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ON SOME PROPERTIES OF THE OPERATORS OF THE GENERALIZED ANGULAR MOMENTUM

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1. INTRODUCTION

In the classical mechanics, the generalized momentum of a particle is related to its velocity by the following relation [1]:

$$m\mathbf{v}=\mathbf{p}-\frac{e}{c}\mathbf{A}(\mathbf{r}). \tag{1}$$

In order to find the operator v in quantum mechanics, we have to commute the vector $\mathbf{r}$ with the Hamiltonian. A simple calculation gives:

$$m\widehat{v}=\widehat{p}-\frac{e}{c}\widehat{A}(r) \tag{2}$$

which is analogous to the classical expression (1).

For the operators of the velocity components, the following expressions are valid:

$$\begin{aligned}\{v_x, v_y\} &= \frac{ieh}{m^2c}H_z,\\ \{v_y, v_z\} &= \frac{ieh}{m^2c}H_x,\\ \{v_z, v_x\} &= \frac{ieh}{m^2c}H_y,\end{aligned} \tag{3}$$

where H_x, H_y, H_z are the projections of the magnetic field H. From eqs. (3) it is obvious that the velocities v_x, v_y, v_z do not commute in a magnetic field.

Now according to the definition of the operator of angular momentum:

$$\widehat{l}=\widehat{r}\times\widehat{p} \tag{4}$$

one can determine the generalized operator of angular momentum:

$$\widehat{L}=\widehat{r}\times\left(\widehat{p}-\frac{e}{c}\widehat{A}(r)\right). \tag{5}$$

This operator when multiplied in $\frac{e}{2mc}$, gives the operator of the magnetic moment [2, 3], i. e.

(6) $$\hat{m} \quad \frac{e}{2mc}\hat{L}.$$

2. PROPERTIES OF THE OPERATORS $\hat{L}_x$, $\hat{L}_y$, $\hat{L}_z$

The relation (5), because of (4) is written as:

(7) $$\hat{L} \quad \hat{l}-\frac{e}{c}\,\hat{r}\times\hat{A}(\mathbf{r}).$$

Operators L_x, L_y, L_z are now defined as follows:

(8) $$L_x=-i\hbar\left\{y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}+\frac{ie}{\hbar c}(zA_y-yA_z)\right\},$$

(9) $$L_y=-i\hbar\left\{z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}+\frac{ie}{\hbar c}(xA_z-zA_x)\right\},$$

(10) $$L_z=-i\hbar\left\{x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}+\frac{ie}{\hbar c}(yA_x-xA_y)\right\},$$

where A_x, A_y, A_z are the projections of the vector potential **A**.

The commutators of the above operators are:

(11) $$L_xL_y-L_yL_x=i\hbar L_z+i\hbar\frac{e}{c}z(\mathbf{r}\,.\,\mathbf{H}),$$

(12) $$L_yL_z-L_zL_y=i\hbar L_x+i\hbar\frac{e}{c}x(\mathbf{r}\,.\,\mathbf{H}),$$

(13) $$L_zL_x-L_xL_z \quad i\hbar L_y+i\hbar\frac{e}{c}y(\mathbf{r}\,.\,\mathbf{H}).$$

It is noticeable that the new operators L_x, L_y, L_z do not form a lie group.

According to the total angular momentum operator l^2, we now define the total generalized angular momentum operator:

(14) $$L^2=L_x^2+L_y^2+L_z^2.$$

After some algebra, eqs. (14) and (8), (9), (10) give:

(15) $$L^2=-\hbar^2\Bigg[\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)^2+\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)^2+\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)^2$$
$$+\frac{2ie}{\hbar c}\Big\{(z(yA_y+xA_x)-(x^2+y^2)A_z)\frac{\partial}{\partial z}+(y(xA_x+zA_z)-(x^2+z^2)A_y)\frac{\partial}{\partial y}$$
$$+(x(zA_z+yA_y)-(y^2+z^2)A_x)\frac{\partial}{\partial x}+xA_x+yA_y+zA_z\Big\}$$
$$+\frac{ie}{\hbar c}\Big\{xy\left(\frac{\partial A_x}{\partial y}+\frac{\partial A_y}{\partial x}\right)+xz\left(\frac{\partial A_z}{\partial x}+\frac{\partial A_x}{\partial z}\right)+yz\left(\frac{\partial A_y}{\partial z}+\frac{\partial A_z}{\partial y}\right)$$
$$-x^2\left(\frac{\partial A_y}{\partial y}+\frac{\partial A_z}{\partial z}\right)-y^2\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_z}{\partial z}\right)-z^2\left(\frac{\partial A_x}{\partial x}+\frac{\partial A_y}{\partial y}\right)\Big\}$$
$$-\frac{e^2}{\hbar^2c^2}\{(zA_y-yA_z)^2+(xA_z-zA_x)^2+(yA_x-xA_y)^2\}\Bigg].$$

The case of a uniform external magnetic field H parallel to the z-axis is of particular interest. For this case one distinguishes three different gauges of the vector potential $\mathbf{A}$:

$$
\begin{aligned}
&\text{a)}\quad \mathbf{A}=\left(-\frac{1}{2}Hy,\ \frac{1}{2}Hx,\ 0\right),\\
(16)\qquad &\text{b)}\quad \mathbf{A}=(-Hy,0,0),\\
&\text{c)}\quad \mathbf{A}=(0,Hx,0)
\end{aligned}
$$

the Operators L_x, L_y, L_z are, for each one of the three above cases: $B=\frac{eH}{\hbar c}$:

$$
\begin{aligned}
&\text{a)}\quad L_x=-i\hbar\left\{y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}+i\frac{B}{2}xz\right\},\\
(17)\qquad &\quad\ \ L_y=-i\hbar\left\{z\frac{\partial}{\partial y}-x\frac{\partial}{\partial z}+i\frac{B}{2}yz\right\},\\
&\quad\ \ L_z=-i\hbar\left\{x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}-i\frac{B}{2}(x^2+y^2)\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\text{b)}\quad L_x=-i\hbar\left\{y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right\},\\
(18)\qquad &\quad\ \ L_y=-i\hbar\left\{z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}+iByz\right\},\\
&\quad\ \ L_z=-i\hbar\left\{x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}-iBy^2\right\},
\end{aligned}
$$

$$
\begin{aligned}
&\text{c)}\quad L_x=-i\hbar\left\{y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}+iBxz\right\},\\
(19)\qquad &\quad\ \ L_y=-i\hbar\left\{z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right\},\\
&\quad\ \ L_z=-i\hbar\left\{x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}-iBx^2\right\}
\end{aligned}
$$

It can be seen that eqs. (18) and (19) can be deduced from eqs. (17) by multiplication from the right by the factors $e^{-iB/2xy}$ and $e^{iB/2xy}$ respectively. Therefore in what follows, only the operators (17) are examined.

The total operator L^2, for the considered case takes the following form:

$$
\begin{aligned}
(20)\qquad L^2=-\hbar^2\Big\{&\left(y\frac{\partial}{\partial z}-z\frac{\partial}{\partial y}\right)^2+\left(z\frac{\partial}{\partial x}-x\frac{\partial}{\partial z}\right)^2+\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)^2\\
&-iB(x^2+y^2+z^2)\left(x\frac{\partial}{\partial y}-y\frac{\partial}{\partial x}\right)-\frac{B^2}{4}(x^2+y^2+z^2)(x^2+y^2)\Big\}.
\end{aligned}
$$

Operators l_z and L^2 commute, i. e.

$$(21)\qquad L^2 l_z-l_z L^2=0$$

as it can be easily proved. Consequently these operators have a common set of eigenfunctions.

In spherical coordinates, these operators are written as

$$l_z = -i\hbar \frac{\partial}{\partial\varphi}, \tag{22}$$

$$L^2 = -\hbar^2 \left\{ \frac{\partial^2}{\partial\vartheta^2} + \operatorname{ctg}\vartheta \frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} - iBr^2 \frac{\partial}{\partial\varphi} - \frac{B^2}{4} r^4 \sin^2\vartheta \right\} \tag{23}$$

The Schrödinger equation

$$\left\{ \frac{\hbar^2}{2m} \left(\nabla - \frac{ie}{\hbar c} A(r) \right)^2 + E - V(r) \right\} \psi(r) = 0 \tag{24}$$

for the symmetrical vector potential takes the form

$$\nabla\psi - iB \left(x\frac{\partial\psi}{\partial y} - y\frac{\partial\psi}{\partial x} \right) + \left\{ \frac{2m}{\hbar^2} (E - V(r)) - \frac{B^2}{4}(x^2+y^2) \right\} \psi = 0. \tag{25}$$

If the potential energy $V(r)$ depends only on the distance r, then the above equation, in spherical coordinates, takes the following form

$$\left\{ \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{L^2}{\hbar^2 r^2} \right\} \psi(r) + \frac{2m}{\hbar^2} (E - V(r)) \psi(r) = 0. \tag{26}$$

Equation (25) for $V(r)=0$ can be solved exactly in cartesian coordinates Landau [4] and Jannussis [5], and in cylindrical coordinates Dingle [6]. The same equation can not be solved exactly in spherical coordinates. Equation (26) will be investigated in the following section concerning the case of a rotator in a magnetic field.

3. THE ROTATOR IN A UNIFORM MAGNETIC FIELD

The moment of inertia will be:

$$I = M_0 r_0^2, \tag{27}$$

where M_0 is the mass of the rotator and r_0 is the distance from the rotation center.

The Schrödinger equation (26), in this case, and for $V(r) = 0$, becomes:

$$-L^2\psi(r_0, \vartheta, \varphi) + 2IE\psi(r_0, \vartheta, \varphi) = 0 \tag{28}$$

or

$$\left(\frac{\partial^2}{\partial\vartheta^2} + \operatorname{ctg}\vartheta \frac{\partial}{\partial\vartheta} + \frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2} - iBr_0^2 \frac{\partial^2}{\partial\varphi^2} - \frac{B^2}{4} r_0^4 \sin^2\vartheta \right) \psi(r_0, \vartheta, \varphi) + \frac{2I}{\hbar^2} E\psi(r_0, \vartheta, \varphi) = 0. \tag{29}$$

The solution of the above equation is of the form

$$\psi(r_0, \vartheta, \varphi) = e^{im\varphi} u_m(r_0, \vartheta). \tag{30}$$

The function $u_m(r_0, \vartheta)$ is a solution of the following differential equation:

$$\left\{\frac{\partial^2}{\partial\vartheta^2} + \operatorname{ctg}\vartheta\frac{\partial}{\partial\vartheta} - \frac{m^2}{\sin^2\vartheta} - \frac{B^2}{4}r_0^4\sin^2\vartheta + mBr_0^2 + \frac{2I}{\hbar^2}E\right\}u_m(\vartheta) = 0 \tag{31}$$

which for $\cos\vartheta = z$, takes the form:

$$(1-z^2)\frac{d^2u_m}{dz^2} - 2z\frac{du_m}{dz} + \left\{\left(mB + E\frac{2M_0}{\hbar^2}\right)r_0^2 - \frac{m^2}{1-z^2} - \frac{B^2}{4}r_0^4(1-z^2)\right\}u_m = 0. \tag{32}$$

The above differential equation is known as the spheroid differential equation [7]. Its regular form as given in [7] is as follows:

$$(1-z^2)\frac{d^2u}{dz^2} - 2z\frac{du}{dz} + \left(\lambda - \frac{\mu^2}{1-z^2} - \gamma^2(1-z^2)\right)u = 0. \tag{33}$$

Comparison of equations (32) and (33) yields:

$$\lambda = \left(B + \frac{2M_0}{\hbar^2}E\right)r_0^2, \quad \mu^2 = m^2, \quad \gamma^2 = \frac{B^2}{4}r_0^4. \tag{34}$$

Therefore the problem can be considered as solved, because the theory of spheroid functions is already known in the literature and the eigenvalues λ are given in a form of power series of γ^2.

Thus one has:

$$\left(mB + \frac{2M_0}{\hbar^2}E\right)r_0^2 = l(l+1) + \frac{B^2r_0^4}{4}\left\{1 + \frac{(2|m|-1)(2|m|+3)}{(2l-1)(2l+3)}\right\}$$
$$+ \frac{1}{2}\left\{\frac{(l-|m|-1)(l-|m|)(l+|m|-1)(l+|m|)}{(2l-3)(2l-1)^2(2l+1)}\right.$$
$$\left. - \frac{(l-|m|+1)(l-|m|+2)(l+|m|+1)(l+|m|+2)}{(2l+1)(2l+3)^2(2l+5)}\right\}\left(\frac{B^2r_0^4}{4}\right)^2 \tag{35}$$

and the eigenvalues of the energy are the following:

$$\mu_0 Hm + E = \frac{\hbar^2}{2I}l(l+1) + \frac{\hbar^2B^2r_0^4}{8I}\left\{1 + \frac{(2|m|-1)(2|m|+3)}{(2l-1)(2l+3)}\right\}$$
$$+ \frac{\hbar^2}{4I}\left(\frac{B^2r_0^4}{4}\right)^2\left\{\frac{(l-|m|-1)(l-|m|)(l+|m|-1)(l+|m|)}{(2l-3)(2l-1)^2(2l+1)}\right.$$
$$\left. - \frac{(l-|m|+1)(l-|m|+2)(l+|m|+1)(l+|m|+2)}{(2l+1)(2l+3)^2(2l+5)}\right\} + \ldots, \tag{36}$$

where $\mu_0 = \frac{e\hbar}{2M_0c}$

The above result can also be applied to a diatomic molecule, when I is replaced by the moment of inertia about the axis of rotation through the center of mass (center of rotation).

4. SOLUTION OF EQUATION (26) BY USE OF THE PERTURBATION METHOD

Equation (26), as it has been said, cannot be exactly solved in spherical coordinates. As this equation describes several physical problems, we think that it is worth to examine here several forms of the dynamic energy $V(r)$. Eq. (26) has been already negotiated for the case of the hydrogen atom by many authors; but some of their theoretical results do not agree to the experimental ones [8], as e. g. in diamagnetism.

We may write eq. (26) in the form

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\Lambda^2}{r^2}+\frac{2M}{\hbar^2}(E-W(r))+\frac{B^2r^2}{4}\cos^2\vartheta\right\}\psi=0 \tag{37}$$

with

$$\Lambda^2=\frac{\partial^2}{\partial\vartheta^2}+\operatorname{ctg}\vartheta\frac{\partial}{\partial\vartheta}+\frac{1}{\sin^2\vartheta}\frac{\partial^2}{\partial\varphi^2}-ir^2\frac{\partial}{\partial\varphi} \tag{38}$$

and

$$W(r)=V(r)+\frac{\hbar^2B^2}{8M}r^2. \tag{39}$$

If we consider the term $\frac{B^2r^2}{4}\cos^2\vartheta$ as a perturbation, then the solution of (37) will be given in a form of series, as:

$$\psi(\mathbf{r})=\psi_0(r)+\frac{B^2}{4}\psi_1(r)+\left(\frac{B^2}{4}\right)^2\psi_2(r)+ \tag{40}$$

$$E=E_0+\frac{B^2}{4}E_1+\left(\frac{B^2}{4}\right)^2E_2+ \tag{41}$$

Inserting (40) and (41) in (37) we have

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\Lambda^2}{r^2}+\frac{2M}{\hbar^2}(E-W(r))\right\}\psi_0(r)=0, \tag{42}$$

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\Lambda^2}{r^2}+\frac{2M}{\hbar^2}(E_0-W(r))\right\}\psi_1(r)+\left(\frac{2M}{\hbar^2}E_1+r^2\cos^2\vartheta\right)\psi_0(r)\quad 0, \tag{43}$$

$$\left\{\frac{\partial^2}{\partial r^2}+\frac{2}{r}\frac{\partial}{\partial r}+\frac{\Lambda^2}{r^2}+\frac{2M}{\hbar^2}(E_0-W(r))\right\}\psi_2+\left(\frac{2}{\hbar^2}ME_1+r^2\cos^2\vartheta\right)\psi_1 \tag{44}$$

$$+\frac{2M}{\hbar^2}E_2\psi_0(r)=0,$$

Equation (42) has a solution of the form

$$\psi_0(r)=F(r)e^{im\varphi}Y_{l,m}(\vartheta), \tag{45}$$

where $Y_{l,m}(\vartheta)$ are the spherical harmonics and $F(r)$ fulfills the following differential equation

$$\left\{\frac{d^2}{dr^2}+\frac{2}{r}\frac{d}{dr}+\frac{2M}{\hbar^2}\left(E_0+\mu Hm-\frac{\hbar^2}{2M}\frac{l(l+1)}{r^2}-\frac{\hbar^2B^2}{8M}r^2-V(r)\right)\right\}F(r)=0. \tag{46}$$

Equation (46) for

$$rF(r)=f(r) \tag{47}$$

takes its regular form

(48) $$\frac{\hbar^2}{2M}\frac{d^2f}{dr^2}+\left(\mathcal{E}_0-V(r)-\frac{\hbar^2 l(l+1)}{2Mr^2}-\frac{M}{2}\,\omega_L^2 r^2\right)f(r)=0,$$

where

(49) $$\mathcal{E}_0=E_0+\mu Hm,\ \text{and}\ \omega_L=\frac{eH}{2Mc}$$

The above equation (48) can be solved exactly for the cases of

(50) $$V(r)=\frac{M}{2}\,\omega^2 r^2$$

(spherical harmonic oscillator) and

(51) $$V(r)=\frac{a}{r^2}$$

Here we will examine the case of the spherical harmonic oscillator. For $V=\frac{M}{2}\omega^2 r^2$, equation (48) is written

(52) $$\frac{\hbar^2}{2M}\frac{d^2f}{dr^2}+\left(\mathcal{E}_0-\frac{M}{2}\,\Omega_0^2 r^2-\frac{\hbar^2 l(l+1)}{2Mr^2}\right)f=0,$$

where it has been put

(53) $$\Omega_0^2=\omega^2+\omega_L^2.$$

The normalised eigenfunctions of eqs. (52) or the $F(r)$ are known [9, 10], and are

(54) $$F(r)=\left(\frac{4M\Omega_0}{\hbar}\right)^{1/4}\left(\frac{\Gamma(n+1)}{\Gamma(n+l+3/2)}\right)^{1/2}\frac{1}{r}\left(\frac{M\Omega_0}{\hbar}\,r^2\right)^{\frac{l+1}{2}}e^{-\frac{M\Omega_0 r^2}{2\hbar}}L_n^{l+1/2}\left(\frac{M\Omega_0}{\hbar}\,r^2\right)$$

with the corresponding eigenvalues

(55) $$\mathcal{E}_0=\hbar\Omega_0(2n+l+3/2).$$

Consequently, the unperturbed eigenfunctions are the following

(56) $$\psi_0(r)=\left(\frac{4M\Omega_0}{\hbar}\right)^{1/4}\left\{\frac{\Gamma(n+1)}{\Gamma(n+l+3/2)}\right\}^{1/2}\frac{1}{r}\left(\frac{M\Omega_0 r^2}{\hbar}\right)^{\frac{l+1}{2}}e^{-\frac{M\Omega_0 r^2}{2\hbar}}L_n^{l+1/2}\left(\frac{M\Omega_0 r^2}{\hbar}\right)\cdot\frac{e^{im\varphi}}{\sqrt{2\pi}}\,Y_{l,m}(\vartheta).$$

The energy E_1 is given by the integral

(57) $$\frac{2M}{\hbar^2}E_1=\frac{1}{2\pi}\int_0^\infty\int_0^{2\pi}\int_0^\pi r^2\cos^2\vartheta\sin\vartheta\, r^2\,dr\,d\varphi\,d\vartheta\,\psi_0(r)\psi_0^*(r)$$

and

(58) $$E_1=-\frac{\hbar^2}{2M}\left(\frac{4M\Omega_0}{\hbar}\right)^{1/2}\frac{\Gamma(n+1)}{\Gamma(n+l+3/2)}\int_0^\infty e^{-\frac{M\Omega_0 r^2}{\hbar}}r^2\left(\frac{M\Omega_0 r^2}{\hbar}\right)^{l+1}\left\{L_n^{l+1}\left(\frac{M\Omega_0 r^2}{\hbar}\right)\right\}^2 dr$$

$$\cdot\int_0^\pi\cos^2\vartheta\, Y_{l,m}(\vartheta)\,Y_{l,m}^*(\vartheta)\sin\vartheta\,d\vartheta.$$

The above integrals can be calculated easily by known methods [2], and we will have

$$(59)\qquad E_1 = -\frac{\hbar^2}{2n}\left(2n+l+\frac{3}{2}\right)\frac{\hbar}{M\Omega_0}\,\frac{2l^2+2l-1-2m^2}{(2l+1)(2l+3)}\,.$$

Consequently the eigenvalues of the energy up to the first term are the following

$$(60)\qquad E = -\mu Hm + \hbar\Omega\left(2n+l+\frac{3}{2}\right)\left\{1-\frac{1}{2}\,\frac{\omega_L^2}{\Omega_0^2}\left(\frac{2l^2+2l-1-2m^2}{(2l-1)(2l+3)}\right)\right\},$$

where $\mu = \frac{e\hbar}{2mc}$ the Bohr magneton and $|m| \leq l$.

If we put $2n+l+1 = N$ 1, 2, .., then eq. (60) is written as

$$(61)\qquad E = -\mu Hm + \hbar\Omega\left(N+\frac{1}{2}\right)\left\{1-\frac{1}{2}\,\frac{\omega_L^2}{\Omega^2}\left(\frac{2l^2+2l-1-2m^2}{(2l-1)(2l+3)}\right)\right\}$$

and $l \leq N-1$.

The case $\Omega = \omega_L$, i.e. $V(r)$ 0, ω 0, yields

$$(62)\qquad E = -\hbar\omega_L\, m + \hbar\omega_L\left(N+\frac{1}{2}\right)\left\{1-\frac{1}{2}\,\frac{2l^2+2l-1-2m^2}{(2l-1)(2l+3)}\right\}.$$

Especially for the evaluation of the eigenvalues it is preferable to use the method of development in spherical harmonic functions. For the solution of (37) we put

$$(63)\qquad \psi(r) = \sum_{l,m}\sum_{m=-l}^{l} R_{l,m}(r)e^{im\varphi}Y_{l,m}(\vartheta)$$

and by use of the recurrence relation of the spherical functions [2] we obtain the following differo-differential equation for the radial function $R_{l,m}(r)$

$$(64)\qquad \left\{\frac{d^2}{dr^2}+\frac{2M}{\hbar^2}\left\{\mathcal{E}-V(r)-\frac{B^2\hbar^2}{8m}\left(1-\frac{2l^2+2l-1-2m^2}{(2l+1)(2l+3)}\right)r^2-\frac{\hbar^2}{2M}\frac{l(l+1)}{r^2}\right\}\right\}R_{l,m}(r)$$

$$+\frac{B^2r^2}{4}\left\{\sqrt{\frac{\{(l+2)^2-m^2\}\{(l+1)^2-m^2\}}{(2l+1)(2l+3)^2(2l+5)}}R_{l+2,m}(r)+\sqrt{\frac{(l^2-m^2)\{(l-1)^2-m^2\}}{(2l+1)(2l-1)^2(2l-3)}}R_{l-2,m}(r)\right\}=0,$$

with

$$(65)\qquad \mathcal{E} = E + \mu Hm.$$

The above equation for the case of the spherical harmonic oscillator, becomes

$$(66)\qquad \left\{\frac{d^2}{dr^2}+\frac{2M}{\hbar^2}\left(\mathcal{E}-\frac{M}{2}\,\Omega^2(l,m)\,r^2-\frac{\hbar^2}{2M}\,\frac{l(l+1)}{r^2}\right)\right\}R_{l,m}(r)$$

$$+\frac{B^2r^2}{4}\left\{\sqrt{\frac{\{(l+2)^2-m^2\}\{(l+1)^2-m^2\}}{(2l+1)(2l+3)^2(2l+5)}}R_{l+2,m}(r)+\sqrt{\frac{(l-m^2)\{(l+1)^2-m^2\}}{(2l+1)(2l-1)^2(2l-3)}}R_{l-2,m}(r)\right\}=0,$$

where

$$(67)\qquad \Omega^2(l,m) = \omega^2+\omega_L^2\left(1-\frac{2l^2+2l-1-2m^2}{(2l-1)(2l+3)}\right).$$

We consider as zero order approximation of (66) the solution of the equation

$$\left\{\frac{d^2}{dr^2}+\frac{2M}{\hbar^2}\left(\mathcal{E}_0-\frac{M}{2}\,\Omega^2(l,m)r^2-\frac{\hbar^2}{2M}\frac{l(l+1)}{r^2}\right)\right\}R^{(0)}_{n,l,m}=0. \tag{68}$$

The solution of the above equation is the following:

$$R^{(0)}_{n,l,m}(r)=\left(\frac{4M\Omega(l,m)}{\hbar}\right)^{1/4}\left(\frac{\Gamma(n+1)}{\Gamma(n+l+3/2)}\right)^{1/2}\left(\frac{M\Omega(l,m)}{\hbar}r^2\right)^{\frac{l+1}{2}}e^{-\frac{M\Omega(l,m)}{2\hbar}r^2}\cdot L_n^{l+1/2}\left(\frac{M\Omega(l,m)}{\hbar}r^2\right) \tag{69}$$

with the corresponding eigenvalues

$$\mathcal{E}_0=\hbar\Omega(l,m)\left(2n+l+\frac{3}{2}\right). \tag{70}$$

The above zero-order eigenvalues contain the eigenvalues (60), as one can see by development of the function $\Omega(l,m)$ in respect to $\frac{\omega_L^2}{\omega^2+\omega_L^2}$. Up to first-order approximation one takes the relation (60).

Furtheron, the study of (66) by the simple method of perturbations or of successive approximations, yields the following eigenvalues of the energy up to the first-order approximations, i. e.

$$\mathcal{E}=\mathcal{E}_0+\hbar\frac{\omega_L^2}{\Omega(l,m)}\mathcal{E}_1, \tag{71}$$

where

$$\mathcal{E}_1=\sqrt{\frac{\{(l+2)^2-m^2\}\{(l+1)^2-m^2\}}{(2l+1)(2l+3)^2(2l+5)}}\,I_1-\sqrt{\frac{(l^2-m^2)\{(l-1)^2-m^2\}}{(2l-1)(2l+1)^2(2l+3)}}\,I_2 \tag{72}$$

and

$$I_1=2^{l+\frac{1}{2}}\left(\frac{\Omega(l+2,m)}{\Omega(l,m)}\right)^{\frac{l+3}{2}+\frac{1}{4}}\left(1+\frac{\Omega(l+2,m)}{\Omega(l,m)}\right)^{-l-\frac{7}{2}}\sqrt{\frac{\Gamma\left(n+l+\frac{7}{2}\right)\Gamma\left(n+l+\frac{3}{2}\right)}{\Gamma(n+1)}} \tag{73}$$

$$\cdot\sum_{n_1,n_2=0}^{n}\binom{n}{n_1}\binom{n}{n_2}2^{n_1+n_2}(-1)^{n_2}\left(-\frac{\Omega(l+2,m)}{\Omega(l,m)}\right)^{n_1}\left(1+\frac{\Omega(l+2,m)}{\Omega(l,m)}\right)^{n_1+n_2}\Gamma\left(l+n_1+n_2+\frac{7}{2}\right)$$

$$I_2=2^{l-\frac{3}{2}}\left(\frac{\Omega(l-2,m)}{\Omega(l,m)}\right)^{\frac{l-1}{2}+\frac{1}{4}}\left(1+\frac{\Omega(l-2,m)}{\Omega(l,m)}\right)^{-l-\frac{3}{2}}\sqrt{\Gamma\left(n+l+\frac{3}{2}\right)\Gamma\left(n+l-\frac{1}{2}\right)} \tag{74}$$

$$\cdot\sum_{n_1,n_2=0}^{n}\binom{n}{n_1}\binom{n}{n_2}2^{n_1+n_2}(-1)^{n_2}\left(-\frac{\Omega(l-2,m)}{\Omega(l,m)}\right)^{n_1}\left(1+\frac{\Omega(l-2,m)}{\Omega(l,m)}\right)^{n_1+n_2}\Gamma\left(l+n_1+n_2+\frac{3}{2}\right)$$

with $\Omega(l,m)$ is given by (67).

The above method is the most indicated for the calculation of the eigenvalues, because the method of the perturbations leads to difficult calculations concerning the eigenvalues. One can see that by comparison of the two methods. The zero-order approximation of the second method, concerning the eigenvalues, gives new and satisfactory results. The eigenvalues

$$E = -\hbar\omega_L m + \hbar\Omega(l, m)\left(N + \frac{1}{2}\right) \tag{75}$$

with $l \leq N-1$, $|m| \leq l$, describe the levels of the energy.

The higher-order approximation terms contribute only very little to the energy. This statement seems true from the behaviour of the function

$$\Omega(l, m) = \Omega_0 \sqrt{1 - \frac{\omega_L^2}{\Omega_0^2} \frac{2l^2 + 2l - 1 - 2m^2}{(2l-1)(2l+3)}} \tag{76}$$

with $\Omega_0^2 = \omega^2 + \omega_L^2$.

The function $\frac{2l^2+2l-1-2m^2}{(2l-1)(2l+3)}$ stays always smaller than $\frac{1}{2}$ for almost every value of l and m.

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ВЪРХУ НЯКОИ СВОЙСТВА НА ОПЕРАТОРИТЕ НА ОБОБЩЕНИЯ ЪГЛОВ МОМЕНТ

А. Янусис, П. Ктенас

(Резюме)

В статията се разглеждат някои свойства на операторите на обобщения ъглов момент. Тези оператори са пропорционални на операторите на магнитния момент.

О НЕКОТОРЫХ СВОЙСТВАХ ОПЕРАТОРОВ ОБОБЩЕННОГО УГЛОВОГО МОМЕНТА

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(Резюме)

В статье рассматриваются некоторые свойства операторов обобщенного углового момента. Указывается, что эти операторы пропорциональны операторам магнитного момента.