази задача. Така цялата проблема се свежда до квадратури и се внася прегледност на силните пулсации на овалите на нулевата скорост.

ОДНО ИНТЕГРАЛЬНОЕ УРАВНЕНИЕ О ПУЛЬСАЦИЯХ ОВАЛОВ НУЛЕВОЙ СКОРОСТИ В АСТЕРОИДНОЙ ЭЛЛИПТИЧЕСКОЙ ЗАДАЧЕ

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РЕЗЮМЕ

Хорошо известная теория устойчивости по Хиллу для специальной астероидной круговой задачи усложняется, если мы перейдем к случаю более общей элиптической астероидной задачи, при которой эксцентрицитет орбиты смущающей планеты не совпадает с нулем. Существовавшие до сих пор теории приводили всегда к классическим семиконвергентным развитиям. В результате применения этих методов, однако, любой критерий устойчивости становится ненадежным.

Автор использует широко известную теорему об однородных функциях и ему удается решить полностью интегральное уравнение, играющее роль в этой задаче. Таким образом, вся проблема сводится к квадратурам и получается наглядность сильных пульсаций овалов нулевой скорости.