БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ ACADÉMIE BULGARE DES SCIENCES
ИЗВЕСТИЯ НА МАТЕМАТИЧЕСКИЯ ИНСТИТУТ
BULLETIN DE L'INSTITUT DE MATHÉMATIQUES
Том (Tome) XIV

ON C_S-CURVES IN A SUBSPACE OF A RIEMANNIAN SPACE

S. C. Rastogi

1. INTRODUCTION

C_S-curves in an Euclidean space of three dimensions were defined by Upadhyay and Trivedi (1970) as such curves on the surface of reference of a rectilinear congruence, through a point of which the line of the congruence is inclined at angles θ, φ and ψ respectively with the tangent, principal normal and binormal towards the curve at that point. Later on these curves were generalised to a subspace of a Riemannian space by Rastogi (1970). In this paper we have obtained the differential equation of C_S-curves in a subspace of a Riemannian space by a method different from Rastogi (1970). We have also proved that union curves, hyper-asymptotic curves and hypernormal curves are particular cases of C_S-curves in a subspace of a Riemannian space.

2. PREREQUISITES

Let a Riemannian space V_n with coordinates x^i, $i=1,\dots,n$, and metric tensor g_{ij} be immersed in a Riemannian space V_m of coordinates y^α, $\alpha=1,\dots,m$, and metric tensor $a_{\alpha\beta}$; then we have [Weatherburn (1957)]

$$g_{ij}=a_{\alpha\beta}\,y^\alpha_{;i}\,y^\beta_{;j}, \tag{2.1}$$

where semi-colon followed by indices denotes tensor derivative with regard to x's.

Let $N^\alpha_{\nu|}$, $\nu=n+1,\dots,m$, be the contravariant components in y's of a system of $m-n$ mutually orthogonal unit normals, to the subspace V_n, then we have

$$a_{\alpha\beta}N^\alpha_{\nu|}N^\beta_{\mu|}=\delta^\nu_\mu \tag{2.2}$$

and

$$a_{\alpha\beta}\,y^\alpha_{;i}\,N^\beta_{\nu|}=0. \tag{2.3}$$

Let $\lambda^\alpha_{\tau|}$ be the contravariant components in the y's of a unit vector in the direction of the curve of the congruence $\lambda_{\tau|}$ then it can be expressed as

(2.4) $$\lambda^a_{\tau|}=t^i_{\tau|}\, y^a_{;i}+\sum_\nu c_{\nu\tau|}\, N^a_{\nu|}.$$

If $\theta_{\nu\tau|}$ is the angle between $\lambda^a_{\tau|}$ and $N^a_{\nu|}$, then

(2.5) $$\cos\theta_{\nu\tau|}\quad c_{\nu\tau|}$$

and

(2.6) $$1-t_{\tau i|}\, t^i_{\tau|}=\sum_\nu \cos^2\theta_{\nu\tau|}.$$

If q^a and p^i are the contravariant components of the curvature vectors in V_m and V_n respectively, then

(2.7) $$q^a= y^a_{;i}\, p^i+\sum_\nu k_{\nu|}\, N^a_{\nu|},$$

where

(2.8) $$k_{\nu|}=\Omega_{\nu,ij}\,\frac{dx^i}{ds}\,\frac{dx^j}{ds}.$$

Let η^a_r, $r=3,\ldots,m$, be the contravariant components of the $m-2$ unit binormals of a curve C in V_m and $\xi^i_{r|}$ be their components in V_n. This vector $\eta^a_{r|}$ can be expressed as

(2.9) $$\eta^a_{r|}=L\xi^i_r\, y^a_{;i}+\sum_\nu M_{\nu|}\,.\,N^a_{\nu|}.$$

Let $\gamma_{\tau r}$ be the angle between $\eta^a_{r|}$ and $N^a_{\tau|}$, then from (2.9) we obtain

(2.10) $$M_{\tau|}=\cos\gamma_{\tau r|}$$

and

(2.11) $$L=\sin\gamma_{\tau r|}.$$

Hence equation (2.9) can be written as

(2.12) $$\eta^a_{r|}=\sin\gamma_{\nu r}\,\xi^i_{r|}\, y^a_{;i}+\sum_\nu\cos\gamma_{\nu r|}\, N^a_\nu.$$

3. C_S-CURVES IN A RIEMANNIAN SPACE

In a Riemannian space we define C_S-curves as the curves for which the contravariant components $\lambda^a_{\tau|}$ of the vector tangential to the curve of the congruence λ_τ are inclined at angles $\varphi_{\tau|}$, $\psi_{\tau|}$ and $\chi_{r|}$ to the tangent vector, to the first curvature vector and to the binormals of the curve.

Consider a geodesic surface determined by the vector $\omega^a_{r|}$ (which lies in another geodesic surface determined by the tangent vector dy^a/ds and the first curvature vector q^a in V_m) and $\eta^a_{r|}$, $r=3,\ldots,m$. If $\lambda^a_{\tau|}$ is a unit vector of this geodesic surface, we can express it as follows:

$$(3.1) \qquad \lambda^{\alpha}_{\tau|} = a_{\tau|}\, dy^{\alpha}/ds + b_{\tau|}\, q^{\alpha} + c_{\tau|}\, \eta^{\alpha}_{r|},$$

where a_{τ}, $b_{\tau|}$ and $c_{\tau|}$ are to be determined.

Comparing equations (2.4) and (3.1) and using (2.7) and (2.12) we obtain

$$(3.2) \qquad t^{i}_{\tau|}\, y^{\alpha}_{;i} + \sum_{\nu} C_{\nu\tau|} N^{\alpha}_{\nu|} = \left(a_{\tau|}\frac{dx^{i}}{ds} + b_{\tau|}\, p^{i} + c_{\tau|}\, \sin\gamma_{\nu r|}\right) y^{\alpha}_{;i}$$
$$+ \left(b_{\tau|} \sum_{\nu} k_{\nu|} + c_{\tau|} \sum_{\nu} \cos\gamma_{\nu r|}\right) N^{\alpha}_{\nu|}.$$

Multiplying equation (3.2) by $a_{\alpha\beta}\, y^{\beta}_{;j}$ and using equations (2.1) and (2.3) we obtain

$$(3.3) \qquad g_{ij}\, t^{i}_{\tau|} = a_{\tau|}\, g_{ij}\frac{dx^{i}}{ds} + b_{\tau|}\, g_{ij}\, p^{i} + c_{\tau|} \,.\, \sin\gamma_{\gamma r|} \,.\, g_{ij}\, \xi^{i}_{r|}.$$

Let the angles between $\lambda^{\alpha}_{\tau|}$ and dy^{α}/ds be $\varphi_{\tau|}$, between $\lambda^{\alpha}_{\tau|}$ and q^{α} be $\psi_{\tau 1}$ and between $\lambda^{\alpha}_{\tau|}$ and $\eta^{\beta}_{r|}$ be $\varkappa_{\tau|}$, then we have from (3.1)

$$(3.4) \qquad a_{\alpha\beta}\lambda^{\alpha}_{\tau|}\, dy^{\beta}/ds = \cos\varphi_{\tau|} = a_{\tau|},$$

$$(3.5) \qquad a_{\alpha\beta}\lambda^{\alpha}_{\tau|}\, q^{\beta} = K_{a} \,.\, \cos\psi_{\tau|} = b_{\tau|} \,.\, K^{2}_{a}$$

and

$$(3.6) \qquad a_{\alpha\beta}\lambda^{\alpha}_{\tau|}\, \eta^{\beta}_{r|} = \cos\chi_{\tau|} = c_{\tau 1},$$

where we have used

$$(3.7)\text{a} \qquad a_{\alpha\beta}\, dy^{\alpha}/ds\, q^{\beta} = 0,$$

$$(3.7)\text{b} \qquad a_{\alpha\beta}\, q^{\alpha}\eta^{\beta}_{r|} = 0,$$

$$(3.7)\text{c} \qquad a_{\alpha\beta}\, dy^{\alpha}/ds\, \eta^{\beta}_{r|} = 0$$

and

$$(3.8)\text{a} \qquad a_{\alpha\beta}\, dy^{\alpha}/ds\, dy^{\beta}/ds = 1,$$

$$(3.8)\text{b} \qquad a_{\alpha\beta}\, q^{\alpha} q^{\beta} = K^{2}_{a}$$

and

$$(3.8)\text{c} \qquad a_{\alpha\beta}\eta^{\alpha}_{r|}\, \eta^{\beta}_{r|} = 1.$$

Putting values of $a_{\tau|}$, $b_{\tau|}$ and $c_{\tau|}$ from (3.4), (3.5) and (3.6) in (3.3) and multiplying it by g^{jk}, summing on i we get

$$(3.9) \qquad p^{k}\cos\psi_{\tau|} - K_{a}(t^{k}_{\tau|} - \cos\varphi_{\tau|}\, dx^{k}/ds - \xi^{k}_{r|}\cos\chi_{\nu|} \,.\, \sin\gamma_{\tau r|}) = 0.$$

Equation (3.9) is the differential equation of C_S-curves in a subspace of a Riemannian space.

In the next section we shall consider some particular cases.

4. PARTICULAR CASES

From equation (3.1) we san easily obtain

(4.1) $$1 = a_{\tau|}^2 + b_{\tau|}^2 K_a^2 + c_{\tau|}^2,$$

which by virtue of (3.4), (3.5) and (3.6) yields

(4.2) $$1 = \cos^2 \varphi_{\tau|} + \cos^2 \psi_{\tau|} + \cos^2 \varkappa_\tau$$

Case I. For $\varkappa_{\tau|} = \pi/2$, equation (4.2) gives

(4.3) $$\cos \psi_{\tau|} = \sin \varphi_\tau .$$

Putting $\chi_{\tau|} = \pi/2$ and $\cos \psi_{\tau|} = \sin \varphi_{\tau|}$ in (3.9) we obtain

(4.4) $$p^k - K_a \operatorname{cosec} \varphi_{\tau|}(t^k_{\tau|} - \cos \varphi_{\tau|} \, dx^k/ds) = 0,$$

which is the differential equation of union curves in a subspace of a Riemannian space [Rastogi (1971)].

Case II. For $\psi_{\tau|} = \pi/2$, equation (3.9) yields

(4.4) $$K_a(t^k_{\tau|} - \cos \varphi_{\tau|} dx^k/ds - \xi^k_{r|} . \cos \chi_{\tau|} . \sin \gamma_{\tau r|}) = 0.$$

For $\psi_\tau = \pi/2$, equation (4.2) gives $\cos \chi_{\tau|} = \sin \varphi_{\tau|}$ and since $K_a \neq 0$, therefore the equation (4.5) reduces to

(4.6) $$\xi^k_{r|} - \operatorname{cosec} \varphi_{\tau|} . \operatorname{cosec} \gamma_{\tau r|}(t^k_{\tau|} - dx^k/ds \cos \varphi_{\tau|}) = 0,$$

which is the differential equation of hyper-asymptotic curves in a subspace of a Riemannian space [Rastogi (1971)].

Case III. For $\varphi_{\tau|} = \pi/2$, equation (4.2) yields $\cos \psi_{\tau|} = \sin \chi_{\tau|}$, which when substituted in (3.9) gives rise to

(4.7) $$p^k - K_a \operatorname{cosec} \chi_{\tau|}(t^k_{\tau|} - \xi^k_{r|} \cos \chi_{\tau|} . \sin \gamma_{\tau r|}) = 0.$$

Equation (4.7) is the differential equation of hypernormal curves in a subspace of a Riemannian space Rastogi (1971).

Case IV. For $p^k = 0$, equation (3.9) yields (4.5).

Thus we have:

Theorem (4.1). For a geodesic in V_n, C_S-curves satisfy one of the following:

i) the curve is a geodesic in V_m,

ii) the curve is a hyper-asymptotic curve.

Acknowledgements. The author is thankful to Dr. M. D. Upadhyay for help in the preparation of this paper.

REFERENCES

1. Rastogi, S. C. A study of certain curves in generalised spaces. Ph. D. Thesis. Lucknow University, 1970.
2. Rastogi, S. C. Union, hyper-asymptotic and hypernormal curves in a subspace of a Riemannian space (in print).
3. Trivedi, H. K. N. Differential Geometry of curves and linear congruences. Ph. D. Thesis. Lucknow University, 1970.
4. Weatherburn, C. E. An introduction to Riemannian Geometry and the Tensor Calculus. London, 1957.

Received 17. III. 1971.

ВЪРХУ C_S-КРИВИТЕ В ЕДНО ПОДПРОСТРАНСТВО НА РИМАНОВО ПРОСТРАНСТВО

С. Растоги

(*Резюме*)

През 1970 г. Упадхиай и Триведи дефинираха C_S-крива в тримерно евклидово пространство като такава крива върху базисната повърхнина на една конгруенция прави, в произволна точка на която правата на конгруенцията през тази точка сключва ъгли θ, φ и ψ съответно с допирателната, главната нормала и бинормалата към кривата в тази точка. По-късно (1970) Растоги направи обобщение на тези криви в случая на подпространство на риманово пространство. В тази работа е получено диференциално уравнение на C_S-кривите в подпространство на риманово пространство чрез метод, различен от този на Растоги (1970). Също така е доказано, че съединителните криви, хипер-асимптотичните криви и хипернормалните криви са частни случаи на C_S-криви в подпространство на риманово пространство.

О C_S-КРИВЫХ В ПОДПРОСТРАНСТВЕ РИМАНОВОГО ПРОСТРАНСТВА

С. Растоги

(*Резюме*)

В 1970 г. Упадхиай и Триведи определили C_S-кривую в трехмерном евклидовом пространстве как кривую на базисной поверхности одной конгруэнции прямых, в произвольной точке которой прямая конгруэнции, проходящая через эту точку, образует углы θ, φ и ψ соответственно с касательной, главной нормалью и бинормалью к кривой в этой точке. Позже (1970) Растоги сделал обобщение этих кривых для случая в подпространстве риманового пространства. В этой работе получено дифференциальное уравнение C_S-кривых в подпространстве риманового пространства по методу, отличному от метода Растоги (1970). Также было доказано, что соединительные, гиперасимптотические и гипернормальные кривые являются частными случаями C_S-кривых в подпространстве риманового пространства.