SOME REMARKS ON UNIVALENT FUNCTIONS

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W. Kaplan [1] introduced the class of „close-to-convex" schlicht functions. The function $f(z)$ which is regular in the unit circle $|z|<1$ is called close-to-convex if there exists a function $\varphi(z)$ which is regular and univalent in $|z|<1$ and which maps the unit circle on a convex domain, and is such that

$$\text{Re}\left\{\frac{f'(z)}{\varphi'(z)}\right\}>0 \quad \text{for} \quad |z|<1. \tag{1}$$

Kaplan proved that every close-to-convex function is univalent further that every function $f(z)$, which is regular and univalent in $|z|<1$ and maps $|z|<1$ on a star-like domain, is close-to-convex in the above sense. He proved that the inequality

$$\int_{\Theta_1}^{\Theta_2} \text{Re}\left\{1+\frac{zf''(z)}{f'(z)}\right\} d\vartheta > -\pi \tag{2}$$

for $\Theta_1<\Theta_2$ (were $z=re^{i\vartheta}$, $r<1$) is necessary and sufficient for $f(z)$ being close-to-convex in $|z|<1$.

The class of close-to-convex functions has been investigated further by Maxwell O. Reade [2], who proved that if $f(z)=z+ +a_2 z^2+ \quad +a_n z^n+ \cdot$ is close-to-convex in $|z|<1$ then the Bieberbach-conjecture is valid for $f(z)$, i. e.

$$|a_n| \leq n \qquad (n=2,\ 3, \ldots). \tag{3}$$

This result generalizes a former result of the author of the present paper [3], which states that the Bieberbach-conjecture is valid for functions $f(z)$ which map the unit circle on a domain whose boundary rotation does not exceed 4π, i. e. for which

$$\int_0^{2\pi} \left| 1 + \text{Re}\left\{\frac{z f''(z)}{f'(z)}\right\}\right| d\vartheta \leq 4\pi. \tag{4}$$

In an other paper [4] Maxwell O. Reade proved also that if $f(z) = z + \sum_{n=2}^{\infty} a_n z^n$ is regular in $|z|<1$ and satisfies

(5)
$$\int_{\vartheta_1}^{\vartheta_2} \left(1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}\right) d\vartheta > -\frac{\pi}{2}$$

for $z=re^{i\vartheta}$, $\vartheta_1<\vartheta_2$, then we have

(6)
$$|a_n| \leq \frac{n+1}{2}$$

This result also generalizes a previous result of the author. It has been proved namely in [3] that (6) holds if $f(z)$ is regular in $|z|<1$ and maps the circles $|z|=r<1$ on such curves which have a boundary rotation $\leq 3\pi$, i. e. for which

(7)
$$\int_0^{2\pi} \left|1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}\right| d\vartheta < 3\pi.$$

As a matter of fact, this result is a special case of the following general result, proved in [3]:

Theorem 1. *If $f(z)=z+\sum_{n=2} a_n z^n$ is regular in $|z|<1$ and maps the circles $|z|=r<1$ on such curves, which have a boundary rotation $\leq \alpha$ $(2\pi\leq\alpha\leq 3\pi)$, i. e. if*

(8)
$$\int_0^{2\pi} \left|1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}\right| d\vartheta \leq \alpha$$

then

(9)
$$|a_n| \leq \prod_{k=2}^{n} \left(1+\frac{\alpha-2\pi}{k\pi}\right) \quad (n=2, 3, \ldots)$$

Though the inequality (9) remains valid for $3\pi<\alpha\leq 4\pi$ it is very rough, and for $\alpha=4\pi$ does not give the inequality $|a_n|\leq n$ (which has been proved in [3] by an other method, namely by applying a theorem of M. S. Robertson [5]), but gives only the very rough estimation $|a_n| < \frac{(n+1)(n+2)}{6}$ which coincides with $|a_n|\leq n$ only for $n=2$. Thus there is a gap beetween the inequality (9), valid if the boundary rotation α is $\leq 3\pi$, and the inequality (3) valid for $\alpha=4\pi$.

The purpose of the present paper is to fill this gap, and establish the following result, which is a counterpart of Theorem 1.

Theorem 2. *If $f(z)=z+\sum_{n=2}^{\infty} a_n z^n$ is regular in $|z|<1$ and if $f(z)$ maps the circles $|z|=r<1$ on curves the boundary rotation of which does not exceed α, where $3\pi\leq\alpha<4\pi$, then we have*

(10)
$$|a_n| \leq 1+\frac{(\alpha-2\pi)}{2\pi}(n-1)$$

$$(n=2, 3, \ldots)$$

Remarks: The inequality (10) reduces clearly for $a=3\pi$ to $|a_n| \leq \frac{n+1}{2}$ and for $a=4\pi$ to $|a_n| < n$, i. e. Theorem 1. is a generalization of these results of the paper [3]. The inequality (10) is valid also for $2\pi \leq a \leq 3\pi$; we excluded these cases from the formulation of the theorem only because if $a < 3\pi$ the estimation (10) is weaker then that given by Theorem 1. As a matter of fact, putting $x = \frac{a-2\pi}{2\pi}$, the following inequality is valid:

$$\prod_{k=2}^{n}\left(1+\frac{2x}{k}\right) \leq 1+(n-1)x \quad \text{if } 0 \leq x \leq \frac{1}{2} \tag{11}$$

and

$$1+(n-1)x \leq \prod_{k=2}^{n}\left(1+\frac{2x}{k}\right) \quad \text{if } \frac{1}{2} \leq x \leq 1, \tag{12}$$

as we have

$$1+(n-1)x = \prod_{k=2}^{n}\left(1+\frac{2x}{2+2x(k-2)}\right) \leq \prod_{k=2}^{n}\left(1+\frac{2x}{k}\right) \quad \text{for } \frac{1}{2} \leq x \leq 1$$

and

$$1+(n-1)x = \prod_{k=2}^{n}\left(1+\frac{2x}{2+2x(k-2)}\right) \geq \prod_{k=2}^{n}\left(1+\frac{2x}{k}\right) \quad \text{for } 0 \leq x \leq \frac{1}{2}.$$

Thus (11) and (12) follows.

To prove Theorem 2. we shall introduce the classes of close-to-convex functions of type $\beta\left(0 < \beta < \frac{\pi}{2}\right)$ these classes being subclasses of the class of close-to-convex functions, introduced by W. Kaplan

We shall say that $f(z)$ is close-to-convex of type β in $|z| < 1$ $\left(0 \leq \beta \leq \frac{\pi}{2}\right)$ if $f(z)$ is regular in $|z| < 1$ and there exists a function $\varphi(z)$ which is regular and univalent in $|z| < 1$, maps the unit circle on a convex domain and is such that

$$\left|\arg \frac{f'(z)}{\varphi'(z)}\right| \leq \beta. \tag{13}$$

Clearly, every function, which is close-to-convex of type $\beta\left(0 \leq \beta < \frac{\pi}{2}\right)$ is also close-to-convex of type γ if $\beta < \gamma \leq \frac{\pi}{2}$. Thus especially every function which is close-to-convex of type $\beta \leq \frac{\pi}{2}$ is close-to-convex in sense of W. Kaplan referred to above. Clearly $f(z)$ is close-to-convex of type 0 if and only if $f(z)$ is univalent in $|z| < 1$ and maps the unit circle on a convex domain.

We shall prove two results (Theorem 3. and 4.) which generalize the results of W. Kaplan and Maxwell O. Reade, and after this shall deduce Theorem 2. from Theorem 4.

Theorem 3. The function $f(z)$ is close-to-convex of type β in the unit circle if and only if

$$\int_{\Theta_1}^{\Theta_2} \left(1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}\right) d\vartheta > -2\beta \tag{14}$$

where $z=re^{i\vartheta}$, $0<r<1$, $\Theta_1<\Theta_2$.

Remark: For $\beta=\frac{\pi}{2}$ Theorem 3. reduces to the characterization of close-to-convex functions given by W. Kaplan.

The proof of Theorem 3. is in every respect analogous to that of the special case $\beta=\frac{\pi}{2}$ given by Kaplan, and therefore may be left to the reader.

Theorem 3. enables us to give a geometric characterization of functions which are close-to-convex of type $\beta\left(0\leqq\beta\leqq\frac{\pi}{2}\right)$. A function $f(z)$ is close-to-convex of type β in the unit circle if and only if it is univalent in the unit circle and maps each circle $|z|=r<1$ on a simple closed curve C_r which has the property that if we go around the curve C_r in the positive (counter-clockwise) direction, the directed tangent of the curve never turns back by more than the angle 2β. By other words if $p(z)=\arg f'(z)$ and $P(r,\vartheta)=p(re^{i\vartheta})+\vartheta$ we have for $\vartheta_1<\vartheta_2$

$$P(r,\vartheta_2)-P(r,\vartheta_1)\geq-2\beta \tag{15}$$

for any $r<1$.

As a matter of fact if $z=re^{i\vartheta}$

$$\frac{\partial P(r,\vartheta)}{\partial\vartheta}=1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\} \tag{16}$$

and thus

$$\int_{\Theta_1}^{\Theta_2}\left(1+\mathrm{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}\right)d\vartheta=P(r,\Theta_2)-P(r,\Theta_1). \tag{17}$$

Theorem 4. If $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ is close-to-convex of type $\beta\left(0\leqq\beta\leqq\frac{\pi}{2}\right)$ in the unit circle, then

$$|a_n|\leq 1+\frac{2\beta}{\pi}(n-1). \tag{18}$$

Remark: For $\beta=\frac{\pi}{2}$ and $\beta=\frac{\pi}{4}$ Theorem 4. reduces to the theorems of Maxwell O. Reade [2], [4] referred to above.

Proof of Theorem 4. According to our supposition there exists a function $\varphi(z)$ regular and univalent in $|z|<1$ which maps the circle $|z|<1$ on a convex domain, and for which

$$\left|\arg\frac{f'(z)}{\varphi'(z)}\right| \leqq \beta \quad \text{for } |z|<1.$$

Now let us put

(19) $$\varphi(z)=\sum_{n=1}^{\infty} a_n z^n.$$

We may suppose without loss of generality, that $|a_1|=1$. Let us put further

(20) $$\frac{f'(z)}{\varphi'(z)}=\sum_{n=0}^{\infty} b_n z^n.$$

Then we have $|b_0|=1$ and

(21) $$f'(z)=1+\sum_{n=2}^{\infty} na_n z^{n-1}=\left(\sum_{n=0}^{\infty} b_n z^n\right)\left(\sum_{n=1}^{\infty} na_n z^{n-1}\right)$$

Thus we have

(22) $$na_n=\sum_{k=2}^{n} ka_k b_{n-k} \qquad (n=2, 3, \ldots)$$

As it is supposed that $\varphi(z)$ maps $|z|<1$ on a convex domain, we have (see [6], Problem IV. 162.)

(23) $$|a_k| \leqq 1 \qquad k=2, 3, \ldots$$

To estimate the coefficients b_k we need the following lemma, which is a straight-forward generalization of a theorem of C. Carathéodory (see [6] Problem III. 235) according to which if $f(z)=1+b_1z+\cdots+b_nz^n+\cdots$ is regular in $|z|<1$ and $\mathrm{Re}\, f(z)>0$ then $|b_n|\leqq 2$.

Lemma: If $g(z)=b_0+b_1z+\cdots+b_nz^n+\cdots$ is regular in $|z|<1$, $|b_0|=1$ and

$$|\arg g(z)|\leqq\beta\leqq\frac{\pi}{2}$$

then

(24) $$|b_n|\leqq\frac{4\beta}{\pi}.$$

Proof of the lemma: Let us put $h(z)=\left(\frac{g(z)}{g(0)}\right)^{\frac{\pi}{2\beta}}$. Then $\mathrm{Re}\{h(z)\}>0$ and $h(0)=1$. Thus if $h(z)=1+c_1z+c_2z^2+\cdots$ then according to the theorem of Carathéodory mentioned above we have

$$|c_n|\leqq 2, \qquad n=1, 2, \ldots$$

As

$$|c_1|=|h'(0)|=\frac{\pi}{2\beta}|g'(0)|=\frac{\pi}{2\beta}|b_1|$$

it follows

(25) $$|b_1|\leqq\frac{4\beta}{\pi}.$$

Now let us put

(26) $$G(z)=\frac{1}{n}\sum_{k=1}^{n} g\left(e^{\frac{2\pi ik}{n}}\cdot z^{\frac{1}{n}}\right)=b_0+\sum_{r=1}^{\infty} b_{rn} z^r$$

then $|\arg G(z)|\leqq\beta$ and thus by (25)

(27) $$|b_n|\leqq\frac{4\beta}{\pi}\cdot$$

This proves our lemma.

Applying our lemma we obtain

(28) $$|b_k|\leqq\frac{4\beta}{\pi}$$

and thus by (22)

(29) $$|na_n|\leqq n+\frac{2\beta}{\pi}n(n-1) \qquad (n=2,\ 3,\ldots)$$

which implies

(30) $$|a_n|\leqq 1+\frac{2\beta}{\pi}(n-1) \qquad (n=2,\ 3,\ldots)$$

Thus Theorem 4. is proved.

Let us mention that the inequality (30) is best possible for $n=2$. As a matter of fact, let us consider

$$f(z)=\int_0^z \frac{1}{(1-\zeta)^2}\left(\frac{1+\zeta}{1-\zeta}\right)^{\frac{2\beta}{\pi}} d\zeta .$$

By choosing $\varphi(z)=\frac{z}{1-z}$ we have $\left|\arg\frac{f'(z)}{\varphi'(z)}\right|\leqq\beta$; thus $f(z)$ is close-to-convex of type β; on the other hand $f(z)=z+\left(1+\frac{2\beta}{\pi}\right)z^2+\cdot$

For other values of n the inequality (30) can be probably improved.

Let us remark, that the inequality

$$|a_2|\leqq 1+\frac{2\beta}{\pi}$$

contains the distortion theorem for the class of close-to-convex functions of type β. It can be brought to the usual form, expressed by

Theorem 5. *If $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ is regular and close-to-convex of type β $\left(0\leqq\beta\leqq\frac{\pi}{2}\right)$ in $|z|<1$, we have for $|z|=$ $=r<1$,*

(31) $$\frac{(1-r)^{\frac{2\beta}{\pi}}}{(1+r)^{\frac{2\beta}{\pi}+2}}\leqq |f'(z)| \leqq \frac{(1+r)^{\frac{2\beta}{\pi}}}{(1-r)^{\frac{2\beta}{\pi}+2}}$$

Proof of Theorem 5. Let us consider

$$F(\zeta)=\frac{f\left(\frac{\zeta+z}{1+\bar{z}\zeta}\right)-f(z)}{f'(z)(1-|z|^2)}. \tag{32}$$

If $f(z)$ is close-to-convex of type β, then clearly $F(\zeta)$ has the same property. As a matter of fact

$$\left|\arg\frac{f'(z)}{\varphi'(z)}\right|\leq\beta \qquad |z|<1$$

implies

$$\left|\arg\frac{F'(\zeta)}{\Phi'(\zeta)}\right|\leq\beta$$

where

$$\Phi(\zeta)=\frac{\varphi\left(\frac{\zeta+z}{1+\zeta\bar{z}}\right)-\varphi(z)}{\varphi'(z)(1-|z|^2)} \tag{33}$$

is univalent, and convex in $|\zeta|<1$, by virtue of a theorem of Carathéodory [7], which states that if $\varphi(z)$ maps the circle $|z|<1$ on a convex domain, then it maps every circle lying in the interior of the circle $|z|<1$ on a convex domain too. But

$$F(0)=0 \qquad F'(0)=1$$

and

$$F''(0)=\frac{f''(z)(1-|z|^2)}{f'(z)}-2\bar{z}. \tag{34}$$

Thus we obtain

$$\left|\frac{zf''(z)}{f'(z)}-\frac{2|z|^2}{1-|z|^2}\right|\leq 2\left(1+\frac{2\beta}{\pi}\right)\frac{|z|}{1-|z|^2}. \tag{35}$$

By a well-known argument it follows that

$$\frac{2|z|-2\left(1+\frac{2\beta}{\pi}\right)}{1-|z|^2}\leq\frac{\partial\log|f'(z)|}{\partial|z|}\leq\frac{2|z|+2\left(1+\frac{2\beta}{\pi}\right)}{1-|z|^2} \tag{36}$$

and thus by integration

$$\frac{(1-r)^{\frac{2\beta}{\pi}}}{(1+r)^{\frac{2\beta}{\pi}+2}}\leq|f'(z)|\leq\frac{(1+r)^{\frac{2\beta}{\pi}}}{(1-r)^{\frac{2\beta}{\pi}+2}}. \tag{37}$$

For $\beta=\frac{\pi}{2}$ (37) reduces to the ordinary distortion theorem, for $\beta=0$ to the distorsion theorem for univalent convex functions. The inequality (37) is clearly best possible as there is equality for

$$f(z)=\int_0^z\frac{1}{(1-\zeta)^2}\left(\frac{1+\zeta}{1-\zeta}\right)^{\frac{2\beta}{\pi}}d\zeta.$$

Now we deduce Theorem 2. from Theorem 4.

Let us suppose that $f(z)$ is regular and univalent in $|z|<1$ and

$$\int_0^{2\pi}\left|1+\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}\right|d\vartheta \leqq a$$

where $3\pi\leqq a\leqq 4\pi$. Then we assert that $f(z)$ is close-to-convex of type $\beta=\frac{a-2\pi}{4}$. This can be shown as follows: as $1+\operatorname{Re}\left\{\frac{zf''(z)}{f'(z)}\right\}$ is harmonic in $|z|<1$ and takes the value 1 in $z=0$, we have

$$\int_0^{2\pi}\left(1+\operatorname{Re}\frac{zf''(z)}{f'(z)}\right)d\vartheta=2\pi. \tag{38}$$

Thus if we put

$$\left(1+\operatorname{Re}\frac{zf''(z)}{f'(z)}\right)=u(z) \tag{39}$$

we have

$$2\int_{\Theta_1}^{\Theta_2}u(z)\,d\vartheta=2\pi-\int_0^{\Theta_1}u(z)\,d\vartheta-\int_{\Theta_2}^{2\pi}u(z)\,d\vartheta-\int_{\Theta_1}^{\Theta_2}(-u(z))\,d\vartheta \tag{40}$$

and thus

$$\int_{\Theta_1}^{\Theta_2}u(z)\,d\vartheta>-2\left(\frac{a-2\pi}{4}\right) \tag{41}$$

from which by Theorem 1. it follows that $f(z)$ is close-to-convex of type $\frac{a-2\pi}{4}$. Thus Theorem 2. is proved.

Finaly let us mention some unsolved problems.

A) What is the exact radius of close-to-convexity of type β? By other words which is the greatest number $R(\beta)$ such that any function $f(z)$ which is regular and univalent in $|z|<1$, is close-to-convex of fype β in the circle $|z|<R(\beta)<1$?

Clearly $R(0)$ is the well-known radius of convexity (Rundungsschranke) $2-\sqrt{3}$ and as every star-shaped domain is close-to-convex evidently $R\left(\frac{\pi}{2}\right)=\operatorname{tgh}\frac{\pi}{4}$. The exact value of $R(\beta)$ is not known for $0<\beta\leqq\frac{\pi}{2}$. I succee ded to prove, by means of the rotation theorem of Golusin [8] that

$$R(\beta)\geq\frac{\sin\frac{\beta}{4}+2-\sqrt{3}}{1+(2-\sqrt{3})\sin\frac{\beta}{4}} \tag{42}$$

for $0\leqq\beta\leqq\frac{\pi}{2}$. I hope to return to this question at an other occasion.

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НЯКОИ ЗАБЕЛЕЖКИ ВЪРХУ ЕДНОЛИСТНИТЕ ФУНКЦИИ

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РЕЗЮМЕ

Регулярната в кръга $|z|<1$ функция $f(z)$ се нарича „почти изпъкнала от тип β“ $\left(0 \leqq \beta \leqq \frac{\pi}{2}\right)$, ако съществува такава функция $\varphi(z)$, регулярна и еднолистна в кръга $|z|<1$, която изобразява този кръг в една изпъкнала област и освен това

(1) $$\left|\arg \frac{f'(z)}{\varphi'(z)}\right|<\beta \qquad \text{за} \quad |z|<1.$$

В работата е доказана следната теорема, която е обобщение на две теореми на М. О. Рийд:

Теорема 4. Ако $f(z)=z+\sum\limits_{n=2}^{\infty} a_n z^n$ е почти изпъкнала от тип $\beta\,(0 \leqq \beta \leqq 2\pi)$ в кръга $|z|<1$, то имаме

(2) $$|a_n| \leqq 1+\frac{2\beta}{\pi}(n-1), \qquad (n=2, 3, \ldots).$$

От това следва

Теорема 2. Ако $f(z)$ е еднолистна в кръга $|z|<1$ и изобразява този кръг в област с контурно въртене α, т. е.

(3) $$\int_0^{2\pi}\left|1+\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right| d\vartheta \quad \alpha \quad 4\pi,$$

дето $z=re^{i\vartheta}$ и $0<r<1$, то имаме

(4) $$|a_n| \leqq 1+\frac{(\alpha-2\pi)}{2\pi}(n-1), \qquad (n=2, 3, \ldots).$$

В една своя предишна работа авторът на настоящата статия доказва неравенството

(5) $$|a_n| \leqq \prod_{k=2}^{n}\left(1+\frac{\alpha-2\pi}{k\pi}\right), \qquad (n=2, 3, \ldots).$$

Лесно се вижда, че за $2\pi \leqq \alpha<3\pi$ е в сила (5), но за $3\pi \leqq \alpha < 4\pi$ (4) е по-силно.

НЕКОТОРЫЕ ЗАМЕЧАНИЯ ОБ ОДНОЛИСТНЫХ ФУНКЦИЯХ

А. Реньи (Будапешт)

РЕЗЮМЕ

Регулярная в круге $|z|<1$ функция $f(z)$ называется „почти выпуклой типа β“ $\left(0\leqq\beta\leqq\frac{\pi}{2}\right)$, если существует такая функция $\varphi(z)$, регулярная и однолистная в круге $|z|<1$, которая отображает этот круг на выпуклую область и, кроме того, имеет место

(1) $$\left|\arg\frac{f'(z)}{\varphi'(z)}\right|<\beta \qquad \text{для } |z|<1.$$

В работе доказана следующая теорема, которая является обобщением двух теорем М. О. Рийда:

Теорема 4. Если $f(z)=z+\sum_{n=2}^{\infty}a_n z^n$ является почти выпуклой типа $\beta\left(0\leqq\beta\leqq\frac{\pi}{2}\right)$ в круге $|z|<1$, то имеем

(2) $$|a_n|\leqq 1+\frac{2\beta}{\pi}(n-1), \quad (n=2,3,\ldots).$$

Из этого следует:

Теорема 2. Если $f(z)$ однолистна в круге $|z|<1$ и отображает этот круг на область с граничным вращением a, т. е. для которой

(3) $$\int_0^{2\pi}\left|1+\mathrm{Re}\left(\frac{zf''(z)}{f'(z)}\right)\right|d\vartheta\leqslant a<4\pi,$$

где $z=re^{i\vartheta}$ и $0<r<1$, то имеет место

(4) $$|a_n|\leqq 1+\frac{(a-2\pi)}{2\pi}(n-1), \quad (n=2,3,\ldots).$$

В одной прежней работе автор настоящей статьи доказал неравенство

(5) $$|a_n|\leqq\prod_{k=1}^{n}\left(1+\frac{a-2\pi}{k\pi}\right), \qquad (n=2,3,\ldots).$$

Легко видеть, что для $2\pi\leqq a<3\pi$ (5), сильнее чем (4), однако, для $3\pi\leqq a<4\pi$ (4) является более сильным.