Theory of Exponential Functionals of Lévy Processes

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Contents

Chapter 1

Introduction

The main purpose of the dissertation is to account for the main contributions of the candidate in the rapidly developing theory of exponential functionals of Lévy processes. We recall that an exponential functional of a possibly killed Lévy process is defined as

$$I_{\Psi} = \int_0^{\infty} e^{-\xi_s} ds.$$

Exponential functionals have played a significant role in probability theory and its applications as the perusal of the chapters of this work will convince the reader. For this purpose there have been a number of studies in this area which have approached these objects from different directions. They have resulted in a number of partial results but there have not arisen any general theory. The main contribution of this work is namely the development of the latter. Sifting through the multitude of available approaches for the study of exponential functionals of Lévy processes, mostly for the derivation of the factorizations presented in Chapters 2 and 3 of this work, it has gradually become clear that the most fruitful method is a link between the Mellin transforms of the exponential functionals and a class of special functions that generalizes the celebrated Gamma function. This link has allowed for the computation of the Mellin transform in terms of these functions which we call Bernstein-Gamma functions. This, in turn, through Mellin inversion opened the door to obtaining a lot of new information about the law, the density and its derivatives, of I_{Ψ}. We emphasize that this is almost invariably achieved in complete generality without restriction to special classes of Lévy processes that constitute I_{Ψ}.

We would also like to stress upon the fact that the results of this work are already used in the spectral theory of positive self-similar Markov processes, which was recently developed by Pierre Patie and the candidate. They can also help to strengthen and recover many published results and we emphasize this in the respective chapters. Most importantly these results seem to have beneficial implications in many other emerging studies such as random processes in random environments, pricing of Asian options and many others.

Each subsequent chapter of this work is essentially a published paper or a preprint.

Chapter 2

A Wiener-Hopf type factorization for the exponential functional of Lévy processes

Abstract: For a Lévy process $\xi = (\xi_t)_{t\geq 0}$ drifting to $-\infty$, we define the so-called exponential functional as follows

$$\mathrm{I}_\xi = \int_0^\infty e^{\xi_t} dt.$$

Under mild conditions on ξ, we show that the following factorization of exponential functionals

$$\mathrm{I}_\xi \stackrel{d}{=} \mathrm{I}_{H^-} \times \mathrm{I}_Y$$

holds, where, $\times$ stands for the product of independent random variables, H^- is the descending ladder height process of ξ and Y is a spectrally positive Lévy process with a negative mean constructed from its ascending ladder height process. As a by-product, we generate an integral or power series representation for the law of I_ξ for a large class of Lévy processes with two-sided jumps and also derive some new distributional properties. The proof of our main result relies on a fine Markovian study of a class of generalized Ornstein-Uhlenbeck processes which is of independent interest on its own. We use and refine an alternative approach of studying the stationary measure of a Markov process which avoids some technicalities and difficulties that appear in the classical method of employing the generator of the dual Markov process.

2.1 Introduction and main results

We are interested in studying the law of the so-called exponential functional of Lévy processes which is defined as follows

$$\mathrm{I}_\xi = \int_0^\infty e^{\xi_t} dt,$$

where $\xi = (\xi_t)_{t\geq 0}$ is a Lévy process starting from 0 and drifting to $-\infty$. Recall that a Lévy process ξ is a process with stationary and independent increments and its law is characterized completely by its Lévy-Khintchine exponent Ψ which takes the following form

$$\log \mathbb{E}\left[e^{z\xi_1}\right] = \Psi(z) = bz + \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{\infty}\left(e^{zy} - 1 - zy\mathbb{I}_{\{|y|<1\}}\right)\Pi(dy), \text{ for any } z \in i\mathbb{R}, \tag{2.1.1}$$

where $\sigma \geq 0, b \in \mathbb{R}$ and Π is a Lévy measure satisfying the condition $\int_{\mathbb{R}}(y^2 \wedge 1)\Pi(dy) < \infty$. See [11] for more information on Lévy processes.

The exponential functional I_ξ has attracted the interest of many researchers over the last two decades. This is mostly due to the prominent role played by the law of I_ξ in the study of important processes, such as self-similar Markov processes, fragmentation and branching processes but also in various settings ranging from astrophysics, biology to financial and insurance mathematics, see the survey paper [21].

So far there are two main approaches which have been developed and used to derive information about the law of the exponential functional. The first one uses the fact that the Mellin transform of I_ξ is a solution to a functional equation, see (2.4.1) below, and is due to Carmona et al. [27] and has been extended by Maulik and Zwart [62]. It is important to note that (2.4.1) is useful only under the additional assumption that ξ possesses some finite, positive exponential moments since then it is defined on a strip in the complex plane. This equation can be solved for exponential functionals of negative of subordinators and spectrally positive Lévy processes yielding some simple expressions for their positive and negative integer moments respectively, which, in both cases, determine the law. Recently, Kuznetsov and Pardo [52] have used some special instances of Lévy processes, for which the solution of the functional equation can directly be guessed and verified from (2.4.1), to derive some information concerning the law of I_ξ. It is worth pointing out that, in general, it is not an easy exercise to invert the Mellin (or moments) transform of I_ξ since a fine analysis of its asymptotic behavior is required. This Mellin transform approach relies on two difficult tasks: to find a solution of the functional equation and to provide a general criterion to ensure the uniqueness of its solution. For instance, this approach does not seem to successfully cope with the whole class of spectrally negative Lévy processes.

The second methodology, which has been developed recently by the second author in [73] and [76], is based on the well-known relation between the law of I_ξ and the distribution of the absorption time of positive self-similar Markov processes which were introduced by Lamperti [57] in the context of limit theorems for Markov processes. Indeed, in [76], it is shown that the law of I_ξ can be expressed as an invariant function of a transient Ornstein-Uhlenbeck companion process to the self-similar Markov process. Using some potential theoretical devices, a power series and a contour integral representation of the density is provided when ξ is a possibly killed spectrally negative Lévy process.

In this paper, starting from a large class of Lévy processes, we show that the law of I_ξ can be factorized into the product of independent exponential functionals associated with two companion Lévy processes, namely the descending ladder height process of ξ and a spectrally positive Lévy process constructed from its ascending ladder height process. It is well-known that these two subordinators appear in the Wiener-Hopf factorization of Lévy processes. The laws of these exponential functionals are uniquely determined either by their positive or negative integer moments. Moreover, whenever the law of any of these can be expanded in series we can in general develop the law of I_ξ in series. Thus, for example, the requirements put on the Lévy measure of ξ in [52] can be relaxed to conditions only on the positive jumps (the Lévy measure on the positive half-line) of ξ thus enlarging considerably the class of Lévy processes ξ, for which we can obtain a series expansion of the law of I_ξ.

Although our main result may have a formal explanation through the Wiener-Hopf factorization combined with the functional equation (2.4.1), the proof is rather complicated and involves a careful study of some generalized Ornstein-Uhlenbeck (for short GOU) processes, different from the ones mentioned above. For this purpose, we deepen a technique used by Carmona et al. [27, Proposition 2.1] and further developed in , which relates the law of I_ξ to the stationary measure of a GOU process. More precisely, we show that the density function of I_ξ, say m_ξ, is, under very mild conditions, the unique function satisfying the equation $\mathcal{L}m_\xi = 0$, where $\mathcal{L}$ is an "integrated infinitesimal" operator, which is strictly of an integral form. The latter allows for a smooth and effortless application of Mellin and Fourier transforms. We believe this method itself will attract some attention as it removes generic difficulties related to the study of the invariant measure via the dual Markov process such as the lack of smoothness properties for the density of the stationary measure and also application of transforms which usually requires the use of Fubini Theorem which is difficult to verify when dealing with non-local operators.

Before stating our main result let us introduce some notation. First, since in our setting ξ drifts to $-\infty$, it is well-known that the ascending (resp. descending) ladder height process $H^+ = (H^+(t))_{t\geq 0}$ (resp. $H^- = -H^{-,*} = (-H^{-,*}(t))_{t\geq 0}$) is a killed (resp. proper) subordinator. Then, we write, for any $z \in i\mathbb{R}$,

$$\phi_+(z) = \log \mathbb{E}\left[\exp(zH^+(1))\right] = \delta_+ z + \int_{(0,\infty)} (e^{zy} - 1)\mu_+(\dot{\mathrm{y}}) - k_+\,, \qquad (2.1.2)$$

where $\delta_+ \geq 0$ is the drift and $k_+ > 0$ is the killing rate. Similarly, with $\delta_- \geq 0$, we have

$$\phi_-(z) = \log \mathbb{E}\left[\exp(zH^-(1))\right] = -\delta_- z - \int_{(0,\infty)} (1 - e^{-zy})\mu_-(\dot{\mathrm{y}})\,. \qquad (2.1.3)$$

We recall that the integrability condition $\int_0^\infty (1 \wedge y)\mu_\pm(dy) < \infty$ holds. The Wiener-Hopf factorization then reads off as follows

$$\Psi(z) = -c\phi_+(z)\phi_-(z) = -\phi_+(z)\phi_-(z), \text{ for any } z \in i\mathbb{R}, \qquad (2.1.4)$$

where we have used the convention that the local times have been normalized in a way that $c = 1$, see (5.3.1) in [34]. We avoid further discussion as we assume (2.1.4) holds with $c = 1$.

Definition 2.1.1. *We denote by $\mathcal{P}$ the set of positive measures on $\mathbb{R}_+$ which admit a non-increasing density.*

Before we formulate the main result of our paper we introduce the two main hypothesis:

$(\mathcal{H}_1)$ Assume further that $-\infty < \mathbb{E}[\xi_1]$ and that one of the following conditions holds:

- $\mathbf{E}_+$ $\mu_+ \in \mathcal{P}$ and there exists $z_+ > 0$ such that for all z with, $\mathtt{Re}(z) \in (0, z_+)$, we have $|\Psi(z)| < \infty$.
- $\mathbf{P}+$ $\Pi_+ \in \mathcal{P}$.

$(\mathcal{H}_2)$ Assume that

- $\mathbf{P}_\pm$ $\mu_+ \in \mathcal{P}$, $k_+ > 0$ and $\mu_- \in \mathcal{P}$.

Then the following result holds.

Theorem 2.1.1. *Assume that ξ is a Lévy process that drifts to $-\infty$ with characteristics of the ladder height processes as in* (2.1.2) *and* (2.1.3). *Let either* $(\mathcal{H}_1)$ *or* $(\mathcal{H}_2)$ *holds. Then, in both cases, there exists a spectrally positive Lévy process Y with a negative mean whose Laplace exponent ψ_+ takes the form*

$$\psi_+(-s) = -s\phi_+(-s) = \delta_+ s^2 + k_+ s + s^2 \int_0^\infty e^{-sy}\mu_+(y,\infty)dy, \; s \geq 0, \tag{2.1.5}$$

and the following factorization holds

$$\mathrm{I}_\xi \stackrel{d}{=} \mathrm{I}_{H^-} \times \mathrm{I}_Y \tag{2.1.6}$$

where $\stackrel{d}{=}$ stands for the identity in law and $\times$ for the product of independent random variables.

Remark 2.1.2. *We mention that the case when the mean is $-\infty$ together with other problems will be treated in a subsequent study as it demands techniques different from the spirit of this paper.*

The result in Theorem 2.1.1 can be looked at from another perspective. Let us have two subordinators with Lévy measures $\mu_\pm$ such that $\mu_+ \in \mathcal{P}$, $k_+ > 0$ and $\mu_- \in \mathcal{P}$. Then according to Vigon's theory of philanthropy, see [98], we can construct a process ξ such that its ladder height processes have exponents as in (2.1.2) and (2.1.3) and hence ξ satisfies the conditions of Theorem 2.1.1. Therefore we will be able to synthesize examples starting from the building blocks, i.e. the ladder height processes. We state this as a separate result.

Corollary 2.1.3. *Let $\mu_\pm$ be the Lévy measures of two subordinators and $\mu_+ \in \mathcal{P}$, $k_+ > 0$ and $\mu_- \in \mathcal{P}$. Then there exists a Lévy process which drifts to $-\infty$ whose ascending and descending ladder height processes have the Laplace exponents respectively* (2.1.2) *and* (2.1.3). *Then all the claims of Theorem 2.1.1 hold and in particular we have the factorization* (2.1.6).

We postpone the proof of the Theorem to the Section 2.4. In the next section, we provide some interesting consequences whose proofs will be given in Section 2.5. Finally, in Section 2.3, we state and prove several results concerning some generalized Ornstein-Uhlenbeck processes. They will be useful for our main proof and since they have an independent interest, we present them in a separate section.

2.2 Some consequences of Theorem 2.1.1

Theorem 2.1.1 allows for a multiple of applications. In this section we discuss only a small part of them but we wish to note that almost all results that have been obtained in the literature under restrictions on all jumps of ξ can now be strengthened by imposing conditions only on positive jumps. This is due to (2.1.6) and the fact that on the right-hand side of the identity the law of the exponential functionals has been determined by its integral moments which admit some simple expressions, see Propositions 2.4.6 and 2.4.7 below.

The factorization allows us to derive some interesting distributional properties. For instance, we can show that the random variable I_ξ is unimodal for a large class of Lévy processes. We recall that a positive random variable (or its distribution function) is said to be unimodal if there exists $a \in \mathbb{R}^+$, the mode, such that its distribution function $F(x)$ and the function $1 - F(x)$ are convex respectively on $(0, a)$ and $(a, +\infty)$. It can be easily shown, see e.g. [88], that the random variable I_Y, as defined in Theorem 2.1.1, is self-decomposable and thus, in particular, unimodal. It is natural to ask whether this property is preserved or not for I_ξ. We emphasize that this is not necessarily true even if I_{H^-} is unimodal itself. Cuculescu and Theodorescu [31] provide a criterion for a positive random variable to be multiplicative strongly unimodal (for short MSU), that is, its product with any independent unimodal random variable remains unimodal. More precisely, they show that either the random variable has a unique mode at 0 and the independent product with any random variable has also an unique mode at 0 or the law of the positive random variable is absolutely continuous with a density m having the property that the mapping $x \to \log m(e^x)$ is concave on $\mathbb{R}$. We also point out that it is easily seen that the MSU property remains unchanged under rescaling and power transformations and we refer to the recent paper [93] for more information about this class of random variables.

We proceed by recalling that as a general result on the exponential functional Bertoin et al. [16, Theorem 3.9] have shown that the law of I_ξ is absolutely continuous with a density which we denote throughout by m_ξ.

In what follows, we show that whenever ξ is a spectrally negative Lévy process (that is $\Pi(dy)\mathbb{I}_{\{y>0\}} \equiv 0$ in (2.1.1) and ξ is not the negative of a subordinator), we recover the power series representation obtained by the second author in [76] for the density of I_ξ. We are now ready to state the first consequence of our main factorization.

Corollary 2.2.1. *Let ξ be a spectrally negative Lévy process with a negative mean.*

1. *Then, we have the following factorization*

$$\mathrm{I}_\xi \stackrel{d}{=} \mathrm{I}_{H^-} \times G_\gamma^{-1}, \tag{2.2.1}$$

where G_γ is a Gamma random variable of parameter $\gamma > 0$, where $\gamma > 0$ satisfies the relation $\Psi(\gamma) = 0$. Consequently, if I_{H^-} is unimodal then I_ξ is unimodal.

2. *The density function of I_ξ has the form*

$$m_\xi(x) = \frac{x^{-\gamma-1}}{\Gamma(\gamma)} \int_0^\infty e^{-y/x} y^\gamma m_{H^-}(y) dy, \; x > 0, \tag{2.2.2}$$

where Γ stands for the Gamma function. In particular, we have

$$\lim_{x\to\infty} x^{\gamma+1} m_\xi(x) = \frac{\mathbb{E}[\mathrm{I}_{H^-}^\gamma]}{\Gamma(\gamma)}.$$

3. *Moreover, for any $1/x < \lim_{s\to\infty} \frac{\Psi(s)}{s}$,*

$$m_\xi(x) \;=\; \frac{\mathbb{E}[\mathrm{I}_{H^-}^\gamma]}{\Gamma(\gamma)\Gamma(\gamma+1)} x^{-\gamma-1} \sum_{n=0}^\infty (-1)^n \frac{\Gamma(n+\gamma+1)}{\prod_{k=1}^n \Psi(k+\gamma)} x^{-n}. \tag{2.2.3}$$

4. *Finally, for any $\beta \geq \gamma+1$, the mapping $x \mapsto x^{-\beta} m_\xi(x^{-1})$ is completely monotone on $\mathbb{R}^+$, and, consequently, the law of the random variable I_ξ^{-1} is infinitely divisible with a decreasing density whenever $\gamma \leq 1$.*

Remark 2.2.2. 1. *From [11, Corollary VII.5] we get that*

$$\lim_{s\to\infty} \frac{\Psi(s)}{s} = \begin{cases} b - \int_{-1}^0 y\Pi(dy) & \text{if } \sigma = 0 \text{ and } \int_{-\infty}^0 (1 \wedge y)\Pi(dy) < \infty, \\ +\infty & \text{otherwise.} \end{cases}$$

Since we excluded the degenerate cases, we easily check that $b - \int_{-1}^0 y\Pi(dy) > 0$.

2. *We point out that in [76], it is proved that the density extends to a function of a complex variable which is analytical on the entire complex plane cut along the negative real axis and admits a power series representation for all $x > 0$.*

To illustrate the results above, we consider $\Psi(s) = -(s-\gamma)\phi_-(s), s > 0$, with $\gamma > 0$, and where for any $\alpha \in (0,1)$,

$$\begin{aligned} -\phi_-(s) \;&=\; s\frac{\Gamma(\alpha(s-1)+1)}{\Gamma(\alpha s+1)} \qquad (2.2.4) \\ &=\; \int_0^\infty (1-e^{-sy}) \frac{(1-\alpha)e^{y/\alpha}}{\alpha\Gamma(\alpha+1)(e^{y/\alpha}-1)^{2-\alpha}} dy = \int_0^\infty (1-e^{-sy}) \pi_\alpha(y) dy \end{aligned}$$

is the Laplace exponent of a subordinator. Observing that the density $\pi_\alpha(y)$ of the Lévy measure of ϕ_- is decreasing, we readily check that Ψ is the Laplace exponent of a spectrally negative Lévy process. Next, using the identity $\mathrm{I}_{H^-} \stackrel{(d)}{=} {G_1}^\alpha$, see e.g. [76], we get

$$\mathrm{I}_\xi \stackrel{(d)}{=} {G_1}^\alpha \times G_\gamma^{-1}$$

which, after some easy computations, yields, for any $x > 0$,

$$\begin{aligned} m_\xi(x) &= \frac{x^{-\gamma-1}}{\Gamma(\gamma)\Gamma(\gamma+1)} \sum_{n=0}^{\infty} \Gamma(\alpha(n+\gamma)+1)\frac{(-x)^{-n}}{n!} && (2.2.5) \\ &= \frac{\Gamma(\alpha\gamma+1)x^{-\gamma-1}}{\Gamma(\gamma)\Gamma(\gamma+1)}\,{}_1F_0((\alpha,\alpha\gamma+1);-x^{-1}), && (2.2.6) \end{aligned}$$

where ${}_1F_0$ stands for the so-called Wright hypergeometric function, see e.g. [24, Section 12.1]. Finally, since ${G_1}^\alpha$ is unimodal, we deduce that I_ξ is unimodal. Actually, we have a stronger result in this case since I_ξ is itself MSU being the product of two independent MSU random variables, showing in particular that the mapping $x \mapsto {}_1F_0((\alpha,\alpha\gamma+1);e^x)$ is log-concave on $\mathbb{R}$ for any $\alpha \in (0,1)$ and $\gamma > 0$.

We now turn to the second application as an illustration of the situation $\mathbf{P}+$ of Theorem 2.1.1. We would like to emphasize that in this case in general we do not require the existence of positive exponential moments. We are not aware of general examples that work without such a restriction as (2.4.1) is always crucially used and it is of real help once it is satisfied on a strip.

Corollary 2.2.3. *Let ξ be a Lévy process with $-\infty < \mathbb{E}[\xi_1] < 0$ and $\sigma^2 > 0$. Moreover assume that*

$$\Pi(dy)\mathbb{I}_{\{y>0\}} = c\lambda e^{-\lambda y}dy,$$

where $c, \lambda > 0$. Then, we have, for any $s > -\lambda$,

$$\psi_+(-s) \;=\; \delta_+ s^2 + k_+ s + c_- \frac{s^2}{\lambda+s},$$

where $c_- = c/\phi_-(\lambda)$ and $\delta_+ > 0$. Consequently, the self-decomposable random variable I_Y admits the following factorization

$$\mathrm{I}_Y \stackrel{d}{=} \delta_+ G_{\theta_2}^{-1} \times B^{-1}(\theta_1, \lambda-\theta_1), \tag{2.2.7}$$

where $0 < \theta_1 < \lambda < \theta_2$ are the two positive roots of the equation $\psi_+(s) = 0$ and B stands for a Beta random variable. Then, assuming that $\theta_2 - \theta_1$ is not an integer, we have, for any $1/x < \lim_{s\to\infty} |\phi_-(s)|$,

$$m_\xi(x) \;=\; \frac{k_+\Gamma(\lambda+1)x^{-1}}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)} \left(\sum_{i=1}^{2} \frac{\mathbb{E}\left[\mathrm{I}_{H^-}^{\theta_i}\right]}{\Gamma(\theta_i+1)} x^{-\theta_i} \mathcal{I}_{\phi_-,i}(\theta_i+1;-x^{-1}) \right),$$

where

$$\mathcal{I}_{\phi_-,i}(\theta_i+1;x) \quad = \quad \sum_{n=0}^{\infty} a_n(\phi_-,\theta_i)\frac{x^n}{n!} \tag{2.2.8}$$

and $a_n(\phi_-,\theta_i) = \prod_{\substack{j=1\\ j\neq i}}^{2} \frac{\Gamma(\theta_j-\theta_i-n)}{\Gamma(\lambda-\theta_i-n)} \frac{\Gamma(n+\theta_i+1)}{\prod_{k=1}^{n}\phi_-(k+\theta_i)}$, $i=1,2$.

Remark 2.2.4. *The assumption $\sigma^2>0$, as well as the restriction on $\theta_2-\theta_1$, have been made in order to avoid dealing with different cases but they can both be easily removed. The latter will affect the series expansion* (2.5.3)*. The computation is easy but lengthy and we leave it out.*

Remark 2.2.5. *The methodology and results we present here can also be extended to the case when the Lévy measure $\Pi(dy)\mathbb{I}_{\{y>0\}}$ is a mixture of exponentials as in [52] but we note that here we have no restrictions on the negative jumps whatsoever.*

We now provide an example of Theorem 2.1.1 in the situation $\mathbf{P}_{\pm}$.

Corollary 2.2.6. *For any $\alpha\in(0,1)$, let us set*

$$\Psi(z) = \frac{\alpha z\Gamma(\alpha(-z+1)+1)}{(1-z)\Gamma(-\alpha z+1)}\phi_+(z),\ z\in i\mathbb{R}, \tag{2.2.9}$$

where ϕ_+ is as in (2.1.2) *with $\mu_+\in\mathcal{P}, k_+>0$. Then Ψ is the Laplace exponent of a Lévy process ξ which drifts to $-\infty$. Moreover, the density of I_ξ admits the following representation*

$$m_\xi(x) \quad = \quad \frac{x^{-1/\alpha}}{\alpha}\int_0^\infty g_\alpha\left((y/x)^{1/\alpha}\right)m_Y(y)y^{1/\alpha-1}dy,\ x>0, \tag{2.2.10}$$

where g_α is the density of a positive α-stable random variable. Furthermore, if we have that $\lim_{s\to\infty}s^{\alpha-1}\phi_+(-s)=0$, then for all $x>0$,

$$m_\xi(x) \quad = \quad \frac{k_+}{\alpha}\sum_{n=1}^{\infty}\frac{\prod_{k=1}^{n}\phi_+(-k)}{\Gamma(-\alpha n)n!}x^n. \tag{2.2.11}$$

Finally, the positive random variable I_{H^-} is MSU if and only if $\alpha\le 1/2$. Hence I_ξ is unimodal for any $\alpha\le 1/2$.

Remark 2.2.7. *The fact that I_{H^-} is MSU if and only if $\alpha\le1/2$ is a consequence of the main result of [93].*

Remark 2.2.8. *Note that this is a very special example of the approach of building the Lévy process from $\phi_\pm$ when $\mu_\pm\in\mathcal{P}$. One could construct many examples like this and this allows for interesting applications in mathematical finance and insurance, see e.g. [85].*

As a specific instance of the previous result, we may consider the case when

$$\phi_+(-s) = -\frac{\Gamma(\alpha' s+1)}{\Gamma(\alpha'(s+1)+1)},\ s \geq 0,$$

with $\alpha' \in (0,1)$. We easily obtain from the identity (2.4.8) below that

$$\mathbb{E}\left[\mathrm{I}_Y^{-m}\right] = \frac{\Gamma(\alpha' m+1-\alpha')}{\Gamma(1-\alpha')},\ m = 1,2,\ldots,$$

that is $\mathrm{I}_Y \stackrel{d}{=} G_{1-\alpha'}^{-\alpha'}$. Hence, as the product of independent MSU random variables, I_ξ is MSU for any $\alpha' \in (0,1)$ and $\alpha \leq 1/2$. Moreover, using the asymptotic behavior of the ratio of gamma functions given in (2.5.7) below, we deduce that for any $\alpha' \in (0, 1-\alpha)$ we have

$$m_\xi(x) \quad = \quad \frac{1}{\Gamma(1-\alpha')\alpha}\sum_{n=1}^{\infty}\frac{\Gamma(\alpha' n+1)}{\Gamma(-\alpha n)n!}(-1)^n x^n, \tag{2.2.12}$$

which is valid for any $x > 0$.

We end this section by describing another interesting factorization of exponential functionals. Indeed, assuming that $\mu_- \in \mathcal{P}$, it is shown in [75, Theorem 1] that there exists a spectrally positive Lévy process $\overline{Y} = (\overline{Y}_t)_{t\geq 0}$ with a negative mean and Laplace exponent given by $\overline{\psi}_+(-s) = -s\phi_-(s+1)$, $s > 0$, such that the following factorization of the exponential law

$$\mathrm{I}_{H^-} \times \mathrm{I}_{\overline{Y}}^{-1} \stackrel{d}{=} G_1 \tag{2.2.13}$$

holds. Hence, combining (2.2.13) with (2.1.6), we obtain that

$$\mathrm{I}_\xi \times \mathrm{I}_{\overline{Y}}^{-1} \stackrel{d}{=} G_1 \times \mathrm{I}_Y.$$

Consequently, we deduce from [90, Theorem 51.6] the following.

Corollary 2.2.9. *If in one of the settings of Theorem 2.1.1, we assume further that $\mu_- \in \mathcal{P}$, then the density of the random variable $\mathrm{I}_\xi \times \mathrm{I}_{\overline{Y}}^{-1}$, where $\mathrm{I}_{\overline{Y}}$ is taken as defined in (2.2.13), is a mixture of exponential distributions and in particular it is infinitely divisible and non-increasing on $\mathbb{R}^+$.*

Considering as above that $\mathrm{I}_{H^-} \stackrel{(d)}{=} G_1^\alpha$ in Corollary 2.2.1 and 2.2.3, we deduce from [75, Section 3.2] that the random variable $S_\alpha^{-\alpha} \times \mathrm{I}_\xi$ is a mixture of exponential distributions, where S_α is a positive stable law of index α.

2.3 Some results on generalized Ornstein-Uhlenbeck processes

The results we present here will be central in the development of the proof of our main theorem. However, they also have some interesting implications in the study of generalized

Ornstein-Uhlenbeck processes (for short GOU), and for this reason we state and prove them in a separate section.

We recall that for a given Lévy process ξ the GOU process U^ξ, is defined, for any $t \geq 0$, $x \geq 0$, by

$$U_t^\xi(x) = xe^{\xi_t} + e^{\xi_t}\int_0^t e^{-\xi_s}ds. \tag{2.3.1}$$

This family of positive strong Markov processes has been intensively studied by Carmona et al. [27] and we refer to [71] for some more recent studies and references. The connection with our current problem is explained as follows. From the identity in law $(\xi_t - \xi_{(t-s)-})_{0\leq s\leq t} = (\xi_s)_{s\leq t}$, we easily deduce that, for any fixed $t \geq 0$,

$$U_t^\xi(x) \stackrel{d}{=} xe^{\xi_t} + \int_0^t e^{\xi_s}ds.$$

Thus, we have if $\lim_{t\to\infty}\xi_t = -\infty$ a.s., that

$$U_\infty^\xi(x) \stackrel{d}{=} \mathrm{I}_\xi$$

and hence the law of I_ξ is the unique stationary measure of U^ξ, see [27, Proposition 2.1].

In the sequel we use the standard notation $C_b(\mathbb{R})$ (resp. $C_b(\mathbb{R}_+)$) to denote the set of bounded and continuous functions on $\mathbb{R}$ (resp. on $\mathbb{R}_+$). Furthermore, we set $\mathcal{V}' = C_b^2(\overline{\mathbb{R}})$, where $C_b^2(\overline{\mathbb{R}})$ is the set of twice continuously differentiable bounded functions which together with its first two derivatives are continuous on $\overline{\mathbb{R}} = [-\infty,\infty]$. Then, we recall that, see e.g. [27] for the special case when ξ is the sum of a Brownian motion and an independent Lévy process with bounded variation and finite exponential moments and [55] for the general case, the infinitesimal generator L^{U^ξ} of U^ξ takes the form

$$L^{U^\xi}f(x) = L^\xi f_e(\ln x) + f'(x),\ x > 0, \tag{2.3.2}$$

whenever $\mathbb{E}[|\xi_1|] < \infty$ and $f_e(x) = f(e^x) \in Dom(L^\xi)$, where L^ξ stands for the infinitesimal generator of the Lévy process ξ, considered in the sense of Itô and Neveu (see [60, p. 628-630]). Recall in this sense $\mathcal{V}' \subset Dom(L^\xi)$ and hence $\mathcal{V} = \{f : \overline{\mathbb{R}}_+ \mapsto \overline{\mathbb{R}} | f_e \in \mathcal{V}'\} \subset Dom(L^{U^\xi})$.

In what follows we often appeal to the quantities, defined for $x > 0$, by

$$\overline{\Pi}(x) := \int_{|y|>x}\Pi(dy);\ \overline{\Pi}_\pm(x) := \int_{y>x}\Pi_\pm(dy), \tag{2.3.3}$$

$$\overline{\overline{\Pi}}(x) := \int_{y>x}\overline{\Pi}(y)dy;\ \overline{\overline{\Pi}}_\pm(x) := \int_{y>x}\overline{\Pi}_\pm(y)dy, \tag{2.3.4}$$

where $\Pi_+(dy) = \Pi(dy)1_{\{y>0\}}$ and $\Pi_-(dy) = \Pi(-dy)1_{\{y>0\}}$. Note that the quantities in (2.3.4) are finite when $\mathbb{E}[|\xi_1|] < \infty$. Moreover, when $\mathbb{E}[\xi_1] < \infty$, (2.1.1) can be rewritten, for all $z \in \mathbb{C}$, where it is well defined, as follows

$$\Psi(z) = \mathbb{E}[\xi_1]z + \frac{\sigma^2}{2}z^2 + z^2\int_0^\infty e^{zy}\overline{\overline{\Pi}}_+(y)dy + z^2\int_0^\infty e^{-zy}\overline{\overline{\Pi}}_-(y)dy. \tag{2.3.5}$$

For the proof of our main theorem we need to study the stationary measure of U^ξ and in particular L^{U^ξ} in detail. To this end, we introduce the following functional space

$$\mathcal{K} = \left\{ f : \overline{\mathbb{R}}_+ \mapsto \overline{\mathbb{R}} |\, f_e \in \mathcal{V}';\ \lim_{x\to-\infty} \Big(|f_e'(x)| + |f_e''(x)|\Big) = 0;\ \int_{\mathbb{R}} \Big(|f_e'(x)| + |f_e''(x)|\Big) dx < \infty \right\},$$

where $f_e(x) = f(e^x)$.

Proposition 2.3.1. *Let U^ξ be a GOU process with $\mathbb{E}\left[|\xi_1|\right] < \infty$. Then $\mathcal{K} \subset Dom(L^{U^\xi})$. Moreover, for any $f \in \mathcal{K}$, we have, for all $x > 0$,*

$$L^{U^\xi} f(x) = \frac{g(x)}{x} + \mathbb{E}\left[\xi_1\right] g(x) + \frac{\sigma^2}{2} x g'(x) + \int_x^\infty g'(y) \overline{\overline{\Pi}}_+\Big(\ln \frac{y}{x}\Big) dy + \int_0^x g'(y) \overline{\overline{\Pi}}_-\Big(\ln \frac{x}{y}\Big) dy, \tag{2.3.6}$$

where $g(x) = x f'(x)$. Finally, for any function h such that $\int_0^\infty (y^{-1} \wedge 1)|h(y)|dy < \infty$ and $f \in \mathcal{K}$ we have

$$(L^{U^\xi} f, h) = (g', \mathcal{L}h), \tag{2.3.7}$$

where $(f_1, f_2) = \int_0^\infty f_1(x) f_2(x) dx$ and

$$\mathcal{L}h(x) = \frac{\sigma^2}{2} x h(x) + \int_x^\infty \left(\frac{1}{y} + \mathbb{E}\left[\xi_1\right]\right) h(y) dy + \int_x^\infty \overline{\overline{\Pi}}_-\Big(\ln \frac{y}{x}\Big) h(y) dy + \int_0^x \overline{\overline{\Pi}}_+\Big(\ln \frac{x}{y}\Big) h(y) dy. \tag{2.3.8}$$

Remark 2.3.2. *There are certain advantages when using the linear operator $\mathcal{L}$ instead of the generator of the dual GOU. Its integral form allows for minimal conditions on the integrability of $|h|$ and requires no smoothness assumptions on h. Moreover, if h is positive, Laplace and Mellin transforms can easily be applied to $\mathcal{L}h(x)$ since the justification of Fubini Theorem is straightforward.*

Proof. Let $f \in \mathcal{K}$ then by the very definition of $\mathcal{K}$ we have that $f_e \in \mathcal{V}'$ and from (2.3.2) we get that $\mathcal{K} \subset Dom(L^{U^\xi})$. Next, (2.3.6) can be found in [55] but can equivalently be recovered from (2.3.2) by simple computations using the expression for L^ξ, which can be found on [11, p. 24]. To get (2.3.7) and (2.3.8), we recall that $g(x) = x f'(x) = f_e'(\ln x)$ and use (2.3.6) combined with a formal application of the Fubini Theorem to write

$$\begin{aligned}
(L^{U^\xi} f, h) &= \int_0^\infty \frac{g(y)}{y} h(y) dy + \frac{\sigma^2}{2} \int_0^\infty y g'(y) h(y) dy + \mathbb{E}\left[\xi_1\right] \int_0^\infty g(y) h(y) dy + \\
&\quad \int_0^\infty \int_0^y g'(v) \overline{\overline{\Pi}}_-\big(\ln \frac{y}{v}\big) dv h(y) dy + \int_0^\infty \int_y^\infty g'(v) \overline{\overline{\Pi}}_+\big(\ln \frac{v}{y}\big) dv h(y) dy \\
&= \int_0^\infty g'(v) \int_v^\infty \frac{h(y)}{y} dy dv \\
&+ \mathbb{E}\left[\xi_1\right] \int_0^\infty g'(v) \int_v^\infty h(y) dy dv + \frac{\sigma^2}{2} \int_0^\infty v g'(v) h(v) dv \\
&+ \int_0^\infty g'(v) \int_v^\infty \overline{\overline{\Pi}}_-\big(\ln \frac{y}{v}\big) h(y) dy dv + \int_0^\infty g'(v) \int_0^v \overline{\overline{\Pi}}_+\big(\ln \frac{v}{y}\big) h(y) dy dv \\
&= (g', \mathcal{L}h).
\end{aligned} \tag{2.3.9}$$

To justify Fubini Theorem, note that $f \in \mathcal{K}$ implies that $\lim_{x\to 0} g(x) = \lim_{x\to 0} f'_e(\ln x) = 0$, $g(x) = \int_0^x g'(v)dv$ and

$$\begin{aligned}
&\int_0^\infty |g'(v)|dv = \int_{\mathbb{R}} |f''_e(y)|dy \leq C(g) < \infty, \\
&|g(x)| + x|g'(x)| = |f'_e(\ln x)| + |f''_e(\ln x)| \leq C(g) < \infty,
\end{aligned} \tag{2.3.10}$$

where $C(g) > 0$. Note that (2.3.10) and the integrability of $(1 \wedge y^{-1})|h(y)|$ imply that

$$\int_0^\infty \Big|\frac{g(y)}{y}\Big| h(y)dy \leq \int_0^\infty \int_0^y |g'(v)|dvy^{-1}|h(y)|dy \leq C(g)\int_0^\infty y^{-1}|h(y)|dy < \infty,$$

and so Fubini Theorem applies to the first term in (2.3.9). The second term in (2.3.9) remains unchanged whereas for the third one we do the same computation noting that only y^{-1} is not present. From (2.3.10) and the fact that $\overline{\overline{\Pi}}_+(1) + \overline{\overline{\Pi}}_-(1) < \infty$ since $\mathbb{E}[|\xi_1|] < \infty$, we note that for the other two terms, we have with the constant $C(g) > 0$ in (2.3.10),

$$\begin{aligned}
\int_0^x |g'(v)|\overline{\overline{\Pi}}_-\Big(\ln\frac{x}{v}\Big)dv &= \int_0^\infty |xe^{-w}g'(xe^{-w})|\overline{\overline{\Pi}}_-(w)dw \\
&\leq \overline{\overline{\Pi}}_-(1)\int_0^\infty |g'(v)|dv + C(g)\int_0^1 \overline{\overline{\Pi}}_-(w)dw < \infty \\
\int_x^\infty |g'(v)|\overline{\overline{\Pi}}_+\Big(\ln\frac{v}{x}\Big)dv &= \int_0^\infty |xe^{w}g'(xe^{w})|\overline{\overline{\Pi}}_+(w)dw \\
&\leq \overline{\overline{\Pi}}_+(1)\int_0^\infty |g'(v)|dv + C(g)\int_0^1 \overline{\overline{\Pi}}_+(w)dw < \infty.
\end{aligned}$$

Therefore we can apply Fubini Theorem which completes the proof of Proposition 2.3.1. □

The next result is known and can be found in [55] but we include it and sketch its proof for sake of completeness and for further discussion.

Theorem 2.3.3. *Let U^ξ be a GOU where $-\infty < \mathbb{E}[\xi_1] < 0$. Then U^ξ has a unique stationary distribution which is absolutely continuous with density m and satisfies*

$$\mathcal{L}m(x) = 0 \text{ for a.e. } x > 0. \tag{2.3.11}$$

Remark 2.3.4. *Note that due to the discussion in Section 2.3, $m = m_\xi$, i.e it equals the density of the law of I_ξ. Therefore all the information we gathered for m_ξ in Section 2.2 is valid here for the density of the stationary measure of U^ξ, i.e. m.*

Remark 2.3.5. *Equation* (2.3.11) *can be very useful. In this instance it is far easier to be studied than an equation coming from the dual process which is standard when stationary distributions are discussed. It does not presuppose any smoothness of m but only its existence. Moreover, as noted above* (2.3.11) *is amenable to various transforms and difficult issues such as interchanging integrals using Fubini Theorem are effortlessly overcome.*

Remark 2.3.6. *It is also interesting to explore other cases when a similar equation to* (2.3.11) *can be obtained. It seems the approach is fairly general but requires special examples to reveal its full potential. For example, if* L *is an infinitesimal generator,* $\mathcal{N}$ *is a differential operator,* $\mathcal{L}$ *is an integral operator and it is possible for all* $f \in C_0^\infty(\mathbb{R}_+)$*, i.e. infinitely differentiable functions with compact support, and a stationary density* u *to write*

$$(Lf, u) = (\mathcal{N}f, \mathcal{L}u) = 0$$

then we can solve the equation in the sense of Schwartz to obtain

$$\tilde{\mathcal{N}}\mathcal{L}u = 0,$$

where $\tilde{\mathcal{N}}$ *is the dual of* $\mathcal{N}$*. If we show that necessarily for probability densities* $\mathcal{L}u = 0$*, then we can use* $\mathcal{L}$ *to study stationarity.*

Proof. From (2.3.7) and the fact that m is the stationary density we get, for all $g(x) = xf'(x)$, with $f \in C_0^\infty(\mathbb{R}_+) \subset \mathcal{K}$,

$$(g', \mathcal{L}m) = 0.$$

Then from Schwartz theory of distributions we get $\mathcal{L}m(x) = C \ln x + D$ a.e.. Integrating (2.3.8) and the right-hand side of the latter from 1 to z, multiply the resulting identity by z^{-1}, subsequently letting $z \to \infty$ and using the fact that m is a probability density we can show that necessarily $C = D = 0$. The latter requires some efforts but they are mainly technical. □

Theorem 2.3.7. *Let* $\overline{m}$ *be a probability density function such that* $\int_0^\infty \overline{m}(y)y^{-1}dy < \infty$ *and* (2.3.11) *holds for* $\overline{m}$ *then*

$$m(x) = \overline{m}(x) \text{ a.e.}, \tag{2.3.12}$$

where m *is the density of the stationary measure of* U^ξ*.*

Remark 2.3.8. *This result is very important in our studies. The fact that we have uniqueness on a large class of probability measures allows us by checking that* (2.3.11) *holds to pin down the density of the stationary measure of* U^ξ *which is of course the density of* I_ξ*. The requirement that* $\int_0^\infty \overline{m}(y)y^{-1}dy < \infty$ *is in fact no restriction whatsoever since the existence of a first negative moment of* I_ξ *is known from the literature, see [19].*

Remark 2.3.9. *Also it is well known that if* $L^{\hat{U}}$ *is the generator of the dual Markov process then* $L^{\hat{U}}\overline{m} = 0$ *does not necessarily have a unique solution when* $L^{\hat{U}}$ *is a non-local operator. Moreover one needs assumptions on the smoothness of* $\overline{m}$ *so as to apply* $L^{\hat{U}}$*. Using* $\mathcal{L}$ *circumvents this problem.*

Proof. Let $(P_t)_{t\geq 0}$ be the semigroup of the GOU U^ξ, that is, for any $f \in C_b(\overline{\mathbb{R}}_+)$,

$$P_t f(x) = \mathbb{E}\left[f\left(U_t^\xi(x)\right)\right], \; x \geq 0, \; t \geq 0.$$

If (2.3.11) holds for some probability density $\overline{m}$ then (2.3.7) is valid, i.e. for all $f \in \mathcal{K}$,

$$(L^{U^\xi} f, \overline{m}) = (g', \mathcal{L}\overline{m}) = 0.$$

Assume for a moment that

$$P_s\mathcal{K} \subset \mathcal{K}, \text{ for all } s > 0, \tag{2.3.13}$$

and, there exists a constant $C(f, \xi) > 0$ such that, for all $s \leq t$,

$$\left| L^{U^\xi} P_s f(x) \right| \leq C(f, \xi)(x^{-1} \wedge 1). \tag{2.3.14}$$

Then integrating out with respect to $\overline{m}(x)$ the standard equation

$$P_t f(x) = f(x) + \int_0^t L^{U^\xi} P_s f(x) ds,$$

we get, for all $f \in \mathcal{K}$,

$$\int_0^\infty P_t f(x)\overline{m}(x)dx = \int_0^\infty f(x)\overline{m}(x)dx.$$

Since $C_0^\infty(\mathbb{R}_+) \subset \mathcal{K}$ and $C_0^\infty(\mathbb{R}_+)$ is separating for $C_0(\mathbb{R}_+)$, the last identity shows that $\overline{m}$ is a density of a stationary measure. Thus by uniqueness of the stationary measure we conclude (2.3.12). Let us prove (2.3.13) and (2.3.14). For $f \in \mathcal{K}$ write

$$g_s(x) := P_s f(x) = \mathbb{E}\left[f\left(U_s^\xi(x)\right)\right] = \mathbb{E}\left[f\left(xe^{\xi_s} + \int_0^s e^{\xi_v} dv\right)\right].$$

Put $\tilde{g}_s(x) = g_s(e^x) = (g_s)_e(x)$. Note that since $f \in \mathcal{K}$ and $0 < e^{x+\xi_s} \leq e^{x+\xi_s} + \int_0^s e^{\xi_v} dv$ we have the following bound

$$\left| e^{x+\xi_s} f'\left(e^{x+\xi_s} + \int_0^s e^{\xi_v} dv\right)\right| + \left| e^{2(x+\xi_s)} f''\left(e^{x+\xi_s} + \int_0^s e^{\xi_v} dv\right)\right| \leq C(f) \tag{2.3.15}$$

which holds uniformly in $x \in \mathbb{R}$ and $s \geq 0$. In view of (2.3.15) the dominated convergence theorem gives

$$\begin{aligned}
&\tilde{g}_s'(x) = \mathbb{E}\left[e^{x+\xi_s} f'\left(e^{x+\xi_s} + \int_0^s e^{\xi_v} dv\right)\right], \\
&\tilde{g}_s''(x) = \mathbb{E}\left[e^{x+\xi_s} f'\left(e^{x+\xi_s} + \int_0^s e^{\xi_v} dv\right)\right] + \mathbb{E}\left[e^{2(x+\xi_s)} f''\left(e^{x+\xi_s} + \int_0^s e^{\xi_v} dv\right)\right], \\
&\max\{|\tilde{g}_s'(x)|, |\tilde{g}_s''(x)|\} \leq C(f).
\end{aligned} \tag{2.3.16}$$

Clearly then from (2.3.15), (2.3.16), the dominated convergence theorem and the fact that $f \in \mathcal{K}$ which implies the existence of $\lim_{x\to\infty} f_e''(x) = b$, we have

$$\lim_{x\to\infty} \tilde{g}_s''(x) = \mathbb{E}\left[\lim_{x\to\infty}\left(e^{x+\xi_s} f'\left(e^{x+\xi_s} + \int_0^s e^{\xi_v} dv\right) + e^{2(x+\xi_s)} f''\left(e^{x+\xi_s} + \int_0^s e^{\xi_v} dv\right)\right)\right] = b.$$

Similarly, we show that $\lim_{x\to\infty}\tilde{g}_s'(x)=\lim_{x\to\infty}f_e'(x)$ and trivially that $\lim_{x\to\pm\infty}\tilde{g}_s(x)=\lim_{x\to\pm\infty}f_e(x)$. Finally using (2.3.15), (2.3.16), $f\in\mathcal{K}$, the dominated convergence theorem and the fact that for all $s>0$ almost surely $\int_0^s e^{\xi_v}dv>0$, we conclude that

$$\begin{aligned}
&\lim_{x\to-\infty}|\tilde{g}_s'(x)|+|\tilde{g}_s''(x)|\\
&\leq 2\mathbb{E}\left[\lim_{x\to-\infty}\left|e^{x+\xi_s}f'\left(e^{x+\xi_s}+\int_0^s e^{\xi_v}dv\right)\right|+\left|e^{2(x+\xi_s)}f''\left(e^{x+\xi_s}+\int_0^s e^{\xi_v}dv\right)\right|\right],
\end{aligned}$$

which together with the limits above confirms that $\tilde{g}_s\in\mathcal{V}'$ and proves that

$$\lim_{x\to-\infty}|\tilde{g}_s'(x)|+|\tilde{g}_s''(x)|=0.$$

Finally since $f\in\mathcal{K}$ and (2.3.15), we check that

$$\int_0^\infty|\tilde{g}_s'(y)|dy\leq\mathbb{E}\left[\int_{\int_0^s e^{\xi_v}dv}^\infty|f'(u)|du\right]\leq\int_0^\infty|f'(u)|du=\int_{\mathbb{R}}|f_e'(u)|du<C(f),$$

and

$$\begin{aligned}
\int_0^\infty|\tilde{g}_s''(y)|dy &\leq E\left[\int_{\int_0^s e^{\xi_v}dv}^\infty\left(u-\int_0^s e^{\xi_v}dv\right)|f''(u)|du\right]\leq\int_0^\infty u|f''(u)|du\\
&\leq 2\int_{\mathbb{R}_+}|f_e'(\ln x)|+|f_e''(\ln x)|\frac{dx}{x}=2\int_{\mathbb{R}}(|f_e'(y)|+|f_e''(y)|)dy<C(f),
\end{aligned}$$

where $C(f)$ is chosen to be the largest constant in all the inequalities above and we have used the trivial inequality $u^2|f''(u)|\leq|f_e'(\ln u)|+|f_e''(\ln u)|$. Thus using all the information above we conclude that $g_s=P_sf\in\mathcal{K}$ and (2.3.13) holds. Next we consider (2.3.14) keeping in mind that all estimates on $\tilde{g}_s$ we used to show that $g_s\in\mathcal{K}$ are uniform in s and x. We use (2.3.6) with $g(x)=xg_s'(x)=\tilde{g}_s'(\ln x)$, the bounds on $\tilde{g}_s$ and its derivatives to get

$$\left|\frac{g(x)}{x}+\mathbb{E}[\xi_1]g(x)+\frac{\sigma^2}{2}xg'(x)\right|\leq C(f)x^{-1}+C(f)|\mathbb{E}[\xi_1]|+C(f)\frac{\sigma^2}{2}\leq C(f,\sigma,\mathbb{E}[\xi_1])(1\wedge x^{-1}).$$

Moreover, as in the proof of Proposition 2.3.1, we can estimate

$$\begin{aligned}
&\left|\int_0^x g'(v)\overline{\overline{\Pi}}_-\big(\ln\frac{x}{v}\big)dv\right|+\left|\int_x^\infty g'(v)\overline{\overline{\Pi}}_+\big(\ln\frac{v}{x}\big)dv\right|\leq\\
&\left(\overline{\overline{\Pi}}_-(1)+\overline{\overline{\Pi}}_+(1)\right)\int_0^\infty|g'(s)|ds+C(f)\left(\int_0^1\overline{\overline{\Pi}}_-(s)ds+\int_0^1\overline{\overline{\Pi}}_+(s)ds\right)=\\
&\left(\overline{\overline{\Pi}}_-(1)+\overline{\overline{\Pi}}_+(1)\right)\int_{-\infty}^\infty|\tilde{g}''(y)|dy+C(f)\left(\int_0^1\overline{\overline{\Pi}}_-(s)ds+\int_0^1\overline{\overline{\Pi}}_+(s)ds\right)<C
\end{aligned}$$

and therefore (2.3.14) holds since

$$L^{U^\xi}g_s(x)=\frac{g(x)}{x}+\mathbb{E}[\xi_1]g(x)+\frac{\sigma^2}{2}xg'(x)+\int_0^x g'(v)\overline{\overline{\Pi}}_-\big(\ln\frac{x}{v}\big)dv+\int_x^\infty g'(v)\overline{\overline{\Pi}}_+\big(\ln\frac{v}{x}\big)dv.$$

This concludes the proof. □

Theorem 2.3.10. *Let $(\xi^{(n)})_{n\geq 1}$ be a sequence of Lévy processes with negative means such that*

$$\lim_{n\to\infty} \xi^{(n)} \stackrel{d}{=} \xi,$$

where ξ is a Lévy process with $\mathbb{E}[\xi_1] < 0$. Moreover, if for each $n \geq 1$, $m^{(n)}$ stands for the law of the stationary measure of the GOU process $U^{\xi^{(n)}}$ defined, for any $t \geq 0$, $x \geq 0$, by

$$U_t^{\xi^{(n)}} = xe^{\xi_t^{(n)}} + e^{\xi_t^{(n)}} \int_0^t e^{-\xi_s^{(n)}} ds$$

and the sequence $(m^{(n)})_{n\geq 1}$ is tight then $(m^{(n)})_{n\geq 1}$ converges weakly to $m^{(0)}$, which is the unique stationary measure of the process U^ξ, i.e.

$$\lim_{n\to\infty} m^{(n)} \stackrel{w}{=} m^{(0)}. \tag{2.3.17}$$

Proof. Without loss of generality we assume using Skorohod-Dudley theorem, see Theorem 3.30 in Chapter 3 in [47], that the convergence $\xi^{(n)} \to \xi$ holds a.s. in the Skorohod space $\mathcal{D}((0,\infty))$. Due to the stationarity properties of $m^{(n)}$, for each $t > 0$, we have, for any $f \in C_b(\overline{\mathbb{R}}_+)$,

$$\left(f, m^{(n)}\right) = \left(P_t^{(n)} f, m^{(n)}\right) = \left(P_t^{(n)} f - P_t f, m^{(n)}\right) + \left(P_t f, m^{(n)}\right),$$

where $P_t^{(n)}$ and P_t are the semigroups of $U_t^{\xi^{(n)}}$ and U_t^ξ. For any $x > 0$,

$$\begin{aligned}\left|\left(P_t^{(n)} f - P_t f, m^{(n)}\right)\right| &\leq 2||f||_\infty m^{(n)}(x,\infty) + \sup_{y\leq x}\left|P_t^{(n)} f(y) - P_t f(y)\right| \\ &\leq 2\big|\big|f\big|\big|_\infty m^{(n)}(x,\infty) + \mathbb{E}\left[\sup_{y\leq x}\left|f\left(U_t^{\xi^{(n)}}(y)\right) - f\left(U_t^\xi(y)\right)\right|\right].\end{aligned} \tag{2.3.18}$$

Taking into account that $(m^{(n)})_{n\geq 1}$ is tight we may fix $\delta > 0$ and find $x > 0$ big enough such that

$$\sup_{n\geq 1} m^{(n)}(x,\infty) < \delta.$$

Also since $f \in C_b(\overline{\mathbb{R}}_+)$ then f is uniformly continuous on $\mathbb{R}_+$. Therefore, to show that

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{y\leq x}\left|f\left(U_t^{\xi^{(n)}}(y)\right) - f\left(U_t^\xi(y)\right)\right|\right] = 0,$$

due to the dominated convergence theorem all we need to show is that

$$\lim_{n\to\infty} \sup_{y\leq x} |U_t^{\xi^{(n)}}(y) - U_t^\xi(y)| = 0. \tag{2.3.19}$$

From the definition of $U^{\xi^{(n)}}$ and U^ξ, we obtain that, for $y \leq x$,

$$\left|U_t^{\xi^{(n)}}(y) - U_t^\xi(y)\right| \leq x\left|e^{\xi_t^{(n)}} - e^{\xi_t}\right| + \left|e^{\xi_t^{(n)}} - e^{\xi_t}\right| \int_0^t e^{-\xi_s^{(n)}} ds + e^{\xi_t}\left|\int_0^t e^{-\xi_s^{(n)}} - e^{-\xi_s} ds\right|.$$

Since $\xi^{(n)} \overset{a.s.}{\to} \xi$ in the Skorohod topology and

$$\mathbb{P}\left(\left\{\exists n \geq 1 : \xi_t^{(n)} - \xi_{t-}^{(n)} > 0\right\} \cap \{\xi_t - \xi_{t-} > 0\}\right) = 0$$

the first term on the right-hand side of the last expression converges a.s. to zero as $n \to \infty$. The a.s. convergence in the Skorohod space implies the existence of changes of times $(\lambda_n)_{n\geq 1}$ such that, for each $n \geq 1$, $\lambda_n(0) = 0$, $\lambda_n(t) = t$, the mapping $s \mapsto \lambda_n(s)$ is increasing and continuous on $[0, t]$, and

$$\lim_{n\to\infty} \sup_{s\leq t} |\lambda_n(s) - s| = \lim_{n\to\infty} \sup_{s\leq t} \left|\lambda_n^{-1}(s) - s\right| = 0 \tag{2.3.20}$$

$$\lim_{n\to\infty} \sup_{s\leq t} \left|\xi^{(n)}_{\lambda_n(s)} - \xi_s\right| = \lim_{n\to\infty} \sup_{s\leq t} \left|\xi^{(n)}_s - \xi_{\lambda_n^{-1}(s)}\right| = 0. \tag{2.3.21}$$

Hence,

$$\left|\int_0^t e^{-\xi_s^{(n)}} - e^{-\xi_s} ds\right| \leq \left|\int_0^t e^{-\xi_s^{(n)}} - e^{-\xi_{\lambda_n^{-1}(s)}} ds\right| + \left|\int_0^t e^{-\xi_{\lambda_n^{-1}(s)}} - e^{-\xi_s} ds\right|.$$

The first term on the right-hand side clearly goes to zero due to (2.3.21) whereas (2.3.20) implies that the second term goes to zero a.s. due to the dominated convergence theorem and the fact that pathwise, for $s \leq t$,

$$\limsup_{n\to\infty} \left|e^{-\xi_{\lambda_n^{-1}(s)}} - e^{-\xi_s}\right| > 0$$

only on the set of jumps of ξ and this set has a zero Lebesgue measure. Thus we conclude that

$$\lim_{n\to\infty} e^{\xi_t} \left|\int_0^t e^{-\xi_s^{(n)}} - e^{-\xi_s} ds\right| = 0.$$

Similarly we observe that

$$\lim_{n\to\infty} \left|e^{\xi_t^{(n)}} - e^{\xi_t}\right| \int_0^t e^{-\xi_s^{(n)}} ds \leq \lim_{n\to\infty} t \left|e^{\xi_t^{(n)}} - e^{\xi_t}\right| e^{\sup_{s\leq t}(-\xi_s^{(n)})} = 0, \tag{2.3.22}$$

where the last identity follows from

$$\sup_{s\leq t} \left|(-\xi_s^{(n)})\right| \leq \sup_{s\leq t} \left|\xi_{\lambda_n^{-1}(s)}\right| + \sup_{s\leq t} \left|\xi_s^{(n)} - \xi_{\lambda_n^{-1}(s)}\right| = \sup_{s\leq t} |\xi_s| + \sup_{s\leq t} \left|\xi_s^{(n)} - \xi_{\lambda_n^{-1}(s)}\right|$$

and an application of (2.3.21). Therefore, (2.3.19) holds and

$$\lim_{n\to\infty} \sup_{y\leq x} \left|f\left(U_t^{(n)}(y)\right) - f\left(U_t^{\xi}(y)\right)\right| = 0.$$

The dominated convergence theorem then easily gives that the right-hand side of (2.3.18) goes to zero and hence

$$\limsup_{n\to\infty} \left|\left(P_t^{(n)} f - P_t f, m^{(n)}\right)\right| \leq 2||f||_\infty \sup_{n\geq 1} m^{(n)}(x, \infty) \leq 2||f||_\infty \delta.$$

As $\delta > 0$ is arbitrary we show that

$$\lim_{n\to\infty} \left| \left(P_t^{(n)} f - P_t f, m^{(n)} \right) \right| = 0.$$

Since $(m^{(n)})_{n\geq 1}$ is tight we choose a subsequence $(m^{(n_k)})_{k\geq 1}$ such that $\lim_{k\to\infty} m^{(n_k)} \stackrel{d}{=} \nu$ with ν a probability measure. Then, for each $t \geq 0$,

$$(f, \nu) = \lim_{k\to\infty} \left(f, m^{(n_k)} \right) = \lim_{k\to\infty} \left(P_t^{(n_k)} f, m^{(n_k)} \right) = \lim_{k\to\infty} \left(P_t f, m^{(n_k)} \right) = (P_t f, \nu).$$

Therefore ν is a stationary measure for U^ξ. But since $m^{(0)}$ is the unique stationary measure we conclude that

$$\lim_{n\to\infty} m^{(n)} \stackrel{w}{=} \nu = m^{(0)}.$$

This translates to the proof of (2.3.17). □

2.4 Proof of Theorem 2.1.1

We start the proof by collecting some useful properties in two trivial lemmas. The first one discusses the properties of Ψ.

Lemma 2.4.1. *[90, Theorem 25.17] The function Ψ, defined in* (2.3.5)*, is always well-defined on $i\mathbb{R}$. Moreover, Ψ is analytic on the strip $\{z \in \mathbb{C};\, -a_- < \mathtt{Re}(z) < a_+\}$, where $a_-, a_+ > 0$ if and only if $\mathbb{E}\left[e^{(-a_-+\epsilon)\xi_1}\right] < \infty$ and $\mathbb{E}\left[e^{(a_+-\epsilon)\xi_1}\right] < \infty$ for all $0 < \epsilon < a_- \wedge a_+$.*

The second lemma concerns the properties of $\phi_\pm$ and is easily obtained using Lemma 2.4.1, (2.1.4) together with the analytical extension and the fact that subordinators have all negative exponential moments.

Lemma 2.4.2. *Let ξ be a Lévy process with $\mathbb{E}[\xi_1] < \infty$. Then ϕ_+ is always analytic on the strip $\{z \in \mathbb{C};\, \mathtt{Re}(z) < 0\}$ and is well-defined on $i\mathbb{R}$. Moreover ϕ_+ is analytic on $\{z \in \mathbb{C};\, \mathtt{Re}(z) < a_+\}$, for $a_+ \geq 0$, if and only if $\mathbb{E}\left[e^{(a_+-\epsilon)\xi_1}\right] < \infty$, for some $\epsilon > 0$. Similarly ϕ_- is always analytic on the strip $\{z \in \mathbb{C};\, \mathtt{Re}(z) > 0\}$ and is well-defined on $i\mathbb{R}$ and ϕ_- is analytic on $\{z \in \mathbb{C};\, \mathtt{Re}(z) < -a_-\}$, for $a_- \geq 0$, if and only if $\mathbb{E}\left[e^{(-a_-+\epsilon)\xi_1}\right] < \infty$, for some $\epsilon > 0$. Finally, the Wiener-Hopf factorization* (2.1.4) *holds on the intersection of the strips where ϕ_+ and ϕ_- are well-defined.*

2.4.1 Proof in the case $\mathbf{E}_+$

We recall that in this part we assume, in particular, that ξ is a Lévy process with a finite negative mean and that there exists $a_+ > 0$ such that $|\Psi(z)| < \infty$ for any $0 < \mathtt{Re}(z) < a_+$. Next, we write $\theta^* = \max(\theta, a_+)$, where $\theta = \inf\{s > 0;\, \Psi(s) = 0\}$ (with the convention that $\inf \emptyset = +\infty$). We also recall from [27], see also [62], that the Mellin transform of I_ξ defined by

$$\mathcal{M}_{m_\xi}(z) = \int_0^\infty x^{z-1} m_\xi(x) dx$$

satisfies, for any $0 < \mathtt{Re}(z) < \theta^*$, the following functional equation

$$\mathcal{M}_{m_\xi}(z+1) = -\frac{z}{\Psi(z)}\mathcal{M}_{m_\xi}(z). \tag{2.4.1}$$

We proceed by proving the following easy result.

Lemma 2.4.3. *If $\mu_+ \in \mathcal{P}$ then there exists a spectrally positive Lévy process Y with Laplace exponent $\psi_+(-s) = -s\phi_+(-s)$, $s \geq 0$, and a negative finite mean $-\phi_+(0)$. Moreover, if ξ has a negative finite mean then $\mathbb{E}\left[(\mathrm{I}_{H^-} \times \mathrm{I}_Y)^{-1}\right] = -\phi_+(0)\phi'_-(0^+) < +\infty$.*

Proof. The first claim follows readily from [11, Theorem VII.4(ii)] and by observing that $\psi'_+(0^-) = \phi_+(0)$. From (2.4.8) we get that $\mathbb{E}\left[\mathrm{I}_Y^{-1}\right] = k_+$. Next, since $-\infty < \mathbb{E}[\xi_1] < 0$, using the dual version of [34, Corollary 4.4.4(iv)], we get that $-\infty < \mathbb{E}[H_1^-] < 0$ and thus $|\phi'_-(0^+)| < \infty$. From the functional equation (2.4.1), we easily deduce that $\mathbb{E}\left[\mathrm{I}_{H^-}^{-1}\right] = -\phi'_-(0^+)$ which completes the proof since the two random variables are independent. □

Lemma 2.4.4. *Assume that ξ has a finite negative mean and condition $\mathbf{E}_+$ holds. Let η be a positive random variable with density $\kappa(x)$, such that $\mathbb{E}\left[\eta^{-1}\right] < \infty$ and $\mathbb{E}\left[\eta^{\delta}\right] < \infty$, for some $\theta^* > \delta > 0$. Then, for any z such that $\mathtt{Re}(z) \in (0,\delta)$,*

$$\mathcal{M}_{\mathcal{L}\kappa}(z) = \int_0^\infty x^{z-1}\mathcal{L}\kappa(x)dx = \frac{\Psi(z)}{z^2}\mathcal{M}_\kappa(z+1) + \frac{1}{z}\mathcal{M}_\kappa(z) \tag{2.4.2}$$

and if $\mathcal{M}_{\mathcal{L}\kappa}(z) = 0$, for $0 < a < \mathtt{Re}(z) < b < \delta$, then $\mathcal{L}\kappa(x) = 0$ a.e..

Furthermore the law of the positive random variable $\mathrm{I}_Y \times \mathrm{I}_{H^-}$, as defined in Theorem 2.1.1, is absolutely continuous with a density, denoted by $\overline{m}$, which satisfies

$$\mathcal{L}\overline{m}(x) = 0 \text{ for a.e. } x > 0. \tag{2.4.3}$$

Remark 2.4.5. *Note that the proof of this lemma shows that we have uniqueness for the probability measures with first negative moment that satisfy (2.4.1). This is a rather indirect approach and seems to be more general than the verification approach of [52], see Proposition 2, where precise knowledge on the rate of decay of the Mellin transform $\mathcal{M}_{m_\xi}(z)$ is needed. In general such an estimate on the decay seems impossible to obtain.*

Proof. We start by proving (2.4.2). Note that since $\int_0^\infty y^{-1}\kappa(y)dy < \infty$, we can use Proposition 2.3.1 to get

$$\begin{aligned}\mathcal{L}\kappa(x) = {} & \frac{\sigma^2}{2}x\kappa(x) + \int_x^\infty \frac{\kappa(y)}{y}dy + \mathbb{E}[\xi_1]\int_x^\infty \kappa(y)dy \\ & + \int_x^\infty \overline{\overline{\Pi}}_-\left(\ln\frac{y}{x}\right)\kappa(y)dy + \int_0^x \overline{\overline{\Pi}}_+\left(\ln\frac{x}{y}\right)\kappa(y)dy.\end{aligned} \tag{2.4.4}$$

As κ is a density, one can use Fubini Theorem to get, after some easy computations, that for any $\epsilon < \mathtt{Re}(z) < \delta < \theta^*$, with $0 < \epsilon < \delta$,

$$\begin{aligned}
\mathcal{M}_{\mathcal{L}\kappa}(z) &= \int_0^\infty x^{z-1}\mathcal{L}\kappa(x)dx \\
&= \mathcal{M}_\kappa(z+1)\left(\frac{\sigma^2}{2} + \frac{\mathbb{E}[\xi_1]}{z} + \int_0^\infty \overline{\overline{\Pi}}_-(y)e^{-zy}dy + \int_0^\infty \overline{\overline{\Pi}}_+(y)e^{zy}dy\right) + \frac{1}{z}\mathcal{M}_\kappa(z) \\
&= \frac{\Psi(z)}{z^2}\mathcal{M}_\kappa(z+1) + \frac{1}{z}\mathcal{M}_\kappa(z).
\end{aligned}$$

Let $\mathcal{M}_{\mathcal{L}\kappa}(z) = 0$ for $\epsilon < \mathtt{Re}(z) < \delta$. We show using that all terms in (2.4.4) are positive except the negative one due to $\mathbb{E}[\xi_1] < 0$ that, with $u = \mathtt{Re}(z)$,

$$\int_0^\infty x^{u-1}\left|\mathcal{L}\kappa(x)\right|dx \leq \mathcal{M}_\kappa(u+1)\left(\frac{\Psi(u)}{u^2} - 2\frac{\mathbb{E}[\xi_1]}{u}\right) + \frac{1}{u}\mathcal{M}_\kappa(u) < \infty.$$

Given the absolute integrability of $x^{z-1}\mathcal{L}\kappa(x)$ along imaginary lines determined by $\epsilon < \mathtt{Re}(z) < \delta$ we can apply the Mellin inversion theorem to the identity $\mathcal{M}_{\mathcal{L}\kappa}(z) = 0$ to get $\mathcal{L}\kappa(x) = 0$ a.e., see Theorem 6 in Section 6 in [25].

Next it is plain that the law of $\mathrm{I}_Y \times \mathrm{I}_{H^-}$ is absolutely continuous since, for any $x > 0$,

$$\overline{m}(x) = \int_0^\infty m_Y\left(\frac{x}{y}\right)y^{-1}m_{H^-}(y)dy. \tag{2.4.5}$$

Furthermore from the Wiener-Hopf factorization (2.1.4) and the definition of ψ_+, we have that

$$\frac{-z}{\Psi(z)} = \frac{-z}{\phi_-(z)}\frac{-z}{\psi_+(z)}$$

which is valid, for any $0 < \mathtt{Re}(z) < \theta^*$. Thus, we deduce from the functional equation (2.4.1) and the independency of Y and H^- that, for any $0 < \mathtt{Re}(z) < \theta^*$,

$$\mathcal{M}_{\overline{m}}(z+1) = -\frac{z}{\Psi(z)}\mathcal{M}_{\overline{m}}(z). \tag{2.4.6}$$

Next, since from Lemma 2.4.3, we have that $\int_0^\infty y^{-1}\overline{m}(y)dy < \infty$, we can use Proposition 2.3.1 and thus (2.4.4) and subsequently (2.4.2) are valid for $\overline{m}$. Moreover due to the representation (2.3.5) of Ψ and relation (2.4.6) we have that for any $\epsilon < \mathtt{Re}(z) < \theta^*$ with $0 < \epsilon < \theta^*/4$,

$$\mathcal{M}_{\mathcal{L}\overline{m}}(z) = \frac{\Psi(z)}{z^2}\mathcal{M}_{\overline{m}}(z+1) + \frac{1}{z}\mathcal{M}_{\overline{m}}(z) = 0$$

and we conclude that $\mathcal{L}\overline{m}(x) = 0$ a.e.

□

We are now ready to complete the proof of Theorem 2.1.1 in the case $\mathbf{E}_+$. Indeed, since m_ξ, the density of I_ξ, is the density of the stationary measure of U^ξ, we have that m_ξ is also solution to (2.3.11). Combining Lemma 2.4.4 with the uniqueness argument of Theorem 2.3.7, we conclude that the factorization (2.1.6) holds.

2.4.2 Proof of the two other cases : P+ and P$_\pm$

We start by providing some results which will be used several times throughout this part.

Proposition 2.4.6 (Carmona et al. [27]). *Let H be the negative of a (possibly killed) subordinator with Laplace exponent ϕ, then the law of I_H is determined by its positive entire moments as follows*

$$\mathbb{E}\left[\mathrm{I}_H^m\right] \;=\; \frac{\Gamma(m+1)}{\prod_{k=1}^m(-\phi(k))},\; m=1,2,\ldots \tag{2.4.7}$$

Proposition 2.4.7 (Bertoin and Yor [20]). *Let Y be an unkilled spectrally positive Lévy process with a negative mean and Laplace exponent ψ_+, then the law of $1/\mathrm{I}_Y$ is determined by its positive entire moments as follows*

$$\mathbb{E}\left[\mathrm{I}_Y^{-m}\right] \;=\; \mathbb{E}\left[-Y_1\right]\frac{\prod_{k=1}^{m-1}\psi_+(-k)}{\Gamma(m)},\; m=1,2,\ldots, \tag{2.4.8}$$

with the convention that the right-hand side is $\mathbb{E}\left[-Y_1\right]$ when $m=1$.

In order to get (2.1.6) in the case when ξ does not have some finite positive exponential moments, we will develop some approximation techniques. However, the exponential functional is not continuous in the Skorohod topology and therefore we have to find some criteria in order to secure the weak convergence of sequences of exponential functionals. This is the aim of the next result.

Lemma 2.4.8. *Let $(\xi^{(n)})_{n\geq1}$ be a sequence of Lévy processes with negative means such that*

$$\lim_{n\to\infty}\xi^{(n)}\stackrel{d}{=}\xi$$

where ξ is a Lévy process with $\mathbb{E}\left[\xi_1\right]<0$. Let us assume further that at least one of the following conditions holds:

1. *for each $n\geq1$, $\xi^{(n)}$ and ξ are unkilled spectrally positive Lévy processes such that $\lim_{n\to\infty}\mathbb{E}\left[\xi_1^{(n)}\right]=\mathbb{E}\left[\xi_1\right]$,*
2. *for each $n\geq1$, $\xi^{(n)}$ and ξ are the negative of unkilled subordinators,*
3. *the sequence $(m_{\xi^{(n)}})_{n\geq1}$ is tight, where $m_{\xi^{(n)}}$ is the law of $\mathrm{I}_{\xi^{(n)}}$.*

Then, in all cases, we have

$$\lim_{n\to\infty}\mathrm{I}_{\xi^{(n)}}\stackrel{d}{=}\mathrm{I}_\xi. \tag{2.4.9}$$

Proof. To prove (2.4.9) in the case (a), we simply observe that writing $\psi_+^{(n)}$ for the Laplace exponent of $\xi_1^{(n)}$, we have, by Lévy continuity Theorem, see e.g. [39, Theorem XIII.1.2], that for all $s\geq0$, $\psi_+^{(n)}(-s)\to\psi_+(-s)$ as $n\to\infty$. Next, putting $M_m^{(n)}$ for the sequence of negative entire moments of $\mathrm{I}_{\xi^{(n)}}$, we easily deduce, from (2.4.8) for all $m = 1, 2\ldots$, that $\lim_{n\to\infty} M_m^{(n)} = M_m$ where M_m is the sequence of negative entire moments of I_ξ. These random variables being moment determinate, see Proposition 2.4.7, we conclude (a) by invoking [39, Examples (b) p.269]. The second case follows by applying a similar line of reasoning to the expression (2.4.7). Finally, the case (c) is a straightforward consequence of (2.3.17) of Theorem 2.3.10. $\square$

Before stating our next result, we need to introduce the following notation. Let us first recall that the reflected processes $\left(R_t^+ = \sup_{0\le s\le t}\xi_s - \xi_t\right)_{t\ge 0}$ and $\left(R_t^- = \xi_t - \inf_{0\le s\le t}\xi_s\right)_{t\ge 0}$ are Feller processes in $[0,\infty)$ which possess local times $L^\pm = (L_t^\pm)_{t\ge0}$ at the level 0. The ascending and descending ladder times, $l^\pm = (l^\pm(t))_{t\ge 0}$, are defined as the right-continuous inverses of $L^\pm$, i.e. for any $t \ge 0$, $l^\pm(t) = \inf\{s > 0;\ L_s^\pm > t\}$ and the ladder height processes $H^+ = (H^+(t))_{t\ge0}$ and $-H^- = (-H^-(t))_{t\ge0}$ by

$$H^+(t) = \xi_{l^+(t)} = \sup_{0\le s\le l^+(t)} \xi_s\,, \qquad \text{whenever } l^+(t) < \infty\,,$$

$$-H^-(t) = \xi_{l^-(t)} = \inf_{0\le s\le l^-(t)} \xi_s\,, \qquad \text{whenever } l^-(t) < \infty\,.$$

Here, we use the convention $\inf\{\varnothing\} = \infty$ and $H^+(t) = \infty$ when $L_\infty^+ \le t$ and $-H^-(t) = -\infty$ when $L_\infty^- \le t$. From [34, p. 27], we have, for $\alpha, \beta \ge 0$,

$$\log \mathbb{E}\left[e^{-\alpha l^+(1) - \beta H^+(1)}\right] = -k(\alpha,\beta) = -k_+ - \eta_+\alpha - \delta_+\beta - \int_0^\infty\int_0^\infty \left(1 - e^{-(\alpha y_1 + \beta y_2)}\right)\mu_+(dy_1, dy_2), \tag{2.4.10}$$

where η_+ is the drift of the subordinator l^+ and $\mu_+(dy_1, dy_2)$ is the Lévy measure of the bivariate subordinator (l^+, H^+). Similarly, for $\alpha, \beta \ge 0$,

$$\log \mathbb{E}\left[e^{-\left(\alpha l^-(1) - \beta H^-(1)\right)}\right] = -k_*(\alpha,\beta) = -\eta_-\alpha - \delta_-\beta - \int_0^\infty\int_0^\infty \left(1 - e^{-(\alpha y_1 + \beta y_2)}\right)\mu_-(dy_1, dy_2), \tag{2.4.11}$$

where η_- is the drift of the subordinator l^- and $\mu_-(dy_1, dy_2)$ is the Lévy measure of the bivariate subordinator $(l^-, -H^-)$.

Lemma 2.4.9. *Let ξ be a Lévy process with triplet (a, σ, Π) and Laplace exponent ψ. Let, for any $n \ge 1$, $\xi^{(n)}$ be the Lévy process with Laplace exponent denoted by $\psi^{(n)}$ and triplet $(a, \sigma, \Pi^{(n)})$ such that $\Pi^{(n)} = \Pi$ on $\mathbb{R}_-$ and on $\mathbb{R}_+$*

$$\Pi^{(n)}(dy) = h^{(n)}(y)\Pi(dy),$$

where for all $y > 0$, $0 \le h^{(n)}(y) \uparrow 1$ as $n \to \infty$ and uniformly for $n \ge 1$ we have that for some $C \ge 0$, $\limsup_{y\to 0} y^{-1}(1 - h_n(y)) \le C$. Then,

$$\lim_{n\to\infty} \xi^{(n)} \stackrel{d}{=} \xi, \tag{2.4.12}$$

and for all $\alpha \geq 0$, $\beta \geq 0$, *we have, as* $n \to \infty$,

$$\begin{aligned} k^{(n)}(\alpha,\beta) &\to k(\alpha,\beta), \\ k_*^{(n)}(\alpha,\beta) &\to k_*(\alpha,\beta), \end{aligned} \tag{2.4.13}$$

where $k^{(n)}(\alpha,\beta)$ *and* $k_*^{(n)}(\alpha,\beta)$ *stand for the bivariate Laplace exponents of the ladder processes of* $\xi^{(n)}$, *normalized such that* $k^{(n)}(1,0) = k_*^{(n)}(1,0) = 1$. *Also* $k(\alpha,\beta)$ *and* $k_*(\alpha,\beta)$ *stand for the bivariate Laplace exponents of the ladder processes of* ξ, *normalized such that* $k(1,0) = k_*(1,0) = 1$.

Remark 2.4.10. *Denote by* $\left(l^+_{(n)}, H^+_{(n)}\right)$ $\left(\textit{resp. } \left(l^-_{(n)}, -H^-_{(n)}\right)\right)$ *the bivariate ascending (resp. descending) ladder processes of* $\xi^{(n)}$ *and* (l^+, H^+) *(resp.* $(l^-, -H^-)$*) the bivariate ascending (resp. descending) ladder processes of* ξ, *then from the Lévy continuity Theorem we deduce that as* $n \to \infty$,

$$\begin{aligned} \left(l^+_{(n)}, H^+_{(n)}\right) &\overset{d}{\to} \left(l^+, H^+\right), \\ \left(l^-_{(n)}, -H^-_{(n)}\right) &\overset{d}{\to} \left(l^-, -H^-\right), \end{aligned} \tag{2.4.14}$$

where in the convergence the sequence of killing rates of the ladder height processes also converge to the killing rate of the limiting process.

Proof. For the sake of completeness and also to include both the compound Poisson case and the cases when $\alpha = 0$ and/or $\beta = 0$, we must improve the proof of Lemma 3.4.2 in [99]. Next, since $\Pi^{(n)}(dx) \overset{v}{\to} \Pi(dx)$, where $\overset{v}{\to}$ stands for the vague convergence, we get (2.4.12) from e.g. [47, Theorem 13.14(i)]. We note the identity

$$\xi \overset{d}{=} \xi^{(n)} + \tilde{\xi}^{(n)}, \tag{2.4.15}$$

where $\tilde{\xi}^{(n)}$ is a subordinator with Lévy measure $\tilde{\Pi}^{(n)}(dy) = (1 - h_n(y))\Pi(dy)$ and no drift, since $1 - h_n(y) = O(y)$ at zero. Then, when ξ is a compound Poisson process we have that $\tilde{\xi}^{(n)}$ is a compound Poisson process and, for all $t > 0$,

$$\mathbb{P}\left(\xi_t^{(n)} = 0\right) = \mathbb{P}\left(\xi_t = 0, t < \tilde{T}^{(n)}\right) + \mathbb{P}\left(\xi_t^{(n)} = 0, t \geq \tilde{T}^{(n)}\right),$$

where $\tilde{T}^{(n)} = \inf\{s > 0;\, \tilde{\xi}_s^{(n)} > 0\}$. Since for all $y > 0$, $h^{(n)}(y) \uparrow 1$, then $\mathbb{P}(t > \tilde{T}^{(n)}) \to 0$ as $n \to \infty$ and

$$\mathbb{P}\left(\xi_t^{(n)} \in dy\right) \mathbb{I}_{\{y \geq 0\}} \overset{v}{\to} \mathbb{P}\left(\xi_t \in dy\right) \mathbb{I}_{\{y \geq 0\}}.$$

When ξ is not a compound Poisson process, the law of $\xi^{(n)}$ does not charge $\{0\}$ and thus as $n \to \infty$

$$\mathbb{P}\left(\xi_t^{(n)} \in dy\right) \mathbb{I}_{\{y > 0\}} \overset{v}{\to} \mathbb{P}\left(\xi_t \in dy\right) \mathbb{I}_{\{y > 0\}}.$$

Henceforth, from the expression

$$k^{(n)}(\alpha,\beta)=\exp\left(\int_0^\infty dt\int_0^\infty\left(e^{-t}-e^{-\alpha t-\beta y}\right)t^{-1}\mathbb{P}(\xi_t^{(n)}\in dy)\right) \tag{2.4.16}$$

which holds for any $\alpha>0$ and $\beta>0$, see e.g. [11, Corollary VI.2.10], we deduce easily that for both cases

$$\lim_{n\to\infty}k^{(n)}(\alpha,\beta)=k(\alpha,\beta). \tag{2.4.17}$$

Moreover, we can write

$$k^{(n)}(\alpha,\beta)=k^{(n)}(0,0)+\tilde{k}^{(n)}(\alpha,\beta), \tag{2.4.18}$$

where $\tilde{k}^{(n)}$ are the Laplace exponents of unkilled bivariate subordinators, see [34, p. 27]. Note from (2.4.16) that

$$k^{(n)}(0,0)=\exp\left(-\int_0^\infty\left(1-e^{-t}\right)\mathbb{P}\left(\xi_t^{(n)}\geq 0\right)\frac{dt}{t}\right).$$

Next from (2.4.15) and the fact that $\tilde{\xi}^{(n)}$ is a subordinator, we have that $\mathbb{P}\left(\xi_t^{(n)}\geq 0\right)\leq\mathbb{P}\left(\xi_t\geq 0\right)$ and appealing to the monotone convergence theorem we get that $k^{(n)}(0,0)\downarrow k(0,0)$. Hence we deduce from (2.4.17) and (2.4.18) that for any $\alpha,\beta>0$, $\tilde{k}^{(n)}(\alpha,\beta)\to\tilde{k}(\alpha,\beta)$ where $\tilde{k}(\alpha,\beta)=k(\alpha,\beta)-k(0,0)$. From the Lévy continuity theorem, we have, writing $\left(\tilde{l}^+_{(n)},\tilde{H}^+_{(n)},\right)$ for the unkilled versions of the ascending bivariate ladder processes, that $\left(\tilde{l}^+_{(n)},\tilde{H}^+_{(n)}\right)\overset{d}{\to}\left(\tilde{l}^+,\tilde{H}^+\right)$, where $\left(\tilde{l}^+,\tilde{H}^+\right)$ stands also for the unkilled version of $\left(\tilde{l}^+,\tilde{H}^+\right)$. These probability distributions being proper, we have that for all $\alpha,\beta\in\mathbb{R}$, $\tilde{k}^{(n)}(i\alpha,i\beta)\to\tilde{k}(i\alpha,i\beta)$, see [39, Theorem XV.3.2]. Hence $k^{(n)}(0,i\beta)\to k(0,i\beta)$ for all $\beta\in\mathbb{R}$ which completes the proof for the ascending ladder height processes. The proof of the convergence of the Laplace exponent of the bivariate descending ladder process follows readily from the identities

$$\begin{aligned}\psi^{(n)}(i\beta)-\alpha&=-k^{(n)}(\alpha,-i\beta)k_*^{(n)}(\alpha,i\beta)\\ \psi(i\beta)-\alpha&=-k(\alpha,-i\beta)k_*(\alpha,i\beta)\end{aligned}$$

and the convergence of $\psi^{(n)}$ to ψ and $k^{(n)}$ to k. □

2.4.2.1 The case P+

We first consider the case when ξ satisfies both the conditions $\mathbf{P}+$ and $\mathbb{E}[\xi_1]>-\infty$. We start by showing that the condition $\mathbf{P}+$ implies that $\mu_+\in\mathcal{P}$. To this end, we shall need the so-called *equation amicale inversée* derived by Vigon, for all $x>0$,

$$\bar{\mu}_+(x)=\int_0^\infty\overline{\Pi}_+(x+y)\mathcal{U}_-(dy), \tag{2.4.19}$$

where $\mathcal{U}_-$ is the renewal measure corresponding to the subordinator H^-, see e.g. [34, Theorem 5.16].

Lemma 2.4.11. *Let us assume that $\overline{\Pi}_+(x)$ has a non-positive derivative $\pi_+(x)$ defined for all $x > 0$ and such that $-\pi_+(x)$ is non-increasing. Then $\bar{\mu}_+(x)$ is differentiable with derivative $u(x)$ such that $-u(x)$ is non-increasing.*

Proof. Fix $x > 0$ and choose $0 < h < x/3$. Then we have the trivial bound using the non-increasing property of $-\pi_+(x)$ and the description (2.4.19) of $\bar{\mu}_+(x)$

$$\begin{aligned}\frac{|\bar{\mu}_+(x \pm h) - \bar{\mu}_+(x)|}{h} &\leq \int_0^\infty \frac{\left|\overline{\Pi}_+(x+y \pm h) - \overline{\Pi}_+(x+y)\right|}{h} \mathcal{U}_-(dy) \\ &\leq \int_0^\infty \left(-\pi_+\left(x+y-h\right)\right)\mathcal{U}_-(dy) \\ &\leq \int_0^\infty \left(-\pi_+\left(\frac{2x}{3}+y\right)\right)\mathcal{U}_-(dy).\end{aligned}$$

We show now that the last expression is finite. Note that

$$\int_0^\infty \left(-\pi_+\left(\frac{2x}{3}+y\right)\right)\mathcal{U}_-(dy) \leq \sum_{n\geq 0} -\pi_+\left(\frac{2x}{3}+n\right)\left(\mathcal{U}_-(n+1) - \mathcal{U}_-(n)\right).$$

From the trivial inequality $\mathcal{U}_-(n+1) - \mathcal{U}_-(n) \leq \mathcal{U}_-(1)$, see [34, Chapter 2, p.11], and since $-\pi_+(x)$ is the non-increasing density of $\overline{\Pi}_+(x)$, we have with $C = \mathcal{U}_-(1) > 0$,

$$\begin{aligned}\int_0^\infty -\pi_+\left(\frac{2x}{3}+y\right)\mathcal{U}_-(dy) &\leq C\sum_{n\geq 0} -\pi_+\left(\frac{2x}{3}+n\right) \\ &\leq -C\pi_+\left(\frac{2x}{3}\right) + C\sum_{n\geq 1}\left(\overline{\Pi}_+\left(\frac{2x}{3}+n-1\right) - \overline{\Pi}_+\left(\frac{2x}{3}+n\right)\right) \\ &\leq -C\pi_+\left(\frac{2x}{3}\right) + C\overline{\Pi}_+\left(\frac{2x}{3}\right) < \infty.\end{aligned}$$

Therefore, for all $x > 0$, the dominated convergence applies and gives

$$u(x) = \int_0^\infty \pi_+(x+y)\mathcal{U}_-(dy).$$

As $-\pi_+(x)$ is non-increasing we deduce that $-u(x)$ is non-increasing as well. $\square$

In the case $\mathbf{P}+$, in comparison to the case $\mathbf{E}_+$, we have that ξ does not necessarily have some positive exponential moments. To circumvent this difficulty we introduce the sequence of Lévy processes $\xi^{(n)}$ obtained from ξ by the following construction: we keep the negative jumps intact and we discard some of the positive ones. More precisely, we thin the positive jumps of ξ to get a Lévy process $\xi^{(n)}$ with $\overline{\Pi}_+^{(n)}$ whose density has the form

$$\pi_+^{(n)}(x) = \pi_+(x)\left(\mathbb{I}_{\{0<x\leq 1\}} + e^{-n^{-1}(x-1)}\mathbb{I}_{\{x>1\}}\right). \tag{2.4.20}$$

Clearly, $-\pi_+^{(n)}(x)$ is non-increasing and $\mathbb{E}\left[e^{s\xi_1^{(n)}}\right] < \infty$, for $s \in (0, n^{-1})$, see (2.4.20). Moreover, since we have only thinned the positive jumps and pointwise $\lim_{n\to\infty} \pi_+^{(n)}(x) = \pi_+(x)$, see (2.4.20),

$$\lim_{n\to\infty} \xi^{(n)} \overset{a.s.}{=} \xi \tag{2.4.21}$$

almost surely in the Skorohod space $\mathcal{D}(0,\infty)$. Finally, since $-\infty < \mathbb{E}\left[\xi_1^{(n)}\right] < \mathbb{E}\left[\xi_1\right] < 0$ and $-\pi_+^{(n)}(x)$ is non-increasing then Lemma 2.4.11 applies and we deduce that the Lévy measure of the ascending ladder height process of $\xi^{(n)}$ has a negative density whose absolute value is non-increasing in x. Then since, for each $n \geq 1$, $\xi^{(n)}$ has some finite positive exponential moments, we have that

$$\mathrm{I}_{\xi^{(n)}} \overset{d}{=} \mathrm{I}_{H^-_{(n)}} \times \mathrm{I}_{Y^{(n)}}. \tag{2.4.22}$$

Since we thinned the positive jumps of ξ, for all $t \geq 0$, $\xi_t^{(n)} \leq \xi_t$ and the monotone convergence theorem together with (2.4.21) imply that

$$\lim_{n\to\infty} \mathrm{I}_{\xi^{(n)}} \overset{a.s.}{=} \mathrm{I}_\xi. \tag{2.4.23}$$

By the choice of the approximating sequence $\xi^{(n)}$ we can first use Lemma 2.4.9 to get

$$\lim_{n\to\infty} H^-_{(n)} \overset{d}{=} H^- \tag{2.4.24}$$

and then Lemma 2.3.10 (b) to obtain that

$$\lim_{n\to\infty} \mathrm{I}_{H^-_{(n)}} \overset{d}{=} \mathrm{I}_{H^-}. \tag{2.4.25}$$

Again from Lemma 2.4.9 we deduce that $k^{(n)}(0,-s) \to k(0,-s)$, for all $s \geq 0$, and $\lim_{n\to\infty} \mathbb{E}\left[Y_1^{(n)}\right] = -\lim_{n\to\infty} k^{(n)}(0,0) = \mathbb{E}\left[Y_1\right]$, so we can apply Lemma 2.3.10 (a) to get that

$$\lim_{n\to\infty} \mathrm{I}_{Y^{(n)}} \overset{d}{=} \mathrm{I}_Y,$$

which completes the proof in this case. □

2.4.2.2 The case $\mathbf{P}_\pm$

First from the philanthropy theory developed by Vigon [99], we know that the conditions $\mu_+ \in \mathcal{P}$ and $\mu_- \in \mathcal{P}$ ensure the existence of a Lévy process ξ with ladder processes H^+ and H^- and such that the Wiener-Hopf factorization (2.1.4) holds on $i\mathbb{R}$. Since we also assume that $k_+ > 0$, this Lévy process necessarily drifts to $-\infty$. Next let us introduce the Laplace exponents

$$\begin{aligned}
\phi_+^{(p)}(z) &= \delta_+ z + \int_{(0,\infty)} (e^{zx} - 1)\mu_+^{(p)}(dx) - k_+\,, &(2.4.26)\\
\phi_-^{(n)}(z) &= -\delta_- z - \int_{(0,\infty)} (1 - e^{-zx})\mu_-^{(n)}(dx), &(2.4.27)
\end{aligned}$$

where we set $\mu_+^{(p)}(dx) = e^{-x/p}\mu_+(dx)$, $p > 0$, and $\mu_-^{(n)}(dx) = e^{-x/n}\mu_+(dx)$, $n > 0$. Plainly, for any $p > 0$, $n > 0$, $\mu_+^{(p)} \in \mathcal{P}$ and $\mu_-^{(n)} \in \mathcal{P}$, hence there exists a Lévy process $\xi^{(p,n)}$ with Laplace exponent $\Psi^{(p,n)}$ satisfying

$$\Psi^{(p,n)}(z) = -\phi_+^{(p)}(z)\phi_-^{(n)}(s), \tag{2.4.28}$$

which is easily seen to be analytic on the strip $-1/n < \mathtt{Re}(z) < 1/p$. Moreover, from [34, Corollary 4.4.4], we have $\mathbb{E}\left[\xi_1^{(p,n)}\right] = -k_+\left(\int_0^\infty xe^{-x/n}\mu_+(dx) + \delta_-\right)$, which is clearly finite and negative. Hence the conditions $\mathbf{E}_+$ are satisfied and we have, with the obvious notation, that

$$\mathrm{I}_{\xi^{(p,n)}} \stackrel{d}{=} \mathrm{I}_{H^-_{(n)}} \times \mathrm{I}_{Y^{(p)}}$$

where for any $p > 0$, $Y^{(p)}$ is a spectrally positive Lévy process with Laplace exponent $\psi_+^{(p)}(-s) = -s\phi_+^{(p)}(-s)$, $s \geq 0$. Let us first deal with the case $n \to \infty$. Since $\phi_-^{(n)}(s) \to \phi_-(s)$, for all $s \geq 0$, we have that

$$\lim_{n\to\infty} H^-_{(n)} \stackrel{d}{=} H^-$$

and from Lemma 2.3.10 (b) we get that

$$\lim_{n\to\infty} \mathrm{I}_{H^-_{(n)}} \stackrel{d}{=} \mathrm{I}_{H^-}.$$

Thus, we deduce that, for any fixed $p > 0$, the sequence $(\mathrm{I}_{\xi^{(p,n)}})_{n\geq 1}$ is tight. Moreover, for any fixed $p > 0$, we also have $\xi^{(p,n)} \stackrel{d}{\to} \xi^{(p)}$, as $n \to \infty$, where $\xi^{(p)}$ has a Laplace exponent $\Psi^{(p)}$ given by

$$\Psi^{(p)}(z) = -\phi_+^{(p)}(z)\phi_-(z). \tag{2.4.29}$$

Indeed this is true by the philanthropy theory. Then from Lemma 2.3.10 (c), we have that

$$\lim_{n\to\infty} \mathrm{I}_{\xi^{(p,n)}} \stackrel{d}{=} \mathrm{I}_{\xi^{(p)}} \stackrel{d}{=} \mathrm{I}_{H^-} \times \mathrm{I}_{Y^{(p)}},$$

which provides a proof of the statement in the case $\mathbf{P}_\pm$ together with the existence of some finite positive exponential moments. Next, as $p \to \infty$, $\phi_+^{(p)}(s) \to \phi_+(s)$, for all $s \geq 0$, and we have that

$$\lim_{p\to\infty} Y^{(p)} \stackrel{d}{=} Y,$$

where Y is a spectrally positive Lévy process with Laplace exponent $\psi_+(-s) = -s\phi_+(-s)$. As $\mathbb{E}\left[Y_1^{(p)}\right] = \phi_+^{(p)}(0) = -k_+$, we can use Lemma 2.3.10 (a) to get

$$\lim_{p\to\infty} \mathrm{I}_{Y^{(p)}} \stackrel{d}{=} \mathrm{I}_Y.$$

As above, we conclude from Lemma 2.3.10 (c) that

$$\lim_{p\to\infty} \mathrm{I}_{\xi^{(p)}} \stackrel{d}{=} \mathrm{I}_\xi \stackrel{d}{=} \mathrm{I}_{H^-} \times \mathrm{I}_Y,$$

which completes the proof of the theorem. $\square$

2.5 Proof of the corollaries

2.5.1 Corollary 2.2.1

First, since ξ is spectrally negative and has a negative mean, it is well known that the function Ψ admits an analytical extension on the right-half plane which is convex on $\mathbb{R}^+$ drifting to ∞, with $\Psi'(0^+) < 0$, and thus there exists $\gamma > 0$ such that $\Psi(\gamma) = 0$. Moreover, the Wiener-Hopf factorization for spectrally negative Lévy processes boils down to

$$\Psi(s) = \frac{\Psi(s)}{s-\gamma}(s-\gamma),\ s > 0.$$

It is not difficult to check that with $\phi_+(s) = s-\gamma$ and $\phi_-(s) = -\frac{\Psi(s)}{s-\gamma}$, we have $\mu_-,\ \mu_+ \in \mathcal{P}$. Observing that $\psi_+(s) = s^2 - \gamma s$ is the Laplace exponent of a scaled Brownian motion with a negative drift γ, it is well-known, see e.g. [102], that

$$\mathrm{I}_Y \stackrel{d}{=} G_\gamma^{-1}.$$

The factorization follows then from Theorem 2.1.1 considered under the condition $\mathbf{P}_\pm$. Since the random variable G_γ^{-1} is MSU, see [32], we have that if I_{H^-} is unimodal then I_ξ is unimodal, which completes the proof of (1). Next, (2) follows easily from the identity

$$m_\xi(x) \quad = \quad \frac{1}{\Gamma(\gamma)}x^{-\gamma-1}\int_0^\infty e^{-y/x}y^\gamma m_{H^-}(y)dy \tag{2.5.1}$$

combined with an argument of monotone convergence.

Further, we recall that Chazal et al. [29, Theorem 4.1] showed, that for any $\beta \geq 0$, $\phi_\beta(s) = \frac{s}{s+\beta}\phi_-(s+\beta)$ is also the Laplace exponent of a negative of a subordinator and with the obvious notation

$$m_{H_\beta^-}(x) \quad = \quad \frac{x^\beta m_{H^-}(x)}{\mathbb{E}\left[\mathrm{I}_{H^-}^\beta\right]}, \quad x > 0. \tag{2.5.2}$$

Then, assuming that $1/x < \lim_{u\to\infty} \Psi(u)/u$, we have, from (2.4.7), (2.5.1) and (2.5.2),

$$\begin{aligned}
m_\xi(x) &= \frac{1}{\Gamma(\gamma)}x^{-\gamma-1}\sum_{n=0}^\infty(-1)^n\frac{x^{-n}}{n!}\int_0^\infty y^{n+\gamma}m_{H^-}(y)dy \\
&= \frac{\mathbb{E}\left[\mathrm{I}_{H^-}^\gamma\right]}{\Gamma(\gamma)}x^{-\gamma-1}\sum_{n=0}^\infty(-1)^n\frac{x^{-n}}{n!}\frac{n!}{\prod_{k=1}^n -\frac{k}{k+\gamma}\phi_-(k+\gamma)} \\
&= \frac{\mathbb{E}\left[\mathrm{I}_{H^-}^\gamma\right]}{\Gamma(\gamma)\Gamma(\gamma+1)}x^{-\gamma-1}\sum_{n=0}^\infty(-1)^n\frac{\Gamma(n+\gamma+1)}{\prod_{k=1}^n -k\phi_-(k+\gamma)}x^{-n} \\
&= \frac{\mathbb{E}\left[\mathrm{I}_{H^-}^\gamma\right]}{\Gamma(\gamma)\Gamma(\gamma+1)}x^{-\gamma-1}\sum_{n=0}^\infty(-1)^n\frac{\Gamma(n+\gamma+1)}{\prod_{k=1}^n \Psi(k+\gamma)}x^{-n},
\end{aligned}$$

where we used an argument of dominated convergence and the identity $-k\phi_-(k+\gamma) = \Psi(k+\gamma)$. Next, again from (2.5.1), we deduce that

$$x^{-\beta}m_\xi(x^{-1}) = \frac{1}{\Gamma(\gamma)}x^{\gamma+1-\beta}\int_0^\infty e^{-xy}y^\gamma m_{H^-}(y)dy$$

from where we easily see that, for any $\beta \geq \gamma+1$, the mapping $x \mapsto x^{-\beta}m_\xi(x^{-1})$ is completely monotone as the product of two Laplace transforms of positive measures. The proof of the Corollary is completed by invoking [90, Theorem 51.6] and noting that I_ξ^{-1} has a density given by $x^{-2}m_\xi(x^{-1})$, i.e. with $\beta = 2$.

2.5.2 Corollary 2.2.3

We first observe from the equation (2.4.19) that, in this case,

$$\begin{aligned}\bar{\mu}_+(x) &= ce^{-\lambda x}\int_0^\infty e^{-\lambda y}\mathcal{U}_-(dy)\\ &= c_-e^{-\lambda x},\end{aligned}$$

where the last identity follows from [34] and we have set $c_- = \frac{c}{\phi_-(\lambda)}$. From (2.1.5), we deduce that Y is a spectrally positive Lévy process with Laplace exponent given, for any $s<\lambda$, by

$$\begin{aligned}\psi_+(s) &= \delta_+s^2 - k_+s + c_-\frac{s^2}{\lambda-s}\\ &= \frac{s}{\lambda-s}\left(-\delta_+s^2-(\delta_+\lambda+k_++c_-)s-k_+\lambda\right),\end{aligned}$$

where $\delta_+>0$ since $\sigma>0$, see [34, Corollary 4.4.4]. Thus, using the continuity and convexity of ψ_+ on $(-\infty,\lambda)$ and on (λ,∞), studying its asymptotic behavior on these intervals and the identity $\psi_+'(0) = -k_+ < 0$, we easily show that the equation $\psi_+(s)=0$ has 3 roots which are real, one is obviously 0 and the two others θ_1 and θ_2 are such that $0<\theta_1<\lambda<\theta_2$. Thus,

$$\psi_+(-s) = \frac{\delta_+s}{\lambda+s}\left(s+\theta_1\right)\left(s+\theta_2\right),\ s>-\lambda\,.$$

Next, from (2.4.8), we have, with $C = k_+\frac{\Gamma(\lambda+1)}{\Gamma(\theta_1+1)\Gamma(\theta_2+1)}$ and for $m=2,\ldots,$ that

$$\mathbb{E}\left[\mathrm{I}_Y^{-m}\right] = C\delta_+^{m-1}\frac{\Gamma(m+\theta_1)\Gamma(m+\theta_2)}{\Gamma(m+\lambda)}$$

from where we easily deduce (3.1.9) by moments identification. Note that a simple computation gives that $\theta_1\theta_2 = \delta_+\lambda k_+$ securing that the distribution of I_Y is proper. Next, the random variable I_Y^{-1} being moment determinate, we have, for $\mathtt{Re}(z)<\theta_1+1$, that

$$\mathbb{E}\left[\mathrm{I}_Y^{z-1}\right] = C\delta_+^{-z}\frac{\Gamma(-z+\theta_1+1)\Gamma(-z+\theta_2+1)}{\Gamma(-z+\lambda+1)}.$$

Applying the inverse Mellin transform, see e.g. [70, Section 3.4.2], we get

$$m_Y\left(\frac{x}{\delta^+}\right) = C\sum_{i=1}^{2} x^{-\theta_i-1}\mathcal{I}_i(-x^{-1}),\ x>0, \tag{2.5.3}$$

where $\mathcal{I}_i(x) = \sum_{n=0}^{\infty} b_{n,i}\frac{x^n}{n!}$, $b_{n,1} = \frac{\Gamma(\theta_2-\theta_1-n)}{\Gamma(\lambda-\theta_1-n)}$ and $b_{n,2} = \frac{\Gamma(\theta_1-\theta_2-n)}{\Gamma(\lambda-\theta_2-n)}$. The proof of the Corollary is completed by following a line of reasoning similar to the proof of Corollary 2.2.1.

2.5.3 Corollary 2.2.6

For any $\alpha\in(0,1)$, let us observe that, for any $s\geq 0$,

$$\phi_-(-s) = \frac{\alpha s\Gamma(\alpha(s+1)+1)}{(1+s)\Gamma(\alpha s+1)} \tag{2.5.4}$$

$$= \int_0^\infty (1-e^{-sy})u_\alpha(y)dy, \tag{2.5.5}$$

where $u_\alpha(y) = \frac{e^{-y}e^{-y/\alpha}}{\Gamma(1-\alpha)(1-e^{-y/\alpha})^{\alpha+1}}$. We easily check that $u_\alpha(y)dy\in\mathcal{P}$ and hence Ψ is a Laplace exponent of a Lévy process which drifts to $-\infty$. Next, we know, see e.g. [75], that

$$\mathrm{I}_{\tilde{H}^-} \stackrel{d}{=} S_\alpha^{-\alpha},$$

where $\tilde{H}^-$ is the negative of the subordinator having Laplace exponent

$$\tilde{\phi}_-(-s) = \frac{\alpha\Gamma(\alpha s+1)}{\Gamma(\alpha(s-1)+1)}.$$

Observing that $\phi_-(-s) = \frac{-s}{-s+1}\tilde{\phi}_-(-s+1)$, we deduce, from (2.5.2), that

$$m_{H^-}(x) = \frac{x^{-1/\alpha}}{\alpha}g_\alpha\left(x^{-1/\alpha}\right),\ x>0, \tag{2.5.6}$$

from which we readily get the expression (2.2.10). Then, we recall the following power series representation of positive stable laws, see e.g. [90, Formula (14.31)],

$$g_\alpha(x) = \sum_{n=1}^{\infty}\frac{(-1)^n}{\Gamma(-\alpha n)n!}x^{-(1+\alpha n)},\ x>0.$$

Then, by means of an argument of dominated convergence justified by the condition $\lim_{s\to\infty} s^{\alpha-1}\phi_+(-s)=0$, we get, for all $x>0$, that

$$m_\xi(x) = \frac{k_+}{\alpha}\sum_{n=1}^{\infty}\frac{(-1)^n}{\Gamma(-\alpha n)n!}x^n\int_0^\infty y^{-(n+1)}f_Y(y)dy$$

$$= \frac{k_+}{\alpha}\sum_{n=1}^{\infty}\frac{\prod_{k=1}^n\phi_+(-k)}{\Gamma(-\alpha n)n!}x^n,$$

where we used the identities (2.4.8), $\mathbb{E}[-Y_1] = k_+$ and $\psi_+(-k) = -k\phi_+(-k)$. The fact that the series is absolutely convergent is justified by using classical criteria combined with the Euler's reflection formula $\Gamma(1-z)\Gamma(z)\sin(\pi z) = \pi$ with the asymptotics

$$\frac{\Gamma(z+a)}{\Gamma(z+b)} = z^{a-b}\left(1 + O\left(|z|^{-1}\right)\right) \qquad \text{as } z \to \infty,\ |arg(z)| < \pi, \tag{2.5.7}$$

see e.g. [58, Chap. 1]. We complete the proof by mentioning that Simon [93] proved recently that the positive stable laws are MSU if and only if $\alpha \leq 1/2$ which implies, from (2.5.6), that I_{H^-} is also MSU in this case.

Chapter 3

Extended factorizations of exponential functionals of Lévy processes

In [68], under mild conditions, a Wiener-Hopf type factorization is derived for the exponential functional of proper Lévy processes. In this paper, we extend this factorization by relaxing a finite moment assumption as well as by considering the exponential functional for killed Lévy processes. As a by-product, we derive some interesting fine distributional properties enjoyed by a large class of this random variable, such as the absolute continuity of its distribution and the smoothness, boundedness or complete monotonicity of its density. This type of results is then used to derive similar properties for the law of maxima and first passage time of some stable Lévy processes. Thus, for example, we show that for any stable process with $\rho \in (0, \frac{1}{\alpha} - 1]$, where $\rho \in [0, 1]$ is the positivity parameter and α is the stable index, then the first passage time has a bounded and non-increasing density on $\mathbb{R}_+$. We also generate many instances of integral or power series representations for the law of the exponential functional of Lévy processes with one or two-sided jumps. The proof of our main results requires different devices from the one developed in [68]. It relies in particular on a generalization of a transform recently introduced in [29] together with some extensions to killed Lévy process of Wiener-Hopf techniques. The factorizations developed here also allow for further applications which we only indicate here.

3.1 Introduction and main results

Let $\xi = (\xi_t)_{t \geq 0}$ be a possibly killed Lévy process starting from 0. We denote by Ψ_q its Lévy-Khintchine exponent which takes the form, for any $z \in i\mathbb{R}$,

$$\Psi_q(z) = bz + \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{\infty} \left(e^{zy} - 1 - zy\mathbb{I}_{\{|y|<1\}}\right) \Pi(dy) - q, \tag{3.1.1}$$

where $q \geq 0$ is the killing rate, $\sigma \geq 0, b \in \mathbb{R}$ and Π is a sigma-finite positive measure satisfying the condition $\int_{\mathbb{R}}(y^2 \wedge 1)\Pi(dy) < \infty$. In this paper, we are interested in both characterizing the distribution and deriving some fine distributional properties of the so-called exponential functional of ξ, which is defined by

$$\mathrm{I}_{\Psi_q} = \int_0^\infty e^{\xi_t} \mathbb{I}_{\{t<e_q\}} dt,$$

where e_q is the lifetime of ξ, i.e. it is an exponential random variable of parameter q (with the convention that $e_0 = \infty$) independent of ξ. When $q = 0$ we simply write $\Psi = \Psi_0$ and we assume that ξ drifts to $-\infty$. The motivation for studying this positive random variable finds its roots in probability theory but has some strong connections with issues coming from other fields of mathematics such as functional and complex analysis. Besides their inherent interest, problems of this type have also ties with other areas of sciences, e.g. astrophysics, biology, insurance and mathematical finance. For more information and motivation as to why exponential functionals are of interest we refer to [21]. It is also worth mentioning that there exists a close connection between the law of the exponential functional of some specific Lévy processes and the one of the maxima of stable processes offering a way to study the fluctuation of these processes from a perspective different from the classical Wiener-Hopf techniques. We refer to [68] for a thorough description of the recent methodologies which have been developed to investigate the distribution of I_{Ψ_q} and [55] for more general forms of the exponential functional. In particular, we mention that, in that paper, it is shown under a mild assumption that, when $q = 0$ and $-\infty < \mathbb{E}[\xi_1] < 0$, the variable I_Ψ factorizes into the product of two independent exponential functionals of Lévy processes defined in terms of the ladder height processes of ξ. The purpose of this paper is to extend this Wiener-Hopf type factorization by first relaxing the finite moment condition on the underlying Lévy processes and then by deriving similar factorization identities for the exponential functional of killed Lévy processes. We emphasize that the approach carried out in [68] can not be used to deal with this generalization. Indeed, therein, the main identity is obtained by means of the functional equation (3.2.9), satisfied by the Mellin transform of I_Ψ combined with the characterization of its distribution as the stationary measure of some generalized Ornstein-Uhlenbeck processes. Indeed the law of I_{Ψ_q}, for any $q > 0$, cannot be identified as a stationary measure of some Markov process anymore whereas when $q = 0$ and the first moment of ξ_1 is not finite, the functional equation (3.2.9) does not hold even on the imaginary line and therefore we cannot directly guess the existence of a probabilistic factorization. In order to circumvent these difficulties our strategy relies on a transformation between Laplace exponents of Lévy processes which allows to establish a connection between the study of the exponential functional for unkilled and killed Lévy processes. This will be achieved by generalizing to our context a mapping recently introduced by Chazal et al. [29] and by providing some interesting results concerning the Wiener-Hopf factorization of killed Lévy processes. We also indicate that our extended factorizations of exponential functionals allow us to identify some fine distributional properties enjoyed by a large class of these random variables, such as the smoothness of their distribution, the monotonicity, complete monotonicity of their density, etc. We will be using these type of results to provide some new distributional properties enjoyed by the density of first passage times for some stable Lévy processes.

The factorizations developed here seem to have further implications. For instance, as

shown in [83] they allow for the development of intertwining relations between semigroups of positive self-similar Markov processes (PSSMP), i.e. $P_t\Lambda = \Lambda Q_t$, where Λ is a suitable kernel. The PSSMP, whose semigroups P_t and Q_t are intertwined, are based precisely on the duals of the Lévy processes associated to Ψ_0 and ψ^0, see (3.1.6). Thus the factorization (3.1.6) gives us the tools to obtain explicit results for the semigroups of PSSMP and thus suggests that the Wiener-Hopf factorization is more natural to the setting of PSSMP. For more information and applications we refer to [83].
In order to state our main result we introduce some notation. First, we recall that the reflected processes $\left(\sup_{0\leq s\leq t}\xi_s - \xi_t\right)_{t\geq 0}$ and $(\xi_t - \inf_{0\leq s\leq t}\xi_s)_{t\geq 0}$ are Feller processes in $[0,\infty)$ which possess local times $(L_t^{\pm})_{t\geq 0}$ at level 0, see [11, Chapter IV]. The ascending and descending ladder times are defined as the right-continuous inverse of $L^{\pm}$, viz. $(L_t^{\pm})^{-1} = \inf\{s > 0;\ L_s^{\pm} > t\}$ and the ladder height processes H^+ and H^- by

$$H_t^+ = \xi_{(L_t^+)^{-1}} = \sup_{0\leq s\leq (L_t^+)^{-1}} \xi_s\,, \qquad \text{whenever } (L_t^+)^{-1} < \infty\,,$$

$$H_t^- = \xi_{(L_t^-)^{-1}} = \inf_{0\leq s\leq (L_t^-)^{-1}} \xi_s\,, \qquad \text{whenever } (L_t^-)^{-1} < \infty\,.$$

Here, we use the convention that $\inf\varnothing = \infty$ and $H_t^{\pm} = \infty$, when $L_\infty^{\pm} \leq t$. From [34, p. 27], we have for $q \geq 0$, $s \geq 0$,

$$\log \mathbb{E}\left[e^{-q(L_1^+)^{-1} - sH_1^+}\right] = -\Phi_+(q,s) = -k_+ - \eta_+ q - \delta_+ s - \int_0^\infty\int_0^\infty \left(1 - e^{-(qy_1+sy_2)}\right)\mu_+(dy_1,dy_2), \tag{3.1.2}$$

where η_+ (resp. δ_+) is the drift of the subordinator $(L^+)^{-1}$ (resp. H^+) and $\mu_+(dy_1,dy_2)$ is the Lévy measure of the bivariate subordinator $((L^+)^{-1}, H^+)$. Similarly, for $q \geq 0$, $s \geq 0$,

$$\log \mathbb{E}\left[e^{-q(L_1^-)^{-1} + sH_1^-}\right] = -\Phi_-(q,s) = -k_- - \eta_- q - \delta_- s - \int_0^\infty\int_0^\infty \left(1 - e^{-(qy_1+sy_2)}\right)\mu_-(dy_1,dy_2), \tag{3.1.3}$$

where η_- (resp. δ_-) is the drift of the subordinator $(L^-)^{-1}$ (resp. $-H^-$) and $\mu_-(dy_1,dy_2)$ is the Lévy measure of the bivariate subordinator $((L^-)^{-1}, -H^-)$. The celebrated Wiener-Hopf factorization then reads off as

$$\Psi_q(z) = -\Phi_+(q,-z)\Phi_-(q,z), \tag{3.1.4}$$

where we set $\Phi_+(1,0) = \Phi_-(1,0) = 1$ as the normalization of the local times. We point out that while it can happen that $((L^+)^{-1}, H^+)$ (resp. $((L^-)^{-1}, -H^-))$) can be increasing renewal processes, see [22, Section 1], this does not affect our definitions.

As in [68] throughout the paper we work with the following set of measures:

$\mathcal{P}$: *the set of positive measures on* $\mathbb{R}_+$ *which admit a non-increasing density.*

Our first theorem is the main result in our paper. Equation (3.1.6) extends [68, (1.6), Theorem 1.2] to the killed case as well to the case when $\mathbb{E}[\xi_1] = -\infty$ and it is the backbone of all our applications.

Theorem 3.1.1. *Let $q \geq 0$ and assume that ξ drifts to $-\infty$, when $q = 0$. Then the law of the random variable I_{Ψ_q} is absolutely continuous with density which we denote by m_{Ψ_q}. Next, assume that one of the following conditions holds:*

1. $\mathbf{P}+$ $\Pi_+(dy) = \Pi(dy)\mathbb{I}_{\{y>0\}} \in \mathcal{P}$,
2. $\mathbf{P}^q_\pm$ $\mu_{q_+}(dy) = \int_0^\infty e^{-qy_1}\mu_+(dy_1, dy) \in \mathcal{P}$, $\mu_{q_-}(dy) = \int_0^\infty e^{-qy_1}\mu_-(dy_1, dy) \in \mathcal{P}$.

Then, in both cases, there exists an unkilled spectrally positive Lévy process with a negative mean such that its Laplace exponent ψ^{q_+} takes the form

$$\psi^{q_+}(-s) = s\Phi_+(q, s) = \delta_+ s^2 + q_+ s + s^2 \int_0^\infty e^{-sy}\mu_{q_+}(y, \infty)dy, \ s \geq 0, \tag{3.1.5}$$

where $q_+ = k_+ + \eta_+ q + \int_0^\infty \int_0^\infty \left(1 - e^{-qy_1}\right)\mu_+(dy_1, dy_2) > 0$. Furthermore, for any $q \geq 0$, we have the factorization

$$\mathrm{I}_{\Psi_q} \stackrel{d}{=} \mathrm{I}_{\phi_{q_-}} \times \mathrm{I}_{\psi^{q_+}}, \tag{3.1.6}$$

where $\times$ stands for the product of independent random variables and $\phi_{q_-}(z) = -\Phi_-(q, z)$ is the Laplace exponent of a negative of a subordinator which is killed at the rate given by the expression
$q_- = k_- + \eta_- q + \int_0^\infty \int_0^\infty \left(1 - e^{-qy_1}\right)\mu_-(dy_1, dy_2) \geq 0$ *and $q_- = 0$ if and only if $q = 0$.*

Remark 3.1.2.

1. *We mention that when $q = 0$, in comparison to [68, Theorem 1.1], here we also include the case when $\mathbb{E}[\xi_1] = -\infty$. We recall that under such a condition, the functional equation* (3.2.9) *below does not even hold on the imaginary line $i\mathbb{R}$.*

2. *We emphasize that the main factorization identity* (3.1.6) *allows to build up many examples of two-sided Lévy processes for which the density of I_{Ψ_q} can be described as a convergent power series. This is due to the fact that the exponential functionals on the right-hand side of the identity are easier to study as we have, for instance, simple expressions for their positive or negative integer moments. More precisely, the positive entire moments of $\mathrm{I}_{\phi_{q_-}}$, for any $q \geq 0$, are given in* (3.2.22) *below and we have from [20] that the law of $1/\mathrm{I}_{\psi^{q_+}}$ is determined by its positive entire moments as follows*

$$\mathbb{E}[\mathrm{I}^{-m}_{\psi^{q_+}}] \;=\; -(\psi^{q_+})'(0^-)\frac{\prod_{k=1}^{m-1}\psi^{q_+}(-k)}{\Gamma(m)}, \ m = 1, 2, \ldots \tag{3.1.7}$$

   with the convention that the right-hand side is $-(\psi^{q_+})'(0^-)$ when $m = 1$. Some specific examples will be detailed in Section 3.3.

3. *Assuming that we start with bivariate Laplace exponents Φ_+ and Φ_- such that their Lévy measures satisfy condition $\mathbf{P}^q_\pm$ with $\Phi_+(q, 0)\Phi_-(q, 0) > 0$, then from Lemma 3.2.10 below we can construct a killed Lévy process with Laplace exponent Ψ_q given by identity* (3.1.4) *and such that factorization* (3.1.6) *holds.*

4. *We point out that* (3.1.6) *holds even when* $\phi_{q_-}(z) - \phi_{q_-}(0) = 0$, *for all* $z \in i\mathbb{R}$, *i.e. when* ξ *is a subordinator. In this case* $I_{\phi_{q_-}} = \int_0^{e_{q_-}} ds = e_{q_-}$.

We postpone the proof of the theorem to Section 3.2. We proceed instead by providing some consequences of our factorization identity (3.1.6) concerning some interesting distributional properties of the exponential functional. Before stating the results, we recall that the density m of a positive random variable is completely monotone if m is infinitely continuously differentiable and $(-1)^n m^{(n)}(x) \geq 0$, for all $x \geq 0$ and $n = 0, 1, \ldots$. Note in particular, that m is non-increasing and thus the distribution of the random variable is unimodal with mode at 0, that is its distribution is concave on $[0, \infty)$.

Corollary 3.1.3. *(i) Let us assume that either condition (1) or (2) of Theorem 3.1.1 and* $|\Psi_q(s)| < +\infty$, *for* $s \in [-1, 0]$, *holds true. Then, for any* $q > 0$, *such that* $\Psi_q(-1) \leq 0$, *the density* m_{Ψ_q} *is non-increasing, continuous and a.e. differentiable on* $\mathbb{R}^+$ *with* $m_{\Psi_q}(0) = q$.

(ii) Let ξ *be a subordinator with Lévy measure* $\Pi_+ \in \mathcal{P}$. *Then, for any* $q > 0$, *the density of* I_{Ψ_q} *is completely monotone and bounded with* $m_{\Psi_q}(0) = q$. *Moreover, recalling that, in this case, the drift* b *of* ξ *is non-negative, we have, for any* $x < 1/b$ *(with the convention that* $1/0 = +\infty$*), that*

$$m_{\Psi_q}(x) = \sum_{n=0}^{\infty} a_n(\Psi_q) \frac{(-x)^n}{n!},$$

where $a_n(\Psi_q) = q \prod_{k=1}^{n} -\Psi_q(-k)$ *with* $a_0(\Psi_q) = q$. *If* $b > 0$, *we have for any* $x > 0$

$$m_{\Psi_q}(x) = (1 + bx)^{-1} \sum_{n=0}^{\infty} \tilde{a}_n(\Psi_q) \left(\frac{bx}{bx+1}\right)^n, \tag{3.1.8}$$

where $\tilde{a}_n(\Psi_q) = \sum_{k=0}^{n} (-1)^k \binom{n}{k} \frac{a_k(\Psi_q)}{k!}$.

(iii) Let ξ *be a spectrally positive Lévy process and we denote, for any* $q > 0$, *by* γ_q *the only positive root of the equation* $\Psi_q(-s) = 0$. *Then, we have*

$$\mathrm{I}_{\Psi_q} \stackrel{d}{=} B^{-1}(1, \gamma_q) \times \mathrm{I}_{\psi^{q+}}, \tag{3.1.9}$$

where $B(1, \gamma_q)$ *is a Beta random variable and* $\psi^{q+}(z) = z\phi_{q+}(z)$. *Moreover, if* $\Psi_q(-1) \leq 0$ *then* I_{Ψ_q} *has a non-increasing density.*

(iv) Finally, let ξ *be a spectrally negative Lévy process. Denoting here, for any* $q > 0$, *by* γ_q *the only positive root of the equation* $\Psi_q(s) = 0$, *we have*

$$m_{\Psi_q}(x) = \frac{x^{-\gamma_q - 1}}{\Gamma(\gamma_q)} \int_0^{\infty} e^{-y/x} y^{\gamma_q} m_{\phi_{q_-}}(y) dy, \ x > 0, \tag{3.1.10}$$

where Γ stands for the Gamma function. In particular, we get the precise asymptotic for the density at infinity, i.e.

$$\lim_{x\to\infty} x^{\gamma_q+1} m_{\Psi_q}(x) = \frac{\mathbb{E}\left[\mathrm{I}_{\phi_{q_-}}^{\gamma_q}\right]}{\Gamma(\gamma_q)}.$$

Then, for any $\beta \geq \gamma_q+1$, the mapping $x \mapsto x^{-\beta} m_{\Psi_q}(x^{-1})$ is completely monotone. In particular, the density of the random variable $\mathrm{I}_{\Psi_q}^{-1}$ is completely monotone, whenever $\gamma_q \leq 1$.

Remark 3.1.4. 1. *We point out that a positive random variable with a completely monotone density is in particular infinitely divisible, see [90, Theorem 51.6].*

2. *A positive random variable with a non-increasing density is strongly multiplicative unimodal (for short SMU), that is the product of this random variable with any independent positive random variable is unimodal and in this case the product has its mode at* 0, *see [32, Proposition 3.6].*

3. *We mention that the two cases (ii) and (iii), i.e. when the Lévy process has no negative jumps, have not been studied in the literature so far.*

4. *One may recover from item (iv) the expression of the density found in [76] in this case. Furthermore, we point out that in [76] it is proved that the density extends actually to a function of a complex variable which is analytical in the entire complex plane cut along the negative real axis and admits a power series representation for all $x > 0$.*

5. *Note that the series* (3.1.8) *is easily amenable to numerical computations since $a_k(\Psi_q)$ can be computed recurrently. We stress that* (3.1.8) *would be difficult to get from* (3.2.9) *because the functional equation holds on a strip in the complex plane which might explain why such simple series has not yet appeared in the literature.*

The proof of this corollary and of the following one will be given in Section 3.3. We will also describe therein some examples illustrating these results. As a by-product of Corollary 3.1.3, we get the following new distributional properties for the law of maximum and first passage times of some stable Lévy processes.

Corollary 3.1.5. *Let $Z = (Z_t)_{t\geq 0}$ be an α-stable Lévy process starting from 0 with $\alpha \in (0,2]$. Let us write $S_1 = \sup_{0\leq t\leq 1} Z_t$ and $T_1 = \inf\{t > 0;\ Z_t \geq 1\}$ and recall that the scaling property of Z yields the identity $T_1 \stackrel{d}{=} S_1^{-\alpha}$. Then, writing $\rho = \mathbb{P}(Z_1 > 0)$, we have the following claims:*

(i) The density of T_1 is bounded and non-increasing for any $\alpha \in (0,1)$ and $\rho \in \left(0, \frac{1}{\alpha} - 1\right]$. In particular, this property holds true for any $\alpha \in (0, \frac{1}{2}]$ and for symmetric stable Lévy processes, i.e. $\rho = \frac{1}{2}$, with $\alpha \in (0, \frac{2}{3}]$.

(ii) *The density of S_1^{α} is completely monotone if $\alpha \in (1,2]$ and $\rho = 1 - \frac{1}{\alpha}$, that is when Z is spectrally positive.*

Remark 3.1.6. *When $\alpha \in (0,1)$ and $\rho = 1$, that is Z is a stable subordinator, we have the obvious identity $Z_1 \stackrel{d}{=} S_1$. Thus, we get from the first item that the density of $Z_1^{-\alpha}$ is non-increasing on $\mathbb{R}^+$ if $\alpha \in (0, \frac{1}{2}]$. From Remark 3.1.4 (2) this means that for these values of α, $Z_1^{-\alpha}$ is SMU with mode at 0. Note that this result is consistent with the main result of Simon in [93] where it is shown that Z_1 is SMU with a positive mode if and only if $\alpha \in (0, \frac{1}{2}]$. The positivity of the mode implies that any non-zero real power of Z_1, and in particular $T_1 \stackrel{d}{=} Z_1^{-\alpha}$, is SMU, see e.g. [93, Section 1].*

3.2 Proof of Theorem 3.1.1

3.2.1 The case P+

We start by extending to two-sided Lévy processes a transformation which has been introduced in [29] in the framework of spectrally negative Lévy processes. This mapping turns out to be very useful in the context of both the Wiener-Hopf factorization of Lévy processes and their exponential functionals. For its statement we need the following notation

$$\overline{\Pi}_-(y) = \int_{-\infty}^{y} \Pi(dr)\mathbb{I}_{\{y<0\}} \text{ and } \overline{\Pi}_+(y) = \int_{y}^{\infty} \Pi(dr)\mathbb{I}_{\{y>0\}}$$

and the following assumption:

$\mathbf{T}_{\beta+}$: There exists $\beta^+ > 0$ such that for all $\beta \in [0, \beta^+)$, $|\Psi(\beta)| < +\infty$ and $e^{\beta y}\overline{\Pi}_+(y)dy \in \mathcal{P}$.

Also if Ψ satisfies $\mathbf{T}_{\beta+}$, we write, for all $q \geq 0$,

$$\beta_q^* = \sup\{\beta > 0;\ \Psi(\beta) - q < 0\} \wedge \beta_+. \tag{3.2.1}$$

Proposition 3.2.1. *Let us assume that $\mathbf{T}_{\beta+}$ holds. Then, for any $\beta \in (0, \beta_+)$, the linear mapping $\mathcal{T}_\beta$ defined by*

$$\mathcal{T}_\beta \Psi_q(s) \ = \ \frac{s}{s+\beta}\Psi_q(s+\beta),\ s \in (-\beta, \beta_+ - \beta),$$

is the Laplace exponent of an unkilled Lévy process $\xi^{(\beta,q)} = (\xi_t^{(\beta,q)})_{t\geq 0}$ with Gaussian coefficient $\frac{\sigma^2}{2}$ and Lévy measure Π^β given by

$$\Pi^\beta(dy) = e^{\beta y}\left(\Pi(dy) - \beta\overline{\Pi}_+(y)dy + \left(\beta\overline{\Pi}_-(y) + q\beta\right)dy\mathbb{I}_{\{y<0\}}\right). \tag{3.2.2}$$

Furthermore, if we assume that ξ drifts to $-\infty$ when $q = 0$, then for any $q \geq 0$, $\beta_q^* > 0$, and, for any $\beta \in (0, \beta_q^*)$, we have

$$-\infty < \mathbb{E}\left[\xi_1^{(\beta,q)}\right] = \frac{\Psi(\beta) - q}{\beta} < 0. \tag{3.2.3}$$

Proof. First, by linearity of the mapping $\mathcal{T}_\beta$, one gets

$$\mathcal{T}_\beta \Psi_q(z) = \mathcal{T}_\beta \Psi(z) - q\frac{z}{z+\beta}, \tag{3.2.4}$$

where we recognize, on the right-hand side, the Laplace exponent of a negative of a compound Poisson process with parameter $q > 0$, whose jumps are exponentially distributed with parameter $\beta > 0$. Next we observe that one can write

$$\Psi(z) = \Psi_-(z) + \Psi_+(z), \tag{3.2.5}$$

where $\Psi_+(z) = \int_0^\infty \left(e^{zy} - 1 - zy\mathbb{I}_{\{|y|<1\}}\right)\Pi_+(dy)$ and Ψ_- is the Laplace exponent of a Lévy process without positive jumps. The description of $\mathcal{T}_\beta\Psi_-$ as the Laplace exponent of a Lévy process without positive jumps follows from [29, Proposition 2.2]. Thus, from the linearity of the transform it remains to study its effect on Ψ_+. An integration by parts gives us that

$$\Psi_+(z) \;=\; z\left(\int_0^\infty \left(e^{zy} - \mathbb{I}_{\{|y|<1\}}\right)\overline{\Pi}_+(y)dy + \overline{\Pi}_+(1)\right).$$

Then

$$\begin{aligned}
\mathcal{T}_\beta\Psi_+(z) &= \frac{z}{z+\beta}\Psi_+(z+\beta) = z\left(\int_0^\infty \left(e^{(z+\beta)y} - \mathbb{I}_{\{|y|<1\}}\right)\overline{\Pi}_+(y)dy + \overline{\Pi}_+(1)\right)\\
&= z\left(\int_0^\infty \left(e^{zy} - \mathbb{I}_{\{|y|<1\}}\right)e^{\beta y}\overline{\Pi}_+(y)dy + \int_0^1 \left(e^{\beta y}-1\right)\overline{\Pi}_+(y)dy + \overline{\Pi}_+(1)\right)\\
&= \int_0^\infty \left(e^{zy} - 1 - zy\mathbb{I}_{\{|y|<1\}}\right)\left(-e^{\beta y}\overline{\Pi}_+(y)\right)' dy\\
&+ z\left(\int_0^1 \left(e^{\beta y}-1\right)\overline{\Pi}_+(y)dy + \overline{\Pi}_+(1)(1-e^{\beta})\right),
\end{aligned}$$

which provides the expression (3.2.2) since the mapping $y \mapsto e^{\beta y}\overline{\Pi}(y)$ is non-increasing by assumption and plainly the condition $\mathbf{T}_{\beta^+}$ gives that $\int_0^\infty (1\wedge y^2)(-e^{\beta y}\overline{\Pi}_+(y))'dy < +\infty$. Next, when $q > 0$ then $\beta_q^* > 0$ since $\Psi(0) = 0$ and the mapping $s \mapsto \Psi(s)$ is continuous on $[0, \beta^+)$. When $q = 0$, the condition $\mathbf{T}_{\beta^+}$ combined with the fact that ξ drifts to $-\infty$ implies that $\Psi'(0+) < 0$, where $\Psi'(0+)$ can be $-\infty$. Clearly then we have that $\Psi(\beta) < 0$, for any $\beta \in (0, \epsilon)$ and some $\epsilon > 0$, and hence $\beta_0^* > 0$. Moreover, we observe that, for any $q \geq 0$,

$$(\mathcal{T}_\beta\Psi_q)'(0^+) = \frac{\Psi(\beta) - q}{\beta},$$

which is clearly finite and negative, for any $\beta \in (0, \beta_q^*)$. The proof of the proposition is completed. □

Remark 3.2.2. *We note that, for any $q > 0$ and any $0 < \beta < \beta^+$, the Lévy process $\xi^{(\beta,q)}$ can be decomposed as $\xi_t^{(\beta,q)} = \xi_t^\beta - N_t^q$, where $(\xi_t^\beta)_{t\geq 0}$ is a Lévy process with Laplace exponent $\mathcal{T}_\beta\Psi$ and $(N_t^q)_{t\geq 0}$ is an independent compound Poisson process with parameter q whose jumps are exponentially distributed with parameter β.*

We shall need the following alternative representation of the bivariate ladder exponents as well as an interesting application of the transform $\mathcal{T}$ in the context of the Wiener-Hopf factorization of Lévy processes.

Proposition 3.2.3. *For any $q > 0$, we have $\phi_{q+}(z) = -\Phi_+(q, -z)$ and $\phi_{q-}(z) = -\Phi_-(q, z)$, where ϕ_{q+} (resp. ϕ_{q-}) is the Laplace exponent of (resp. the negative of) a subordinator. More precisely, they take the form*

$$\begin{aligned} \phi_{q+}(z) &= -q_+ + \delta_+ z + \int_0^\infty (e^{zy} - 1)\, \mu_{q+}(dy), \\ \phi_{q-}(z) &= -q_- - \delta_- z - \int_0^\infty \left(1 - e^{-zy}\right) \mu_{q-}(dy), \end{aligned}$$

where we recall that $\mu_{q\pm}(dy) = \int_0^\infty e^{-qy_1} \mu_\pm(dy_1, dy)$ and $q_\pm = k_\pm + \eta_\pm q + \int_0^\infty \int_0^\infty (1 - e^{-qy_1})\, \mu_\pm(dy_1, dy_2) > 0$. Consequently, the Wiener-Hopf factorization (3.1.4) *takes the form*

$$\Psi_q(z) = -\phi_{q+}(z)\phi_{q-}(z). \tag{3.2.6}$$

Moreover, assume that $\mathbf{T}_{\beta^+}$ holds and that ξ drifts to $-\infty$ when $q = 0$. Then, for any $\beta \in (0, \beta_q^*)$, we have

$$\mathcal{T}_\beta \Psi_q(z) = -\phi_{q+}(z + \beta)\mathcal{T}_\beta \phi_{q-}(z). \tag{3.2.7}$$

Proof. Since, for any $q > 0$, we have that

$$\begin{aligned} \int_0^\infty \int_0^\infty \left(1 - e^{-(qy_1 + zy_2)}\right) \mu_\pm(dy_1, dy_2) &= \int_0^\infty \left(1 - e^{-zy_2}\right) \mu_{q\pm}(dy_2) \\ &+ \int_0^\infty \int_0^\infty \left(1 - e^{-qy_1}\right) \mu_\pm(dy_1, dy_2), \end{aligned}$$

we deduce the first claim from the fact that $q_\pm > 0$, for any $q > 0$.
Next, we have, under the $\mathbf{T}_{\beta^+}$ condition that $s \mapsto \phi_{q+}(s)$ is well-defined on $(-\infty, \beta^+)$, see [68, Lemma 4.2]. Also, for any $\beta \in (0, \beta_q^*)$, $\phi_{q+}(\beta) < 0$ as clearly both $\Psi_q(\beta) < 0$ and $\phi_{q-}(\beta) < 0$. Thus, for such β the mapping $s \mapsto \phi_{q+}(s+\beta)$ is the Laplace exponent of a killed subordinator. Moreover, it is not difficult to check that, for any fixed $q \geq 0$ and $\beta \in (0, \beta_q^*)$, $z \mapsto \mathcal{T}_\beta \phi_{q-}(z)$ is the Laplace exponent of the negative of a proper subordinator. Moreover, since $\beta \in (0, \beta_q^*)$ we deduce, from the item (2) of Proposition 3.2.1, that the proper Lévy process with characteristic exponent $\mathcal{T}_\beta \Psi_q$ drifts to $-\infty$ and hence its descending ladder height process is also the negative of a proper subordinator, see e.g. [34]. We conclude by observing the identities

$$\mathcal{T}_\beta \Psi_q(z + \beta) = \frac{z}{z + \beta} \Psi_q(z + \beta) = -\phi_{q+}(z + \beta) \frac{z}{z + \beta} \phi_{q-}(z + \beta) = -\phi_{q+}(z + \beta)\mathcal{T}_\beta \phi_{q-}(z)$$

and by invoking the uniqueness for the Wiener-Hopf factors, see [90, Theorem 45.2 (i)]. $\square$

The $\mathcal{T}_\beta$ transform turns out to be also very useful in proving the following claim which shows, in particular, that the family of exponential functional of Lévy processes is invariant under some length-biased transforms. In particular, the law of I_{Ψ_q} admits such a representation in terms of a perpetual exponential functional. Although, a similar result was given in [29] for one-sided Lévy processes, its extension requires deeper arguments.

Theorem 3.2.4. *Let us assume that ξ drifts to $-\infty$ when $q = 0$. Then the following claims hold:*

1. *The law of the random variable I_{Ψ_q} is absolutely continuous.*

2. *Assume further that $\mathbf{T}_{\beta+}$ holds. Then, for every $\beta \in (0, \beta_q^*)$, there exists a proper Lévy process with a finite negative mean and Laplace exponent $\mathcal{T}_\beta\Psi_q$, such that*

$$m_{\Psi_q}(x) = \mathbb{E}\left[\mathrm{I}_{\Psi_q}^{\beta}\right] x^{-\beta} m_{\mathcal{T}_\beta\Psi_q}(x) \text{ a.e., for } x > 0, \tag{3.2.8}$$

where $m_{\mathcal{T}_\beta\Psi_q}$ is the density of $\mathrm{I}_{\mathcal{T}_\beta\Psi_q}$.

3. *Finally, for any $q \geq 0$, we have*

$$\lim_{\beta\to 0} \mathrm{I}_{\mathcal{T}\Psi_q} \stackrel{d}{=} \mathrm{I}_{\Psi_q}.$$

Remark 3.2.5. *We point out that in the recently accepted paper [69]item (1) has been proved in the same setting with a different approach.*

Proof. First, we point out that the absolute continuity of I_{Ψ_q} in the case $q = 0$ is well-known and can be found in [16, Theorem 3.9]. Thus, we assume that $q > 0$. The case when ξ is with infinitely many jumps can be recovered from [16, Theorem 3.9 (b)]. Indeed, for any $v > 0$, denote by $g(s) = e^s$ and $dY_t^{(v)} = \mathbb{I}_{\{t<v\}}dt$. Since $g(s)$ is strictly increasing we have that condition (3.12) in [16] is satisfied. Moreover, for $\epsilon < v$, we have that the density of the absolutely continuous part of the measure $dY_t^{(v)}$ restricted to $[0, \epsilon]$ has a density which equals 1. According to [16, Theorem 3.9 (b)] this suffices to show that $\int_0^v e^{\xi_s} ds$ has a law which is absolutely continuous with respect to the Lebesgue measure. Then for any Borel set $A \subset (0, \infty)$ we have that

$$\mathbb{P}\left(\mathrm{I}_{\Psi_q} \in A\right) = q \int_0^\infty e^{-qt} \mathbb{P}\left(\int_0^t e^{\xi_s} ds \in A\right) dt$$

and our statement follows in this case. Next, assume that $\xi = \xi^{(1)} + B$ where $\xi^{(1)}$ is a compound Poisson process and B a Brownian motion with given mean and variance, which can be both zero. We denote by $(T_n)_{n\geq 1}$ (resp. $(Y_n)_{n\geq 1}$) the sequence of inter-arrival times (resp. the sequence) of the jumps of $\xi^{(1)}$. Define the measures Υ and $\tilde{\Upsilon}$ respectively on $\mathbb{R}_+^{\mathbb{N}_+} = \{\omega = (t_1, t_2, ...) : \ t_i > 0, \ \text{for } i \geq 1\}$ and $\mathbb{R}^{\mathbb{N}_+} = \{\tilde{\omega} = (y_1, ...) : y_i \in \mathbb{R}, \ \text{for } i \geq 1\}$

to be induced by the sequences $(T_n)_{n\geq 1}$ and $(Y_n)_{n\geq 1}$. For any ω and $\tilde{\omega}$, we set $S_0(\omega) = \tilde{S}_0(\tilde{\omega}) = 0$ and we write $S_j(\omega) = \sum_{i=1}^{j} t_i$, $\tilde{S}_j(\tilde{\omega}) = \sum_{i=1}^{j} y_i$, and

$$P_j(\omega) = \mathbb{P}(S_j(\omega) \leq e_q < S_{j+1}(\omega)) = \mathbb{P}(A_j(\omega)).$$

Denote by

$$\Gamma_{j,\omega}(dx) = \mathbb{P}\left(e_q \in dx; A_j(\omega)|\omega\right) = P_j(\omega)\mathbb{P}\left(e_q \in dx|A_j(\omega)\right)$$

and note that $\Gamma_{j,\omega}$ are absolutely continuous with respect to the Lebesgue measure. We also set

$$\mathrm{I}_k(\omega) = e^{\tilde{S}_{k-1}(\tilde{\omega})}\int_{S_{k-1}(\omega)}^{S_k(\omega)} e^{B_s}ds.$$

Now, we pick $A \subset \mathbb{R}_+$ such that the Lebesgue measure of A is zero and write

$$\mathbb{P}\left(\mathrm{I}_{\Psi_q} \in A\right) = \int_{\mathbb{R}_+^{\mathbb{N}_+}\times\mathbb{R}^{\mathbb{N}_+}} \sum_{j=1}^{\infty} \mathbb{P}\left(\sum_{k=1}^{j} \mathrm{I}_k(\omega) + e^{\tilde{S}_k(\tilde{\omega})}\int_{S_k(\omega)}^{e_q} e^{B_s}ds \in A;\ A_j(\omega)\middle|\, \omega, \tilde{\omega}\right)\Upsilon(d\omega)\tilde{\Upsilon}(d\tilde{\omega})$$

$$= \int_{\mathbb{R}_+^{\mathbb{N}_+}\times\mathbb{R}^{\mathbb{N}_+}} \sum_{j=1}^{\infty} \mathbb{P}\left(\int_{S_k(\omega)}^{e_q} e^{B_s}ds \in e^{-\tilde{S}_k(\tilde{\omega})}\left(A - \sum_{k=1}^{j}\mathrm{I}_k(\omega)\right);\ A_j(\omega)\middle|\, \omega, \tilde{\omega}\right)\Upsilon(d\omega)\tilde{\Upsilon}(d\tilde{\omega}).$$

Next, denote by $\mathbb{D}_k = \mathbb{D}_{S_k(\omega)}$ the full set of continuous functions images of the Brownian motion up to time $S_k(\omega)$ and note that

$$\mathbb{P}\left(\int_{S_k(\omega)}^{e_q} e^{B_s}ds \in e^{-\tilde{S}_k(\tilde{\omega})}\left(A - \sum_{k=1}^{j}\mathrm{I}_k(\omega)\right);\ A_j(\omega)\middle|\, \omega, \tilde{\omega}\right)$$

$$= \int_{f\in\mathbb{D}_k} \mathbb{P}\left(\int_0^{e_q - S_k(\omega)} e^{B'_s}ds \in \tilde{A}_j(\omega);\ A_j(\omega)\middle|\, \omega, \tilde{\omega}, (B_s)_{s\leq S_k(\omega)} = (f_s)_{s\leq S_k(\omega)}\right)\Theta(df),$$

where B' is an independent copy of B, $\tilde{A}_j(\omega) = \left\{e^{-\tilde{S}_k(\tilde{\omega}) - f_{S_k(\omega)}}\left(A - \sum_{k=1}^{j}\mathrm{I}_k(\omega)\right)\right\}$ and Θ is the measure on $\mathbb{D}_k$ induced by B. Furthermore since the sets $\tilde{A}_j(\omega)$ have zero Lebesgue measure it suffices to show that $\int_0^{e_q - S_k(\omega)} e^{B'_s}ds$ is absolutely continuous on $A_j(\omega)$. Indeed it follows because the law of $\int_0^t e^{B'_s}ds$ is absolutely continuous for any $t > 0$ and non trivial Brownian motion, see [102]. When $B'_s = as$ the same claims follows as the measures $\Gamma_{j,\omega}$ are absolutely continuous and hence so is $\int_0^{e_q - S_k(\omega)} e^{as}ds$. With this ends the proof of the absolute continuity of the law of I_{Ψ_q}.
Next from [27, Proposition 3.1.(i)] for $q > 0$ and [62, Lemma 2.1] for the case $q = 0$ the following equation

$$\mathbb{E}\left[\mathrm{I}_{\Psi_q}^z\right] \quad = \quad -\frac{z}{\Psi_q(z)}\,\mathbb{E}\left[\mathrm{I}_{\Psi_q}^{z-1}\right] \tag{3.2.9}$$

holds for any $z \in \mathbb{C}$ such that $0 < \mathtt{Re}(z) < \beta_q^*$, where we recall from Proposition 3.2.1, that for any $q \geq 0$, $\beta_q^* > 0$ is valid. We point out that all quantities involved are finite since for every $q \geq 0$ for which I_{Ψ_q} is well-defined, we have using [89, Lemma 2] and a monotone argument, that

$$\mathbb{E}\left[\mathrm{I}_{\Psi_q}^{\rho}\right] < \infty, \quad \text{for all } \rho \in \left(-1, \beta_q^*\right). \tag{3.2.10}$$

Thus, for any $\beta \in (0, \beta_q^*)$, we have, for any $-\beta < \mathtt{Re}(z) < \beta_q^* - \beta$,

$$\mathbb{E}\left[\mathrm{I}_{\Psi_q}^{z+\beta}\right] = -\frac{z+\beta}{\Psi_q(z+\beta)}\,\mathbb{E}\left[\mathrm{I}_{\Psi_q}^{z+\beta-1}\right]. \tag{3.2.11}$$

We note in particular from (3.2.10) that $\mathbb{E}\left[\mathrm{I}_{\Psi_q}^{\beta}\right] < \infty$. On the other hand, we have from Proposition 3.2.1, that for any $q \geq 0$ and any $\beta \in (0, \beta_q^*)$, $\mathcal{T}\Psi_q$ is the Laplace exponent of a Lévy process with a finite negative mean and thus the random variable $\mathrm{I}_{\mathcal{T}_\beta \Psi_q}$ is well-defined. We deduce from (3.2.9) and the definition of the transformation $\mathcal{T}_\beta$, that, for any $-\beta < \mathtt{Re}(z) < \beta_q^* - \beta$,

$$\begin{aligned}
\mathbb{E}\left[\mathrm{I}_{\mathcal{T}_\beta \Psi_q}^{z}\right] &= -\frac{z}{\mathcal{T}_\beta \Psi_q(z)}\,\mathbb{E}\left[\mathrm{I}_{\mathcal{T}_\beta \Psi_q}^{z-1}\right] \\
&= -\frac{z+\beta}{\Psi_q(z+\beta)}\,\mathbb{E}\left[\mathrm{I}_{\mathcal{T}_\beta \Psi_q}^{z-1}\right].
\end{aligned} \tag{3.2.12}$$

Next, since the distribution of I_{Ψ_q} is absolutely continuous, we have that the function $m_{\beta,q}$ given by

$$m_{\beta,q}(x) = \left(\mathbb{E}[\mathrm{I}_{\Psi_q}^{\beta}]\right)^{-1} x^{\beta} m_{\Psi_q}(x), \text{ a.e. for } x > 0, \tag{3.2.13}$$

is well-defined and determines a probability density function. We denote by $\mathrm{I}_{\beta,q}$ the random variable with density $m_{\beta,q}$. Then, clearly from (3.2.10) and (3.2.11), the functional equation

$$\mathbb{E}\left[\mathrm{I}_{\beta,q}^{z}\right] = -\frac{z+\beta}{\Psi_q(z+\beta)}\mathbb{E}\left[\mathrm{I}_{\beta,q}^{z-1}\right] = -\frac{z}{\mathcal{T}_\beta \Psi_q(z)}\mathbb{E}\left[\mathrm{I}_{\beta,q}^{z-1}\right]$$

holds for $-\beta < \mathtt{Re}(z) < \beta_q^* - \beta$ and $\mathbb{E}\left[\mathrm{I}_{\beta,q}^{-1}\right] < \infty$ and $\mathbb{E}\left[\mathrm{I}_{\beta,q}^{\delta}\right] < \infty$, for some $\delta > 0$. Thanks to the existence of these moments we can use [68, Lemma 4.4] to deduce that in the notation of [68], $\mathcal{L}m_{\beta,q}(x) = 0$ a.e. Moreover, as $\mathbb{E}\left[\mathrm{I}_{\beta,q}^{-1}\right] < \infty$ we can apply [68, Theorem 3.7] to get in fact that (3.2.8) holds. Indeed, otherwise, both the law of $\mathrm{I}_{\mathcal{T}_\beta \Psi_q}$ and $\mathrm{I}_{\beta,q}$ will be different stationary measures of the generalized Ornstein-Uhlenbeck process associated to the Lévy process with exponent $\mathcal{T}_\beta \Psi_q$ as defined in [68, Theorem 3.7] which is impossible. The proof of the last claim follows readily from the limit $\lim_{\beta \to 0} \mathbb{E}\left[\mathrm{I}_{\Psi_q}^{\beta}\right] = 1$ combined with identity (3.2.8). □

The next result is in the spirit of the result of [68, Theorem 1] in the case $\mathbf{E}_+$ and thus can be seen as its extension.

Proposition 3.2.6. *Let us assume that* $\mathbf{T}_{\beta+}$ *holds and* $e^{\beta y}\mu_+^q(dy) \in \mathcal{P}$, *for some* $\beta \in (0, \beta^+)$. *Then,*

$$\mathrm{I}_{\Psi_q} \stackrel{d}{=} \mathrm{I}_{\phi_{q_-}} \times \mathrm{I}_{\psi^{q_+}}, \tag{3.2.14}$$

where $\psi^{q_+}(z) = z\phi_{q_+}(z)$.

Proof. First, from Proposition 3.2.1 we get that, for any $\beta \in (0, \beta_q^*)$, $\mathcal{T}\Psi_q$ is the Laplace exponent of a Lévy process with a finite negative mean and thus $\mathrm{I}_{\mathcal{T}\Psi_q}$ is well-defined. Next, $\mathbf{T}_{\beta+}$ trivially holds for ϕ_{q_-} since it corresponds to the Laplace exponent of a negative of a subordinator. Clearly $e^{\beta y}\mu_{q_+}(dy) \in \mathcal{P}$ implies $\mu_{q_+}(dy) \in \mathcal{P}$, thus ψ^{q+} is the Laplace exponent of an unkilled spectrally positive Lévy process whose tail of the Lévy measure has the form $\overline{\Pi}^{q+}(y)dy = \mu_{q_+}(dy), y > 0$, see [68, Lemma 4.3]. Therefore, as $e^{\beta y}\overline{\Pi}^{q+}(y)dy = e^{\beta y}\mu_{q_+}(dy) \in \mathcal{P}$ and $s \mapsto \phi_{q_+}(s)$ is well-defined on $(-\infty, \beta^+)$ which implies that $|\psi^{q+}(s)| < +\infty$, for any $s \in (-\infty, \beta^+)$, we have that ψ^{q+} satisfies the condition $\mathbf{T}_{\beta+}$. Thus $\mathcal{T}\psi^{q+}(s) = \frac{s}{s+\beta}\psi^{q+}(s+\beta) = s\phi_{q_+}(s+\beta)$ is the Laplace exponent of a proper spectrally positive Lévy process with a finite negative mean $\phi_{q_+}(\beta)$. Next, since $e^{\beta y}\mu_{q_+}(dy) \in \mathcal{P}$, we have that the unkilled Lévy process with Laplace exponent $\mathcal{T}\Psi_q$ satisfies the condition $\mathbf{E}_+$ of [68, Theorem 1]. From (3.2.7) of Proposition 3.2.3 we deduce that

$$\frac{\mathcal{T}\Psi_q(z)}{z} = -\frac{z\phi_{q_+}(z+\beta)}{z}\frac{\mathcal{T}\phi_{q_-}(z)}{z} = -\frac{\mathcal{T}\psi^{q+}(z)}{z}\frac{\mathcal{T}\phi_{q_-}(z)}{z}$$

and from [68, Theorem 1] that

$$\mathrm{I}_{\mathcal{T}\Psi_q} \stackrel{d}{=} \mathrm{I}_{\mathcal{T}\phi_{q_-}} \times \mathrm{I}_{\mathcal{T}\psi^{q+}}. \tag{3.2.15}$$

Then, from Theorem 3.2.4, we get that

$$\lim_{\beta\to 0} \mathrm{I}_{\mathcal{T}\Psi_q} \stackrel{d}{=} \mathrm{I}_{\Psi_q},\ \lim_{\beta\to 0} \mathrm{I}_{\mathcal{T}\phi_{q_-}} \stackrel{d}{=} \mathrm{I}_{\phi_{q_-}} \text{ and } \lim_{\beta\to 0} \mathrm{I}_{\mathcal{T}\psi^{q+}} \stackrel{d}{=} \mathrm{I}_{\psi^{q+}},$$

which completes the proof. □

Next we provide a killed version of the Vigon's equation amicale, see [34, Theorem 16].

Proposition 3.2.7. *Let us assume that $\mathbf{T}_{\beta+}$ holds.*

1. *Then, we have*
$$\overline{\mu}_{q_+}(y) = \int_0^\infty \overline{\Pi}_+(r+y)\mathcal{U}_-^{(q)}(dr),\ y > 0, \tag{3.2.16}$$
where $\int_0^\infty e^{-sy}\mathcal{U}_-^{(q)}(dy) = \frac{1}{\phi_{q_-}(z)}$.

2. *Moreover ϕ_{q_+} satisfies the condition $\mathbf{T}_{\beta+}$. Finally, if for some $\beta \in (0, \beta^+)$, $e^{\beta y}\Pi_+(dy) \in \mathcal{P}$ then $e^{\beta y}\mu_{q_+}(dy) \in \mathcal{P}$.*

Proof. We consider only the case when $q > 0$ since when $q = 0$ we are in the setting of the classical Vigon's equation amicale. Next, the latter applied to the unkilled Lévy process $\xi^{(\beta,q)}$ as defined in Proposition 3.2.1 with $\beta \in (0, \beta_q^*)$, yields, with the obvious notation,

$$\overline{\mu}^{\beta}_{q_+}(y) = \int_0^\infty \overline{\Pi}^{\beta}_+(y+r)\mathcal{U}_-^{(\beta,q)}(dr), \tag{3.2.17}$$

where, from (3.2.2), we have

$$\overline{\Pi}_+^{\beta}(y) = \int_y^{\infty} \Pi^{\beta}(dr)\mathbb{I}_{\{y>0\}} = \int_y^{\infty} e^{\beta r}\left(\Pi(dr) - \beta\overline{\Pi}_+(r)dr\right)\mathbb{I}_{\{y>0\}} \tag{3.2.18}$$

and from Proposition 3.2.3 and [11, p. 74]

$$\int_0^{\infty} e^{-zy}\mathcal{U}_-^{(\beta,q)}(dy) = \frac{1}{\mathcal{T}_{\beta}\phi_{q-}(z)} = \frac{1}{\frac{z}{z+\beta}\phi_{q-}(z+\beta)} = \frac{1}{\phi_{q-}(z+\beta)} + \beta\frac{1}{z\phi_{q-}(z+\beta)}.$$

From the latter we immediately deduce by comparing the Laplace transforms that

$$\mathcal{U}_-^{(\beta,q)}(dy) = e^{-\beta y}\mathcal{U}_-^{(q)}(dy) + \beta\int_0^y e^{-\beta r}\mathcal{U}_-^{(q)}(dr)dy. \tag{3.2.19}$$

Plugging (3.2.19) into (3.2.17), we get, for all $y > 0$,

$$\overline{\mu}_{q+}^{\beta}(y) = \int_0^{\infty} \overline{\Pi}_+^{\beta}(y+r)e^{-\beta r}\mathcal{U}_-^{(q)}(dr) + \beta\int_0^{\infty} \overline{\Pi}_+^{\beta}(y+r)\int_0^r e^{-\beta v}\mathcal{U}_-^{(q)}(dv)dr.$$

Next, we have, using identity (3.2.2) and the fact that condition $\mathbf{T}_{\beta_+}$ holds, the existence of a constant $C > 0$ such that, for all β small enough,

$$\int_y^{\infty} \overline{\Pi}_+^{\beta}(r)dr \le \int_y^{\infty}\int_r^{\infty} e^{\beta v}\Pi_+(dv)dr = \int_y^{\infty} re^{\beta r}\Pi_+(dr) \le C.$$

Using this inequality and recalling that, for any $q > 0$, $\mathcal{U}_-^{(q)}$ is a positive finite measure on $\mathbb{R}^+$, as a potential measure of a negative of a killed subordinator, that is a transient Markov process, we obtain, with $\overline{\mathcal{U}}^{(q)} = \mathcal{U}_-^{(q)}(0,\infty)$, that

$$\int_0^{\infty} \overline{\Pi}_+^{\beta}(y+r)\int_0^r e^{-\beta v}\mathcal{U}_-^{(q)}(dv)dr \;\le\; \overline{\mathcal{U}}^{(q)}\int_y^{\infty} \overline{\Pi}_+^{\beta}(r)dr \le \overline{\mathcal{U}}^{(q)}\int_y^{\infty} re^{\beta r}\Pi_+(dr) \le C_q,$$

where the constant $C_q > 0$ is also uniform for all β small enough. This gives us

$$\lim_{\beta\to 0}\overline{\mu}_{q+}^{\beta}(y) = \lim_{\beta\to 0}\int_0^{\infty} \overline{\Pi}_+^{\beta}(y+r)e^{-\beta r}\mathcal{U}_-^{(q)}(dr). \tag{3.2.20}$$

Since for all β small enough $\int_y^{\infty} e^{\beta r}\overline{\Pi}_+(r)dr \le C_1$, with $C_1 > 0$, we have, from (3.2.18), at the points of continuity of $\Pi_+(dy)$, that

$$\lim_{\beta\to 0}\overline{\Pi}_+^{\beta}(y) = \lim_{\beta\to 0}\left(\int_y^{\infty} e^{\beta r}\Pi_+(dr) - \beta\int_y^{\infty} e^{\beta r}\overline{\Pi}_+(r)dr\right) = \overline{\Pi}_+(y).$$

Since $\mathcal{U}_-^{(q)}$ defines a finite measure, we conclude from (3.2.20) that $\lim_{\beta\to 0}\overline{\mu}_{q+}^{\beta}(y) = \overline{\mu}_+^{q}(y)$ and hence (3.2.16) holds. Next, the fact that the mapping $s \mapsto \phi_{q+}(s)$ is well defined on

$(0, \beta_+)$ follows readily from [68, Lemma 4.2] since Ψ_q satisfies the condition $\mathbf{T}_{\beta^+}$. Then, for any $0 < \beta < \beta^+$, (3.2.16) gives us that

$$e^{\beta y}\overline{\mu}_{q_+}(y) = \int_0^\infty e^{\beta(y+r)}\overline{\Pi}_+(y+r)e^{-\beta r}\mathcal{U}_-^{(q)}(dr).$$

The claim that $e^{\beta y}\overline{\mu}_{q_+}(y) \in \mathcal{P}$ now follows from the fact that for every fixed $r > 0$, the mapping $y \mapsto e^{\beta(y+r)}\overline{\Pi}_+(y+r)$ is non-increasing on $\mathbb{R}^+$. Hence ϕ_{q_+} also satisfies $\mathbf{T}_{\beta_+}$. Assume now that $e^{\beta y}\Pi_+(dy) \in \mathcal{P}$, then one may write $\Pi_+(dy) = \pi_+(y)dy$ and the equation

$$e^{\beta y}\mu_{q_+}(dy) = \int_0^\infty e^{\beta(y+r)}\pi_+(y+r)e^{-\beta r}\mathcal{U}_-^{(q)}(dr)dy,$$

which is a differentiated version of (3.2.16) shows that $e^{\beta y}\mu_{q_+}(dy) \in \mathcal{P}$. To rigorously justify the exchange of differentiation and integration in the differentiated version above note that under $\mathbf{T}_{\beta^+}$ the differentiated version is clearly valid if $q > 0$ since $\mathcal{U}_-^{(q)}$ defines a finite measure. Moreover, when $q = 0$ and $\beta > 0$, $e^{-\beta r}\mathcal{U}_-(dr)$ is a finite measure due to the sublinearity of the potential function $\mathcal{U}_-((0, r))$, see [11, p 74]. Finally when both $q = 0$ and $\beta = 0$ the differentiated version follows from [68, Lemma 4.11]. $\square$

In order to complete the proof of Theorem 3.1.1 in the case $\mathbf{P}+$ we will resort to some approximation procedures for which we need the following results.

Lemma 3.2.8. *(a) Let $(\phi_-^{(n)})_{n\geq 1}$ be a sequence of Laplace exponents of negative of possibly killed subordinators. Assume that for all $s \geq 0$, $\lim_{n\to\infty} \phi_-^{(n)}(s) = \phi_-(s)$, where ϕ_- is the Laplace exponent of a negative of a possibly killed subordinator. Then*

$$\lim_{n\to\infty} \mathrm{I}_{\phi_-^{(n)}} \stackrel{d}{=} \mathrm{I}_{\phi_-}.$$

(b) Let $(\Psi^{(n)})_{n\geq 1}$ be a sequence of characteristic exponents of Lévy processes such that, for all $z \in i\mathbb{R}$,

$$\lim_{n\to\infty} \Psi^{(n)}(z) = \Psi(z), \tag{3.2.21}$$

where Ψ is the characteristic exponent of a Lévy process. Assume further that for all $n \geq 1$, $\Psi^{(n)}(0) = \Psi(0) = 0$. Then, for all $q > 0$,

$$\lim_{n\to\infty} \mathrm{I}_{\Psi_q^{(n)}} \stackrel{d}{=} \mathrm{I}_{\Psi_q}.$$

Remark 3.2.9. *A case similar to (a) was treated in Lemma 4.8 in [68]. However there it is assumed that the subordinators are proper. Note that case (b) is far simpler than Lemma 4.8 in [68] as we are strictly in the killed case and the exponential functional up to a finite time horizon is continuous in the Skorohod topology.*

Proof. First we use the fact that the law of the exponential functional of a negative of a possibly killed subordinator is moment determinate. More specifically, Carmona et al. [27], showed, writing $\mathbb{E}\left[\mathrm{I}^m_{\phi_-^{(n)}}\right] = M_m^{(n)}$, that

$$M_m^{(n)} \quad = \quad \frac{\Gamma(m+1)}{\prod_{k=1}^m -\phi_-^{(n)}(k)},\ m = 1, 2, \dots\ . \tag{3.2.22}$$

From the convergence of the Laplace exponents, we deduce that, for all integers $m \geq 1$, $\lim_{n\to\infty} M_m^{(n)} = \frac{\Gamma(m+1)}{\prod_{k=1}^m -\phi_-(k)}$, which is the m-th moment of the exponential functional I_{ϕ_-}. Item (a) follows then from [39, Examples (b) p.269]. Next, (3.2.21) combined with $\Psi^{(n)}(0) = \Psi(0) = 0$, implies that the corresponding sequence of Lévy processes $\left(\xi^{(n)}\right)_{n\geq 1}$ converges in distribution to a Lévy process ξ. Using Skorohod-Dudley's theorem, we assume that the convergence holds a.s. on the Skorohod space $\mathcal{D}((0,\infty))$ and check that, for any $t > 0$,

$$\int_0^t e^{\xi_s^{(n)}} ds \xrightarrow{d} \int_0^t e^{\xi_s} ds.$$

Then applying Portmanteau's theorem, for any fixed $t, x \geq 0$, we have that

$$\limsup_{n\to\infty} \mathbb{P}\left(\int_0^t e^{\xi_s^{(n)}} ds \leq x\right) \leq \mathbb{P}\left(\int_0^t e^{\xi_s} ds \leq x\right)$$

$$\liminf_{n\to\infty} \mathbb{P}\left(\int_0^t e^{\xi_s^{(n)}} ds < x\right) \geq \mathbb{P}\left(\int_0^t e^{\xi_s} ds < x\right).$$

Hence since, for any $q > 0$ and $A \subset \mathbb{R}_+$,

$$\mathbb{P}\left(\mathrm{I}_{\Psi_q} \in A\right) = q\int_0^\infty e^{-qt}\mathbb{P}\left(\int_0^t e^{\xi_s} ds \in A\right) dt$$

and $qe^{-qt}dt$ defines a finite measure, we have from the reverse Fatou's lemma (resp. Fatou's lemma) that

$$\limsup_{n\to\infty} \int_0^\infty dtqe^{-qt}\mathbb{P}\left(\int_0^t e^{\xi_s^{(n)}} ds \leq x\right) \leq \int_0^\infty dtqe^{-qt}\mathbb{P}\left(\int_0^t e^{\xi_s} ds \leq x\right) = \mathbb{P}\left(\mathrm{I}_{\Psi_q} \leq x\right)$$

$$\liminf_{n\to\infty} \int_0^\infty dtqe^{-qt}\mathbb{P}\left(\int_0^t e^{\xi_s^{(n)}} ds < x\right) \geq \int_0^\infty dtqe^{-qt}\mathbb{P}\left(\int_0^t e^{\xi_s} ds < x\right) = \mathbb{P}\left(\mathrm{I}_{\Psi_q} < x\right).$$

This suffices since from Theorem 3.2.4 we know that $\mathbb{P}\left(\mathrm{I}_{\Psi_q} = x\right) = 0$, for all $x \geq 0$. □

Now, we have all the ingredients to complete the proof of Theorem 3.1.1 in the case **P**+. Let us consider, for any $\delta > 0$, the Lévy process $\xi^{(\delta)} = (\xi_t^{(\delta)})_{t\geq 0}$, with Laplace exponent $\Psi^{(\delta)}$, constructed from ξ by tilting the positive jumps. More precisely, we modify the Lévy measure of ξ as follows

$$\Pi^{(\delta)}(dy) = \Pi(dy)\mathbb{I}_{\{y<0\}} + e^{-\delta y}\Pi_+(dy)$$

and leave the Gaussian coefficient and the linear term untouched. From [90, Theorem 25.17], we have that $|\Psi^{(\delta)}(s)| < +\infty$, for any $s \in (0,\delta)$. For $\Psi_q^{(\delta)}$, we define $\beta_\delta^*(q)$ as in (3.2.1) and choose β such that $0 < \beta < \delta \wedge \beta_\delta^*(q) = \delta'$. Then, since $\Pi_+(dy) = \pi_+(y)dy \in \mathcal{P}$ the mapping defined on $\mathbb{R}^+$ by

$$y \mapsto e^{\beta y} \int_y^\infty \Pi^{(\delta)}(dr) = e^{(\beta-\delta)y} \int_0^\infty e^{-\delta r}\pi_+(r+y)dr$$

is plainly non-increasing. Hence $\Psi^{(\delta)}$ satisfies the condition $\mathbf{T}_{\delta'}$. Moreover, $e^{\beta y}\Pi_+^{(\delta)}(dy) \in \mathcal{P}$ and hence from the item (2) of Proposition 3.2.7, we have with the obvious notation $e^{\beta y}\mu_{q+}^{(\delta)}(dy) \in \mathcal{P}$. Thus, the Lévy process with characteristic exponent $\Psi_q^{(\delta)}$ satisfies the conditions of Lemma 3.2.6 and we deduce that

$$\mathrm{I}_{\Psi_q^{(\delta)}} \stackrel{d}{=} \mathrm{I}_{\phi_{q-}^{(\delta)}} \times \mathrm{I}_{\psi^{(\delta),q_+}},$$

where we have set $\Psi_q^{(\delta)}(z) = -\phi_{q-}^{(\delta)}(z)\phi_{q+}^{(\delta)}(z)$ and $\psi^{(\delta),q_+}(z) = z\phi_{q+}^{(\delta)}(z)$. Next, since as $\delta \to 0$, $\Pi^{(\delta)}(dy) \stackrel{v}{\to} \Pi(dy)$, where $\stackrel{v}{\to}$ stands for the vague convergence, we have that $\lim_{\delta\to 0}\xi^{(\delta)} \stackrel{d}{=} \xi$, see [47, Theorem 13.14] Putting $h^{(\delta)}(y) = e^{-\delta y}$ we see that the assumptions of [68, Lemma 4.9] are satisfied (note that the only case which [68, Lemma 4.9] does not encompass, i.e. when $q = 0$ and ξ does not drift to $-\infty$, is ruled out by our assumptions) and thus we have, for all $s \geq 0$,

$$\lim_{\delta\to 0}\phi_{q-}^{(\delta)}(s) = \phi_{q_-}(s) \text{ and } \lim_{\delta\to 0}\phi_{q+}^{(\delta)}(-s) = \phi_{q_+}(-s). \tag{3.2.23}$$

From (3.2.23) combined with Lemma 3.2.8 (a) we get that

$$\lim_{\delta\to 0}\mathrm{I}_{\phi_{q-}^{(\delta)}} \stackrel{d}{=} \mathrm{I}_{\phi_{q_-}}.$$

Next from (3.2.23), we deduce that for any $s \geq 0$, $\lim_{\delta\to 0}\psi^{(\delta),q_+}(-s) = \lim_{\delta\to 0} -s\phi_{q+}^{(\delta)}(-s) = -s\phi_{q_+}(-s) = \psi^{q_+}(-s)$ and $\lim_{\delta\to 0}(\psi^{(\delta),q_+})'(0^-) = \lim_{\delta\to 0}\phi_{q+}^{(\delta)}(0) = \phi_{q_+}(0) = (\psi^{q_+})'(0^-)$. Thus, we can apply [68, Lemma 4.8 (a)] to get

$$\lim_{\delta\to 0}\mathrm{I}_{\psi^{(\delta),q_+}} \stackrel{d}{=} \mathrm{I}_{\psi^{q_+}}.$$

Finally, since $\lim_{\delta\to 0}\xi^{(\delta)} \stackrel{d}{=} \xi$, Lemma 3.2.8 (b) implies, when $q > 0$, that

$$\lim_{\delta\to 0}\mathrm{I}_{\Psi_q^{(\delta)}} \stackrel{d}{=} \mathrm{I}_{\Psi_q}$$

and the case when $q > 0$ is finished. When $q = 0$ due to the considerations above we have already shown that

$$\lim_{\delta\to 0}\mathrm{I}_{\phi_{q-}^{(\delta)}} \times \mathrm{I}_{\psi^{(\delta),q_+}} = \mathrm{I}_{\phi_{q_-}} \times \mathrm{I}_{\psi^{q_+}}.$$

It remains to show that $\mathrm{I}_{\Psi^{(\delta)}} \stackrel{d}{\to} \mathrm{I}_{\Psi}$, as $\delta \to 0$. From the construction of $\xi^{(\delta)}$ we can write

$$\xi_t = \xi_t^{(\delta)} + \tilde{\xi}_t^{(\delta)},\ t \geq 0,$$

where $\tilde{\xi}^{(\delta)} = (\tilde{\xi}_t^{(\delta)})_{t\geq 0}$ is a subordinator with zero drift and Lévy measure $(1-e^{-\delta y})\Pi(dy)\mathbb{I}_{\{y>0\}}$ which is taken independent of $\xi^{(\delta)}$. Therefore $\xi_t \geq \xi_t^{(\delta)}$, for all $t \geq 0$, and hence we conclude that $\lim_{\delta\to 0} \mathrm{I}_{\Psi^{(\delta)}} \stackrel{d}{=} \mathrm{I}_{\Psi}$ from the monotone convergence theorem. This completes the proof of Theorem 3.1.1 in the case $\mathbf{P}+$.

3.2.2 The case $\mathbf{P}_{\pm}^{\mathbf{q}}$

Since the case $q = 0$ was treated in [68], we assume in the sequel that $q > 0$. In what follows, we provide a necessary condition on the Lévy measures of the characteristic exponent of bivariate subordinators in order that they correspond to the Wiener-Hopf factors of a killed Lévy process. We mention that Vigon [99] provides such a criteria for proper Lévy processes and our condition relies heavily on his approach.

Lemma 3.2.10. *Let us consider ϕ_{q+} and ϕ_{q-} as defined in Proposition 3.2.3. Assume that $\mu_{q+} \in \mathcal{P}$ and $\mu_{q-} \in \mathcal{P}$ with $q = q_+q_- > 0$.*

1. *There exists a characteristic exponent of a killed Lévy process Ψ_q such that*

$$\Psi_q(z) = -\Phi_+(q, -z)\Phi_-(q, z) = -\phi_{q+}(z)\phi_{q-}(z).$$

2. *If in addition for any $0 < \beta < \beta_+$, for some $\beta_+ > 0$, $-\infty < \phi_{q+}(\beta) < 0$ and $e^{\beta y}\mu_{q+}(dy) \in \mathcal{P}$, then Ψ_q satisfies the condition $\mathbf{T}_{\beta+}$.*

Proof. From Proposition 3.2.3, writing $\phi_{q\pm}(z) = \phi_{\pm}(z) - q_{\pm}$, we observe that

$$-\Phi_+(q, -z)\Phi_-(q, z) \;=\; -\phi_{q+}(z)\phi_{q-}(z) = -(\phi_+(z) - q_+)\phi_-(z) + q_-\phi_+(z) - q_+q_-.$$

Then, from Vigon's philantropy theory, we know that $-(\phi_+(z) - q_+)\phi_-(z)$ is the characteristic exponent of an unkilled Lévy process that drifts to $-\infty$. It is also clear that $q_-\phi_+(z)$ is the characteristic exponent of an unkilled subordinator. From the inequality $q_+q_- > 0$ we complete the proof of the first item. Next, from the form of ϕ_{q-} in Proposition 3.2.3 and carefully using the same techniques as in deriving (3.2.2) we deduce that $\mathcal{T}\phi_{q-}$ is the Laplace exponent of a negative of an unkilled subordinator whose Lévy measure has the form $\mu_{q-}^{\beta}(dy) = e^{-\beta y}(\mu_{q-}(dy) + \beta\overline{\mu}_{q-}(y)dy)$. Similarly, due to our assumption, i.e. $-\infty < \phi_{q+}(\beta) < 0$, the mapping $s \mapsto \phi_{q+}(s + \beta)$ is the Laplace exponent of a killed subordinator with Lévy measure $e^{\beta y}\mu_{q+}(dy)$. As $\mu_{q-} \in \mathcal{P}$, we easily check that $\mu_{q-}^{\beta}(dy) \in \mathcal{P}$ and since, by assumption, $e^{\beta y}\mu_{q+}(dy) \in \mathcal{P}$, we have from the first item that there exists a characteristic exponent Ψ^{β}, of an unkilled Lévy process drifting to $-\infty$, which is defined by

$$\Psi_q^{\beta}(z) = -\phi_{q+}(z + \beta)\mathcal{T}\phi_{q-}(z) = -\phi_{q+}(z + \beta)\frac{z}{z + \beta}\phi_{q-}(z + \beta).$$

Moreover, as

$$\mathcal{T}_\beta \Psi_q(z) = \frac{z}{z+\beta}\Psi_q(z+\beta) = -\phi_{q+}(z+\beta)\frac{z}{z+\beta}\phi_{q-}(z+\beta) = -\phi_{q+}(z+\beta)\mathcal{T}\phi_{q-}(z),$$

we deduce, by means of an uniqueness argument, that $\mathcal{T}\Psi_q = \Psi_q^\beta$. Then, by the mere definition of condition $\mathbf{T}_{\beta+}$ we check that Ψ_q satisfies condition $\mathbf{T}_{\beta+}$. □

We are ready to complete the proof of Theorem 3.1.1. First, we set, for any $\delta > 0$,

$$\phi_{q+}^{(\delta)}(z) = \phi_{q+}(z-\delta) - \phi_{q+}(-\delta) + \phi_{q+}(0). \tag{3.2.24}$$

This is the Laplace exponent of a subordinator with drift δ_+, killing rate $-\phi_{q+}(0) = q_+ > 0$ and Lévy measure $\mu_{q+}^{(\delta)}(dy) = e^{-\delta y}\mu_{q+}(dy)$. Next we choose $\delta > 0$ so small that $\phi_{q+}^{(\delta)}(\delta) < 0$. Since, by assumption $\mu_{q\pm} \in \mathcal{P}$ plainly $\mu_{q+}^{(\delta)} \in \mathcal{P}$, and thus according to item (1) of Lemma 3.2.10, there exists a characteristic exponent $\Psi_q^{(\delta)}$ of a killed Lévy process such that

$$\Psi_q^{(\delta)}(z) = -\phi_{q+}^{(\delta)}(z)\phi_{q-}(z). \tag{3.2.25}$$

Moreover since we have that $|\phi_{q+}^{(\delta)}(s)| < +\infty$, for any $s < \delta$, we get from [68, Lemma 4.2] that $|\Psi_q^{(\delta)}(s)| < +\infty$, for any $0 < s < \delta$. Also, since $\phi_{q+}^{(\delta)}$ is increasing on $(-\infty, \delta)$, we get from our choice of δ that, for any $0 < \beta < \delta$, $-\infty < \phi_{q+}^{(\delta)}(\beta) < 0$. As for any $0 < \beta < \delta$, $e^{\beta y}\mu_{q+}^{(\delta)}(dy) \in \mathcal{P}$, we deduce from item (2) of Lemma 3.2.10, that $\Psi_q^{(\delta)}$ satisfies the condition $\mathbf{T}_\delta$. Hence, we can apply Proposition 3.2.6 to get the identity

$$\mathrm{I}_{\Psi_q^{(\delta)}} \stackrel{d}{=} \mathrm{I}_{\phi_{q-}} \times \mathrm{I}_{\psi^{(\delta),q_+}},$$

where $\psi^{(\delta),q_+}(z) = z\phi_{q+}^{(\delta)}(z)$. Next, on the one hand, we have, from (3.2.24), that for any $s \geq 0$, $\lim_{\delta\to 0}\phi_{q+}^{(\delta)}(s) = \phi_{q+}(s)$ and thus $\lim_{\delta\to 0}\psi^{(\delta),q_+}(s) = \psi^{q_+}(s)$ together with $\lim_{\delta\to 0}(\psi^{(\delta),q_+})'(0^-) = \lim_{\delta\to 0}\phi_{q+}^{(\delta)}(0) = \phi_{q+}(0) = (\psi^{q_+})'(0^-)$. Thus, we can use [68, Lemma 4.8(a)] to get $\lim_{\delta\to 0}\mathrm{I}_{\psi^{(\delta),q_+}} \stackrel{d}{=} \mathrm{I}_{\psi^{q_+}}$. On the other hand, we deduce from (3.2.25) that, for any $z \in i\mathbb{R}$, $\lim_{\delta\to 0}\Psi_q^{(\delta)}(z) = \Psi_q(z)$ and for any $\delta \geq 0$, $\Psi_q^{(\delta)}(0) = \Psi_q(0)$. Hence, from from Lemma 3.2.8 (b), we have $\lim_{\delta\to 0}\mathrm{I}_{\Psi_q^{(\delta)}} \stackrel{d}{=} \mathrm{I}_{\Psi_q}$, which completes the proof of Theorem 3.1.1.

3.3 Proof of the corollaries and some examples

3.3.1 Proof of Corollary 3.1.3

From the Wiener-Hopf factorization (3.2.6) and the assumptions we have that $-\infty < \Psi_q(-1) = -\phi_{q+}(-1)\phi_{q-}(-1) \leq 0$. Then we get from [68, Lemma 4.1] that the mapping $s \mapsto \phi_{q-}(s)$ is well-defined on $[-1,\infty)$ and since $\phi_{q+}(-1) < 0$, we conclude that $\phi_{q-}(-1) \leq 0$. Thus, $\tilde{\phi}_{q-}(s) = \phi_{q-}(s-1)$ is a Laplace of a negative of a possibly killed subordinator and so $\mathcal{T}_1\tilde{\phi}_{q_-}$ is the Laplace exponent of a negative of a proper subordinator. From (3.2.22), we have, for $m = 1, 2, \ldots,$

$$\mathbb{E}\left[\mathrm{I}^m_{\phi_{q_-}}\right] = \frac{\Gamma(m+1)}{\prod_{k=1}^m -\phi_{q_-}(k)} = \frac{1}{m+1}\frac{\Gamma(m+1)}{\prod_{k=1}^m -\frac{k}{k+1}\tilde{\phi}_{q_-}(k+1)} = \frac{1}{m+1}\frac{\Gamma(m+1)}{\prod_{k=1}^m -\mathcal{T}_1\tilde{\phi}_{q_-}(k)}.$$

By moment identification and moment determinacy of $\mathrm{I}_{\phi_{q_-}}$, see [27], we deduce that

$$\mathrm{I}_{\phi_{q_-}} \stackrel{d}{=} U \times \mathrm{I}_{\mathcal{T}_1\tilde{\phi}_{q_-}}, \tag{3.3.1}$$

where U stands for an uniform random variable on $(0,1)$. Thus, from Khintchine Theorem, see e.g. [39, Theorem p.158], we have that $m_{\phi_{q_-}}$ is non-increasing on $\mathbb{R}^+$. We also get, from (3.3.1), that

$$m_{\phi_{q_-}}(x) = \int_x^\infty m_{\mathcal{T}_1\tilde{\phi}_{q_-}}(y)dy/y,$$

which combined with (3.2.8) and (3.2.22) yields $m_{\phi_{q_-}}(0) = -\tilde{\phi}_{q_-}(1) = -\phi_{q_-}(0) > 0$ since when $q = 0$ we assume that ξ drifts to $-\infty$ and hence the descending ladder height process is the negative of a killed subordinator. Since we also suppose that either one of the two conditions of Theorem 3.1.1 applies, we conclude that

$$\mathrm{I}_{\Psi_q} \stackrel{d}{=} U \times \mathrm{I}_{\mathcal{T}_1\tilde{\phi}_{q_-}} \times \mathrm{I}_{\psi^{q+}} = U \times V \tag{3.3.2}$$

which gives that m_{Ψ_q} is non-increasing on $\mathbb{R}^+$ and hence a.e. differentiable on $\mathbb{R}^+$. Moreover, since

$$m_{\Psi_q}(x) = \int_0^\infty m_{\phi_{q_-}}(x/y)m_{\psi^{q+}}(y)dy/y = \int_x^\infty m_V(y)dy/y,$$

we deduce from the discussion above and an argument of dominated converge that

$$\begin{aligned} m_{\Psi_q}(0) &= -\phi_{q_-}(0)\int_0^\infty m_{\psi^{q+}}(y)dy/y \\ &= \phi_{q_-}(0)\phi_{q_+}(0) = q \end{aligned}$$

where the last line follows from (3.1.7). To prove the claim of continuity in item (i) note that from the second integral representation $m_{\Psi_q}(x)$ is continuous.
In order to prove the first statement of item (ii) we show that, for any $q > 0$, we have the following factorization

$$\mathrm{I}_{\Psi_q} \stackrel{d}{=} e_1 \times \mathrm{I}_{\psi^{q+}}, \tag{3.3.3}$$

where $\psi^{q+}(z) = z\Psi_q(z)$. Indeed, this identity follows readily from Theorem 3.1.1, since, in this case, $\mu_{q+} \in \mathcal{P}$, $\phi_{q_-}(z) \equiv 1$ and thus $\mathrm{I}_{\phi_{q_-}} = \int_0^{e_1} e^0 ds = e_1$. Thus, I_{Ψ_q} is a mixture of exponential distributions and the complete monotonicity property of its density follows from [90, Theorem 53.2]. Moreover, from (3.3.3), we deduce that

$$m_{\Psi_q}(x) = \int_0^\infty e^{-x/y} m_{\psi^{q+}}(y)dy/y$$

and for any $x < \lim_{s\to\infty} -s\Psi_q(-s) = 1/b > 0$, we get

$$\begin{aligned} m_{\Psi_q}(x) &= \sum_{n=0}^{\infty} \frac{1}{n!}(-x)^n \int_0^{\infty} y^{-n-1} m_{\psi^{q+}}(y)dy \\ &= q\left(1 + \sum_{n=1}^{\infty} \frac{\prod_{k=1}^{n} -\Psi_q(-k)}{n!}(-x)^n\right), \end{aligned}$$

where we have used an argument of dominated convergence and (3.1.7). Next, assume that $b > 0$ and thus the previous power series defines a function analytical on the disc of radius b. Since the mapping $x \mapsto m_{\Psi_q}(x)$ is the Laplace transform of some positive measure, its first singularity occurs on the negative real line, see e.g. [101, Chap. 2], which means at the point $-b$. Following the proof of [76, Proposition 2.1], we can then apply the Euler transform, see e.g. [87], to obtain the power series representation (3.1.8) which actually defines an analytical function on the half-plane $\texttt{Re}(z) > -(2b)^{-1}$. The proof of the claims of (ii) is completed after observing from the power series representations that $m_{\Psi_q}(0) = q$. Item (iii) follows easily from the Wiener-Hopf factorization for spectrally positive Lévy processes which yields the identity

$$\Psi_q(s) = -\frac{\Psi(s) - q}{s + \gamma_q}(-s - \gamma_q).$$

Thus, in this case, we have $\mathrm{I}_{\phi_{q_-}} = \int_0^{e_{\gamma_q}} e^{-s}ds = 1 - e^{-e_{\gamma_q}}$ which can easily be seen to be a $B^{-1}(1, \gamma_q)$, which provides the factorization from Theorem 3.1.1. We complete the proof of this item by recalling that in this case the mapping $s \mapsto \Psi_q(s)$ is well-defined on $\mathbb{R}^-$ and $\mu_{q+} \in \mathcal{P}$, see e.g. [99, Remark p. 103]. Finally, the proof of the item (iv) goes along the lines of the one of [68, Corollary 2.1].

3.3.2 Proof of Corollary 3.1.5

According, for instance, to [52, Proposition 4], we have

$$T_1 \stackrel{d}{=} \int_0^{e_q} e^{\xi_t}dt, \tag{3.3.4}$$

where we have used the well known identity $T_1 \stackrel{d}{=} S_1^{-\alpha}$. Set $q = \frac{\Gamma(\alpha)}{\Gamma(\alpha\rho)\Gamma(1-\alpha\rho)} > 0$ and note that ξ is a Lévy process with Laplace exponent Ψ^α given, for any $-1/\alpha < \texttt{Re}(z) < 1$, by

$$\Psi^\alpha(z) - q = -\frac{\Gamma(\alpha - \alpha z)\Gamma(\alpha z + 1)}{\Gamma(\alpha\rho - \alpha z)\Gamma(\alpha z + 1 - \alpha\rho)}.$$

First, let us consider the case when $\alpha \in (0,1)$. We observe that $|\Psi^\alpha(s)| < +\infty$, for any $s \in [-1, 0]$. Also we check that $\Psi^\alpha(-1) - q \leq 0$ if $\frac{\Gamma(2\alpha)\Gamma(1-\alpha)}{\Gamma(\alpha(\rho+1))\Gamma(1-\alpha(\rho+1)} \geq 0$ which is the case when $1 - \alpha(\rho + 1) \geq 0$. Moreover, we know from [26] that for $0 < \alpha < 1$,

the density of the Lévy measure of ξ restricted on $\mathbb{R}^+$ takes the form, up to a positive constant, $e^y(e^y-1)^{-\alpha-1}, y>0$, which is easily seen to be decreasing on $\mathbb{R}^+$. Hence, we can apply Corollary 3.1.3 (i) to get that the density of T_1 is bounded and non-increasing. The boundedness property could have also been observed when $\rho<1$ from [35, Remark 5] which states that the density of T_1 has a finite limit at zero. Recalling that when $\rho=1$, we have $T_1 \stackrel{d}{=} Z_1^{-\alpha}$, we could also easily check from the Humbert-Pollard series representation of the density of Z_1, see e.g. [90, 14.35], that the density of T_1 has also a finite limit at 0. The remaining part of the statement follows trivially.
Next, we assume that $\alpha \in (1,2]$ and $\rho = 1-\frac{1}{\alpha}$, that is Z is spectrally positive and thus ξ is a spectrally negative Lévy process with Laplace exponent, given, for any $\mathtt{Re}(z) > \frac{1}{\alpha}-1$, by

$$\Psi^\alpha(z) - q \quad = \quad \frac{\Gamma(\alpha z+1)}{\Gamma(\alpha z+1-\alpha)}.$$

Since $\Psi^\alpha\left(1-\frac{1}{\alpha}\right)-q=0$ we have $0<\gamma_q = 1-\frac{1}{\alpha} \leq \frac{1}{2}$, we deduce from Corollary 3.1.3 (iv) that the density of $1/T_1$ is completely monotone which means that the density of S_1^α is completely monotone. Note that the law of S_1 has been computed explicitly as a power series by Bernyk et al. [10]. We end up the paper by pointing out that in the Brownian motion case, i.e. $\alpha=2$, the density of S_1^2 is well-known to be $m_{S_1^2}(x) = \frac{e^{-\frac{x}{2}}}{\sqrt{2\pi x}}$ and we get $m_{S_1^2}(x) = \frac{1}{\sqrt{2\pi}}\int_{1/2}^\infty \frac{e^{-xr}}{\sqrt{r-1/2}}dr$.

Chapter 4

Exponential functional of Lévy processes: generalized Weierstrass products and Wiener-Hopf factorization

Abstract: In this note, we state a representation of the Mellin transform of the exponential functional of Lévy processes in terms of generalized Weierstrass products. As by-product, we obtain a multiplicative Wiener-Hopf factorization generalizing previous results obtained by the authors in [78] as well as smoothness properties of its distribution.

4.1 Introduction

Let $\xi = (\xi_t)_{t\geq 0}$ be a possibly killed real-valued Lévy process with a positive mean if it is conservative. That means that ξ is a process with stationary and independent increments with $m = \mathbb{E}[\xi_1] > 0$ if not killed at an independent exponential time. We refer to the excellent monographs [11] and [90] for background. The law of ξ is characterized by its characteristic exponent, i.e. $\ln \mathbb{E}\left[e^{z\xi_t}\right] = \Psi(z)t$ where $\Psi : i\mathbb{R} \to \mathbb{C}$ admits, in our context, the following Lévy-Khintchine representation

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + bz + \int_{\mathbb{R}} \left(e^{zr} - 1 - zr\mathbb{I}_{\{|r|<1\}}\right) \Pi(dr) - q, \tag{4.1.1}$$

where $q \geq 0$ is the killing rate, $\sigma \geq 0$, $b \in \mathbb{R}$, $m = b + \int_{|r|>1} r\Pi(dr) \in (0, \infty]$ if $q = 0$, and, the Lévy measure Π satisfies the integrability condition $\int_{\mathbb{R}}(1 \wedge r^2)\,\Pi(dr) < +\infty$. We use the convention that $\xi_t = \infty$ for any $t \geq \mathbf{e_q}$, where $\mathbf{e_q}$ stands for an independent (of ξ) exponential variable of parameter $q > 0$. The aim of this note is to describe the distribution of the positive random variable

$$I_\Psi = \int_0^\infty e^{-\xi_t} dt$$

which is called the exponential functional of the Lévy process ξ. More specifically, we provide a representation on a strip of the Mellin transform of this variable in terms of generalized Weierstrass products. We deduce from this result a multiplicative Wiener-Hopf factorization as well as some smoothness properties of its distribution. We refer to the book of Yor [102] and the paper of Bertoin and Yor [21] for references on the topic and for motivations for studying the law of I_Ψ. We would also like to mention that from these publications, there have been numerous investigations of fine distributional properties of this random variable revealing new interesting connections with several fields. For instance, we indicate that its analysis is intimately connected with the study of special functions, see e.g. Kuznetsov and Pardo [52] for Barnes multiple gamma functions and also Patie [76] Patie and Savov [78] for generalization of hypergeometric functions. Hirsch and Yor [46] established an interesting connection between the exponential functional of some subordinators and the class of multiplicative infinitely divisible random variables. Haas and Rivero [43] study Yaglom limits of positive self-similar Markov processes and extreme value theory by means of specific properties of I_Ψ. Pardo et al. [68] and Patie and Savov [78] derived a first Wiener-Hopf type factorization for its distribution by resorting to probabilistic devices. Finally, in [83] the law of the variable I_Ψ turns out to be a key object in the development of the spectral theory of a class of non-selfadjoint invariant Feller semigroups.

4.2 Main results

We shall provide a representation of the Mellin transform of the positive random variable I_Ψ which we denote, for some $z \in \mathbb{C}$, as follows

$$\mathcal{M}_{I_\Psi}(z) = \mathbb{E}\left[I_\Psi^{z-1}\right].$$

Before stating the main result and its main consequences, we introduce a few further notation. First, by defining the following subspaces of the negative of continuous negative definite functions

$$\mathcal{N} = \{\Psi \text{ of the form (5.7.1)}\},$$

we have $I_\Psi < +\infty$ a.s. if and only if $\Psi \in \mathcal{N}$. Indeed, this is plain if $q > 0$ and if $q = 0$, by the strong law of large numbers for Lévy processes we have the equivalence

$$I_\Psi < +\infty \text{ a.s.} \iff m > 0.$$

We have that $\Psi \in \mathcal{N}$ admits an analytical extension in the strip $\mathbb{C}_{(a,b)} = \{z \in \mathbb{C};\ a < \mathtt{Re}(z) < b\}$ with $a < 0 < b$, if and only if $\left|\mathbb{E}\left[e^{z\xi_1}\right]\right| < \infty$, for all $z \in \mathbb{C}_{(a,b)}$. Under this condition, the restriction of Ψ on the real interval (a, b) is clearly convex and zero free on $(0, b)$. Next, with the usual convention $\inf\{\emptyset\} = \infty$, we set $\theta_\Psi = \inf_{u>0}\{\Psi(-u) = 0\} \in (0, \infty]$, $a_\Psi = \sup_{u>0}\{\Psi \text{ is analytical on } \mathbb{C}_{(-u,0)}\} \in [0, \infty)$ and

$$d_\Psi \quad = \quad a_\Psi \wedge \theta_\Psi \in [0, \infty).$$

We shall also need the space of Bernstein functions $\mathcal{B}$ defined as the set of functions $\phi : \mathbb{C}_{(0,\infty)} \to \mathbb{C}$ which admit the representation

$$\phi(z) = \kappa + \delta z + \int_0^\infty (1 - e^{-zr})\mu(dr) \tag{4.2.1}$$

where $\kappa, \delta \geq 0$ and μ is a Lévy measure such that $\int_0^\infty (1 \wedge r)\mu(dr) < \infty$. It is easy to see that if $\phi \in \mathcal{B}$ then $-\phi \in \mathcal{N}$ and, in such a case, we simply write I_ϕ for $\mathrm{I}_{-\phi}$. We proceed by recalling that the analytical form of the Wiener-Hopf factorization of Lévy processes leads, for any $\Psi \in \mathcal{N}$,

$$\Psi(z) = -\phi_+(-z)\phi_-(z),\ z \in i\mathbb{R},$$

where $\phi_\pm \in \mathcal{B}$ with $\phi_+(0) = 0$ if $q = 0$ and $\phi_\pm(0) > 0$ otherwise. We point out that in the case $q > 0$, the parameters of the Wiener-Hopf factors depend on q. In particular, in this case we write simply, in the representation (4.2.1), with the obvious notation, $\mu_\pm(dr) = \int_0^\infty e^{-qr_1}\mu_{\pm,q}(dr_1, dr)$, where $\mu_{\pm,q}(dr_1, dr)$ is the Lévy measure of the bivariate ascending and descending ladder height and time processes. Finally, for a function $\phi : \mathbb{C} \to \mathbb{C}$, we write formally the generalized Weierstrass product

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^\infty \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)}z}$$

where

$$\gamma_\phi = \lim_{n\to\infty} \left(\sum_{k=1}^n \frac{\phi'(k)}{\phi(k)} - \log\phi(n) \right).$$

We observe that if $\phi(z) = z$, then W_ϕ corresponds to the Weierstrass product representation of the Gamma function Γ, valid on $\mathbb{C}/\{0, -1, -2, \ldots\}$, and γ_ϕ is the Euler-Mascheroni constant, see e.g. [58], justifying both the terminology and notation. We are now ready to state our main result which provides an explicit representation, in terms of generalized Weierstrass products, of the Mellin transform of I_Ψ for general Lévy processes.

Theorem 4.2.1. *For any $\phi \in \mathcal{B}$, we have $|\gamma_\phi| < \infty$. Moreover, for any $\Psi \in \mathcal{N}$, we have*

$$\mathcal{M}_{I_\Psi}(z) = \phi_-(0)\Gamma(z)\frac{W_{\phi_-}(1-z)}{W_{\phi_+}(z)}, \quad z \in \mathbb{C}_{(0,d_\Psi+1)}, \tag{4.2.2}$$

where the product W_{ϕ_+} (resp. W_{ϕ_-}) is absolutely convergent on $\mathbb{C}_{(0,\infty)}$ (resp. $\mathbb{C}_{(-d_{\phi_-},\infty)}$).

We point out that besides its probabilistic nature, this result seems to have some analytical interests. Indeed, on the one hand, it reveals that the class of Bernstein functions appears to be a natural set to generalize the Weierstrass product of the Gamma function. In this vein, we mention that writing $\phi(z) = \frac{\Gamma(z+\alpha)}{\Gamma(z)}$, one has $\phi \in \mathcal{B}$ for any $0 < \alpha < 1$ and one recovers the Weierstrass product representation of double Gamma functions, see e.g. [53]. On the other hand, in the case $0 < m < \infty$, we are able to show, by developing a precised study of the asymptotic decay of the Mellin transform $\mathcal{M}_{I_\Psi}$ on imaginary lines in its strip of definition, that the right-hand side of (5.2.47) is the unique solution of the functional equation, derived by Maulik and Zwart [62],

$$\mathcal{M}_{I_\Psi}(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_{I_\Psi}(z), \quad \mathcal{M}_{I_\Psi}(1) = 1, \tag{4.2.3}$$

which holds on the domain $\{z \in \mathbb{C}; \Psi(\mathtt{Re}(-z)) \leq 0\}$. We emphasize that our representation remains valid in the case $m = \infty$ although, to the best of our knowledge, there does not exist any characterization of the Mellin transform in this case.
By Mellin identification, we obtain as a straightforward consequence of the representation (5.2.47), the following factorization of the distribution of the exponential functional.

Corollary 4.2.2. *For any $\Psi \in \mathcal{N}$, we have the following multiplicative Wiener-Hopf factorization*

$$I_\Psi \stackrel{d}{=} I_{\phi_+} \times X_{\phi_-} \tag{4.2.4}$$

where $\times$ stands for the product of two independent random variables and X_{ϕ_-} is a positive variable whose distribution is determinate by its negative entire moments as follows, $\mathbb{E}\left[X_{\phi_-}^{-1}\right] = \phi_-(0)$, and for any $n \geq 2$, $\mathbb{E}\left[X_{\phi_-}^{-n}\right] = \phi_-(0)\prod_{k=1}^{n-1}\phi_-(k)$. Finally, we have $\psi(z) = z\phi_-(z) \in \{\Psi \in \mathcal{N}; q = 0$ and $\Pi(dr)\mathbb{I}_{\{r>0\}} \equiv 0\}$ together with the identity in distribution

$$X_{\phi_-} \stackrel{d}{=} I_\psi \tag{4.2.5}$$

if and only if the Lévy measure μ_- in the representation (4.2.1) *of ϕ_- has a non-increasing density.*

It is worth mentioning that recently Hirsch and Yor [46] provide an infinite product representation, which differs from our Weierstrass product, of the Mellin transform of the variable I_{ϕ_+} when $q = 0$, which is stated valid on the positive real line. They also have a similar type of representation for the positive moments of a variable, introduced by Bertoin and Yor [18], whose distribution is closely connected to the one $X_{\phi_-}^{-1}$ when $\phi_-(0) = 0$ a situation excluded in our work. We also point out that the factorization (5.2.44) is a generalization of the result obtained first by Pardo et al. [68] and improved in Patie and Savov [78] under the sufficient conditions that either the Lévy measure $\Pi(r)\mathbb{I}_{\{r<0\}}$ or both Lévy measures in the representation (4.2.1) of ϕ_+ and ϕ_-, have a decreasing density. In fact, in the aforementioned papers the identity (5.2.44) was obtained with $X_{\phi_-} \stackrel{d}{=} I_\psi$, and, the methodologies developed therein stem heavily on the connection between the distribution of the exponential functional and the stationary measure of a family of Markov processes and additional specific properties of the exponential functional. It is not clear to us how these techniques could be used to derive both the general version of the factorization (5.2.44) and the necessary condition for the identity (4.2.5) to hold. We also point out that, in [82], this Wiener-Hopf factorization allows us to derive general intertwining relationships between the Feller semigroups of positive self-similar Markov processes, extending the work of Carmona et al. [28]. For at least this purpose, it would be interesting to give an interpretation of X_{ϕ_-}, in the most general situation, in terms of the underlying Lévy process ξ. We would also like to take the opportunity to raise the question whether one can provide a pathwise interpretation of the multiplicative Wiener-Hopf factorization (5.2.44). In [82], relying on the representation (5.2.47), as mentioned above, we develop a detailed study of the asymptotic behavior of the modulus of the Mellin transform $\mathcal{M}_{I_\Psi}(z)$ along the imaginary lines. In particular, we provide sufficient conditions for its asymptotic decay to be nearly exponential, that is decay faster than any inverse polynomials. These estimates combined with Mellin inversion theorem allows to deduce smoothness properties of the distribution of I_Ψ. We recall that, in the case $q = 0$, Bertoin et al. [16], see also [78] for the case $q > 0$, have shown that the distribution of the positive variable I_Ψ for any $\Psi \in \mathcal{N}$ is absolutely continuous with a density which we denote by p_ψ. To state our last results, we write, for any $\phi \in \mathcal{B}$ and $b \geq 0$,

$$H_\phi(b) = \int_0^\infty \ln\left(\frac{|\phi(b(y+i))|}{\phi(by)}\right) dy,$$

$\underline{H}_\phi = \liminf_{b\to\infty} H_\phi(b)$ and $\overline{H}_\phi = \limsup_{b\to\infty} H_\phi(b)$.

Corollary 4.2.3. *The following assertions hold true.*

1. *We have for all* $b \geq 0$, $0 \leq H_{\phi_\pm}(b) \leq \frac{\pi}{2}$.
2. $p_\Psi \in C_0^\infty(\mathbb{R}^+)$ *if for any* $n \in \mathbb{N}$
$$\limsup_{b\to\infty} b^n e^{-b\left(\frac{\pi}{2} - H_{\phi_+}(b) + H_{\phi_-}(b)\right)} = 0. \tag{4.2.6}$$
3. (4.2.6) *holds if, for instance, at least one the following conditions is satisfied.*
   (a) $\delta_- > 0$, *where* δ_- *is the drift term of* ϕ_-.
   (b) *For all* λ *big enough and some* $\alpha > 0$, $\limsup_{x\to 0} \frac{\overline{\Pi}_-(\lambda x)}{\overline{\Pi}_-(x)} < \frac{1}{\lambda^{1+\alpha}}$, *where* $\overline{\Pi}_-$ *is the tail of the measure* $\Pi(r)\mathbb{I}_{\{r<0\}}$.
   (c) $\exists \epsilon > 0$ *such that* $\liminf_{x\to 0} \overline{\mu}_-(x)x^\epsilon > 0$ *where* $\overline{\mu}_-$ *is the tail of* μ_- *the measure associated to* ϕ_-.
   (d) *If* μ_- *has a decreasing density* g_- *and either* $\liminf_{b\to\infty} \frac{g_-\left(\frac{1}{b}\right)}{\phi_-(b)b} > 0$ *or* $\phi_-(b)ln(b) = o\left(g_-\left(\frac{1}{b}\right)\right)$ *as* $b \to \infty$.
4. *Finally, if* $0 < z_0 = \underline{H}_{\phi_-} + \frac{\pi}{2} - \overline{H}_{\phi_+} < \pi$ *then the mapping* $z \mapsto p_\Psi\left(\frac{1}{z}\right)$ *is analytical in the sector* $\arg z < z_0$. *This is, for instance, true if the condition* (3a), (3b) *or the first condition in* (3d) *above holds.*

Note that the condition on $\overline{\Pi}_-$ in (*3b*) is very natural if we have a process of paths of unbounded variation. Since $\overline{\Pi}_-(r) \overset{0}{\sim} r^{-1}$ at worst we see that the ratio is justified if we assume a bit heavier tails. We conclude this note by pointing out that if $\Pi \equiv 0$, which corresponds to the Brownian motion case with positive drift m, i.e. $\Psi(z) = \frac{z^2}{2} + mz$, then $\overline{H}_{\phi_-} = \underline{H}_{\phi_+} = \frac{\pi}{2}$ and hence $z_0 = \frac{\pi}{2}$ in (*4*). Indeed, it is well known that in this case $p_\Psi\left(\frac{1}{z}\right) = \frac{z^m}{\Gamma(m)}e^{-z}$ and the Mellin transform of I_Ψ is $\frac{\Gamma(m+1-z)}{\Gamma(m)}$ whose exponential rate of decay along complex lines of the type $a + ib$ is precisely of the rate of $\frac{\pi}{2}|b|$. However, the rate of decay of the Mellin transform does not describe the sector of analyticity of $p_\Psi\left(\frac{1}{z}\right)$ which is obviously the whole complex plane in the case m is an integer. Such phenomena could also be observed for any $\Psi \in \mathcal{N}$ with $\overline{\Pi}_- \equiv 0$ and $\int_0^1 r\Pi(dr) = \infty$ for which we also know that $p_\Psi\left(\frac{1}{z}\right)$ is an entire function, see [76].

Chapter 5

Bernstein-gamma functions and exponential functionals of Lévy processes

Abstract: In this work we analyse the solution to the recurrence equation

$$\mathcal{M}_\Psi(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_\Psi(z)$$

defined on a subset of the imaginary line and where Ψ runs through the set of all negative definite functions. Using the analytic Wiener-Hopf method we furnish the solution to this equation as a product of functions that extend the classical gamma function. These latter functions, being in bijection with the class of Bernstein functions, we call the Bernstein-gamma functions. Using their Weierstrass product representation we establish universal Stirling type asymptotic which is explicit in terms of the constituting Bernstein function. This allows the thorough understanding of the decay of $|\mathcal{M}_\Psi(z)|$ along imaginary lines and an access to quantities important for many theoretical and applied studies in probability and analysis.

This functional equation appears as a central object in several recent studies ranging from analysis and spectral theory to probability theory. In this paper, as an application of the results above, we investigate from a global perspective the exponential functionals of Lévy processes whose Mellin transform satisfies the equation above. Although these variables have been intensively studied our new approach based on a combination of probabilistic and analytical techniques enables us to derive comprehensive properties and strengthen several results on the law of these random variables for some classes of Lévy processes that could be found in the literature. These encompass smoothness for its density, regularity and analytical properties, large and small asymptotic behaviour, including asymptotic expansions, bounds, and Mellin-Barnes representations of its successive derivatives. In some cases we also study the weak convergence of exponential functionals on a finite time horizon when the latter expands to infinity. As a result of new Wiener-Hopf and infinite product factorizations of the law of the exponential functional we deliver important intertwining

relation between members of the class of positive self-similar semigroups. Some of the results presented in this paper have been announced in the note [79].

5.1 Introduction

The main aim of this work is to develop an in-depth analysis of the solutions to the functional equation defined for any negative definite function Ψ by

$$\mathcal{M}_\Psi(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_\Psi(z) \tag{5.1.1}$$

and valid (at least) on the domain $i\mathbb{R} \setminus (\mathcal{Z}_0(\Psi) \cup \{0\})$, where we put the set $\mathcal{Z}_0(\Psi) = \{z \in i\mathbb{R} : \Psi(-z) = 0\}$ and the negative definite functions are defined in (5.2.1).

The foremost motivation underlying this study is the methodology underpinning an approach developed by the authors for understanding the spectral decomposition of at least some non-self-adjoint Markov semigroups. This program has been carried out for a class of generalized Laguerre semigroups and thereby via a deterministic mapping for an equivalent class of positive self-similar semigroups, see [83], and this study has revealed that the solutions to the recurrence equations of type (5.1.1) play a central role in obtaining and quantitatively characterizing the spectral representation of the entire class of positive self-similar semigroups. A natural approach to derive and understand the solution to an equation defined on a subset of $i\mathbb{R}$, in this instance (5.1.1), stems from the classical Wiener-Hopf method. It is well-known that for any $\Psi \in \overline{\mathcal{N}}$, where $\overline{\mathcal{N}}$ stands for the space of negative definite functions, we have the analytic Wiener-Hopf factorization

$$\Psi(z) = -\phi_+(-z)\phi_-(z),\ z \in i\mathbb{R}, \tag{5.1.2}$$

where $\phi_+, \phi_- \in \mathcal{B}$, that is $\phi_\pm$ are Bernstein functions, see (5.2.2). Exploiting (5.1.2) the derivation and characterization of the solution of (5.1.1) can be reduced to considering an equation of the type

$$W_\phi(z+1) = \phi(z)W_\phi(z), \quad \phi \in \mathcal{B}, \tag{5.1.3}$$

for $z \in \mathbb{C}_{(0,\infty)} = \{z \in \mathbb{C} : \mathtt{Re}(z) > 0\}$. In turn the solution to (5.1.3) can be represented on $\mathbb{C}_{(0,\infty)}$ as an infinite Weierstrass product involving $\phi \in \mathcal{B}$, see [83, Chapter 6]. Here, in Theorem 5.3.1 we manage to characterize the main properties of W_ϕ, as a meromorphic function on an identifiable complex strip via a couple of global parameters pertaining to all $\phi \in \mathcal{B}$. Also, new and informative asymptotic representations of W_ϕ are offered and contained in Theorem 5.3.2. From them the asymptotic of W_ϕ along $a + i\mathbb{R}$ can be related to the geometry of the image of $\mathbb{C}_{(0,\infty)}$ via $\phi \in \mathcal{B}$ and in many instances this asymptotic can be precisely computed or well-estimated as illustrated in Proposition 5.3.15. All results are reminiscent of the Stirling asymptotic for the gamma function which solves (5.1.3) with $\phi(z) = z$. For this purpose we call the functions W_ϕ Bernstein-gamma functions. Due to their ubiquitous, albeit often unrecognised, presence in many theoretical studies they are an important class of special functions. If $\mathbb{R}^+ = (0,\infty)$ the restriction of (5.1.3) on $\mathbb{R}^+$ has been considered in a larger generality by [100] and for the class of Bernstein functions by [46]. More information on the literature can be found in Section 5.3.

These novel results on the general solution of (5.1.3), that is W_ϕ, allow for an asymptotic representation and a complete characterization of the solution of (5.1.1), that is $\mathcal{M}_\Psi$, as a meromorphic function on an identifiable strip, in terms of four global parameters describing the analytical properties of ϕ_+, ϕ_- and thereby of Ψ as stated by Theorem 5.2.1. In Theorem 5.2.5 we also conduct asymptotic analysis of $|\mathcal{M}_\Psi(z)|$. We wish to emphasize that (5.1.2) does not fully reduce the study of (5.1.1) to the decoupled investigation of (5.1.3) for ϕ_+ and ϕ_-. In fact the usage of the interplay between ϕ_+ and ϕ_- induced by (5.1.2) is the key to getting sharp and exhaustive results on the properties of $\mathcal{M}_\Psi$ as illustrated by (5.2.16) of Theorem 5.2.5. The latter gives complete and quantifiable information as to the rate of polynomial decay of $|\mathcal{M}_\Psi(z)|$ along complex lines of the type $a + i\mathbb{R}$, where a is fixed.

As a major application of our results on the solutions of functional equations of the type (5.1.1) we develop and present a general and unified study of the exponential functionals of Lévy processes. To facilitate the discussion of our main motivation, aims and achievements in light of the existing body of literature we recall that a possibly killed Lévy process $\xi = (\xi_t)_{t\geq 0}$ is a.s. right-continuous, real-valued stochastic process which possesses stationary and independent increments that is killed at an independent of itself exponential random variable (time) of parameter $q \geq 0$, that is $\mathbf{e_q}$ and $\xi_t = \infty$ for any $t \geq \mathbf{e_q}$. Note that $\mathbf{e}_0 = \infty$. The law of a possibly killed Lévy process ξ is characterized via its characteristic exponent, i.e. $\mathbb{E}\left[e^{z\xi_t}\right] = e^{\Psi(z)t}$, $\Psi \in \overline{\mathcal{N}}$, and there is a bijection between the class of possibly killed Lévy processes and $\overline{\mathcal{N}}$. Denote the exponential functional of the Lévy process ξ by

$$I_\Psi(t) = \int_0^t e^{-\xi_s} ds, \quad t \geq 0,$$

and its associated perpetuity by

$$I_\Psi = \int_0^\infty e^{-\xi_s} ds = \int_0^{\mathbf{e_q}} e^{-\xi_s} ds. \tag{5.1.4}$$

Its study has been initiated by Urbanik in [97] and proceeded by M. Yor with various co-authors [21, 46, 102]. There is also a number of subsequent and intermediate contributions to the study of these random variables, a small sample of which comprises of [4, 51, 68, 76, 78, 79, 83]. This is due to the fact that the exponential functionals appear and play a crucial role in various theoretical and applied contexts such as the spectral theory of some non-reversible Markov semigroups ([80, 83]), the study of random planar maps ([14]), limit theorems of Markov chains ([15]), positive self-similar Markov processes ([13, 19, 26, 74]), financial and insurance mathematics ([77, 45]), branching processes with immigration ([72]), fragmentation processes ([21]), random affine equations, perpetuities to name but a few. Starting from [62] it has become gradually evident that studying the Mellin transform of the exponential functional is the right tool in many contexts. For particular subclasses, that include and allow the study of the supremum of the stable process, this transform has been evaluated and sometimes via inversion the law of exponential functional has been obtained, see [10, 44, 50, 51, 52]. In this paper, as a consequence of the detailed study of $\mathcal{M}_\Psi$ and W_ϕ above, and the fact that whenever $I_\Psi < \infty$ the Mellin transform of I_Ψ, that is $\mathcal{M}_{I_\Psi}$, satisfies $\mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z)$ at least on $\texttt{Re}(z) \in (0,1)$, we obtain, refine and complement various results on the law of I_Ψ. Deriving complete and quantifiable information on the decay of $|\mathcal{M}_\Psi(z)|$ along complex lines allows us to show that the law of I_Ψ is infinitely differentiable unless ξ is a compound Poisson process with strictly positive drift in which case (5.2.16) of Theorem 5.2.5 evaluates the minimum number of smooth derivatives the law of I_Ψ possesses. Under no restriction we provide a Mellin-Barnes representation for the law of I_Ψ and thereby bounds for the law of I_Ψ and its derivatives. In Theorem 5.2.7(4) and Corollary 5.2.12 we show that polynomial small asymptotic expansion is possible if and only if the Lévy process is killed, in which case we obtain explicit evaluation of the terms of this expansion. In Theorem 5.2.14 general results on the tail of the law are offered. These include the computation of the Pareto index for any exponential functional and under Cramer's condition, depending on the decay of $|\mathcal{M}_{I_\Psi}(z)|$ and under minute additional requirements, the elucidation of the tail asymptotic and its extension to the level of the density and its derivatives. The latter for example immediately recovers the asymptotic behaviour of the density of the supremum of a stable Lévy process as investigated in [10, 35, 49, 76]. In Theorem 5.2.19 general results have also been derived for the law at zero. Finally, when $\lim\limits_{t\to\infty} I_\Psi(t) = \int_0^\infty e^{-\xi_s}ds = \infty$ and under the celebrated Spitzer's condition imposed on ξ we establish the weak convergence of $\mathbb{P}\left(I_\Psi(t) \in dx\right)$ after proper rescaling in time and space. This result is particularly relevant in the world of random processes in random environments, where such information strengthens significantly the results of [59, 67]. We proceed by showing that the Wiener-Hopf type factorization of the law of I_Ψ which was proved in [68, 78] under various conditions, holds in fact in complete generality, see Theorem 5.2.27 which also contains additional interesting factorizations. By means of a classical relation between the entrance law of positive self-similar Markov processes and the law of the exponential functional of Lévy processes, we compute explicitly the Mellin transform of the former, see Theorem 5.2.29(1). Moreover, exploiting this relation and the Wiener-Hopf decomposition of the law of I_Ψ mentioned earlier, we derive some original intertwining relations between positive self-similar semigroups, see Theorem 5.2.29(2).

The outcome of this paper seems to reaffirm the power of complex analytical tools in probability theory. Even departing from a completely general perspective the Mellin transform is the key tool for understanding the exponential functional of a Lévy process. The reason for the latter is the possibility to represent the Mellin transform as a product combination of identifiable Bernstein-gamma functions and thus access quantifiable information about it as a meromorphic function and its asymptotic behaviour in a complex strip. However, as it can be most notably seen in the proofs of Theorem 5.2.5, Theorem 5.2.14 and Theorem 5.2.19, the most precise results depend on mixing analytical tools with probabilistic techniques and the properties of Lévy processes.

The paper is structured as follows: Section 5.2 is dedicated to the main results and their statements; Section 5.3 introduces and studies in detail the Bernstein-gamma functions; Section 5.4 considers the proofs of the results related to the functional equation (5.1.1);

Section 5.5 furnishes the proofs for the results regarding the exponential functionals of Lévy processes; Section 5.6 deals with the factorizations of the law of the exponential functional and the intertwinings between positive self-similar semigroups; the Appendix provides some additional information on Lévy processes and results on them that cannot be easily detected in the literature, e.g. the version of *équation amicale inversée* for killed Lévy processes.

5.2 Main Results

5.2.1 Wiener-Hopf factorization, Bernstein-Weierstrass representation and asymptotic analysis of the solution of (5.1.1)

We start by introducing some notation. We use $\mathbb{N}$ for the set of non-negative integers and the standard notation $\mathtt{C}^k(\mathbb{K})$ for the k times differentiable functions on some complex or real domain $\mathbb{K}$. The space $\mathtt{C}_0^k(\mathbb{R}^+)$ stands for the k times differentiable functions which together with their k derivatives vanish at infinity, whereas $\mathtt{C}_b^k(\mathbb{R}^+)$ requires only boundedness. For any $z \in \mathbb{C}$ set $z = |z|e^{i \arg z}$ with the branch of the argument function defined via the convention $\arg : \mathbb{C} \mapsto (-\pi, \pi]$. For any $-\infty \leq a < b \leq \infty$, we denote by $\mathbb{C}_{(a,b)} = \{z \in \mathbb{C} : a < \mathtt{Re}(z) < b\}$ and for any $a \in (-\infty, \infty)$ we set $\mathbb{C}_a = \{z \in \mathbb{C} : \mathtt{Re}(z) = a\}$. We use $\mathtt{A}_{(a,b)}$ for the set of holomorphic functions on $\mathbb{C}_{(a,b)}$, whereas if $-\infty < a$ then $\mathtt{A}_{[a,b)}$ stands for the holomorphic functions on $\mathbb{C}_{(a,b)}$ that can be extended continuously to $\mathbb{C}_a$. Similarly, we have the spaces $\mathtt{A}_{[a,b]}$ and $\mathtt{A}_{(a,b]}$. Finally, we use $\mathtt{M}_{(a,b)}$ for the set of meromorphic functions on $\mathbb{C}_{(a,b)}$. It is well-known that $\Psi \in \overline{\mathcal{N}}$, that is Ψ is a negative definite function, if and only if $\Psi : i\mathbb{R} \to \mathbb{C}$ and it admits the following Lévy-Khintchine representation

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + cz + \int_{-\infty}^{\infty} \left(e^{zr} - 1 - zr\mathbb{I}_{\{|r|<1\}}\right) \Pi(dr) - q, \tag{5.2.1}$$

where $q \geq 0$, $\sigma^2 \geq 0$, $c \in \mathbb{R}$, and, the sigma-finite measure Π satisfies the integrability condition $\int_{-\infty}^{\infty}(1 \wedge r^2)\,\Pi(dr) < \infty$. The class of Bernstein functions $\mathcal{B}$ consists of all functions $\phi \not\equiv 0$ that can be represented as follows

$$\phi(z) = \phi(0) + \mathtt{d}z + \int_0^{\infty} \left(1 - e^{-zy}\right) \mu(dy) = \phi(0) + \mathtt{d}z + z\int_0^{\infty} e^{-zy}\bar{\mu}(y)\,dy, \ z \in \mathbb{C}_{[0,\infty)}, \tag{5.2.2}$$

where $\mathbb{C}_{[0,\infty)} = \{z \in \mathbb{C} : \mathtt{Re}(z) \geq 0\}$, $\phi(0) \geq 0$, $\mathtt{d} \geq 0$, μ is a sigma-finite measure satisfying $\int_0^{\infty}(1 \wedge y)\mu(dy) < \infty$ and $\bar{\mu}(y) = \int_y^{\infty} \mu(dr)$, $y \geq 0$. With any function $\phi \in \mathcal{B}$ since $\phi \in \mathtt{A}_{[0,\infty)}$, see (5.2.2), we associate the quantities

$$\mathfrak{u}_\phi = \sup\{u \leq 0 : \phi(u) = 0\} \in [-\infty, 0] \tag{5.2.3}$$

$$\mathfrak{a}_\phi = \inf\ \{u < 0 : \phi \in \mathtt{A}_{(u,\infty)}\} \in [-\infty, 0] \tag{5.2.4}$$

$$\overline{\mathfrak{a}}_\phi = \max\{\mathfrak{a}_\phi, \mathfrak{u}_\phi\} = \sup\{u \leq 0 : \ \phi(u) = -\infty \text{ or } \phi(u) = 0\} \in [-\infty, 0], \tag{5.2.5}$$

which are well defined thanks to the form of ϕ, see (5.2.2), and the convention $\sup\emptyset=-\infty$ and $\inf\emptyset=0$. Note that ϕ is a non-zero constant if and only if (iff) $\overline{\mathfrak{a}}_\phi=-\infty$. Indeed, otherwise, if $\mathfrak{a}_\phi=-\infty$ then necessarily from (5.2.2), $\lim\limits_{a\to\infty}\phi(-a)=-\infty$. Hence, $\mathfrak{u}_\phi>-\infty$ and $\overline{\mathfrak{a}}_\phi\in(-\infty,0]$. For these quantities associated to ϕ_+,ϕ_- in (5.1.2) for the sake of clarity we drop the subscript ϕ and use $\mathfrak{u}_+,\mathfrak{u}_-,\mathfrak{a}_+,\mathfrak{a}_-,\overline{\mathfrak{a}}_+,\overline{\mathfrak{a}}_-$. For $\phi\in\mathcal{B}$, we write the generalized Weierstrass product

$$W_\phi(z)=\frac{e^{-\gamma_\phi z}}{\phi(z)}\prod_{k=1}^{\infty}\frac{\phi(k)}{\phi(k+z)}e^{\frac{\phi'(k)}{\phi(k)}z},\ z\in\mathbb{C}_{(0,\infty)},$$

where

$$\gamma_\phi=\lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\phi'(k)}{\phi(k)}-\log\phi(n)\right)\in\left[-\ln\phi(1),\frac{\phi'(1)}{\phi(1)}-\ln\phi(1)\right].$$

Both W_ϕ,γ_ϕ are known to exist and observe that if $\phi(z)=z$, then W_ϕ corresponds to the Weierstrass product representation of the celebrated gamma function Γ, valid on $\mathbb{C}/\{0,-1,-2,\ldots\}$, and γ_ϕ is the Euler-Mascheroni constant, see e.g. [58], justifying both the terminology and notation. Note also that when $z=n\in\mathbb{N}$ then $W_\phi(n+1)=\prod_{k=1}^{n}\phi(k)$. We are ready to state the first of our central results which, for any $\Psi\in\overline{\mathcal{N}}$, provides an explicit representation of $\mathcal{M}_\Psi$ in terms of generalized Bernstein-gamma functions.

Theorem 5.2.1. *Let $\Psi\in\overline{\mathcal{N}}$ and recall that $\Psi(z)=-\phi_+(-z)\phi_-(z)$, $z\in i\mathbb{R}$. Then, the mapping $\mathcal{M}_\Psi$ defined by*

$$\mathcal{M}_\Psi(z)=\frac{\Gamma(z)}{W_{\phi_+}(z)}W_{\phi_-}(1-z)\tag{5.2.6}$$

satisfies the recurrence relation (5.1.1) at least on $i\mathbb{R}\setminus(\mathcal{Z}_0(\Psi)\cup\{0\})$ where we recall that $\mathcal{Z}_0(\Psi)=\{z\in i\mathbb{R}:\ \Psi(-z)\neq0\}$. Setting $\mathfrak{a}_\Psi=\mathfrak{a}_+\mathbb{I}_{\{\overline{\mathfrak{a}}_+=0\}}\leq0$, we have that

$$\mathcal{M}_\Psi\in\mathtt{A}_{(\mathfrak{a}_\Psi,\,1-\overline{\mathfrak{a}}_-)}.\tag{5.2.7}$$

If $\mathfrak{a}_\Psi=0$, then $\mathcal{M}_\Psi$ extends continuously to $i\mathbb{R}\setminus\{0\}$ if $\phi_+'(0^+)=\infty$ or $\overline{\mathfrak{a}}_+<0$, and otherwise $\mathcal{M}_\Psi\in\mathtt{A}_{[0,1-\overline{\mathfrak{a}}_-)}$. In any case

$$\mathcal{M}_\Psi\in\mathtt{M}_{(\mathfrak{a}_+,\,1-\mathfrak{a}_-)}.\tag{5.2.8}$$

Let $\mathfrak{a}_+\leq\overline{\mathfrak{a}}_+<0$. If $\mathfrak{u}_+=-\infty$ or $-\mathfrak{u}_+\notin\mathbb{N}$ then on $\mathbb{C}_{(\mathfrak{a}_+,1-\overline{\mathfrak{a}}_-)}$, $\mathcal{M}_\Psi$ has simple poles at all points $-n$ such that $-n>\mathfrak{a}_+$, $n\in\mathbb{N}$. Otherwise, on $\mathbb{C}_{(\mathfrak{a}_+,1-\overline{\mathfrak{a}}_-)}$, $\mathcal{M}_\Psi$ has simple poles at all points $-n$ such that $n\in\mathbb{N}\setminus\{|\mathfrak{u}_+|,|\mathfrak{u}_+|+1,\ldots\}$. In both cases the residues are of values $\phi_+(0)\frac{\prod_{k=1}^{n}\Psi(k)}{n!}$ at each of those $-n$ where we apply the convention $\prod_{k=1}^{0}=1$.

This theorem is proved in Section 5.4.1.

Remark 5.2.2. *If $\phi_+\equiv\phi_-$, then, for any $z\in\mathbb{C}_{(\overline{\mathfrak{a}}_-\vee\mathfrak{a}_\Psi,(1-\overline{\mathfrak{a}}_-)\wedge(1-\mathfrak{a}_\Psi))}$*

$$\mathcal{M}_\Psi(z)\mathcal{M}_\Psi(1-z)=\frac{\pi}{\sin\pi z},\tag{5.2.9}$$

and hence $\mathcal{M}_\Psi\left(\frac{1}{2}\right) = \sqrt{\pi}$. *Thus,* (5.2.9) *is a generalized version of the reflection formula for the classical gamma function and offers further benefits for the understanding of the behaviour of* $\mathcal{M}_\Psi$*, see e.g. Theorem 5.2.5*(2).

Remark 5.2.3. *In view of the comprehensive asymptotic representation of* $|W_\phi(z)|$ *for any* $\phi \in \mathcal{B}$*, see Theorem 5.3.2*(5.3.19), (5.2.6) *also provides asymptotic expansion for* $|\mathcal{M}_\Psi(z)|$.

Remark 5.2.4. *We note, from* (5.1.2) *and* (5.2.3)-(5.2.5)*, that the quantities* $\mathfrak{a}_+, \mathfrak{a}_-, \mathfrak{u}_+, \mathfrak{u}_-, \overline{\mathfrak{a}}_+, \overline{\mathfrak{a}}_-$ *can be computed from the analytical properties of* Ψ*. For example, if* $\Psi \notin \mathtt{A}_{(u,0)}$ *for any* $u < 0$ *then* $\mathfrak{a}_- = 0$*. Similarly, if* $\overline{\lim}_{t\to\infty} \xi_t = -\underline{\lim}_{t\to\infty} \xi_t = \infty$ *a.s. then clearly* $\overline{\mathfrak{a}}_- = \overline{\mathfrak{a}}_+ = 0$ *as* $\phi_+(0) = \phi_-(0) = 0$*. The latter indicates that the ladder height processes of* ξ *are unkilled, see e.g. [11, Chapetr VI].*

In view of the fact that, for any $\phi \in \mathcal{B}$, W_ϕ has a Stirling type asymptotic representation, see Theorem 5.3.2 below, and (5.2.6) holds, we proceed with a definition of two classes that will encapsulate different modes of decay of $|\mathcal{M}_\Psi(z)|$ along complex lines. Put $\mathtt{I}_\Psi = (0, 1-\overline{\mathfrak{a}}_-)$ and for any $\beta \in [0,\infty]$, we write

$$\begin{aligned}\overline{\mathcal{N}}_\beta = & \left\{\Psi \in \overline{\mathcal{N}} : \lim_{|b|\to\infty} |b|^{\beta-\varepsilon} \left|\mathcal{M}_\Psi(a+ib)\right| = 0,\ \forall a \in \mathtt{I}_\Psi, \forall \varepsilon \in (0,\beta)\right\} \\ & \bigcap \left\{\Psi \in \overline{\mathcal{N}} : \lim_{|b|\to\infty} |b|^{\beta+\varepsilon} \left|\mathcal{M}_\Psi(a+ib)\right| = \infty,\ \forall a \in \mathtt{I}_\Psi, \forall \varepsilon \in (0,\beta)\right\},\end{aligned} \tag{5.2.10}$$

where if $\beta = \infty$ we understand

$$\overline{\mathcal{N}}_\infty = \left\{\Psi \in \overline{\mathcal{N}} : \lim_{|b|\to\infty} |b|^{\beta} \left|\mathcal{M}_\Psi(a+ib)\right| = 0,\ \forall a \in \mathtt{I}_\Psi, \forall \beta \geq 0\right\}. \tag{5.2.11}$$

Moreover, for any $\Theta > 0$ we set

$$\overline{\mathcal{N}}(\Theta) = \left\{\Psi \in \overline{\mathcal{N}} : \overline{\lim_{|b|\to\infty}} \frac{\ln\left|\mathcal{M}_\Psi(a+ib)\right|}{|b|} \leq -\Theta,\ \forall a \in \mathtt{I}_\Psi\right\}. \tag{5.2.12}$$

Finally, we shall also need the set of regularly varying functions at 0. For this purpose we introduce some more notation. We use in the standard manner $f \stackrel{a}{\sim} g$ (resp. $f \stackrel{a}{=} \mathrm{O}(g)$) for any $a \in [-\infty,\infty]$, to denote that $\lim_{x\to a} \frac{f(x)}{g(x)} = 1$ (resp. $\overline{\lim}_{x\to a} \left|\frac{f(x)}{g(x)}\right| = C < \infty$). The notation $\mathrm{o}(.)$ specifies that in the previous relations the constants are zero. We shall drop the superscripts if it is explicitly stated or clear that $x \to a$. We say that, for some $\alpha \in [0,1)$, $f \in RV_\alpha \iff f(y) \stackrel{0}{\sim} y^\alpha \ell(y)$, where $\ell \in SV = RV_0$ is a slowly varying function, that is, $\ell(cy) \stackrel{0}{\sim} \ell(y)$ for any $c > 0$. Furthermore, $\ell \in SV$ is said to be quasi-monotone, if ℓ is of bounded variation in a neighbourhood of zero and for any $\gamma > 0$

$$\int_0^x y^\gamma |d\ell(y)| \stackrel{0}{=} \mathrm{O}\left(x^\gamma \ell(x)\right). \tag{5.2.13}$$

With this notion we set

$$R_\alpha = \left\{ f \in RV_\alpha : \; y \mapsto \ell(y) = \frac{f(y)}{y^\alpha} \text{ is quasi-monotone} \right\}$$

and define, after recalling that $\bar{\mu}(y) = \int_y^\infty \mu(dr)$, see (5.2.2),

$$\mathcal{B}_{R_\alpha} = \left\{ \phi \in \mathcal{B} : \; \mathtt{d} = 0 \text{ and } \bar{\mu} \in R_\alpha \right\}. \tag{5.2.14}$$

Next, we define the class of Bernstein functions with a positive drift that is

$$\mathcal{B}_P = \left\{ \phi \in \mathcal{B} : \; \mathtt{d} > 0 \right\}. \tag{5.2.15}$$

Finally, we denote by μ_+, μ_- the measures associated to $\phi_+, \phi_- \in \mathcal{B}$ stemming from (5.1.2) and $\Pi_+(dy) = \Pi(dy)\mathbb{I}_{\{y>0\}}, \Pi_-(dy) = \Pi(-dy)\mathbb{I}_{\{y>0\}}$ for the measure in (5.2.1). Finally, $f(x^\pm)$ will stand throughout for the right, respectively left, limit at x. We provide an exhaustive claim concerning the decay of $|\mathcal{M}_\Psi|$ along complex lines.

Theorem 5.2.5. *Let $\Psi \in \overline{\mathcal{N}}$.*

1. *$\Psi \in \overline{\mathcal{N}}_{\mathtt{N}_\Psi}$ with*

$$\mathtt{N}_\Psi = \begin{cases} \frac{\upsilon_-(0^+)}{\phi_-(0)+\bar{\mu}_-(0)} + \frac{\phi_+(0)+\bar{\mu}_+(0)}{\mathtt{d}_+} < \infty & \text{if } \mathtt{d}_+ > 0, \mathtt{d}_- = 0 \text{ and } \overline{\Pi}(0) = \int_{-\infty}^\infty \Pi(dy) < \infty, \\ \infty & \text{otherwise,} \end{cases} \tag{5.2.16}$$

where we have used implicitly the fact that if $\mathtt{d}_+ > 0$ holds then $\mu_-(dy) = \upsilon_-(y)dy\mathbb{I}_{\{y>0\}}$ with $\upsilon_- \in \mathrm{C}([0,\infty))$.

2. *Moreover, if $\phi_- \in \mathcal{B}_P$, that is $\mathtt{d}_- > 0$, or $\arg\phi_+ \equiv \arg\phi_-$ then $\Psi \in \overline{\mathcal{N}}\left(\frac{\pi}{2}\right)$. If $\phi_- \in \mathcal{B}_{R_\alpha}$ or $\phi_+ \in \mathcal{B}_{R_{1-\alpha}}$, with $\alpha \in (0,1)$, see (5.2.14) for the definition of regularly varying functions, then $\Psi \in \overline{\mathcal{N}}\left(\frac{\pi}{2}\alpha\right)$. Finally, if $\Theta_\pm = \frac{\pi}{2} + \underline{\Theta}_{\phi_-} - \overline{\Theta}_{\phi_+} > 0$, where $\underline{\Theta}_\phi = \varliminf_{b\to\infty} \frac{\int_0^{|b|} \arg\phi(1+iu)du}{|b|}$ and $\overline{\Theta}_\phi = \varlimsup_{b\to\infty} \frac{\int_0^{|b|} \arg\phi(1+iu)du}{|b|}$, then $\Psi \in \overline{\mathcal{N}}(\Theta_\pm)$.*

This theorem is proved in Section 5.4.2.

Remark 5.2.6. *It is known from [34, Chapter V, (5.3.11)] when $\Psi(0) = 0$ and Proposition 5.7.2 in generality that in (5.2.16) $\upsilon_-(0^+) = \int_0^\infty u_+(y)\Pi_-(dy)$, where u_+ is the potential density discussed prior to Proposition 5.4.3. This formula also shows that in this case $\upsilon_-(0^+) \in [0,\infty)$ and thus $\mathtt{N}_\Psi \in (0,\infty)$.*

5.2.2 Exponential functional of Lévy processes

We introduce the subclasses of $\overline{\mathcal{N}}$

$$\mathcal{N} = \left\{\Psi \in \overline{\mathcal{N}} : \ \Psi(z) = -\phi_+(-z)\phi_-(z), z \in i\mathbb{R}, \text{ with } \phi_-(0) > 0\right\} \tag{5.2.17}$$

and

$$\mathcal{N}_\dagger = \left\{\Psi \in \overline{\mathcal{N}} : \ \Psi(0) = -\phi_+(0)\phi_-(0) = -q < 0\right\} \subseteq \mathcal{N}. \tag{5.2.18}$$

We note that

$$I_\Psi = \int_0^{\mathbf{e_q}} e^{-\xi_s} ds < \infty \text{ a.s.} \quad \Longleftrightarrow \quad \Psi \in \mathcal{N} \Longleftrightarrow \quad \phi_-(0) > 0, \tag{5.2.19}$$

which is evident when $q = -\Psi(0) > 0$ and is due to the strong law of large numbers when $q = 0$. The latter includes the case $\mathbb{E}\left[\xi_1 \mathbb{I}_{\{\xi_1>0\}}\right] = \mathbb{E}\left[-\xi_1 \mathbb{I}_{\{\xi_1<0\}}\right] = \infty$ but yet a.s. $\lim_{t\to\infty} \xi_t = \infty$. For an analytical criterion for the validity of the latter there is the celebrated Erickson's test, see [34, Section 6.7]. Let us write for any $x \geq 0$,

$$F_\Psi(x) = \mathbb{P}(I_\Psi \leq x).$$

From [16], we know that the law of I_Ψ is absolute continuous with a density denoted by f_Ψ, i.e. $F^{(1)}_\Psi(x) = f_\Psi(x)$ a.e.. Introduce the Mellin transform of the positive random variable I_Ψ denoted formally, for some $z \in \mathbb{C}$, as follows

$$\mathcal{M}_{I_\Psi}(z) = \mathbb{E}\left[I_\Psi^{z-1}\right] = \int_0^\infty x^{z-1} f_\Psi(x) dx. \tag{5.2.20}$$

We also use the ceiling function $\lceil . \rceil : [0,\infty) \mapsto \mathbb{N}$, that is $\lceil x \rceil = \min\{n \in \mathbb{N} : \ n > x\}$.

5.2.2.1 Regularity, analyticity and representations of the density and its successive derivatives

We start our results on the exponential functional of Lévy processes by providing a result that can be regarded as a corollary to Theorem 5.2.5.

Theorem 5.2.7. *Let* $\Psi \in \mathcal{N}$.

1. *We have*

$$\mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z) = \phi_-(0)\frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) \in \mathtt{A}_{\left(\mathfrak{a}_+\mathbb{I}_{\{\overline{\mathfrak{a}}_+=0\}},\, 1-\overline{\mathfrak{a}}_-\right)} \cap \mathtt{M}_{(\mathfrak{a}_+,\, 1-\mathfrak{a}_-)} \tag{5.2.21}$$

and $\mathcal{M}_{I_\Psi}$ *satisfies the recurrence relation* (5.1.1) *at least on* $i\mathbb{R} \setminus (\mathcal{Z}_0(\Psi) \cup \{0\})$.

2. *If* $\phi_- \equiv 1$ *then* $\operatorname{Supp} I_\Psi = \left[0, \frac{1}{\mathtt{d}_+}\right]$*, unless* $\phi_+(z) = \mathtt{d}_+ z$*,* $\mathtt{d}_+ \in (0,\infty)$*, in which case* $\operatorname{Supp} I_\Psi = \left\{\frac{1}{\mathtt{d}_+}\right\}$ *and if* $\phi_- \not\equiv 1$ *and* $\phi_+(z) = z$ *then* $\operatorname{Supp} I_\Psi = \left[\frac{1}{\phi_-(\infty)}, \infty\right]$*, where we use the convention that* $\frac{1}{\infty} = 0$*. In all other cases* $\operatorname{Supp} I_\Psi = [0,\infty]$.

3. $F_\Psi \in \mathrm{C}_0^{\lceil \mathtt{N}_\Psi \rceil - 1}(\mathbb{R}^+)$ *and if* $\mathtt{N}_\Psi > 1$ *(resp.* $\mathtt{N}_\Psi > \frac{1}{2}$*) for any* $n = 0, \dots, \lceil \mathtt{N}_\Psi \rceil - 2$ *and* $a \in \left(\mathfrak{a}_+ \mathbb{I}_{\{\overline{\mathfrak{a}}_+ = 0\}}, 1 - \overline{\mathfrak{a}}_-\right)$,

$$f_\Psi^{(n)}(x) = (-1)^n \frac{\phi_-(0)}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z-n} \frac{\Gamma(z+1)}{\Gamma(z+1-n)} \mathcal{M}_\Psi(z) dz, \tag{5.2.22}$$

where the integral is absolutely convergent for any $x > 0$ *(resp. is defined in the* L^2*-sense, as in the book of Titchmarsh [95]).*

4. *Let* $\Psi \in \mathcal{N}_\dagger$*, i.e.* $\Psi(0) = -q < 0$ *and let* $\mathtt{N}_+ = |\mathfrak{u}_+| \, \mathbb{I}_{\{|\mathfrak{u}_+| \in \mathbb{N}\}} + (\lceil |\mathfrak{a}_+| + 1 \rceil) \, \mathbb{I}_{\{|\mathfrak{u}_+| \notin \mathbb{N}\}}$*. Then, we have, for any* $0 \leq n < \mathtt{N}_\Psi$*, any* $\mathbb{N} \ni M < \mathtt{N}_+$*,* $a \in ((-M-1) \vee (\mathfrak{a}_+ - 1), -M)$ *and* $x > 0$*,*

$$F_\Psi^{(n)}(x) = q \sum_{k=1 \vee n}^{M} \frac{W_\Psi(k-1)}{(k-n)!} x^{k-n} + (-1)^{n+1} \frac{\phi_-(0)}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z} \frac{\Gamma(z)}{\Gamma(z+1-n)} \mathcal{M}_\Psi(z+1) dz, \tag{5.2.23}$$

where, by analogy to the notation above, we have set $W_\Psi(k-1) = \prod_{j=1}^{k-1} \Psi(j)$*, and by convention* $\prod_1^0 = 1$ *and the sum vanishes if* $1 \vee n > M$*.*

5. *If* $\Psi \in \overline{\mathcal{N}}(\Theta) \cap \mathcal{N}$*,* $\Theta \in (0, \pi]$*, then* f_Ψ *is in fact even holomorphic in the sector* $\mathbb{C}(\Theta) = \{z \in \mathbb{C} : |\arg z| < \Theta\}$*.*

This theorem is proved in Section 5.5.

Remark 5.2.8. *Item* (3) *confirms the conjecture that* $f_\Psi \in \mathtt{C}_0^\infty(\mathbb{R}^+)$ *if there is an infinite activity in the underlying Lévy process, that is, when either* $\sigma^2 > 0$ *and/or* $\int_{-\infty}^{\infty} \Pi(dy) = \infty$ *in* (5.2.1)*. Indeed, from Theorem 5.2.5*(1)*, under each of these conditions in any case,* $\Psi \in \overline{\mathcal{N}}_\infty \cap \mathcal{N} = \mathcal{N}_\infty$*. The surprising fact is that* $\Psi \in \mathcal{N}_\infty$ *and hence* $f_\Psi \in \mathtt{C}_0^\infty(\mathbb{R}^+)$ *even when the possibly killed underlying Lévy process is a pure compound Poisson process or it is a compound Poisson process with a negative drift, that is* $\phi_- \in \mathcal{B}_P$*, whereas if* $\phi_+ \in \mathcal{B}_P$ *then we only know that* $\Psi \in \mathcal{N}_{\mathtt{N}_\Psi}$*,* $\mathtt{N}_\Psi < \infty$*. Finally, if* $\phi_- \equiv 1$*,* $\phi_+(z) = q + z$*,* $q > 0$*, then* $\xi_t = t$ *is killed at rate* q *and* $F_\Psi(x) = 1 - (1-x)^q$*,* $x \in (0,1)$*, which since* $\mathtt{N}_\Psi = q$*, see* (5.2.16)*, confirms that the claim* $F_\Psi \in \mathrm{C}_0^{\lceil \mathtt{N}_\Psi \rceil - 1}(\mathbb{R}^+)$ *is sharp unless* $q \in \mathbb{N}$*.*

Remark 5.2.9. *Let* $\phi_+(z) = z \in \mathcal{B}_P$ *and* $\phi_-(0) > 0$ *so that* $\Psi(z) = z\phi_-(z) \in \mathcal{N}$*. Then* $I_\Psi = \int_0^\infty e^{-\xi_t} dt$ *is a self-decomposable random variable, see [83, Chapter 5]. The rate of decay of the Fourier transform of* I_Ψ *has been computed as* λ *in the notation of [91]. One can check that* $\lambda = \mathtt{N}_\Psi$*. For all exponential functionals this work establishes and evaluates in terms of* $\mathtt{N}_\Psi$ *the properties pertaining to the specific case of self-decomposable random variables* I_Ψ *related to* $\Psi(z) = z\phi_-(z) \in \mathcal{N}$ *modulo to the discussion of whether and how precisely the smoothness of* F_Ψ *breaks down at the* $\lceil \mathtt{N}_\Psi \rceil$*-derivative.*

Remark 5.2.10. *Note that if* $\overline{\mathfrak{a}}_+ = 0$*,* (5.2.21) *combined with* $\mathcal{M}_{I_\Psi}(z) = \mathbb{E}\left[I_\Psi^{z-1}\right]$*, see* (5.2.20)*, and Theorem5.3.1*(4) *evaluates all negative moments* I_Ψ *up to order* $-1 + \mathfrak{a}_+$*. This recovers and extends the computation of [20, Proposition 2], which deals with the entire negative moments of* I_Ψ *when* $\mathbb{E}[\xi_1] = \Psi'(0^+) \in (0, \infty)$*.*

Remark 5.2.11. *If $\overline{\mathtt{a}}_{-} < 0$ then it can be verified from* (5.2.21) *that $\mathcal{M}_{I_\Psi}$ is the unique solution on the domain $\mathbb{C}_{(0,-\overline{\mathtt{a}}_{-})}$ of the functional equation* (5.1.1)*, derived for the case $\Psi \in \mathcal{N} \setminus \mathcal{N}_\dagger$, that is $q = 0$, in [62] and when $\Psi \in \mathcal{N}$ in [4]. If $q = 0$ and $\Psi'(0^+) \in (0,\infty)$ then according to Theorem 5.2.1, $\mathcal{M}_{I_\Psi} \in \mathtt{A}_{[0,1]}$ and again* (5.2.21) *is a solution to* (5.1.1) *which in this case holds only on $i\mathbb{R}$. Many theoretical papers on exponential functionals of Lévy processes depend on* (5.1.1) *for $\mathcal{M}_{I_\Psi}$ to hold on a strip and on the availability of an explicit form of Ψ, e.g. [45, 52], to derive an expression for $\mathcal{M}_{I_\Psi}$. Here,* (5.2.21) *provides an immediate representation of $\mathcal{M}_{I_\Psi}$ in terms of Bernstein-gamma functions.*

Item (4) of Theorem 5.2.7 can be refined as follows.

Corollary 5.2.12. *Let $\Psi \in \mathcal{N}_\dagger$, $|\mathfrak{a}_+| = \infty$ and $-\mathfrak{u}_+ \notin \mathbb{N}$. Then*

$$F(x) \approx q \sum_{k=1}^{\infty} \frac{W_\Psi(k-1)}{k!} x^k$$

is the asymptotic expansion of F_Ψ at zero, that is, for any $N \in \mathbb{N}$, $F(x) - q\sum_{k=1}^{N} \frac{W_\Psi(k-1)}{k!} x^k \overset{0}{=} \mathrm{o}\left(x^N\right)$. The asymptotic expansion cannot be a convergent series for any $x > 0$ unless $\phi_+ \equiv 1$ or $\phi_+(z) = \phi_+(0) + \mathtt{d}_+$ and $\phi_-(\infty) < \infty$ and then in the first case it converges for $x < \frac{1}{\mathtt{d}_-}$ and diverges for $x > \frac{1}{\mathtt{d}_-}$, and in the second it converges for $x < \frac{1}{\mathtt{d}_+\phi_-(\infty)}$ and diverges for $x > \frac{1}{\mathtt{d}_+\phi_-(\infty)}$.

Remark 5.2.13. *When $\phi_+ \equiv 1$ it is well-known from [78, Corollary 1.3] and implicitly from [69] that the asymptotic expansion is convergent if and only if $x < \frac{1}{\mathtt{d}_+}$.*

5.2.2.2 Large asymptotic behaviour of the distribution and its successive derivatives.

We proceed by discussing the large asymptotic of the law and the density of I_Ψ in the general setting. For this purpose we introduce the well-known in the literature *non-lattice subclass* of $\mathcal{B}$ and $\mathcal{N}$. Let, for some $a \in \mathbb{R}$,

$$\mathcal{Z}_a(\Psi) = \{z \in \mathbb{C}_a : \Psi(z) = 0\} = \{z \in \mathbb{C}_a : \Psi(\overline{z}) = 0\}, \tag{5.2.24}$$

where the latter identity follows easily from $\overline{\Psi(z)} = \Psi(\overline{z})$, see (5.2.1), and clearly for $a = 0$ $\mathcal{Z}_0(\Psi) = \{z \in i\mathbb{R} : \Psi(z) = 0\} = \{z \in i\mathbb{R} : \Psi(-z) = 0\}$. Then for any $\Psi \in \mathcal{N}$, $\phi \in \mathcal{B}$ set $\Psi^\sharp(z) = \Psi(z) - \Psi(0) \in \mathcal{N} \setminus \mathcal{N}_\dagger$ and $\phi^\sharp(z) = \phi(z) - \phi(0) \in \mathcal{B} \setminus \mathcal{B}_\dagger$, where $\mathcal{B}_\dagger = \{\phi \in \mathcal{B} : \phi(0) > 0\}$. Then the *non-lattice subclass* is defined as follows

$$\Psi \in \mathcal{N}_{\mathcal{Z}} \iff \mathcal{Z}_0\left(\Psi^\sharp\right) = \{0\}$$

with identical definition and meaning for $\mathcal{B}_{\mathcal{Z}}$. We note that we use the terminology *non-lattice class* since the underlying Lévy processes does not live on a sublattice of $\mathbb{R}$, e.g. if $\Psi \in \mathcal{N}_{\mathcal{Z}}$ then the support of the distribution of the underlying Lévy process ξ is either $\mathbb{R}$

or $\mathbb{R}^+$. It is easily seen that $\Psi \in \mathcal{N}_{\mathcal{Z}} \iff \phi_- \in \mathcal{B}_{\mathcal{Z}}$ and $\phi_+ \in \mathcal{B}_{\mathcal{Z}}$, see (5.1.2). Next, if $\mathfrak{u}_- \in (-\infty, 0)$, see (5.2.3), we introduce the *weak non-lattice* class as follows

$$\begin{aligned} \Psi \in \mathcal{N}_{\mathcal{W}} &\iff \mathfrak{u}_- \in (-\infty,0) \text{ and } \exists\, k \in \mathbb{N} \text{ such that } \varliminf_{|b|\to\infty} |b|^k \left|\Psi\left(\mathfrak{u}_- + ib\right)\right| > 0 \\ &\iff \mathfrak{u}_- \in (-\infty,0) \text{ and } \exists\, k \in \mathbb{N} \text{ such that } \varliminf_{|b|\to\infty} |b|^k \left|\phi_-\left(\mathfrak{u}_- + ib\right)\right| > 0. \end{aligned} \quad (5.2.25)$$

We note that the *weak non-lattice* class seems not to have been introduced in the literature yet. Clearly, $\mathcal{N} \setminus \mathcal{N}_{\mathcal{Z}} \subseteq \mathcal{N} \setminus \mathcal{N}_{\mathcal{W}}$ since $\Psi^\sharp \in \mathcal{N} \setminus \mathcal{N}_{\mathcal{Z}}$ vanishes on $\left\{k \in \mathbb{N} : \frac{2\pi i}{h}\right\}$, where $h > 0$ is the span of the lattice that supports the distribution of the unkilled Lévy process $\xi^\sharp$. We phrase our first main result which virtually encompasses all exponential functionals. We write throughout $\overline{F}_\Psi(x) = 1 - F_\Psi(x)$ for the tail of I_Ψ.

Theorem 5.2.14. *Let $\Psi \in \mathcal{N}$.*

1. *If $|\overline{\mathfrak{a}}_-| < \infty$ (resp. $|\overline{\mathfrak{a}}_-| = \infty$, that is $-\Psi(-z) = \phi_+(z) \in \mathcal{B}$), then for any $\underline{d} < |\overline{\mathfrak{a}}_-| < \overline{d}$ (resp. $\underline{d} < \infty$), we have that*

$$\varlimsup_{x\to\infty} x^{\underline{d}}\, \overline{F}_\Psi(x) = 0, \quad (5.2.26)$$

$$\varliminf_{x\to\infty} x^{\overline{d}}\, \overline{F}_\Psi(x) = \infty. \quad (5.2.27)$$

Therefore, in all cases,

$$\lim_{x\to\infty} \frac{\log \overline{F}_\Psi(x)}{\log x} = \overline{\mathfrak{a}}_-. \quad (5.2.28)$$

2. *If in addition $\Psi \in \mathcal{N}_{\mathcal{Z}}$, $\overline{\mathfrak{a}}_- = \mathfrak{u}_- < 0$ and $|\Psi'(\mathfrak{u}_-^+)| < \infty$ then*

$$\overline{F}_\Psi(x) \overset{\infty}{\sim} \frac{\phi_-(0)\Gamma\left(-\mathfrak{u}_-\right) W_{\phi_-}\left(1+\mathfrak{u}_-\right)}{\phi_-'\left(\mathfrak{u}_-^+\right) W_{\phi_+}\left(1-\mathfrak{u}_-\right)} x^{\mathfrak{u}_-}. \quad (5.2.29)$$

Moreover, if $\Psi \in \mathcal{N}_\infty \cap \mathcal{N}_{\mathcal{W}}$ (resp. $\Psi \in \mathcal{N}_{\mathtt{N}_\Psi}$, $\mathtt{N}_\Psi < \infty$) then for every $n \in \mathbb{N}$ (resp. $n \le \lceil \mathtt{N}_\Psi \rceil - 2$)

$$f_\Psi^{(n)}(x) \overset{\infty}{\sim} (-1)^n \frac{\phi_-(0)\Gamma\left(n+1-\mathfrak{u}_-\right) W_{\phi_-}\left(1+\mathfrak{u}_-\right)}{\phi_-'\left(\mathfrak{u}_-^+\right) W_{\phi_+}\left(1-\mathfrak{u}_-\right)} x^{-n-1+\mathfrak{u}_-}. \quad (5.2.30)$$

This theorem is proved in Section 5.5.5.

Remark 5.2.15. *When $\Psi \in \mathcal{N}_{\mathcal{Z}}$, $\overline{\mathfrak{a}}_- = \mathfrak{u}_- < 0$ and $|\Psi'(\mathfrak{u}_-^+)| < \infty$, whose collective validity is referred to as the Cramer condition for the underlying Lévy process, it is well-known that $\lim_{x\to\infty} x^{-\mathfrak{u}_-}\overline{F}_\Psi(x) = C > 0$, see [88, Lemma 4], that is* (5.2.29) *holds. Here, we evaluate explicitly C too. We emphasize that the ability to refine the tail result to densities at the expense of the minute requirement $\Psi \in \mathcal{N}_{\mathcal{W}}$ comes from the representation* (5.2.21)*, which* *measurably and almost invariably ensures fast decay of $|\mathcal{M}_{I_\Psi}|$ along imaginary lines, see Theorem 5.2.5*(5.2.16)*. This decay given by* (5.2.16) *can be extended to the line $\mathbb{C}_{1-\mathtt{u}_-}$ only when $\Psi \in \mathcal{N}_{\mathcal{W}}$. We stress that the latter requirement is not needed for $\overline{F}_\Psi(x)$ since it is non-increasing and one can apply the general and powerful Wiener-Ikehara theorem. Note that a good decay is never available for the quantity $\mathbb{E}\left[e^{-z\underline{\xi_\infty}}\right] = \phi_-(0)/\phi_-(-z)$, $z \in \mathbb{C}_0$, where $\underline{\xi_\infty} = \inf_{s\geq 0}\xi_s$, and therefore for the tail $\mathbb{P}\left(\underline{\xi_\infty} > x\right)$, since then $|\phi_-(z)| \overset{\infty}{\approx} \mathtt{d}_-|z| + \mathrm{o}(|z|)$, see Proposition 5.3.13*(3)*.*

Remark 5.2.16. *The claim of item* (1) *is general. It is again a result of the decay $|\mathcal{M}_{I_\Psi}|$ along complex lines and a monotone probabilistic approximation with Lévy processes that fall within the setting of item* (2)*. Relation* (5.2.28) *is a strengthening of [4, Lemma 2] in that it quantifies precisely and estimates from below by $-\infty$ the rate of the power decay of $\overline{F}_\Psi(x)$ as $x \to \infty$. Since I_Ψ is also a perpetuity with thin tails, see [41], we provide very precise estimates for the tail behaviour of this class of perpetuities.*

Remark 5.2.17. *Note that in item* (1) *the case $-\Psi(-z) = \phi_+(z) \in \mathcal{B}$, see* (5.1.2)*, corresponds to the Lévy process ξ behind Ψ being a possibly killed subordinator. This assumption is lacking in item* (2) *because the existence of $-\mathtt{u}_- \in (0,\infty)$ precludes the case $-\Psi(-z) = \phi_+(z)$.*

Remark 5.2.18. *Since the supremum of a stable Lévy process can be related to a specific exponential functional for which* (5.2.30) *is valid then our result recovers the mains statements about the asymptotic of the density of the supremum of a Lévy process and its derivatives that appear in [35, 49].*

Under specific conditions, see [76, 78] for the class of (possibly killed) spectrally negative Lévy processes and [49, 51] for some special instances, the density f_Ψ can be expanded into a converging series. This is achieved by a subtle pushing to infinity of the contour of the Mellin inversion when the analytic extension $\mathcal{M}_{I_\Psi} \in \mathtt{M}_{(0,\infty)}$ is available. Our Theorem 5.2.5, which ensures a priori knowledge for the decay of $|\mathcal{M}_{I_\Psi}(z)|$, allows for various asymptotic expansions or evaluation of the speed of convergence of f_Ψ at infinity as long as $\Psi \in \mathcal{N}_{\mathcal{W}}$ and $\mathcal{M}_{I_\Psi}$ extends analytically to the right of $\mathbb{C}_{1-\mathtt{u}_-}$. However, for sake of generality, we leave aside the study of some additional examples.

5.2.2.3 Small asymptotic behaviour of the distribution and its successive derivatives.

We proceed with the small asymptotic behaviour.

Theorem 5.2.19. *Let $\Psi \in \mathcal{N}$ then*

$$\lim_{x\to 0}\frac{F_\Psi(x)}{x} = -\Psi(0) \tag{5.2.31}$$

with $f_\Psi(0^+) = \Psi(0)$. Hence, if f_Ψ is continuous at zero or $\Psi \in \mathcal{N}_{\mathtt{N}_\Psi}$ with $\mathtt{N}_\Psi > 1$, we have that

$$\lim_{x\to 0} f_\Psi(x) = f_\Psi(0) = -\Psi(0). \tag{5.2.32}$$

This theorem is proved in Section 5.5.6.

Remark 5.2.20. *We stress that* (5.2.31) *when* $\Psi(0) < 0$ *has appeared in [4, Theorem 7(i)] but its proof is essentially based on the non-trivial [69, Theorem 2.5]. Our proof is analytic and confirms our intuition that the function* W_ϕ *on which all quantities are based is good enough to be thoroughly investigated from analytical perspective and yield results for the random variables whose Mellin transforms it represents. We note that* (5.2.32) *is a new result for general exponential functionals* I_Ψ*. For exponential functionals based on an increasing Lévy process it has already appeared in [69, Theorem 2.5] without any assumptions. This can be deducted from* (5.2.31) *if we additionally know that* f_Ψ *is continuous at zero. When* $\mathtt{N}_\Psi \leq 1$ *we know from Theorem 5.2.5*(5.2.16) *that* $\bar{\mu}_+(0) \leq \mathtt{d}_+ < \infty$ *and then it is a trivial exercise to prove from the functional equation for* f_Ψ *in [69, Theorem 2.4], that* f_Ψ *is continuous at zero and henceforth* (5.2.31) *yields* $f_\Psi(0) = -\Psi(0)$.

Remark 5.2.21. *Denote by* $\mathcal{N}_- = \{\Psi \in \mathcal{N} : \phi_+(z) = \phi_+(0) + \mathtt{d}_+ z\}$ *the class of the so-called spectrally negative Lévy processes, that is Lévy processes that do not jump upwards and assume that* $\phi_+(0) = 0$*. Then* $\Psi(z) = z\phi_-(z)$ *and if* $\phi_-(\infty) = \infty$ *the small-time asymptotic of* f_Ψ *and its derivatives, according to [83] reads off with* φ_- *such that* $\phi_-(\varphi_-(u)) = u$ *as follows*

$$f_\Psi^{(n)}(x) \stackrel{0}{\sim} \frac{C_{\phi_-}\phi_-(0)}{\sqrt{2\pi}} \frac{\varphi_-^n\left(\frac{1}{x}\right)}{x^n} \sqrt{\varphi_-'\left(\frac{1}{x}\right)} e^{-\int_{\phi_-(0)}^{\frac{1}{x}} \varphi_-(r)\frac{dr}{r}}. \tag{5.2.33}$$

If $\phi_-(\infty) < \infty$ *then according to Theorem 5.2.7*(2) *we have that* $\operatorname{Supp} f_\Psi \in \left[\frac{1}{\phi_-(\infty)}, \infty\right)$. *Comparing the small asymptotic behaviour of* f_Ψ *in* (5.2.33) *with the one in* (5.2.32) *reveals that a simple killing of the underlying Lévy process leads to a dramatic change.*

5.2.2.4 Finiteness of negative moments and asymptotic behaviour for the exponential functionals on a finite time horizon.

For any $\Psi \in \overline{\mathcal{N}}$ and $t \geq 0$ let

$$I_\Psi(t) = \int_0^t e^{-\xi_s} ds.$$

We have the following claim which furnishes necessary and sufficient conditions for finiteness of negative moments of $I_\Psi(t)$.

Theorem 5.2.22. *Let* $\Psi \in \overline{\mathcal{N}} \setminus \mathcal{N}_\dagger$*. Then, for any* $t > 0$,

$$\mathbb{E}\left[I_\Psi^{-a}(t)\right] < \infty \quad \Longleftarrow \quad a \in (0, 1 - \mathfrak{a}_+) \tag{5.2.34}$$

$$\mathbb{E}\left[I_\Psi^{-1+\mathfrak{a}_+}(t)\right] < \infty \quad \Longleftrightarrow \quad |\Psi(-\mathfrak{a}_+)| < \infty \Longleftrightarrow |\phi_+(\mathfrak{a}_+)| < \infty \tag{5.2.35}$$

$$\mathbb{E}\left[I_\Psi^{-1}(t)\right] < \infty \quad \Longleftrightarrow \quad \left|\Psi'(0^+)\right| < \infty \tag{5.2.36}$$

$$\mathbb{E}\left[I_\Psi^{-a}(t)\right] = \infty \quad \Longleftarrow \quad a > 1 - \mathfrak{a}_+. \tag{5.2.37}$$

Finally, we have that for any $a \in (0, 1 - \mathfrak{a}_+)$

$$\lim_{t \to 0} t^a \mathbb{E}\left[I_\Psi^{-a}(t)\right] = 1. \tag{5.2.38}$$

This theorem is proved in Section 5.5.7

Remark 5.2.23. *Some results as to the finiteness of* $\mathbb{E}\left[I_{\Psi}^{-a}(t)\right]$ *appear in the recent preprint [67] but the authors limit their attention on the range* $a \in (0, -\mathfrak{a}_+)$ *which is substantially easier to prove via the relation* (5.5.32).

Next, we consider the case

$$\mathcal{N}^c = \left\{\Psi \in \overline{\mathcal{N}} : \ \phi_-(0) = 0\right\} = \overline{\mathcal{N}} \setminus \mathcal{N}, \tag{5.2.39}$$

that is all conservative Lévy processes such that $\underline{\lim}_{t\to\infty} \xi_t = -\infty$ a.s. and thus $\mathcal{N}^c \cap \mathcal{N}_\dagger = \emptyset$. In the setting of the next claim for any $\Psi \in \mathcal{N}^c$ we use superscript r for $\Psi^r(z) = \Psi(z) - r = -\phi_+^r(-z)\phi_-^r(z)$ and all related quantities. Recall that RV_α stands for the class of regularly varying functions of index $\alpha \in \mathbb{R}$ at zero. Then the following result elucidates the behaviour of the measures $\mathbb{P}\left(I_\Psi(t) \in dx\right)$ as $t \to \infty$ in quite a general framework.

Theorem 5.2.24. *Let* $\Psi \in \mathcal{N}^c$.

1. *Then for any* $a \in (\mathfrak{a}_+ - 1, 0)$ *such that* $-a \notin \mathbb{N}$ *we get for any* $x > 0$, *any* $\mathbb{N} \ni n < 1 - \mathfrak{a}_+$ *and any* $a \in (n, \max\{n+1, 1-\mathfrak{a}_+\})$

$$\frac{e-1}{e}\overline{\lim_{t\to\infty}}\frac{\mathbb{P}\left(I_\Psi(t) \le x\right)}{\kappa_-\left(\frac{1}{t}\right)} \le \phi_+(0)\sum_{k=1}^{n\wedge \mathtt{N}_+}\frac{|W_\Psi(k-1)|}{k!}x^k + \frac{x^a}{2\pi}\int_{-\infty}^{\infty}\frac{|\mathcal{M}_\Psi(-a+1+ib)|}{\sqrt{a^2+b^2}}db, \tag{5.2.40}$$

where $\kappa_-(\frac{1}{t}) = \phi_-^{\frac{1}{t}}(0)$, $\mathtt{N}_+ = |\mathfrak{u}_+|\,\mathbb{I}_{\{|\mathfrak{u}_+|\in\mathbb{N}\}} + (\lceil|\mathfrak{a}_+| + 1\rceil)\,\mathbb{I}_{\{|\mathfrak{u}_+|\notin\mathbb{N}\}}$, $W_\Psi(k-1) = \prod_{j=1}^{k-1}\Psi(j)$ *and we use the convention* $\sum_1^0 = 0$.

2. *Let now* $-\underline{\lim}_{t\to\infty}\xi_t = \overline{\lim}_{t\to\infty}\xi_t = \infty$ *a.s. or alternatively* $\phi_+(0) = \phi_-(0) = 0$ *and assume also that* $\lim_{t\to\infty}\mathbb{P}(\xi_t < 0) = \rho \in [0,1)$, *that is the celebrated Spitzer's condition holds. Then* $\kappa_-(r) = \phi_-^r(0) \in RV_\rho$ *and for any* $a \in (0, 1-\mathfrak{a}_+)$ *and any* $f \in \mathtt{C}_b(\mathbb{R}^+)$

$$\lim_{t\to\infty}\frac{\mathbb{E}\left[I_\Psi^{-a}(t) f\left(I_\Psi(t)\right)\right]}{\kappa_-\left(\frac{1}{t}\right)} = \int_0^\infty f(x)\vartheta_a(dx), \tag{5.2.41}$$

where ϑ_a *is a finite positive measure on* $(0,\infty)$ *such that for any* $c \in (a-1+\mathfrak{a}_+, 0)$ *and* $x > 0$

$$\vartheta_a(0,x) = -\frac{x^{-c}}{2\pi\Gamma(1-\rho)}\int_{-\infty}^{\infty}x^{-ib}\frac{\mathcal{M}_\Psi(c+1-a+ib)}{c+ib}db \tag{5.2.42}$$

and

$$\vartheta_a\left(\mathbb{R}^+\right) = \frac{1}{\Gamma(1-\rho)}\frac{\Gamma(1-a)}{W_{\phi_+}(1-a)}W_{\phi_-}(a). \tag{5.2.43}$$

This theorem is proved in Section 5.5.8.

Remark 5.2.25. *Note that if* $\mathbb{E}[\xi_1] = 0$, $\mathbb{E}[\xi_1^2] < \infty$ *then we have that* $\kappa_-(r) \overset{0}{\sim} Cr^{\frac{1}{2}}$ *and* C *can be elucidated to a degree from the Fristed's formula, [11, Chapter VI], which evaluates* $\kappa_-(r)$. *Therefore, if* $f(x) = x^{-a} f_1(x)$ *with* $a \in (0, 1-\mathfrak{a}_+)$ *and* $f_1 \in \mathtt{C}_b(\mathbb{R}^+)$ *then* $\lim_{t\to\infty} C\sqrt{t}\mathbb{E}[f(I_t)] = \int_0^\infty f_1(x)\vartheta_a(dx)$. *This result has been at the core of two recent preprints which deal with the large temporal asymptotic behaviour of extinction and explosion probabilities of continuous state branching processes in Lévy random environment, see [59, 67]. However, there the authors need to impose some stringent restrictions such as* f *being ultimately non-increasing and of a specific form, and the Lévy process* ξ *to have some two-sided exponential moments. Thus, our Theorem 5.2.24 can significantly extend the aforementioned results of [59, 67] and furnish new ones when* $\mathbb{E}[\xi_1^2] = \infty$ *and* $\lim_{t\to\infty} \mathbb{P}(\xi_t < 0) = \rho \in [0, 1)$.

Remark 5.2.26. *The main reason and necessity behind the imposition of additional conditions in [59, 67] seem to be due to the fact that the results are obtained through discretization of* ξ *and then either by reduction to more general results on specific random sums, see [67], or through reduction to a similar problem for random walks, see [59]. We refer to [1, 48] for the results in the random walk scenario.*

5.2.3 Intertwining relations of self-similar semigroups and factorization of laws

By Mellin identification, we obtain as a straightforward consequence of the representation (5.2.21), the following probabilistic factorizations of the distribution of the exponential functional.

Theorem 5.2.27. *For any* $\Psi \in \mathcal{N}$, *the following multiplicative Wiener-Hopf factorizations of the exponential functional hold*

$$I_\Psi \overset{d}{=} I_{\phi_+} \times X_{\phi_-} \overset{d}{=} \bigotimes_{k=0}^{\infty} \left(C_k \mathfrak{B}_k X_\Psi \times \mathfrak{B}_{-k} Y_\Psi\right), \tag{5.2.44}$$

where $\times$ *stands for the product of independent random variables. The law of the positive variables* X_Ψ, Y_Ψ *are given by*

$$\begin{aligned} \mathbb{P}(X_\Psi \in dx) &= \frac{1}{\phi_+(1)}\left(\bar{\mu}_+(-\ln x)dx + \phi_+(0)dx + \mathtt{d}_+\delta_1(dx)\right),\ x \in (0,1) \\ \mathbb{P}(Y_\Psi \in dx) &= \phi_-(0)\Upsilon_-(dx),\ x > 1 \end{aligned} \tag{5.2.45}$$

where $\Upsilon_-(dv) = U_-(d\ln(v))$, $v > 1$ *is the image of the potential measure* U_- *by the mapping* $y \mapsto \ln y$,

$$C_0 = e^{\gamma_{\phi_+} + \gamma_{\phi_-} - \gamma + 1 - \frac{\phi_+'(1)}{\phi_+(1)}}, C_k = e^{\frac{1}{k+1} - \frac{\phi_+'(k+1)}{\phi_+(k+1)} - \frac{\phi_-'(k)}{\phi_-(k)}}, k = 1, 2, \dots,$$

where γ is the Euler-Mascheroni constant, and for integer $k, \mathfrak{B}_k X$ is the variable defined by

$$\mathbb{E}\left[f(\mathfrak{B}_k X)\right] = \frac{\mathbb{E}\left[X^k f(X)\right]}{\mathbb{E}[X^k]}.$$

Remark 5.2.28. *Note that the first factorization in* (5.2.44) *is proved in [68, 78] under the assumptions $\Pi(dx)\mathbb{I}_{\{x>0\}} = \pi_+(x)dx$ and π_+ is non-decreasing on $\mathbb{R}^+$ with the stronger relation $X_{\phi_-} = I_\psi$ with $\psi(z) = z\phi_-(z) \in \mathcal{N}_-$. It has been announced in generality in [79] and building on it in [4] the authors derive a new three term factorization of I_Ψ. The second factorization of* (5.2.44) *is new in such generality. When $\Psi(z) = -\phi_+(-z) \in \mathcal{B}$ that is ξ is a subordinator then* (5.2.44) *is contained in [2, Theorem 3]. For the class of meromorphic Lévy processes, I_Ψ has been factorized in an infinite product of independent Beta random variables, see e.g. [45].*

Next, we discuss some immediate results for the positive self-similar Markov process whose semigroups we call for brevity the positive self-similar semigroups and denote by $K^\Psi = \left(K_t^\Psi\right)_{t\geq 0}$. The dependence on $\Psi \in \overline{\mathcal{N}}$ is due to the celebrated Lamperti transformation which identifies a bijection between the positive self-similar semigroups and $\overline{\mathcal{N}}$, see [56]. We recall that for some $\alpha > 0$, any $f \in \mathtt{C}_0([0,\infty))$ and any $x, c > 0$, $K_t^\Psi f(cx) = K^\Psi_{c^{-\alpha}t} d_c f(x)$, where $d_c f(x) = f(cx)$ is the dilatation operator. Without loss of generality we consider $\alpha = 1$ and we introduce the set $\mathcal{N}_m = \{\Psi \in \mathcal{N} : q = 0, \phi_+'(0^+) < \infty, \mathcal{Z}_0(\Psi) = \{0\}\}$. From (5.1.2), (5.2.17) and Theorem 5.3.1(2) it is clear that from probabilistic perspective the class $\mathcal{N}_m$ stands for the conservative Lévy processes that do not live on a lattice which either drift to infinity and possess finite positive mean or oscillate but the ascending ladder height process has a finite mean, that is $\phi_+'(0^+) < \infty$. It is well-known from [17] that $\Psi \in \mathcal{N}_m$ if and only if K^Ψ possesses an entrance law from zero. More specifically, there exists a family of probability measures $\nu^\Psi = \left(\nu_t^\Psi\right)_{t>0}$ such that for any $f \in \mathtt{C}_0([0,\infty))$ and any $t, s > 0$, $\nu^\Psi_{t+s} f = \int_0^\infty f(x)\nu^\Psi_{t+s}(dx) = \int_0^\infty K_s^\Psi f(x)\nu_t^\Psi(dx) = \nu_t^\Psi K_s^\Psi f$. We denote by V_Ψ the random variable whose law is ν_1^Ψ.

Theorem 5.2.29. *Let $\Psi \in \mathcal{N}_m$.*

1. *Then the Mellin transform $\mathcal{M}_{V_\Psi}$ of V_Ψ is the unique solution to the following functional equation with initial condition $\mathcal{M}_{V_\Psi}(1) = 1$*

$$\mathcal{M}_{V_\Psi}(z+1) = \frac{\Psi(z)}{z}\mathcal{M}_{V_\Psi}(z),\ z \in \mathbb{C}_{(\overline{\mathfrak{a}}_-,1)}, \tag{5.2.46}$$

   and admits the representation

$$\mathcal{M}_{V_\Psi}(z) = \frac{1}{\phi_+'(0^+)}\frac{\Gamma(1-z)}{W_{\phi_+}(1-z)}W_{\phi_-}(z), \quad z \in \mathbb{C}_{(\overline{\mathfrak{a}}_-,1)}. \tag{5.2.47}$$

2. *Let in addition $\Pi(dx)\mathbb{I}_{\{x>0\}} = \pi_+(x)dx$, with π_+ non-increasing on $\mathbb{R}^+$, see* (5.2.1)*. Then $\Lambda_{\phi_+} f(x) = \mathbb{E}\left[f\left(xV_{\phi_+}\right)\right]$ is a continuous linear operator from $\mathtt{C}_0([0,\infty))$, endowed*

with the uniform topology, into itself and we have the following intertwining identity on $\mathtt{C}_0([0,\infty))$

$$K_t^{\psi}\Lambda_{\phi_+}f = \Lambda_{\phi_+}K_t^{\Psi}f, \quad t \geq 0, \tag{5.2.48}$$

where $\psi(z) = z\phi_-(z) \in \mathcal{N}$.

This theorem is proved in Section 5.6.2.

Remark 5.2.30. *The literature on intertwining of Markov semigroups is very rich and reveals that it is useful in a variety of contexts, see e.g. Diaconis and Fill [33] in relation with strong uniform times, by Carmona, Petit and Yor [28] in relation to the so-called selfsimilar saw tooth-processes, by Borodin and Corwin [23] in the context of Macdonald processes, by Pal and Shkolnikov [66] for linking diffusion operators, and, by Patie and Simon [84] to relate classical fractional operators. In this direction, it seems that the family of intertwining relations* (5.2.48) *are the first instances involving a Markov processes with two-sided jumps.*

Remark 5.2.31. *More recently, this type of commutation relations have proved to be a natural concept in some new developments of spectral theory. We refer to the work of Miclo [63] where it is shown that the notions of isospectrality and intertwining of some self-adjoint Markov operators are equivalent leading to an alternative view of the work of Bérard [7] on isospectral compact Riemanian manifolds, see also Arendt et al. [3] for similar developments that enable them to provide counterexamples to the famous Kac's problem. Intertwining is also the central idea in the recent works from the authors [80, 83] on the spectral analysis of classes of non-self-adjoint and non-local Markov semigroups. We also emphasize that the intertwining relation* (5.2.48) *and more generally the analytical properties of the solution of the recurrence equation* (5.1.1) *presented in this paper are critical in the spectral theory of the entire class of positive self-similar Markov semigroups developed in [81].*

5.3 The class of Bernstein-Gamma functions

Perhaps the most celebrated special function is the gamma function Γ introduced by Euler in [38]. Amongst its various properties is the fact that it satisfies the recurrence equation

$$\Gamma(z+1) = z\Gamma(z), \quad \Gamma(1) = 1, \tag{5.3.1}$$

valid on $\mathbb{C} \setminus \mathbb{N}^-$. For example, the relation (5.3.1) allows for the derivation of both the Weierstrass product representation of $\Gamma(z)$ and the precise Stirling asymptotic expression for the behaviour of the gamma function as $|z| \to \infty$ which are given respectively by

$$\begin{aligned}\Gamma(z) &= \frac{e^{-\gamma z}}{z}\prod_{k=1}^{\infty}\frac{k}{k+z}e^{\frac{1}{k}z},\\ \Gamma(z) &= \sqrt{2\pi}e^{z\log z - z - \frac{1}{2}\log z}\left(1 + \mathrm{O}\left(\frac{1}{z}\right)\right),\end{aligned} \tag{5.3.2}$$

where $\gamma = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{1}{k} - \ln n\right)$ is the Euler-Mascheroni constant, see e.g. [58]. Recall from (5.2.2) that $\phi \in \mathcal{B}$ if and only if $\phi \not\equiv 0$ and, for $z \in \mathbb{C}_{[0,\infty)}$,

$$\phi(z) = \phi(0) + \mathtt{d}z + \int_0^\infty \left(1 - e^{-zy}\right)\mu(dy) = \phi(0) + \mathtt{d}z + z\int_0^\infty e^{-zy}\bar{\mu}(y)dy, \qquad (5.3.3)$$

$\phi(0) \geq 0$, $\mathtt{d} \geq 0$ and μ is a sigma-finite measure satisfying $\int_0^\infty (1\wedge y)\mu(dy) < \infty$. Also here and hereafter we denote the tail of a measure λ by $\bar{\lambda}(x) = \int_{|y|\geq x}^{\infty}\lambda(dy)$, provided it exists. Due to its importance the class $\mathcal{B}$ has been studied extensively in several monographs and papers, see e.g. [83, 92]. Here, we use it to introduce the class of Bernstein-gamma functions denoted by $\mathcal{W}_{\mathcal{B}}$, which appear in any main result above.

Definition 5.3.1. *We say that $W_\phi \in \mathcal{W}_{\mathcal{B}}$ if and only if for some $\phi \in \mathcal{B}$*

$$W_\phi(z+1) = \phi(z)W_\phi(z), \ \mathtt{Re}(z) > 0; \quad W_\phi(1) = 1; \qquad (5.3.4)$$

and there exists a positive random variable Y_ϕ such that $W_\phi(z+1) = \mathbb{E}\left[Y_\phi^z\right]$, $\mathtt{Re}(z) \geq 0$.

Note that when, in (5.3.3), $\phi(z) = z \in \mathcal{B}$ then W_ϕ boils down to the gamma function with Y_ϕ a standard exponential random variable. This yields to the well known integral representation of the gamma function $\Gamma(z) = \int_0^\infty x^{z-1}e^{-x}dx$ valid on $\mathtt{Re}(z) > 0$. The functions $W_\phi \in \mathcal{W}_{\mathcal{B}}$ have already appeared explicitly, see [2, 46, 79, 83] or implicitly, [21, 62] in the literature. However, with the exception of [83, Chapter 6] we are not aware of other studies that focus on the understanding of W_ϕ as a holomorphic function on the complex half-plane $\mathbb{C}_{(0,\infty)}$. The latter is of significant importance at least for the following reasons. First, the class $\mathcal{W}_{\mathcal{B}}$ arises in the spectral study of Markov semigroups and the quantification of its analytic properties in terms of $\phi \in \mathcal{B}$ virtually opens the door to obtaining explicit information about most of the spectral objects and quantities of interest, see [83]. Then, the class $\mathcal{W}_{\mathcal{B}}$ appears in the full explicit description of $\mathcal{M}_\Psi$ and hence of $\mathcal{M}_{I_\Psi}$, see (5.2.6), and thus the understanding of its analytic properties yields detailed information about the law of those exponential functionals. Also, the class $\mathcal{W}_{\mathcal{B}}$ contains some well-known special functions, e.g. the Barnes-gamma function and the q-gamma function related to the q-calculus, see [12, 30], [83, Remark 6.4], and the derivation of the analytic properties of $\mathcal{W}_{\mathcal{B}}$ in general will render many special computations and efforts to direct application of the results concerning the functions comprising $\mathcal{W}_{\mathcal{B}}$. Equations of the type (5.3.4) have been considered on $\mathbb{R}^+$ in greater generality. For example when ϕ is merely a log-concave function on $\mathbb{R}^+$, Webster [100] has provided comprehensive results on the solution to (5.3.4), which we use readily throughout this work when possible since $\phi \in \mathcal{B}$ is a log-concave function on $\mathbb{R}^+$ itself.

In this Section, we start by stating the main results of our work concerning the class $\mathcal{W}_{\mathcal{B}}$ and postpone their proofs to the subsections 5.3.1-5.3.7. In particular, we derive and state representations, asymptotic and analytical properties of W_ϕ. To do so we introduce some notation. Similarly to (5.2.24) for Ψ, we write and have, for any $a \geq \mathfrak{a}_\phi$,

$$\mathcal{Z}_a(\phi) = \{z \in \mathbb{C}_a : \phi(z) = 0\} = \{z \in \mathbb{C}_a : \phi(\overline{z}) = 0\} \qquad (5.3.5)$$

and as in (5.2.3), (5.2.4) and (5.2.5),

$$\mathtt{u}_\phi = \sup\{u \le 0 : \phi(u) = 0\} \in [-\infty, 0] \tag{5.3.6}$$
$$\mathfrak{a}_\phi = \inf\ \{u < 0 : \phi \in \mathtt{A}_{(u,\infty)}\} \in [-\infty, 0] \tag{5.3.7}$$
$$\bar{\mathfrak{a}}_\phi = \max\{\mathfrak{a}_\phi, \mathtt{u}_\phi\} = \sup\{u \le 0 :\ \phi(u) = -\infty \text{ or } \phi(u) = 0\} \in [-\infty, 0]\,. \tag{5.3.8}$$

The next theorem contains some easy but very useful results which stem from the existing literature. Before we state them we recall from [83, Chapter 6] that the class $\mathcal{B}$ is in bijection with $\mathcal{W}_{\mathcal{B}}$ via the absolutely convergent product on (at least) $\mathbb{C}_{(0,\infty)}$

$$W_\phi\left(z\right) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi\left(k+z\right)} e^{\frac{\phi'(k)}{\phi(k)}z}, \tag{5.3.9}$$

where

$$\gamma_\phi = \lim_{n\to\infty}\left(\sum_{k=1}^{n}\frac{\phi'(k)}{\phi(k)} - \ln\phi(n)\right) \in \left[-\ln\phi(1), \frac{\phi'(1)}{\phi(1)} - \ln\phi(1)\right]. \tag{5.3.10}$$

Theorem 5.3.1. *Let $\phi \in \mathcal{B}$.*

1. *$W_\phi \in \mathtt{A}_{\left(\bar{\mathfrak{a}}_\phi,\infty\right)} \cap \mathtt{M}_{\left(\mathfrak{a}_\phi,\infty\right)}$ and W_ϕ is zero-free on $\mathbb{C}_{\left(\mathfrak{a}_\phi,\infty\right)}$. If $\phi(0) > 0$ then $W_\phi \in \mathtt{A}_{[0,\infty)}$ and W_ϕ is zero-free on $\mathbb{C}_{[0,\infty)}$. If $\phi(0) = 0$ (resp. $\phi(0) = 0$ and $\phi'(0^+) < \infty$) then W_ϕ (resp. $z \mapsto W_\phi\left(z\right) - \frac{1}{\phi'(0^+)z}$) extends continuously to $i\mathbb{R} \setminus \mathcal{Z}_0(\phi)$ (resp. $(i\mathbb{R} \setminus \mathcal{Z}_0(\phi)) \cup \{0\}$) and if $\mathfrak{z} \in \mathcal{Z}_0(\phi)$ then $\lim\limits_{\substack{\mathtt{Re}(z)\ge 0\\ z\to\mathfrak{z}}} \phi(z)W_\phi\left(z\right) = W_\phi\left(\mathfrak{z}+1\right)$.*

2. *There exists $\mathfrak{z} \in \mathcal{Z}_0(\phi)$ with $\mathfrak{z} \neq 0$ if and only if $\phi(0) = \mathtt{d} = 0$ and $\mu = \sum_{n=0}^{\infty} c_n \delta_{\bar{h}k_n}$ with $\sum_{n=1}^{\infty} c_n < \infty$, $\bar{h} > 0$, $k_n \in \mathbb{N}$ and $c_n \ge 0$ for all $n \in \mathbb{N}$. In this case, the mappings $z \mapsto e^{-z\ln\phi(\infty)}W_\phi(z)$ and $z \mapsto |W_\phi(z)|$ are periodic with period $\frac{2\pi i}{\bar{h}}$ on $\mathbb{C}_{(0,\infty)}$.*

3. *Assume that $\mathtt{u}_\phi \in (-\infty, 0)$, $\mathcal{Z}_{\mathtt{u}_\phi}(\phi) = \{\mathtt{u}_\phi\}$ and $\left|\phi'(\mathtt{u}_\phi^+)\right| < \infty$, the latter being always true if $\mathtt{u}_\phi > \mathfrak{a}_\phi$. Then $z \mapsto W_\phi(z) - \frac{W_\phi(1+\mathtt{u}_\phi)}{\phi'(\mathtt{u}_\phi^+)(z-\mathtt{u}_\phi)} \in \mathtt{A}_{\left[\mathtt{u}_\phi,\infty\right)}$. In this setting $\phi'(\mathtt{u}_\phi^+) = \mathtt{d} + \int_0^\infty y e^{-\mathtt{u}_\phi y}\mu(dy) \in (0, \infty]$.*

4. *Assume that $\mathfrak{a}_\phi < \mathtt{u}_\phi \le 0$ and put $N_{\mathfrak{a}_\phi} = \max\{n \in \mathbb{N} : \mathtt{u}_\phi - n > \mathfrak{a}_\phi\} \in \mathbb{N} \cup \{\infty\}$. Then there exists an open set $\mathcal{O} \subset \mathbb{C}$ such that $[\mathtt{u}_\phi - N_{\mathfrak{a}_\phi}, 1] \subset \mathcal{O}$, if $N_{\mathfrak{a}_\phi} < \infty$, and $(-\infty, 1] \subset \mathcal{O}$, if $N_{\mathfrak{a}_\phi} = \infty$, and W_ϕ is meromorphic on $\mathcal{O}$ with simple poles at $\{\mathtt{u}_\phi - k\}_{0\le k<N_{\mathfrak{a}_\phi}+1}$ and residues $\left\{\mathfrak{R}_k = \frac{W_\phi\left(1+\mathtt{u}_\phi\right)}{\phi'(\mathtt{u}_\phi)\prod_{j=1}^{k}\phi(\mathtt{u}_\phi - j)}\right\}_{0\le k<N_{\mathfrak{a}_\phi}+1}$ with $\prod_1^0 = 1$.*

Theorem 5.3.1 and especially the representation (5.3.9) allow for the understanding of the asymptotic behaviour of $|W_\phi\left(z\right)|$, as $|z| \to \infty$. We employ the floor and ceiling functions $\lfloor u \rfloor = \max\{n \in \mathbb{N} :\ n \le u\}$ and $\lceil u \rceil = \min\{n \in \mathbb{N} :\ u > n\}$. To be able to state the main asymptotic results we use the following notation. For any $\phi \in \mathcal{B}$ we introduce several important functions that describe the asymptotic behaviour of $|W_\phi(z)|$ in detail. Define formally, for any $z = a + ib \in \mathbb{C}_{(0,\infty)}$,

$$A_\phi(z) = \int_0^{|b|} \arg\phi(a+iu)\,du. \tag{5.3.11}$$

The function A_ϕ describes the asymptotics of $|W_\phi(z)|$ along imaginary lines of the type $\mathbb{C}_a$, $a > 0$. The next functions defined for $a > 0$

$$G_\phi(a) = \int_1^{1+a} \ln\phi(u)du,\ H_\phi(a) = \int_1^{1+a} \frac{u\phi'(u)}{\phi(u)}du \text{ and } H^*_\phi(a) = a\left(\frac{\phi(a+1)-\phi(a)}{\phi(a)}\right) \tag{5.3.12}$$

appear in the asymptotic behaviour of $W_\phi(x)$ for large $x > 0$ which in turn can be seen as an extension of the Stirling formula for the classical gamma function. Finally, we introduce the functions that control the error coming from the approximations. For $z = a + ib \in \mathbb{C}_{(0,\infty)}$, writing $\mathrm{P}(u) = (u - \lfloor u \rfloor)(1 - (u - \lfloor u \rfloor))$, we set

$$E_\phi(z) = \frac{1}{2}\int_0^\infty \mathrm{P}(u)\left(\ln\frac{|\phi(u+z)|}{\phi(u+a)}\right)'' du, \tag{5.3.13}$$

$$R_\phi(a) = \frac{1}{2}\int_1^\infty \mathrm{P}(u)\left(\ln\frac{|\phi(u+a)|}{\phi(u)}\right)'' du, \tag{5.3.14}$$

and

$$T_\phi = \frac{1}{2}\int_1^\infty \mathrm{P}(u)\left(\left(\frac{\phi'(u)}{\phi(u)}\right)^2 - \frac{\phi''(u)}{\phi(u)}\right) du. \tag{5.3.15}$$

Finally, we introduce subclasses of $\mathcal{B}$ equivalent to $\overline{\mathcal{N}}_\beta$ and $\overline{\mathcal{N}}(\Theta)$, see (5.2.10) and (5.2.12). For any $\beta \in [0,\infty]$

$$\begin{aligned}\mathcal{B}_\beta = &\left\{\phi \in \mathcal{B} : \lim_{|b|\to\infty} |b|^{\beta-\varepsilon}|W_\phi(a+ib)| = 0,\ \forall a > \overline{\mathfrak{a}}_\phi, \forall \varepsilon \in (0,\beta)\right\} \\ &\bigcap\left\{\phi \in \mathcal{B} : \lim_{|b|\to\infty} |b|^{\beta+\varepsilon}|W_\phi(a+ib)| = \infty,\ \forall a > \overline{\mathfrak{a}}_\phi, \forall \varepsilon \in (0,\beta)\right\}\end{aligned} \tag{5.3.16}$$

and any $\theta \in \left(0, \frac{\pi}{2}\right]$

$$\mathcal{B}(\theta) = \left\{\phi \in \mathcal{B} : \overline{\lim_{|b|\to\infty}} \frac{\ln|W_\phi(a+ib)|}{|b|} \le -\theta,\ \forall a > \overline{\mathfrak{a}}_\phi\right\}. \tag{5.3.17}$$

We now state our second main result which can be thought of as the Stirling asymptotic for the Bernstein-gamma functions as recalled in (5.3.2).

Theorem 5.3.2. *1. For any $a > 0$, we have that*

$$\sup_{\phi\in\mathcal{B}}\sup_{z\in\mathbb{C}_{(a,\infty)}}|E_\phi(z)| < \infty \text{ and } \sup_{\phi\in\mathcal{B}}\sup_{c>a}|R_\phi(c)| < \infty. \tag{5.3.18}$$

Moreover, for any $\phi \in \mathcal{B}$ and any $z = a + ib \in \mathbb{C}_{(0,\infty)}$, we have that

$$|W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a)-A_\phi(z)} e^{-E_\phi(z)-R_\phi(a)} \tag{5.3.19}$$

with

$$\Theta_\phi(a+ib) = \frac{1}{|b|}A_\phi(a+ib) = \frac{1}{|b|}\int_a^\infty \ln\left(\frac{|\phi(u+i|b|)|}{\phi(u)}\right)du \in \left[0,\frac{\pi}{2}\right] \tag{5.3.20}$$

As a result, for any $b \in \mathbb{R}$, $a \mapsto A_\phi(a+ib)$ is non-increasing on $\mathbb{R}^+$ and if $\phi^r(z) = r + \mathtt{d}z + \int_0^\infty (1-e^{-zy})\mu(dy)$, $r \geq 0$, then $r \mapsto A_{\phi^r}(z)$ is non-increasing for any $z \in \mathbb{C}_{(0,\infty)}$.

2. *For any fixed $b \in \mathbb{R}$ and large a, we have*

$$|W_\phi(a+i|b|)| = \frac{e^{-T_\phi}}{\sqrt{|\phi(z)|\,\phi(1)}} e^{a\ln\phi(a)-H_\phi(a)+H^*_\phi(a)-A_\phi(z)}\left(1+\mathrm{O}\left(\frac{1}{a}\right)\right), \tag{5.3.21}$$

where $\lim\limits_{a\to\infty} A_\phi(a+ib) = 0$, $T_\phi = \lim_{a\to\infty}(E_\phi(a+ib)+R_\phi(a))$ is defined in (5.3.15) and

$$0 \leq \varliminf_{a\to\infty}\frac{H_\phi(a)}{a} \leq \varlimsup_{a\to\infty}\frac{H_\phi(a)}{a} \leq 1 \text{ and } 0 \leq \varliminf_{a\to\infty} H^*_\phi(a) \leq \varlimsup_{a\to\infty} H^*_\phi(a) \leq 1. \tag{5.3.22}$$

3. (a) *If $\phi \in \mathcal{B}_P$, then, for any $a > 0$ fixed and $b \in \mathbb{R}$,*

$$A_\phi(a+i|b|) = \frac{\pi}{2}|b| - \left(a + \frac{\phi(0)}{\mathtt{d}}\right)\ln|b| - H(|b|), \tag{5.3.23}$$

where $H(|b|) \stackrel{\infty}{=} \mathrm{o}(|b|)$ and

$$\varliminf_{b\to\infty}\frac{\mathtt{d}\,H(b)}{\ln(b)\bar{\mu}\left(\frac{1}{b}\right)} \geq 1.$$

Thus $\mathcal{B}_P \subseteq \mathcal{B}\left(\frac{\pi}{2}\right)$.

(b) *Next, let $\phi \in \mathcal{B}_{R_\alpha}$, see (5.2.14), with $\alpha \in (0,1)$. Then, for any fixed $a > 0$,*

$$A_\phi(a+i|b|) \stackrel{\infty}{=} \frac{\pi}{2}\alpha|b|\left(1+\mathrm{o}(1)\right) \tag{5.3.24}$$

and thus $\mathcal{B}_{R_\alpha} \subseteq \mathcal{B}\left(\frac{\pi}{2}\alpha\right)$.

(c) *Let* $\phi \in \mathcal{B}_P^c$, *that is* $\mathtt{d} = 0$, *such that* $\mu(dy) = \upsilon(y)dy$. *If* $\upsilon(0^+) < \infty$ *exists and* $||\upsilon||_\infty = \sup_{x\geq 0} \upsilon(x) < \infty$, *then* $\phi \in \mathcal{B}_{\mathtt{N}_\phi}$ *with* $\mathtt{N}_\phi = \frac{\upsilon(0^+)}{\phi(\infty)}$. *If* $\upsilon(0^+) = \infty$, $\upsilon(y) = \upsilon_1(y) + \upsilon_2(y)$, $\upsilon_1, \upsilon_2 \in \mathrm{L}^1(\mathbb{R}^+)$, $\upsilon_1 \geq 0$ *is non-increasing in* $\mathbb{R}^+$, $\int_0^\infty \upsilon_2(y)dy \geq 0$ *and* $|\upsilon_2(y)| \leq \left(\int_y^\infty \upsilon_1(r)dr\right) \vee C$ *for some* $C > 0$, *then* $\phi \in \mathcal{B}_\infty$.

Remark 5.3.3. *Note the beautiful dependence of* (5.3.19) *on the geometry of* $\phi\left(\mathbb{C}_{(0,\infty)}\right) \subseteq \mathbb{C}_{(0,\infty)}$, *see* (5.3.33) *for this inclusion. The more* ϕ *shrinks* $\mathbb{C}_{(0,\infty)}$ *the smaller the contribution of* A_ϕ. *In fact* $\frac{1}{b}A_\phi(a+ib)$, *as* $b \to \infty$, *measures the fluctuations of the average angle along the contour* $\phi(\mathbb{C}_a)$, *which are necessarily of lesser order than those of* $\arg\phi(a+ib)$, $b \to \infty$, *along* $\mathbb{C}_a$.

Remark 5.3.4. *When* $z = a$, *modulo to the specifications of the constants, the central result* (5.3.19) *has appeared for* log*-concave functions in [100, Theorem 6.3] and for Bernstein functions in [83, Theorem 5.1]. Here, we provide an explicit representation of the terms of the asymptotics of* $|W_\phi(a+ib)|$, *as* $a \to \infty$, *which depend on the real part of a solely.*

Remark 5.3.5. *Since* $W_\phi \in \mathtt{A}_{\left(\bar{\mathfrak{a}}_\phi,\infty\right)} \cap \mathtt{M}_{\left(\mathfrak{a}_\phi,\infty\right)}$ *one can extend, via* (5.3.4), *the estimate* (5.3.19), *away from the poles residing in* $\mathbb{C}_{\left(\mathfrak{a}_\phi,\bar{\mathfrak{a}}_\phi\right]}$, *to* $z \in \mathbb{C}_{\left(\mathfrak{a}_\phi,\infty\right)}$. *Indeed, setting, for any* $c \in \mathbb{R}$, $c^{\rightarrow} = (\lfloor -c \rfloor + 1)\,\mathbb{I}_{\{c\leq 0\}}$, *then, for* $a > \mathfrak{a}_\phi$ *and* $z = a+ib$ *not a pole, one has*

$$W_\phi(a+ib) = W_\phi\left(a + a^{\rightarrow} + ib\right) \prod_{j=0}^{a^{\rightarrow}-1} \frac{1}{\phi(a+j+ib)} \tag{5.3.25}$$

with the convention that $\prod_0^{-1} = 1$.

Remark 5.3.6. *With the help of additional notation and arguments the remainder term* $H(b)$ *in* (5.3.23) *can be much better understood, see Proposition 5.3.15. When the convolutions of* $\bar{\mu}$ *can be evaluated full asymptotic expansion of* A_ϕ *can be achieved, see Remark 5.3.16 below.*

Remark 5.3.7. *The requirements for the case* (3c) *might seem stringent but in fact what they impose is that in a small positive neighbourhood of* 0, *the density can be decomposed as a non-increasing, integrable away from zero function and an oscillating error function* υ_2 *that is of smaller order than* $\int_y^\infty \upsilon_1(r)dr$. *This is obviously the case when the density* υ *itself is non-increasing.*

The next theorem contains alternative representations of W_ϕ, which modulo to an easy extension to $\mathbb{C}_{(0,\infty)}$ is due to [46] as well as a number of mappings that can be useful in a variety of contexts.

Theorem 5.3.8. *Let* $\phi, \underline{\phi} \in \mathcal{B}$.

1. $z \mapsto \log W_\phi(z+1) \in \mathcal{N}$ *with*

$$\log W_\phi(z+1) = (\ln\phi(1))\,z + \int_0^\infty \left(e^{-zy} - 1 - z(e^{-y}-1)\right) \frac{\kappa(dy)}{y(e^y-1)} \tag{5.3.26}$$

$\kappa(dy)=\int_0^y U\left(dy-r\right)\left(r\mu(dr)+\delta_{\mathtt{d}}(dr)\right)$, *where U is the potential measure associated to ϕ, see Proposition 5.3.13*(5)*.*

2. $z\mapsto \log\Big(W_\phi(z+1)W_{\underline{\phi}}(1-z)\Big)\in\mathcal{N}$

3. *If $u\mapsto \frac{\phi}{\underline{\phi}}(u)$ is a non-zero completely monotone function then there exists a positive variable I which is moment determinate and such that, for all $n\geq 0$, $\mathbb{E}\left[I^n\right]=\frac{W_{\underline{\phi}}(n+1)}{W_\phi(n+1)}$. Next, assume that $\phi\in\mathcal{B}_{\mathtt{N}_\phi}$ and $\underline{\phi}\in\mathcal{B}_{\mathtt{N}_{\underline{\phi}}}$. If $\mathtt{N}=\mathtt{N}_{\underline{\phi}}-\mathtt{N}_\phi>\frac{1}{2}$, then the law of I is absolutely continuous with density $f_I\in L^2\left(\mathbb{R}^+\right)$ and if $\mathtt{N}>1$ then $f_I\in \mathrm{C}_0^{\lceil \mathtt{N}_\Psi\rceil-2}\left(\mathbb{R}^+\right)$.*

4. *$u\mapsto\left(\frac{\underline{\phi}'}{\underline{\phi}}-\frac{\phi'}{\phi}\right)(u)$ is completely monotone if and only if $z\mapsto\log\frac{W_\phi}{W_{\underline{\phi}}}(z+1)\in\mathcal{N}$, that is, with the notation of the previous item, $\log I$ is infinitely divisible on $\mathbb{R}$. An equivalent condition is that the measure $U_\phi\star\mu_*$ is absolutely continuous with respect to the measure $U_{\underline{\phi}}\star\underline{\mu}_*$ with a density $h\leq 1$.*

Remark 5.3.9. *Note that in the trivial case $\phi(z)=z$ then $\kappa(dy)=dy$ and the representation* (5.3.26) *yields to the classical Malmstén formula for the gamma function, see [37]. With the recurrence equation* (5.3.4)*, the Weierstrass product* (5.3.9) *and the Mellin transform of a positive random variable, see Definition 5.3.1, this integral provides a fourth representation that the set of functions W_ϕ share with the classical gamma function, justifying our choice to name them the Bernstein-gamma functions.*

Remark 5.3.10. *We mention that when $\phi(0)=0$ the representation* (5.3.26) *for $z\in\mathbb{R}^+$ appears in [46, Theorem 3.1] and for any $\phi\in\mathcal{B}$ in [8, Theorem 2.2]. We also emphasize that with the aim of getting detailed information regarding bounds and asymptotic behaviours of $|W_\phi(z)|$, see Theorem 5.3.2, we found the Weierstrass product representation more informative to work with. However, as it is aptly illustrated by the authors of [46], the integral representation is useful for other purposes such as, for instance, for proving the multiplicative infinite divisibility property of some random variables.*

Remark 5.3.11. *The existence of Bernstein functions whose ratios are completely monotone, that is, the condition in item* (3)*, has been observed by the authors in [83].*

The final claim shows that the mapping $\phi\in\mathcal{B}\mapsto W_\phi\in\mathcal{W}_{\mathcal{B}}$ is continuous with respect to the pointwise topology in $\mathcal{B}$. This handy result is widely used throughout.

Lemma 5.3.12. *Let $(\phi_n)_{n\geq 0}$, $\underline{\phi}\in\mathcal{B}$ and $\lim_{n\to\infty}\phi_n(a)=\underline{\phi}(a)$ for all $a>0$. Then $\lim_{n\to\infty}W_{\phi_n}(z)=W_{\underline{\phi}}(z)$, $z\in\mathbb{C}_{(0,\infty)}$.*

Before providing the proofs of the previous claims, we collect some classical results concerning the set of Bernstein functions $\mathcal{B}$ that will be useful also in several remaining parts of the paper. For thorough information on these functions, we refer to the excellent monograph [92]. Then we have the following claims which can be found in [83, Section 4].

Proposition 5.3.13. *Let $\phi\in\mathcal{B}$.*

1. *For any* $z \in \mathbb{C}_{(0,\infty)}$,

$$\phi'(z) = \mathtt{d} + \int_0^\infty y e^{-zy}\mu(dy) = \mathtt{d} + \int_0^\infty e^{-zy}\bar{\mu}(y)dy - z\int_0^\infty e^{-zy}y\bar{\mu}(y)dy. \quad (5.3.27)$$

2. *For any* $u \in \mathbb{R}^+$,

$$0 \le u\phi'(u) \le \phi(u) \quad \text{and} \quad |\phi''(u)| \le 2\frac{\phi(u)}{u^2}. \quad (5.3.28)$$

3. $\phi(u) \overset{\infty}{=} \mathtt{d}u + \mathrm{o}(u)$ *and* $\phi'(u) \overset{\infty}{=} \mathtt{d} + \mathrm{o}(1)$. *Fix* $a > \mathfrak{a}_\phi$, *then* $|\phi(a+ib)| = |a+ib|(\mathtt{d} + \mathrm{o}(1))$ *as* $|b| \to \infty$.

4. *If* $\phi(\infty) < \infty$ *and* μ *is absolutely continuous then for any fixed* $a > \mathfrak{a}_\phi$, $\lim_{|b|\to\infty} \phi(a+ib) = \phi(\infty)$.

5. *The mapping* $u \mapsto \frac{1}{\phi(u)}$, $u \in \mathbb{R}^+$, *is completely monotone, i.e. there exists a positive measure* U, *whose support is contained in* $[0,\infty)$, *called the potential measure, such that the Laplace transform of* U *is given via the identity*

$$\frac{1}{\phi(u)} = \int_0^\infty e^{-uy}U(dy).$$

6. *In any case,*

$$\lim_{u\to\infty} \frac{\phi(u \pm a)}{\phi(u)} = 1 \quad \text{uniformly for } a\text{-compact intervals on } \mathbb{R}^+. \quad (5.3.29)$$

5.3.1 Proof of Theorem 5.3.1

We start with item 1. The fact that $W_\phi \in \mathtt{A}_{(\bar{\mathfrak{a}}_\phi,\infty)}$ is a consequence of [83, Theorem 6.1] and is essentially due to the recurrence equation (5.3.4) and the fact that ϕ is zero-free on $\mathbb{C}_{(\bar{\mathfrak{a}}_\phi,\infty)}$. Also, $W_\phi \in \mathtt{M}_{(\mathfrak{a}_\phi,\infty)}$ comes from the observation that $0 \not\equiv \phi \in \mathtt{A}_{(\mathfrak{a}_\phi,\infty)}$, that is ϕ can only have zeros of finite order, and (5.3.4) which allows a recurrent meromorphic extension to $\mathbb{C}_{(\mathfrak{a}_\phi,\infty)}$. W_ϕ is zero-free on $\mathbb{C}_{(\bar{\mathfrak{a}}_\phi,\infty)}$ follows from [83, Theorem 6.1 and Corollary 7.8] whereas the fact that W_ϕ extends to zero-free on $\mathbb{C}_{(\mathfrak{a}_\phi,\infty)}$ thanks to $\phi \in \mathtt{A}_{(\mathfrak{a}_\phi,\infty)}$ and (5.3.4). If $\phi(0) > 0$ then $\mathcal{Z}_0(\phi) = \emptyset$ and hence the facts that W_ϕ is zero-free on $\mathbb{C}_{[0,\infty)}$ and $W_\phi \in \mathtt{A}_{[0,\infty)}$ are immediate from W_ϕ being zero-free on $(0,\infty)$ and (5.3.4). However, when $\phi(0) = 0$ relation (5.3.4) ensures that W_ϕ extends continuously to $i\mathbb{R} \setminus \mathcal{Z}_0(\phi)$ and clearly if $\mathfrak{z} \in \mathcal{Z}_0(\phi)$ then $\lim_{\mathrm{Re}(z)\ge 0, z\to\mathfrak{z}} \phi(z)W_\phi(z) = W_\phi(\mathfrak{z}+1)$. Finally, let us assume that $\phi'(0^+) = \mathtt{d} + \int_0^\infty y\mu(dy) < \infty$ and $\{0\} \in \mathcal{Z}_0(\phi)$, that is $\phi(0) = 0$. From the assumption $\phi'(0^+) < \infty$ and the dominated convergence theorem, we get that ϕ' extends to $i\mathbb{R}$, see

(5.3.27). Therefore, from (5.3.4) and the assumption $\phi(0)=0$ we get, for any $z \in \mathbb{C}_{(0,\infty)}$, that

$$W_\phi(z+1)=\phi(z)W_\phi(z)=(\phi(z)-\phi(0))W_\phi(z)=\left(\phi'(0^+)z+\mathrm{o}(|z|)\right)W_\phi(z).$$

Clearly, then the mapping $z\mapsto W_\phi(z)-\frac{1}{\phi'(0^+)z}$ extends continuously to $i\mathbb{R}\setminus(\mathcal{Z}_0(\phi)\setminus\{0\})$ provided $\phi'(0^+)=\mathtt{d}+\int_0^\infty y\mu(dy)>0$, which is apparently true. Let us deal with item (2). We note that

$$e^{-\phi(z)}=\mathbb{E}\left[e^{-z\xi_1}\right],\ z\in\mathbb{C}_{[0,\infty)},$$

where $\xi=(\xi_t)_{t\geq 0}$ is a non-decreasing Lévy process (subordinator) as $-\phi(-z)=\Psi(z)\in\overline{\mathcal{N}}$, see (5.2.1). Thus, if $\phi(z_0)=0$ then $\mathbb{E}\left[e^{-z_0\xi_1}\right]=1$. If in addition, $z_0\in\mathbb{C}_{[0,\infty)}\setminus\{0\}$ then $\phi(0)=0$ and $z_0\in i\mathbb{R}$. Next, $\phi(z_0)=0$ also triggers that ξ lives on a lattice of size, say $\bar{h}>0$, which immediately gives that $\mathtt{d}=0$ and $\mu=\sum_{n=1}^\infty c_n\delta_{x_n}$ with $\sum_{n=1}^\infty c_n<\infty$ and $\forall n\in\mathbb{N}$ we have that $x_n=\bar{h}k_n$, $k_n\in\mathbb{N}$, $c_n\geq 0$. Finally, $\bar{h}$ can be chosen to be the largest such that ξ lives on $\left(\bar{h}n\right)_{n\in\mathbb{N}}$. Thus,

$$\phi(z)=\sum_{n=1}^\infty c_n\left(1-e^{-z\bar{h}k_n}\right)$$

and we conclude that ϕ is periodic with period $\frac{2\pi i}{\bar{h}}$ on $\mathbb{C}_{(0,\infty)}$. Next, note that

$$\phi(\infty)=\lim_{u\to\infty}\phi(u)=\lim_{u\to\infty}\sum_{n=1}^\infty c_n\left(1-e^{-uhk_n}\right)=\sum_{n=1}^\infty c_n<\infty. \tag{5.3.30}$$

Then (5.3.10) implies that

$$\lim_{n\to\infty}\sum_{k=1}^n\frac{\phi'(k)}{\phi(k)}=\sum_{k=1}^\infty\frac{\phi'(k)}{\phi(k)}=\gamma_\phi+\ln\phi(\infty).$$

Thus, from (5.3.9) we get that

$$W_\phi(z)=\frac{e^{-\gamma_\phi z}}{\phi(z)}\prod_{k=1}^\infty\frac{\phi(k)}{\phi(k+z)}e^{\frac{\phi'(k)}{\phi(k)}z}=\frac{e^{z\ln\phi(\infty)}}{\phi(z)}\prod_{k=1}^\infty\frac{\phi(k)}{\phi(k+z)}.$$

Hence, the claim for the $\frac{2\pi i}{\bar{h}}$ periodicity of the mappings $z\mapsto e^{-z\ln\phi(\infty)}W_\phi(z)$ and $z\mapsto|W_\phi(z)|$ follows immediately from the periodicity of ϕ. Thus, item (2) is proved. Item (3) follows in the same manner as item (1) noting that when $\mathfrak{u}_\phi<0$ then $\bar{\mathfrak{a}}_\phi=\mathfrak{u}_\phi$, see (5.3.6) and (5.3.8). The last item (4) is an immediate result from (5.3.4) and the fact that $\phi'>0$ on $(\mathfrak{a}_\phi,\infty)$, see (5.3.27), that is $\mathfrak{u}_\phi$ is the unique zero of order one of ϕ on $(\mathfrak{a}_\phi,\infty)$. This ends the proof of Theorem 5.3.1.

5.3.2 Proof of Theorem 5.3.2(1)

First, the proof and claim of [83, Proposition 6.10(2)] show that, for any $a > 0$ and any $\phi \in \mathcal{B}$, $\sup_{z\in\mathbb{C}_{(a,\infty)}} |E_\phi(z)| \leq \frac{19}{8}a$ from where we get the first global bound in (5.3.18). Next, we know from the proof of [83, Proposition 6.10] and see in particular the expressions obtained for [83, (6.33) and (6.34)], that, for any $z = a+ib \in \mathbb{C}_{(0,\infty)}$, $b > 0$,

$$\begin{aligned} |W_\phi(z)| &= W_\phi(a)\,\frac{\phi(a)}{|\phi(z)|}\sqrt{\left|\frac{\phi(z)}{\phi(a)}\right|}e^{-\int_0^\infty \ln\left|\frac{\phi(u+a+ib)}{\phi(a+bu)}\right|du}e^{-E_\phi(z)} \\ &= W_\phi(a)\sqrt{\left|\frac{\phi(a)}{\phi(z)}\right|}e^{-b\Theta_\phi(z)}e^{-E_\phi(z)}. \end{aligned} \tag{5.3.31}$$

We note that the term $-E_\phi(z)$ is the limit in n of the error terms $E^B_\phi(n,a) - E^B_\phi(n,a+ib)$ in the notation of the proof of [83, Proposition 6.10]. Thus, the last three terms of the second expression in (5.3.31) are in fact the quantity $\frac{\phi(a)}{|\phi(z)|}Z_\phi(z)$ in the notation of [83, Proposition 6.10, (6.33)]. Let $\log_0$ stand for the branch of the logarithm such that $\arg z \in (-\pi,\pi]$, that is, it coincides with our definition of the argument function. We note from (5.3.3) and (5.3.8) that, for any $z = a+ib \in (\overline{\mathtt{a}}_\phi,\infty)$,

$$\mathtt{Re}\,(\phi(a+ib)) = \phi(0) + \mathtt{d}a + \int_0^\infty \left(1 - e^{-ay}\cos(by)\right)\mu(dy) \geq \phi(a) > 0, \tag{5.3.32}$$

and thus

$$\phi:\ \mathbb{C}_{(0,\infty)} \to \mathbb{C}_{(0,\infty)}. \tag{5.3.33}$$

Therefore, $\log_0\phi \in \mathtt{A}_{(0,\infty)}$. By means of the integral expression in (5.3.20), an application of the Cauchy integral theorem to $\log_0\phi$ on the closed rectangular contour with vertices $a+ib, u+ib, u$ and a, for any $z = a+ib \in \mathbb{C}_{(0,\infty)}$, $b > 0$, $u > a$, yields that

$$\begin{aligned} b\Theta_\phi(a+ib) &= \int_a^\infty \ln\left(\frac{|\phi(y+ib)|}{\phi(y)}\right)dy = \lim_{u\to\infty}\int_a^u \ln\left(\frac{|\phi(y+ib)|}{\phi(y)}\right)dy \\ &= \lim_{u\to\infty}\mathtt{Re}\left(\int_a^u \log_0\frac{\phi(y+ib)}{\phi(y)}dy\right) \\ &\overset{Cauchy}{=} \lim_{u\to\infty}\mathtt{Re}\left(\int_{u\to u+ib}\log_0\phi(z)dz\right) - \mathtt{Re}\left(\int_{a\to a+ib}\log_0\phi(z)dz\right) \\ &= \int_0^b \arg\phi(a+iy)dy - \lim_{u\to\infty}\int_0^b \arg\phi(u+iy)dy. \end{aligned}$$

We investigate the last limit. Note that (5.3.3) gives that for $z = a+ib \in \mathbb{C}_{(0,\infty)}$, $b > 0$,

$$\mathtt{Im}\,(\phi(a+ib)) = \mathtt{d}b + \int_0^\infty e^{-ay}\sin(by)\,\mu(dy). \tag{5.3.34}$$

From (5.3.34) if $\underline{b} \in (0,b)$ then by the dominated convergence theorem

$$\overline{\lim_{a\to\infty}} \left|\mathtt{Im}\left(\phi(a+i\underline{b})\right)\right| \leq \lim_{a\to\infty} \left(\mathtt{d}b + b\int_0^\infty e^{-ay} y\mu(dy)\right) = \mathtt{d}b.$$

Similarly, from (5.3.32), we have that

$$\underline{\lim}_{a\to\infty} \mathtt{Re}\left(\phi(a+i\underline{b})\right) = \infty\mathbb{I}_{\{\mathtt{d}>0\}} + \left(\phi(0)+\mu\left(0,\infty\right)\right)\mathbb{I}_{\{\mathtt{d}=0\}}.$$

From the last two relations we conclude that

$$\begin{aligned}\overline{\lim_{a\to\infty}} \frac{\left|\mathtt{Im}\left(\phi(a+i\underline{b})\right)\right|}{\mathtt{Re}\left(\phi(a+i\underline{b})\right)} &\leq \frac{\mathtt{d}b}{\underline{\lim}_{a\to\infty} \mathtt{Re}\left(\phi(a+i\underline{b})\right)} \\ &= \lim_{a\to\infty} \frac{\mathtt{d}b}{2\left(\infty\mathbb{I}_{\{\mathtt{d}>0\}} + \left(\phi(0)+\mu\left(0,\infty\right)\right)\mathbb{I}_{\{\mathtt{d}=0\}}\right)} = 0.\end{aligned}$$

Therefore, since, from (5.3.32), $\mathtt{Re}\left(\phi(a+i\underline{b})\right) > 0$, we get that

$$\lim_{a\to\infty} \arg\phi\left(a+iy\right) = 0, \text{ uniformly on } y\text{-compact sets,} \tag{5.3.35}$$

and the second term on the right-hand side of the last relation of the equation above (5.3.34) vanishes. Since $A_\phi\left(a+ib\right) = \int_0^b \arg\phi(a+iy)dy$ we conclude that $b\Theta_\phi\left(a+ib\right) = A_\phi\left(a+ib\right)$, for $a+ib \in \mathbb{C}_{(0,\infty)}, b>0$, and thus prove (5.3.20), as from [83, Proposition 6.10(1)] we have that $\Theta\left(a+ib\right) \in \left[0,\frac{\pi}{2}\right]$. Henceforth, we conclude the alternative expression to (5.3.31), for $z = a+ib \in \mathbb{C}_{(0,\infty)}, b>0$,

$$|W_\phi(z)| = W_\phi\left(a\right)\sqrt{\left|\frac{\phi(a)}{\phi(z)}\right|}e^{-A_\phi(z)}e^{-E_\phi(z)}. \tag{5.3.36}$$

Since $\left|\overline{W_\phi\left(z\right)}\right| = \left|W_\phi\left(\bar{z}\right)\right|$ we conclude (5.3.36), for any $z = a+ib \in \mathbb{C}_{(0,\infty)}, b \neq 0$. Since from (5.3.13), $E_\phi\left(\mathtt{Re}(z)\right) = 0$, we deduct that (5.3.36) holds for $z = a \in \mathbb{R}^+$ too. Next, let us investigate $W_\phi(a)$ in (5.3.36). Recall that, for $a>0$, from (5.3.9) and (5.3.10) we get that

$$\begin{aligned}W_\phi\left(a\right) &= \frac{e^{-\gamma_\phi a}}{\phi(a)}\prod_{k=1}^\infty \frac{\phi(k)}{\phi\left(k+a\right)}e^{\frac{\phi'(k)}{\phi(k)}a} \\ &= \frac{1}{\phi(a)}\lim_{n\to\infty}\prod_{k=1}^n \frac{\phi(k)}{\phi\left(k+a\right)}e^{a\ln\phi(n)}e^{a\left(\sum_{k=1}^n \frac{\phi'(k)}{\phi(k)} - \ln\phi(n) - \gamma_\phi\right)} \\ &= \frac{1}{\phi(a)}\lim_{n\to\infty}\prod_{k=1}^n \frac{\phi(k)}{\phi\left(k+a\right)}e^{a\ln\phi(n)} = \frac{1}{\phi(a)}\lim_{n\to\infty} e^{-S_n(a)+a\ln\phi(n)},\end{aligned} \tag{5.3.37}$$

where $S_n(a) = \sum_{k=1}^n \ln \frac{\phi(a+k)}{\phi(k)}$. Then, we get, from [65, Section 8.2, (2.01), (2.03)] applied to the function $\ln \frac{\phi(a+u)}{\phi(u)}$, $u > 0$, with $m = 1$ in their notation, that

$$\begin{aligned} S_n(a) &= \int_1^n \ln \frac{\phi\,(a+u)}{\phi(u)} du + \frac{1}{2} \ln \frac{\phi(1+a)}{\phi(1)} + \frac{1}{2} \ln \frac{\phi(n+a)}{\phi(n)} + R_2(n,a) \\ &= \frac{1}{2} \ln \frac{\phi(1+a)}{\phi(1)} - \int_1^{a+1} \ln \phi(u) du + \int_0^a \ln \phi(n+u) du + \frac{1}{2} \ln \frac{\phi(n+a)}{\phi(n)} + R_2(n,a), \end{aligned} \tag{5.3.38}$$

where, recalling that $\mathrm{P}(u) = (u - \lfloor u \rfloor)(1 - (u - \lfloor u \rfloor))$, for any $a > 0$,

$$R_2(n,a) = \frac{1}{2} \int_1^n \mathrm{P}(u) \left(\ln \frac{\phi\,(a+u)}{\phi(u)} \right)'' du. \tag{5.3.39}$$

Using Proposition 5.3.13(5.3.29) and (5.3.38), we get that

$$\begin{aligned} \lim_{n\to\infty} \left(S_n(a) - a \ln \phi(n) \right) &= \frac{1}{2} \ln \frac{\phi(1+a)}{\phi(1)} - \int_1^{a+1} \ln \phi(u) du + \lim_{n\to\infty} \left(\int_0^a \ln \frac{\phi(n+u)}{\phi(n)} du + R_2(n,a) \right) \\ &= \frac{1}{2} \ln \frac{\phi(1+a)}{\phi(1)} - G_\phi(a) + \lim_{n\to\infty} R_2(n,a). \end{aligned}$$

Let us show that $R_\phi(a) = \lim_{n\to\infty} R_2(n,a)$ exists. It follows from (5.3.39), $\sup_{u>0} |\mathrm{P}(u)| \leq \frac{1}{4}$ and the dominated convergence theorem since

$$\sup_{n\geq 1} \sup_{\phi\in\mathcal{B}} |R_2(n,a)| \leq \frac{1}{4} \sup_{\phi\in\mathcal{B}} \int_1^\infty \left(\left(\frac{\phi'(u)}{\phi(u)} \right)^2 + \left| \frac{\phi''(u)}{\phi(u)} \right| \right) du < 2, \tag{5.3.40}$$

where the finiteness follows from (5.3.28). Therefore, from (5.3.37),(5.3.39) and the existence of $R_\phi(a)$, we get that

$$W_\phi(a) = \frac{1}{\phi(a)} \sqrt{\frac{\phi(1)}{\phi(1+a)}} e^{G_\phi(a) - R_\phi(a)}.$$

Substituting this in (5.3.36) we prove (5.3.19). From (5.3.40) we also obtain the second global bound in (5.3.18). To conclude item (1) it remains to prove the two monotonicity properties. The monotonicity, for fixed $b \in \mathbb{R}$, of the mapping $a \mapsto A_\phi\,(a+ib)$ on $\mathbb{R}^+$ follows from (5.3.20) right away since for all $u > 0, b \geq 0$, $\ln \left(\frac{|\phi(u+i|b|)|}{\phi(u)} \right) \geq 0$, see [83, Proposition 6.10, (6.32)]. Otherwise, set $0 \leq q \leq q' < \infty$ and for any $r \geq 0$, $\phi^r(z) = r + \phi(0) + \phi^\sharp(z)$. Then, the mapping

$$r \mapsto \ln \left(\frac{|\phi^r\,(u+i|b|)|}{\phi^r(u)} \right) = \ln \left| 1 + \frac{\phi^\sharp(u+i|b|) - \phi^\sharp(u)}{r + \phi(0) + \phi^\sharp(u)} \right|$$

is non-increasing on $\mathbb{R}^+$ and (5.3.20) closes the proof of item (1).

We proceed by investigating in detail the principal functions that control the asymptotic behaviour of $|W_\phi(z)|$, $z = a + ib \in \mathbb{C}_{(0,\infty)}$ in (5.3.19). We start with the large asymptotic behaviour for large a and fixed b.

5.3.3 Proof of Theorem 5.3.2(2)

Note that, for $z = a+ib \in \mathbb{C}_{(0,\infty)}$, some elementary algebra yields that

$$\begin{aligned} E_\phi(z) + R_\phi(a) &= \frac{1}{2}\int_0^1 \mathrm{P}(u)\left(\frac{\ln|\phi(u+z)|}{\ln\phi(u+a)}\right)'' du + \frac{1}{2}\int_1^\infty \mathrm{P}(u)\left(\frac{\ln|\phi(u+z)|}{\ln\phi(u)}\right)'' du \\ &:= \overline{E}_\phi(z) + \overline{R}_\phi(z). \end{aligned}$$

Then, noting that $(\ln|\phi(u+z)|)'' \leq \left|(\log_0 \phi(u+z))''\right|$, we easily obtain the estimate

$$\left|\overline{E}_\phi(z)\right| \leq \int_0^1 \left(\left|\frac{\phi'(u+z)}{\phi(u+z)}\right|^2 + \left(\frac{\phi'(u+a)}{\phi(u+a)}\right)^2 + \left|\frac{\phi''(u+z)}{\phi(u+z)}\right| + \left|\frac{\phi''(u+a)}{\phi(u+a)}\right|\right) du.$$

Clearly, from the proof of [83, Lemma 4.5(2)] combined with (5.3.28), we get that

$$\left|\frac{\phi'(a+ib)}{\phi(a+ib)}\right| \leq \sqrt{10}\frac{\phi'(a)}{\phi(a)} \leq \frac{\sqrt{10}}{a} \tag{5.3.41}$$

and

$$\left|\frac{\phi''(a+ib)}{\phi(a+ib)}\right| \leq \sqrt{10}\frac{\phi''(a)}{\phi(a)} \leq \frac{2\sqrt{10}}{a^2}, \tag{5.3.42}$$

from where we deduce that $\lim_{a\to\infty}\left|\overline{E}_\phi(z)\right| = 0$. Next, we look into $\overline{R}_\phi(z)$. Note that

$$\begin{aligned} \overline{R}_\phi(z) &= \frac{1}{2}\int_1^\infty \mathrm{P}(u)\left(\ln|\phi(u+z)|\right)'' du - \frac{1}{2}\int_1^\infty \mathrm{P}(u)\left(\ln\phi(u)\right)'' du \\ &= \overline{R}_1(z) + \overline{R}_2(z). \end{aligned}$$

Clearly, $\overline{R}_2$ is the right-hand side of (5.3.15) and we proceed to show that $\lim_{a\to\infty}\left|\overline{R}_1(z)\right| = 0$. To do so we simply repeat the work done for $\overline{E}_\phi(z)$ to conclude that

$$\left|\overline{R}_1(z)\right| \leq \frac{1}{8}\int_1^\infty \left|\frac{\phi'(u+z)}{\phi(u+z)}\right|^2 + \left|\frac{\phi''(u+z)}{\phi(u+z)}\right| du.$$

Since each integrand converges to zero as $a \to \infty$, see (5.3.41) and (5.3.42), we invoke again (5.3.41) and (5.3.42) to ensure that the dominated convergence theorem is applicable. Therefore, for any fixed $b \in \mathbb{R}$,

$$\lim_{a\to\infty} e^{-E_\phi(a+ib)-R_\phi(a)} = e^{-T_\phi}$$

and the very first term in the product on the right-hand side of (5.3.21) is established. For any $\phi \in \mathcal{B}$, we deduce by performing an integration by parts in the expression of G_ϕ in (5.3.12) that, for any $a > 0$,

$$G_\phi(a) = (a+1)\ln\phi(a+1) - \ln\phi(1) - H_\phi(a). \tag{5.3.43}$$

Next, note that, for large a,

$$a\ln\frac{\phi(a+1)}{\phi(a)} = a\ln\left(1+\frac{\phi(a+1)-\phi(a)}{\phi(a)}\right) = H^*_\phi(a) + a\,\mathrm{O}\left(\left(\frac{\phi(a+1)-\phi(a)}{\phi(a)}\right)^2\right),$$

where the last asymptotic relation follows from $\lim_{a\to\infty}\frac{\phi(a+1)}{\phi(a)} = 1$, see (5.3.29). However, from (5.3.28) we get that $a\frac{\phi'(a)}{\phi(a)} \leq 1$, $a>0$, and, since ϕ' is non-increasing, we conclude that

$$a\ln\frac{\phi(1+a)}{\phi(a)} = H^*_\phi(a) + \mathrm{O}\left(\frac{1}{a}\right).$$

Thus, putting pieces together we deduce from (5.3.43) that

$$G_\phi(a) = a\ln\phi(a) - H_\phi(a) + H^*_\phi(a) + \ln\phi(a) - \ln\phi(1) + \mathrm{O}\left(\frac{1}{a}\right). \tag{5.3.44}$$

This combined with (5.3.19) and (5.3.15) leads to the remaining terms in the product in (5.3.21) which completes the proof of Theorem 5.3.1(2).

5.3.4 Proof of Theorem 5.3.2(3c)

The proof is based on the observations of Remark 5.4.4 which simply crystallize the main ingredients of the lengthy proof of Proposition 5.4.3 stated below.

Finally, we discuss the term $A_\phi(z)$ in (5.3.19) which governs the asymptotics along complex lines $\mathbb{C}_a$, $a>\overline{\mathtt{a}}_\phi$, see Remark 5.3.3. It can be simplified for two general subclasses of $\mathcal{B}$, namely the case when $\mathtt{d}>0$, that is $\phi\in\mathcal{B}_P$, and the class $\mathcal{B}_{R_\alpha}$, see (5.2.14). We start with the former case.

5.3.5 Proof of Theorem 5.3.2(3a)

We start the proof by introducing some notation and quantities. For a measure (or a function) μ on $\mathbb{R}$, $\mathcal{F}_\mu(-ib) = \int_{-\infty}^{\infty} e^{-iby}\mu(dy)$ stands for its Fourier transform. Also, we use $\lambda*\gamma$ to denote the convolution between measures and/or functions. Next, for any measure λ on $\mathbb{R}$ such that $||\lambda||_{TV} := \int_{-\infty}^{\infty}|\lambda(dy)| < 1$ we define

$$\mathfrak{L}_\lambda(dy) = \sum_{n=1}^{\infty}\frac{\lambda^{*n}(dy)}{n} \text{ and } \underline{\mathfrak{L}}_\lambda(dy) = \sum_{n=1}^{\infty}(-1)^{n-1}\frac{\lambda^{*n}(dy)}{n}. \tag{5.3.45}$$

Clearly, $||\mathfrak{L}_\lambda||_{TV}<\infty$ and $||\underline{\mathfrak{L}}_\lambda||_{TV}<\infty$. Finally, we use for a measure (resp. function) $\lambda_a(dy) = e^{-ay}\lambda(dy)$ (resp. $\lambda_a(y) = e^{-ay}\lambda(y)$) with $a\in\mathbb{R}$. Let from now on λ be a measure on $\mathbb{R}^+$. If $||\lambda_a||_{TV}<1$ for some $a>0$, then by virtue of the fact that $\lambda_a^{*n}(dy) = e^{-ay}\lambda^{*n}(dy)$, $y\in(0,\infty)$, (5.3.45) holds locally on $(0,\infty)$ for $\lambda_0=\lambda$. Let next $\lambda(dy) = \lambda(y)dy$, $y\in(0,\infty)$, and $\lambda\in(\mathrm{L}^1(\mathbb{R}^+),*)$, i.e. the C^* algebra of the integrable functions on

$\mathbb{R}^+$, that is $\mathrm{L}^1(\mathbb{R}^+)$ considered as a subalgebra of $(\mathrm{L}^1(\mathbb{R}),*)$ which is endowed with the convolution operation as a multiplication. Note that formally

$$-\log_0(1-\mathcal{F}_\lambda) = \mathcal{F}_{\mathfrak{L}_\lambda} = \sum_{n=1}^{\infty} \frac{\mathcal{F}_\lambda^n}{n} \quad \text{and} \quad \log_0(1+\mathcal{F}_\lambda) = \mathcal{F}_{\underline{\mathfrak{L}}_\lambda} = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{\mathcal{F}_\lambda^n}{n}.$$

Then, the Wiener-Lévy theorem for normed algebras will be shown below to yield that

$$\exists \underline{\mathfrak{L}}_\lambda(dy) = \underline{\mathfrak{L}}_\lambda(y)dy,\ y \in (0,\infty)\ \text{s.t.}\ \underline{\mathfrak{L}}_\lambda \in \left(\mathrm{L}^1\left(\mathbb{R}^+\right),*\right) \iff \mathrm{Supp}\,\mathcal{F}_\lambda \cap (-\infty,-1] = \emptyset, \tag{5.3.46}$$

$$\exists \mathfrak{L}_\lambda(dy) = \mathfrak{L}_\lambda(y)dy,\ y \in (0,\infty)\ \text{s.t.}\ \mathfrak{L}_\lambda \in \left(\mathrm{L}^1\left(\mathbb{R}^+\right),*\right) \iff \mathrm{Supp}\,\mathcal{F}_\lambda \cap [1,\infty) = \emptyset, \tag{5.3.47}$$

that is $\underline{\mathfrak{L}}_\lambda$ (resp. $\mathfrak{L}_\lambda$) is an element of $(\mathrm{L}^1(\mathbb{R}^+),*)$ if the support of $\mathcal{F}_\lambda$ is a strict subset of the domain of analyticity of the function $\log_0(1+z)$ (resp. $\log_0(1-z)$). We then have the following claim.

Proposition 5.3.14. *Let λ be a real-valued measure on $(0,\infty)$. If $||\lambda||_{TV} < 1$ then both $||\mathfrak{L}_\lambda||_{TV} < \infty$ and $||\underline{\mathfrak{L}}_\lambda||_{TV} < \infty$. Furthermore, let $\lambda \in (\mathrm{L}^1(\mathbb{R}^+),*)$. If $\mathrm{Supp}\,\mathcal{F}_\lambda \cap (-\infty,-1] = \emptyset$ (resp. $\mathrm{Supp}\,\mathcal{F}_\lambda \cap [1,\infty) = \emptyset$) then $\underline{\mathfrak{L}}_\lambda \in (\mathrm{L}^1(\mathbb{R}^+),*)$ (resp. $\mathfrak{L}_\lambda \in (\mathrm{L}^1(\mathbb{R}^+),*)$) and*

$$\log_0(1+\mathcal{F}_\lambda(-ib)) = \mathcal{F}_{\underline{\mathfrak{L}}_\lambda}(-ib) \ \textit{if}\ \underline{\mathfrak{L}}_\lambda \in \left(\mathrm{L}^1\left(\mathbb{R}^+\right),*\right) \tag{5.3.48}$$

$$-\log_0(1-\mathcal{F}_\lambda(-ib)) = \mathcal{F}_{\mathfrak{L}_\lambda}(-ib) \ \textit{if}\ \mathfrak{L}_\lambda \in \left(\mathrm{L}^1\left(\mathbb{R}^+\right),*\right). \tag{5.3.49}$$

Proof. Let $||\lambda||_{TV} < 1$. Since, for all $n \in \mathbb{N}$, $||\lambda^{*n}||_{TV} \leq ||\lambda||_{TV}^n$ and $\sup_{b\in\mathbb{R}} |\mathcal{F}_\lambda^n(ib)| \leq ||\lambda||_{TV}^n$, then (5.3.45), (5.3.46), (5.3.47) and the Taylor expansion about zero of $\log_0(1 \pm z)$ yield all results in the case $||\lambda||_{TV} < 1$. Let the support of $\mathcal{F}_\lambda$ does not intersect with $[1,\infty)$ (resp. $(-\infty,-1]$), which is the region where $\log_0(1-z)$ (resp. $\log_0(1+z)$) is not holomorphic. Then we get from the Wiener-Lévy theorem, see [64, Section 7.7] that $\mathfrak{L}_\lambda = c_1\delta_0 + \lambda^+$ (resp. $\underline{\mathfrak{L}}_\lambda = c_2\delta_0 + \lambda^-$), where the Dirac mass δ_0 completes $(\mathrm{L}^1(\mathbb{R}),*)$ to a semistable Banach algebra with unity, $c_1, c_2 \in \mathbb{R}$ and $\lambda^\pm \in \mathrm{L}^1(\mathbb{R})$. Since the Taylor expansion of $\log_0(1 \pm z)$ is approximated with monomials of order greater or equal to 1 and $\mathcal{F}_{\delta_0} \equiv 1$ then $c_1 = c_2 = 0$. Also, considering $a > 0$ big enough such that $||\lambda_a||_{TV} < 1$ then (5.3.45) confirms that $\mathfrak{L}_{\lambda_a}, \underline{\mathfrak{L}}_{\lambda_a} \in \mathrm{L}^1(\mathbb{R}^+)$. Since $\mathfrak{L}_{\lambda_a}(dy) = e^{-ay}\mathfrak{L}_\lambda(dy)$ and $\underline{\mathfrak{L}}_{\lambda_a}(dy) = e^{-ay}\underline{\mathfrak{L}}_\lambda(dy)$ we conclude that $\lambda^\pm \in \mathrm{L}^1(\mathbb{R}^+)$ and hence all the remaining claims. □

Let us assume that $\mathtt{d} > 0$. Then, for each $z = a + ib, a > 0$, we have from the second expression in (5.3.3) that

$$\phi(z) = \mathtt{d}z\left(1 + \frac{\phi(0)}{\mathtt{d}z} + \mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right), \tag{5.3.50}$$

where we have set $\bar{\mu}_{a,\mathtt{d}}(y) = \frac{1}{\mathtt{d}}e^{-ay}\bar{\mu}(y)$, $y \in (0,\infty)$. With the preceding notation we can state the next result which with the help of Lemma 5.4.1 and the substitution

$$H(b) := \int_0^\infty \frac{1-\cos(by)}{y}\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy)$$

concludes the proof of Theorem 5.3.1(3a).

Proposition 5.3.15. *Assume that $\phi \in \mathcal{B}_P$ and $z = a+ib \in \mathbb{C}_a$, with $a > 0$ fixed. We have*

$$\begin{aligned}A_\phi(z) &= |b|\arctan\left(\frac{|b|}{a}\right) - \frac{\left(a + \frac{\phi(0)}{\mathtt{d}}\right)}{2}\ln\left(1+\frac{b^2}{a^2}\right) - \int_0^\infty \frac{1-\cos(by)}{y}\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy) + \bar{A}_\phi(a)\\ &\overset{\infty}{\approx} \frac{\pi}{2}|b| - \left(a + \frac{\phi(0)}{\mathtt{d}}\right)\ln|b| - \mathrm{o}\left(|b|\right),\end{aligned} \tag{5.3.51}$$

where $\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}$ is related to $\bar{\mu}_{a,\mathtt{d}}(y) = \mathtt{d}^{-1}e^{-ay}\bar{\mu}(y)$ via (5.3.46) *and $\left|\bar{A}_\phi(a)\right| < \infty$ for all $a > 0$. For all $a > 0$ big enough, $||\bar{\mu}_{a,\mathtt{d}}||_{TV} < 1$ and* (5.3.45) *relates $\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy)$ to $\bar{\mu}_{a,\mathtt{d}}(y)\,dy$, $y \in (0,\infty)$. Also, for those $a > 0$ such that $||\bar{\mu}_{a,\mathtt{d}}||_{TV} < 1$, we have that*

$$\int_0^\infty \frac{1-\cos(by)}{y}\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy) \quad = \quad \int_0^{|b|}\arctan\left(\frac{\mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(iu)\right)}{1+\mathtt{Re}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(iu)\right)}\right)du \tag{5.3.52}$$

with, as $b \to \infty$,

$$\arctan\left(\frac{\mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(ib)\right)}{1+\mathtt{Re}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(ib)\right)}\right) = \mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(ib)\right)\left(1+\mathrm{O}\left(\left(\mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(ib)\right)\right)^2\right)\right) \tag{5.3.53}$$

and

$$\lim_{b\to\infty}\frac{\mathtt{d}}{\ln(b)\bar{\mu}\left(\frac{1}{b}\right)}\int_0^\infty \frac{1-\cos(by)}{y}\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy) \geq 1. \tag{5.3.54}$$

Remark 5.3.16. *We note that* (5.3.52) *coupled with* (5.3.53) *give a more tractable way to compute A_ϕ in* (5.3.23)*. When $||\bar{\mu}_{a,\mathtt{d}}||_{TV} < 1$ thanks to* (5.3.45) *we have that*

$$\int_0^\infty \frac{1-\cos(by)}{y}\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy) = \int_0^\infty \frac{1-\cos(by)}{y}\sum_{n=1}^\infty(-1)^{n-1}\frac{\bar{\mu}^{*n}_{a,\mathtt{d}}(dy)}{n}$$

and thus via (5.3.51) *we have precise information for the asymptotic expansion of A_ϕ when the convolutions of $\bar{\mu}_{a,\mathtt{d}}(y) = \mathtt{d}^{-1}e^{-ay}\bar{\mu}(y)$ or equivalently of $\bar{\mu}$ are accessible. For example, if $\bar{\mu}_{a,\mathtt{d}}(y) \overset{0}{\sim} \mathtt{d}^{-1}y^{-\alpha}$, $\alpha \in (0,1)$, then with $B(a,b)$, $a,b > 0$, standing for the classical Beta function and $C_n = \mathtt{d}^{-n}\prod_{j=1}^{n-1}B(j-j\alpha, 1-\alpha)$, $\bar{\mu}^{*n}_{a,\mathtt{d}}(y) \overset{0}{\sim} C_n y^{n-1-n\alpha}$ and, with the obvious notation for asymptotic behaviour of densities of measures,*

$$\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy) \overset{0}{\sim} \left(\sum_{\frac{1}{1-\alpha} > n \geq 1}(-1)^{n-1}C_n y^{n-1-n\alpha} + \mathrm{o}(1)\right)dy.$$

A substitution in (5.3.23) *and elementary calculations yield that*

$$\int_0^\infty \frac{1-\cos(by)}{y}\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy) \overset{\infty}{\approx} \sum_{\frac{1}{1-\alpha}>n\geq 1} (-1)^{n-1}\tilde{C}_n b^{1-n+n\alpha} + \mathrm{O}\left(1\right)$$

with $\tilde{C}_n = C_n \int_0^\infty \frac{1-\cos(v)}{v^{2-n(1-\alpha)}}dv$, $n \in \left[1, \frac{1}{1-\alpha}\right)$, *and the asymptotic expansion of* A_ϕ *follows.*

Proof. Let for the course of the proof $z = a+ib \in \mathbb{C}_a$, $a>0$, and since $|W_\phi(a+ib)| = |W_\phi\left(a-ib\right)|$ without loss of generality we assume throughout that $b>0$. From (5.3.50) we have modulo to $(-\pi,\pi]$

$$\arg\phi\left(z\right) = \arg z + \arg\left(1+\frac{\phi(0)}{\mathtt{d}z} + \mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right). \tag{5.3.55}$$

However, an application of the Riemann-Lebesgue lemma to the function $\bar{\mu}_{a,\mathtt{d}} \in \left(\mathrm{L}^1\left(\mathbb{R}^+\right),*\right)$ yields, as $b\to\infty$, that

$$\left|\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right| = \mathrm{o}\left(1\right). \tag{5.3.56}$$

Therefore, for all b big enough,

$$\arg\left(1+\frac{\phi(0)}{\mathtt{d}\left(a+ib\right)} + \mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right) = \arg\left(1+\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right) - \frac{\phi(0)b}{\mathtt{d}\left(a^2+b^2\right)} + \mathrm{O}\left(\frac{1}{b^2}\right). \tag{5.3.57}$$

Also

$$\begin{aligned}\mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}\left(-ib\right)\right) &= -\int_0^\infty \sin\left(by\right)\bar{\mu}_{a,\mathtt{d}}(y)dy \\ &= -\frac{1}{b}\int_0^\infty \left(1-\cos(by)\right)\mu_{a,\mathtt{d}}(dy) < 0,\end{aligned} \tag{5.3.58}$$

since for $a>0$, $y\mapsto \bar{\mu}_{a,\mathtt{d}}(y) = \mathtt{d}^{-1}e^{-ay}\bar{\mu}(y)$ is strictly decreasing on $\mathbb{R}^+$ and $\mu_{a,\mathtt{d}}(dy) = d\bar{\mu}_{a,\mathtt{d}}(y)$, $y\in(0,\infty)$, is not supported on a lattice. Therefore, from (5.3.56), (5.3.58) and the fact that $b\mapsto \mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)$ is continuous we deduct that $\mathrm{Supp}\,\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}} \cap (-\infty,-1] = \emptyset$. Therefore, Proposition 5.3.14 gives that $\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}} \in \left(\mathrm{L}^1\left(\mathbb{R}^+\right),*\right)$ and from (5.3.48), for all $b>0$,

$$\begin{aligned}\arg\left(1+\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right) &= \mathtt{Im}\left(\log_0\left(1+\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right)\right) \\ &= \mathtt{Im}\left(\mathcal{F}_{\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}}(-ib)\right) = -\int_0^\infty \sin(by)\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy).\end{aligned} \tag{5.3.59}$$

Then, from (5.3.11) due to (5.3.55),(5.3.57) and (5.3.59) we deduce that, for any $a,b>0$,

$$\begin{aligned}A_\phi(a+ib) &= \int_0^b \arg\left(a+iu\right)du + \int_0^b \arg\left(1+\frac{\phi(0)}{\mathtt{d}\left(a+iu\right)} + \mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-iu)\right)du + \tilde{A}_\phi(a) \\ &= \int_0^b \arctan\left(\frac{u}{a}\right)du - \frac{\phi(0)}{2\mathtt{d}}\ln\left(1+\frac{b^2}{a^2}\right) - \int_0^\infty \frac{1-\cos(by)}{y}\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy) + \bar{A}_\phi(a),\end{aligned}$$

where $\tilde{A}_\phi(a)$ is the error induced when $\arg\phi(z) \neq \arg z + \arg\left(1 + \frac{\phi(0)}{\mathtt{d}z} + \mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right) \notin (-\pi,\pi]$ in (5.3.55) which can happen only for small u thanks to (5.3.56) and (5.3.57), and $\bar{A}_\phi(a)$ is the sum of $\tilde{A}_\phi(a)$ and the integral of the error term in (5.3.57) and clearly $\left|\bar{A}_\phi(a)\right| < \infty$. Then the first relation in (5.3.51) follows by a simple integration by parts. The asymptotic relation in (5.3.51) comes from $\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}} \in (\mathrm{L}^1(\mathbb{R}^+), *)$ and the auxiliary claim that for any $h \in (\mathrm{L}^1(\mathbb{R}^+), *)$

$$\left|\int_0^\infty \frac{1-\cos(by)}{y} h(y)dy\right| = \mathrm{o}(|b|), \tag{5.3.60}$$

which follows from the Riemann-Lebesgue lemma invoked in the middle term of

$$\left|\int_0^\infty \frac{1-\cos(by)}{y} h(y)dy\right| = \left|\int_0^b \int_0^\infty \sin(uy)\, h(y)dydu\right| \leq \mathrm{o}(1)\left|\int_0^b du\right|.$$

Finally, since $\lim_{a\to\infty}\int_0^\infty e^{-ay}\bar{\mu}(y)dy = 0$ then $\lim_{a\to\infty} ||\bar{\mu}_{a,\mathtt{d}}||_{TV} = 0$ and thus eventually, for some a large enough, $||\bar{\mu}_{a,\mathtt{d}}||_{TV} < 1$ and $\sup_{b\in\mathbb{R}}\left|\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(ib)\right| < 1$. Choose such $a > 0$. Then, from (5.3.56) with $z \in \mathbb{C}_a$, $b > 0$,

$$\arg\left(1 + \mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right) = \arctan\left(\frac{\mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right)}{1 + \mathtt{Re}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right)}\right) = -\arctan\left(\frac{\mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(ib)\right)}{1 + \mathtt{Re}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(ib)\right)}\right), \tag{5.3.61}$$

and using the latter in (5.3.59) then (5.3.52) follows upon simple integration of (5.3.59). The asymptotic relation (5.3.53) follows from (5.3.56), i.e. $\left|\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(-ib)\right| = \mathrm{o}(1)$ combined with the Taylor expansion of $\arctan x$. Next, consider the bound (5.3.54). Since from (5.3.58) $\mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(ib)\right) > 0$ and, as $b \to \infty$, $\left|\mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(ib)\right)\right| = \mathrm{o}(1)$ we get from (5.3.52), (5.3.53), (5.3.61) and (5.3.59) that

$$\begin{aligned}
\int_0^\infty \frac{1-\cos(by)}{y}\underline{\mathfrak{L}}_{\bar{\mu}_{a,\mathtt{d}}}(dy) &= \int_0^{|b|} \arctan\left(\frac{\mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(iu)\right)}{1 + \mathtt{Re}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(iu)\right)}\right) du \\
&\stackrel{\infty}{\approx} (1 + \mathrm{o}(1)) \int_0^b \mathtt{Im}\left(\mathcal{F}_{\bar{\mu}_{a,\mathtt{d}}}(iu)\right) du \\
&\stackrel{5.3.58}{=} (1 + \mathrm{o}(1)) \int_0^\infty \frac{1-\cos(by)}{y}\bar{\mu}_{a,\mathtt{d}}(y)dy \\
&\geq (1 + \mathrm{o}(1)) \int_0^1 \frac{1-\cos(by)}{y}\bar{\mu}_{a,\mathtt{d}}(y)dy \\
&\geq (1 + \mathrm{o}(1))\, \bar{\mu}_{a,\mathtt{d}}\left(\frac{1}{b}\right) \int_1^b \frac{1-\cos(y)}{y} dy \stackrel{\infty}{\approx} \bar{\mu}_{a,\mathtt{d}}\left(\frac{1}{b}\right) \ln(b).
\end{aligned}$$

This proves (5.3.54) since $\bar{\mu}_{a,\mathtt{d}}(b^{-1}) = e^{-ab^{-1}}\mathtt{d}^{-1}\bar{\mu}(b^{-1}) \stackrel{\infty}{\approx} \mathtt{d}^{-1}\bar{\mu}(b^{-1})$. □

5.3.6 Proof of Theorem 5.3.2(3b)

Let $\phi \in \mathcal{B}_{R_\alpha}$ with $\alpha \in (0,1)$ and let $z = a + ib \in \mathbb{C}_a$, $a > 0$. Then, there exists $\alpha \in (0,1)$ such that $\bar{\mu}(y) = y^{-\alpha}\ell(y)$ and ℓ is quasi-monotone, see (5.2.13) and (5.2.14). Recall the second relation of (5.3.3) which, since in this setting $\mathtt{d} = 0$, takes the form

$$\phi(z) = \phi(0) + z\int_0^\infty e^{-iby}e^{-ay}\bar{\mu}(y)dy = \phi(0) + z\int_0^\infty e^{-iby}y^{-\alpha}e^{-ay}\ell(y)dy. \tag{5.3.62}$$

Since the mapping $y \mapsto \ell(y)e^{-ay}$ is clearly quasi-monotone we conclude from [94, Theorem 1.39] that, for fixed $a > 0$ and $b \to \infty$,

$$\int_0^\infty e^{-iby}y^{-\alpha}e^{-ay}\ell(y)dy \overset{\infty}{\sim} \Gamma(1-\alpha)\left(be^{\frac{i\pi}{2}}\right)^{\alpha-1}\ell\left(\frac{1}{b}\right).$$

Therefore, from (5.3.62) and the last relation we obtain, as $b \to \infty$, that

$$\arg\phi(z) = \arg z + \arg\left(\int_0^\infty e^{-iby}y^{-\alpha}e^{-ay}\ell(y)dy + \frac{\phi(0)}{z}\right) \overset{\infty}{\sim} \arg z + \frac{\pi(\alpha-1)}{2} \overset{\infty}{\sim} \frac{\pi}{2}\alpha,$$

which proves (5.3.24) by using the definition of A_ϕ in (5.3.11). This together with Lemma 5.4.1 establishes the claim.

5.3.7 Proof of Lemma 5.3.12

Let Y_{ϕ_n}, $Y_{\underline{\phi}}$, $n \in \mathbb{N}$, be the random variables associated to $W_{\phi_n}, W_{\underline{\phi}}$, $n \in \mathbb{N}$, see Definition 5.3.1. Clearly, since for any $\phi \in \mathcal{B}$

$$\mathbb{E}\left[e^{tY_\phi}\right] = \sum_{k=0}^\infty t^k\frac{\mathbb{E}\left[Y_\phi^k\right]}{k!} = \sum_{k=0}^\infty t^k\frac{W_\phi(k+1)}{k!} = \sum_{k=0}^\infty t^k\frac{\prod_{j=1}^k\phi(j)}{k!}$$

and Proposition 5.3.13(3) holds, we conclude that $\mathbb{E}\left[e^{tY_\phi}\right]$ is well defined for $t < \frac{1}{\mathtt{d}} \in (0,\infty]$. However, $\lim_{n\to\infty}\phi_n(a) = \underline{\phi}(a)$ implies that $\mathtt{d}^* = \sup_{n\geq 0}\mathtt{d}_n < \infty$, where $\mathtt{d}_n$ are the linear terms in (5.3.3). Therefore, $\mathbb{E}\left[e^{zY_{\underline{\phi}}}\right]$, $\mathbb{E}\left[e^{zY_{\phi_n}}\right]$, $n \in \mathbb{N}$, are analytic in $\mathbb{C}_{\left(-\infty,\min\left\{\frac{1}{\mathtt{d}},\frac{1}{\mathtt{d}^*}\right\}\right)} \supsetneq \mathbb{C}_{(-\infty,0]}$. Moreover, for any $k \in \mathbb{N}$,

$$\begin{aligned}\lim_{n\to\infty}\mathbb{E}\left[Y_{\phi_n}^k\right] &= \lim_{n\to\infty}W_{\phi_n}(k+1)\\ &= \lim_{n\to\infty}\prod_{j=1}^k\phi_n(j) = W_{\underline{\phi}}(k+1) = \mathbb{E}\left[Y_{\underline{\phi}}^k\right].\end{aligned}$$

The last two observations trigger that $\lim_{n\to\infty}Y_{\phi_n} \stackrel{d}{=} Y_{\underline{\phi}}$, see [39, p.269, Example (b)]. Therefore, $\lim_{n\to\infty}W_{\phi_n}(z) = W_{\underline{\phi}}(z)$, $z \in \mathbb{C}_{(0,\infty)}$, which concludes the proof.

5.3.8 Proof of Theorem 5.3.8

The first item is proved, for any $\phi \in \mathcal{B}$, by Berg in [8, Theorem 2.2]. The second item follows readily from the first one after recalling that if $\Psi \in \mathcal{N}$ then $-\Psi(-z) \in \mathcal{N}$. For the item (3), since $\frac{\phi}{\underline{\phi}}$ is completely monotone, and we recall that for any $n \in \mathbb{N}$, $W_\phi(n+1) = \prod_{k=1}^n \phi(k)$, we first get from [9, Theorem 1.3] that the sequence $\left(f_n = \frac{W_{\underline{\phi}}(n+1)}{W_\phi(n+1)}\right)_{n\geq 0}$ is the moment sequence of a positive variable I. Next, from the recurrence equation (5.3.4) combined with the estimates stated in (3), we deduce that $\frac{f_{n+1}}{f_n} = \frac{\underline{\phi}(n)}{\phi(n)} \stackrel{\infty}{=} O(n)$. Thus, there exists $A > 0$ such that for any $a < A$,

$$\mathbb{E}\left[e^{aI}\right] = \sum_{n=0}^{\infty} \frac{f_n}{n!} a^n < \infty,$$

implying that I is moment determinate. Next, with the notation of the statement, if $\mathtt{N} = \mathtt{N}_{\underline{\phi}} - \mathtt{N}_\phi > \frac{1}{2}$ then, as $\left|\mathcal{M}_I\left(i|b| - \frac{1}{2}\right)\right| = \left|\frac{W_{\underline{\phi}}(i|b|+\frac{1}{2})}{W_\phi(i|b|+\frac{1}{2})}\right| \stackrel{\infty}{=} O(|b|^{\mathtt{N}})$ and, from Theorem 5.3.1, for any $\phi \in \mathcal{B}$, $W_\phi \in \mathtt{A}_{(0,\infty)}$ and is zero-free on $\mathbb{C}_{(0,\infty)}$, we obtain that $b \mapsto \mathcal{M}_I\left(ib - \frac{1}{2}\right) \in L^2(\mathbb{R})$ and hence by the Parseval identity for Mellin transform we conclude that $f_I \in L^2(\mathbb{R}^+)$. Finally if $\mathtt{N} > 1$ then the result follows from a similar estimate for the Mellin transform which allows to use a Mellin inversion technique to prove the claim in this case. For the last item, we first observe, from (5.3.27) and Proposition 5.3.13(5), that for any $\phi \in \mathcal{B}$ and $u > 0$,

$$\frac{\phi'(u)}{\phi(u)} = \int_0^\infty e^{-uy}\kappa(dy) \tag{5.3.63}$$

where we recall that $\kappa(dy) = \int_0^y U\left(dy - r\right)\left(r\mu(dr) + \delta_{\mathtt{d}}(dr)\right)$. Thus,

$$\underline{\Psi}(z) = \ln \frac{W_\phi(z+1)}{W_{\underline{\phi}}(z+1)} = \ln \frac{\phi(1)}{\underline{\phi}(1)} z + \int_0^\infty \left(e^{-zy} - 1 - z(e^{-y}-1)\right) \frac{\kappa(dy) - \underline{\kappa}(dy)}{y(e^y - 1)} \tag{5.3.64}$$

where, as in (5.3.63), we have set $\frac{\underline{\phi}'(u)}{\underline{\phi}(u)} = \int_0^\infty e^{-uy}\underline{\kappa}(dy)$. Next, since plainly $\frac{\underline{\phi}'(1)}{\underline{\phi}(1)} - \frac{\phi'(1)}{\phi(1)} < \infty$, we have that the measure $e^{-y}K(dy) = e^{-y}\left(\kappa(dy) - \underline{\kappa}(dy)\right)$ is finite on $\mathbb{R}^+$. Thus, by the Lévy-Khintchine formula, see [11], $\underline{\Psi} \in \mathcal{N}$ if and only if K is a positive measure, which by Bernstein theorem, see e.g. [39], is equivalent to the mapping $u \mapsto \left(\frac{\underline{\phi}'}{\underline{\phi}} - \frac{\phi'}{\phi}\right)(u)$ to be completely monotone. The last equivalent condition being immediate from the definition of K, the proof of the theorem is completed.

5.4 The functional equation (5.1.1)

5.4.1 Proof of Theorem 5.2.1

Recall that by definition $\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z)$, see (5.2.6). From (5.1.2) and (5.1.3) it is clear that formally for $z \in i\mathbb{R}$

$$\mathcal{M}_\Psi(z+1) = \frac{\Gamma(z+1)}{W_{\phi_+}(z+1)} W_{\phi_-}(-z) = \frac{z\Gamma(z)}{\phi_+(z) W_{\phi_+}(z)} \frac{W_{\phi_-}(1-z)}{\phi_-(-z)} = \frac{-z}{\Psi(-z)} \mathcal{M}_\Psi(z). \quad (5.4.1)$$

However, from Theorem 5.3.1(1) it is clear that W_{ϕ_+} (resp. W_{ϕ_-}) extend continuously to $i\mathbb{R} \setminus \mathcal{Z}_0(\phi_+)$ (resp. $i\mathbb{R} \setminus \mathcal{Z}_0(\phi_-)$). Since from (5.1.2) we have that $\mathcal{Z}_0(\Psi) = \mathcal{Z}_0(\phi_+) \cup \mathcal{Z}_0(\phi_-)$, see (5.2.24) and (5.3.5) for the definition of the sets of zeros, we conclude that $\mathcal{M}_\Psi$ satisfies (5.4.1) on $i\mathbb{R} \setminus \mathcal{Z}_0(\Psi)$. The fact that $\mathcal{M}_\Psi \in \mathtt{A}_{(0,1-\bar{\mathfrak{a}}_-)} \cap \mathtt{M}_{(\mathfrak{a}_+,1-\mathfrak{a}_-)}$ then follows from the facts that $W_\phi \in \mathtt{A}_{(\bar{\mathfrak{a}}_\phi,\infty)} \cap \mathtt{M}_{(\mathfrak{a}_\phi,\infty)}$ and W_ϕ is zero-free on $\mathbb{C}_{(\mathfrak{a}_\phi,\infty)}$ for any $\phi \in \mathcal{B}$, see Theorem 5.3.1(1), which lead to

$$\frac{1}{W_{\phi_+}} \in \mathtt{A}_{(\mathfrak{a}_+,\infty)} \text{ and } W_{\phi_-}(1-\cdot) \in \mathtt{A}_{(-\infty,1-\bar{\mathfrak{a}}_-)}. \quad (5.4.2)$$

Thus, (5.2.8), that is $\mathcal{M}_\Psi \in \mathtt{M}_{(\mathfrak{a}_+,1-\mathfrak{a}_-)}$, in general, and (5.2.7), that is $\mathcal{M}_\Psi \in \mathtt{A}_{(\mathfrak{a}_\Psi,1-\bar{\mathfrak{a}}_-)}$, when $\mathfrak{a}_\Psi = \mathfrak{a}_+ \mathbb{I}_{\{\bar{\mathfrak{a}}_+=0\}} = 0$ follow. Note that when $0 = \bar{\mathfrak{a}}_+ > \mathfrak{a}_+$, i.e. $\mathfrak{a}_\Psi = \mathfrak{a}_+ < 0$, then necessarily $\phi_+'(a^+) = \mathtt{d} + \int_0^\infty y e^{ay} \mu_+(dy) \in (0,\infty)$ for any $a > \mathfrak{a}_+$, see (5.3.27), and $\bar{\mathfrak{a}}_+ = 0$, see (5.3.8), is the only zero of ϕ_+ on $(\mathfrak{a}_+,\infty)$. Therefore, Theorem 5.3.1(4) applies and yields that at $z=0$, W_{ϕ_+} has a simple pole, which through (5.3.4) and $\phi_+ < 0$ on $(\mathfrak{a}_+,0)$ is propagated to all $n \in \mathbb{N}$ such that $-n > \mathfrak{a}_+$. These simple poles however are simple zeros for $\frac{1}{W_{\phi_+}} \in \mathtt{A}_{(\mathfrak{a}_+,\infty)}$ which cancel the poles of Γ. Thus, $\mathcal{M}_\Psi \in \mathtt{A}_{(\mathfrak{a}_\Psi,1-\bar{\mathfrak{a}}_-)}$ and (5.2.7) is established. For $z \in \mathbb{C}_{(0,\infty)}$ we have that

$$\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) = \frac{\phi_+(z)}{z} \frac{\Gamma(z+1)}{W_{\phi_+}(z+1)} W_{\phi_-}(1-z) \quad (5.4.3)$$

From Theorem 5.3.1(1) if $\phi_+(0) > 0$, that is $\mathfrak{a}_+ < 0$, then $W_{\phi_+} \in \mathtt{A}_{[0,\infty)}$ and W_{ϕ_+} is zero-free on $\mathbb{C}_{[0,\infty)}$, and the pole of Γ at zero is uncontested, see (5.4.3). Therefore, $\mathcal{M}_\Psi$ extends continuously to $i\mathbb{R} \setminus \{0\}$ in this case. The same follows from (5.4.3) when $\phi_+'(0^+) = \infty$. Let next $\phi_+(0) = 0 = \mathfrak{a}_+$ and $\phi_+'(0^+) < \infty$. Then (5.4.3) shows that the claim $\mathcal{M}_\Psi \in \mathtt{A}_{[0,1-\mathfrak{a}_-)}$ clearly follows. The fact that $\mathcal{M}_\Psi \in \mathtt{M}_{(\mathfrak{a}_+,1-\mathfrak{a}_-)}$, that is (5.2.8) is apparent from (5.4.2). We proceed with the final assertions. Let $\mathfrak{a}_+ \leq \bar{\mathfrak{a}}_+ < 0$. If $-\mathfrak{u}_+ \notin \mathbb{N}$ then $\mathfrak{u}_+$ is the only zero of ϕ_+ on $(\mathfrak{a}_+,\infty)$ and if $\mathfrak{u}_+ = -\infty$ since $\bar{\mathfrak{a}}_+ < 0$ then ϕ_+ has no zeros on $(\mathfrak{a}_+,\infty)$ at all. Henceforth, from (5.3.4) we see that W_{ϕ_+} does not possess poles at the negative integers. Thus, the poles of the function Γ are uncontested. If $-\mathfrak{u}_+ \in \mathbb{N}$ and $\mathfrak{u}_+ = \mathfrak{a}_+$ there is nothing to prove, whereas if $\mathfrak{u}_+ > \mathfrak{a}_+$ then Theorem 5.3.1(3) shows that W_{ϕ_+} has a simple pole at $\mathfrak{u}_+$. Thus $\frac{1}{W_{\phi_+}}$ has a simple zero at $\mathfrak{u}_+$. Then, (5.3.4) propagates the zeros to all $\mathfrak{a}_+ < -n \leq \mathfrak{u}_+$ cancelling the poles of Γ at those locations. The values of the residues are easily computed via the recurrent equation (5.1.3) for W_{ϕ_+}, W_{ϕ_-}, the Wiener-Hopf factorization (5.1.2), the form of $\mathcal{M}_\Psi$, see (5.2.6), and the residues of the gamma function which are of value $\frac{(-1)^n}{n!}$ at $-n$.

5.4.2 Proof of Theorem 5.2.5

Before we commence the proof we introduce some more notation. We use $f \asymp g$ to denote the existence of two positive constants $0 < C_1 < C_2 < \infty$ such that $C_1 \leq \underline{\lim}_{x\to a} \left|\frac{f(x)}{g(x)}\right| \leq \overline{\lim}_{x\to a} \left|\frac{f(x)}{g(x)}\right| \leq C_2$, where a is usually 0 or ∞. The relation $f \lesssim g$, that will be employed from now on, requires only that $\overline{\lim}_{x\to a} \left|\frac{f(x)}{g(x)}\right| \leq C_2 < \infty$.

We recall from (5.2.15) that $\mathcal{B}_P = \{\phi \in \mathcal{B} : \mathtt{d} > 0\}$ and $\mathcal{B}_P^c$ is its complement. Appealing to various auxiliary results below we consider Theorem 5.2.5(1) first. Throughout the proof we use (5.2.6), that is

$$\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z) \in \mathtt{A}_{(0,1-\bar{\mathtt{a}}_-)}. \tag{5.4.4}$$

We note from Definition 5.3.1 and (5.2.44) of Theorem 5.2.27 that $W_{\phi_-}(z)$ and $\frac{\Gamma(z)}{W_{\phi_+}(z)}$ are Mellin transforms of positive random variables. Therefore the bounds

$$|\mathcal{M}_\Psi(z)| \leq \frac{\Gamma(a)}{W_{\phi_+}(a)} \; |W_{\phi_-}(1-z)| \tag{5.4.5}$$

$$|\mathcal{M}_\Psi(z)| \leq \frac{|\Gamma(z)|}{|W_{\phi_+}(z)|} \, |W_{\phi_-}(1-a)| \tag{5.4.6}$$

hold for $z \in \mathbb{C}_a$, $a \in (0, 1-\bar{\mathtt{a}}_-)$. From (5.4.5) and (5.4.6) we have that $\Psi \in \overline{\mathcal{N}}_\infty$, see (5.2.11), if and only if either $\phi_- \in \mathcal{B}_\infty$ and/or $\left|\frac{\Gamma(z)}{W_{\phi_+}(z)}\right|$ decays faster than any polynomial along $a + i\mathbb{R}$, $\forall a \in (0, 1-\bar{\mathtt{a}}_-)$. The former certainly holds if $\phi_- \in \mathcal{B}_P$, that is $\mathtt{d}_- > 0$, since from Proposition 5.4.3(1) we have the even stronger $\phi_- \in \mathcal{B}\left(\frac{\pi}{2}\right) \subset \mathcal{B}_\infty$, and the latter if $\phi_+ \in \mathcal{B}_P^c$, see Proposition 5.4.2(5.4.11). Also, if $\phi_+ \in \mathcal{B}_P$ then Proposition 5.4.2(5.4.11) holds for all $u > 0$ iff $\bar{\mu}_+(0) = \infty$ and hence from (5.4.6) we deduce that $\Psi \in \overline{\mathcal{N}}_\infty$. Next, from Proposition 5.4.3(2) if $\phi_+ \in \mathcal{B}_P$, $\bar{\mu}_+(0) < \infty$ then

$$\phi_- \in \mathcal{B}_\infty \iff \Psi \in \overline{\mathcal{N}}_\infty \iff \overline{\Pi}_-(0) = \infty.$$

Therefore to confirm the second line of (5.2.16) we ought to check only that if $\Psi \in \overline{\mathcal{N}}$, $\phi_+ \in \mathcal{B}_P$ and $\phi_- \in \mathcal{B}_P^c$ then

$$\overline{\Pi}(0) = \infty \iff \overline{\Pi}_-(0) = \infty \text{ or } \bar{\mu}_+(0) = \infty. \tag{5.4.7}$$

However, if $\overline{\Pi}_-(0) < \infty$ and $\bar{\mu}_+(0) < \infty$ then since $\phi_+ \in \mathcal{B}_P$ the Lévy process is a positive linear drift plus compound Poisson process which proves the backward direction of (5.4.7). The forward part is identical. In fact the expression for $\mathtt{N}_\Psi$ in the first line of (5.2.16) is derived as the sum of the rate of polynomial decay of $\left|\frac{\Gamma(z)}{W_{\phi_+}(z)}\right|$ in Proposition 5.4.2 and of $|W_{\phi_-}(z)|$ in Proposition 5.4.3(3) coupled with (5.4.4). The assertions $\phi_- \in \mathcal{B}_P \implies \Psi \in \overline{\mathcal{N}}\left(\frac{\pi}{2}\right)$ and $\phi_- \in \mathcal{B}_{R_\alpha}, \phi_+ \in \mathcal{B}_{R_{1-\alpha}} \implies \Psi \in \overline{\mathcal{N}}\left(\frac{\pi}{2}\alpha\right)$ of item (2) follow from (5.4.5) and

(5.4.6) with the help of items (3a) and (3b) of Theorem 5.3.2. Let $\arg\phi_+ = \arg\phi_-$ hold and choose $a = \frac{1}{2}$. Then from (5.3.19) and (5.4.4) we see that modulo two constants, as $b \to \infty$,

$$\left|\mathcal{M}_\Psi\left(\frac{1}{2}+ib\right)\right| \asymp \frac{\sqrt{\left|\phi_+\left(\frac{1}{2}+ib\right)\right|}}{\sqrt{\left|\phi_-\left(\frac{1}{2}-ib\right)\right|}}\left|\Gamma\left(\frac{1}{2}+ib\right)\right|.$$

From (5.3.32), that is $\mathtt{Re}\left(\phi_-\left(\frac{1}{2}-ib\right)\right) > \phi_-\left(\frac{1}{2}\right) > 0$, from Proposition 5.3.13(3) and the standard asymptotic for the gamma function

$$|\Gamma(a+ib)| = \sqrt{2\pi}|b|^{a-\frac{1}{2}}e^{-\frac{\pi}{2}|b|}\left(1+\mathrm{o}(1)\right), \tag{5.4.8}$$

as $b \to \infty$ and a fixed, see [42, 8.328.1], we see that $\Psi \in \overline{\mathcal{N}}\left(\frac{\pi}{2}\right)$. Finally, the last claim of $\Psi \in \overline{\mathcal{N}}(\Theta_\pm)$ follows readily from (5.4.4) and (5.3.19). This ends the proof. ☐

The next sequence of results are used in the proof above. Recall the classes $\mathcal{B}_\infty$ and $\mathcal{B}(\theta)$, see (5.3.16) and (5.3.17). We start with the following useful lemma.

Lemma 5.4.1. *Let $\phi \in \mathcal{B}$ and assume that there exists $\widehat{a} > \overline{\mathfrak{a}}_\phi$ such that for all $n \in \mathbb{N}$, $\overline{\lim}_{|b|\to\infty} |b|^n\left|W_\phi(\widehat{a}+ib)\right| = 0$ (resp. there exists $\theta \in \left(0, \frac{\pi}{2}\right]$ such that $\overline{\lim}_{|b|\to\infty} \frac{\ln\left|W_\phi(\widehat{a}+ib)\right|}{|b|} \leq -\theta$) then $\phi \in \mathcal{B}_\infty$ (resp. $\phi \in \mathcal{B}(\theta)$).*

Proof. To prove the claims we rely on the Lindelöf's theorem combined with the functional equation (5.3.4) in Definition 5.3.1. First, assume that $\exists\,\widehat{a} > \overline{\mathfrak{a}}_\phi$ such that $\overline{\lim}_{|b|\to\infty} |b|^n\left|W_\phi(\widehat{a}+ib)\right| = 0, \forall n \in \mathbb{N}$ and since $\left|W_\phi(\widehat{a}+ib)\right| = \left|W_\phi(\widehat{a}-ib)\right|$ consider $b > 0$ only. The recurrent equation (5.3.4) and $\left|\phi(\widehat{a}+ib)\right| = \mathtt{d}b + \mathrm{o}(|b|)$, as $b \to \infty$ and $\widehat{a} > \overline{\mathfrak{a}}_\phi$ fixed, see Proposition 5.3.13(3), yield that $\overline{\lim}_{|b|\to\infty} |b|^n\left|W_\phi(1+\widehat{a}+ib)\right| = 0, \forall n \in \mathbb{N}$. Then, we apply the Lindelöf's theorem to the strip $\mathbb{C}^+_{[\widehat{a},\,\widehat{a}+1]} = \mathbb{C}_{[\widehat{a},\,\widehat{a}+1]} \cap \{b \geq 0\}$ and to the functions $f_n(z) = z^n W_\phi(z)$, $n \in \mathbb{N}$, which are holomorphic on $\mathbb{C}^+_{[\widehat{a},\,\widehat{a}+1]}$. Indeed, from our assumptions and the observation above, we have that, for every $n \in \mathbb{N}$ and some finite constants $C_n > 0$,

$$\sup_{z\in\partial\mathbb{C}^+_{[\widehat{a},\,\widehat{a}+1]}} |f_n(z)| \leq C_n$$

and clearly from Definition 5.3.1

$$\sup_{z\in\mathbb{C}^+_{[\widehat{a},\,\widehat{a}+1]}} \left|\frac{f_n(z)}{z^n}\right| = \sup_{z\in\mathbb{C}^+_{[\widehat{a},\,\widehat{a}+1]}} \left|W_\phi(z)\right| = \sup_{v\in[\widehat{a},\,\widehat{a}+1]} W_\phi(v) < \infty.$$

Thus, we conclude from the Lindelöf's strip theorem that

$$\sup_{z\in\mathbb{C}^+_{[\widehat{a},\,\widehat{a}+1]}} |f_n(z)| = \sup_{z\in\mathbb{C}^+_{[\widehat{a},\,\widehat{a}+1]}} \left|z^n W_\phi(z)\right| \leq C_n,$$

see [40, Theorem 1.0.1], which is a discussion of the celebrated paper by Phragmén and Lindelöf, that is [86]. Finally, (5.3.4) and $|\phi(a+ib)| = \mathtt{d}b + \mathrm{o}(b)$, as $b \to \infty$, allows us to deduce that

$$\overline{\lim_{|b|\to\infty}} |b|^n |W_\phi(a+ib)| = 0, \forall n \in \mathbb{N}, \forall a \geq \widehat{a}.$$

To conclude the claim for $a \in (\overline{\mathfrak{a}}_\phi, \widehat{a})$ we use (5.3.4) in the opposite direction and (5.3.32), that is $\mathtt{Re}(\phi(a+ib)) \geq \phi(a) > 0$, for any $a > \overline{\mathfrak{a}}_\phi$, to get that

$$\frac{1}{\phi(a)} |W_\phi(1+a+ib)| \geq \frac{1}{|\phi(a+ib)|} |W_\phi(1+a+ib)| = |W_\phi(a+ib)|. \tag{5.4.9}$$

Thus, $\phi \in \mathcal{B}_\infty$. Next, assume that $\exists \widehat{a} > \overline{\mathfrak{a}}_\phi$, $\theta \in \left(0, \frac{\pi}{2}\right]$ such that $\overline{\lim_{|b|\to\infty}} \frac{\ln|W_\phi(\widehat{a}+ib)|}{|b|} \leq -\theta$. Then, arguing as above, we conclude that this relation holds for $\widehat{a}+1$ too. Then, on $\mathbb{C}^+_{[\widehat{a},\widehat{a}+1]}$, for the function $f_\varepsilon(z) = W_\phi(z)e^{-i(\theta-\varepsilon)z}$, $\varepsilon \in (0,\theta)$, we have from Definition 5.3.1 with some $C > 0$, $D = D(\widehat{a}) > 0$, that

$$\sup_{z\in\partial\mathbb{C}^+_{[\widehat{a},\,\widehat{a}+1]}} |f_\varepsilon(z)| \leq C \quad \text{and} \quad \sup_{b\geq 0} \sup_{v\in[\widehat{a},\widehat{a}+1]} |f_\varepsilon(v+ib)| \leq De^{(\theta-\varepsilon)b} \leq De^{\frac{\pi}{2}b}.$$

This suffices to apply [40, Theorem 1.0.1] with $\tilde{f}_\varepsilon(z) = f_\varepsilon\left(e^{i\frac{\pi}{2}}z\right)$. Therefore, we conclude that $|f_\varepsilon(z)| \leq C$ on $\mathbb{C}^+_{[\widehat{a},\widehat{a}+1]}$ and thus, for all $v \in [\widehat{a}, \widehat{a}+1]$,

$$\overline{\lim_{b\to\infty}} \frac{\ln|W_\phi(v+ib)|}{b} \leq -\theta + \varepsilon.$$

Sending $\varepsilon \to 0$ we conclude that, for all $v \in [\widehat{a}, \widehat{a}+1]$,

$$\overline{\lim_{b\to\infty}} \frac{\ln|W_\phi(v+ib)|}{b} \leq -\theta. \tag{5.4.10}$$

Therefore, from the identities $|W_\phi(a+ib)| = |W_\phi(a-ib)|$ and (5.3.4), the relations $|\phi(a+ib)| = \mathtt{d}b + \mathrm{o}(b)$, as $b \to \infty$ and $a > \overline{\mathfrak{a}}_\phi$ fixed, see Proposition 5.3.13(3), and (5.4.9) we deduct that (5.4.10) holds for $a > \overline{\mathfrak{a}}_\phi$. Thus, we deduce that $\phi \in \mathcal{B}(\theta)$ and conclude the entire proof. $\square$

The proof of Theorem 5.2.5 via (5.4.5) and (5.4.6) hinges upon the assertions of Proposition 5.4.2 and Proposition 5.4.3. Let us examine $\left|\frac{\Gamma(z)}{W_{\phi_+}(z)}\right|$, that is (5.4.6), first.

Proposition 5.4.2. *Let $\phi \in \mathcal{B}^c_P$ then for any $u \geq 0$ and $a > 0$ fixed*

$$\lim_{|b|\to\infty} |b|^u \left|\frac{\Gamma(a+ib)}{W_\phi(a+ib)}\right| = 0. \tag{5.4.11}$$

If $\phi \in \mathcal{B}_P$ then (5.4.11) holds for any $u < \frac{1}{\mathtt{d}}(\phi(0) + \bar{\mu}(0)) \in (0,\infty]$. In fact, if $\bar{\mu}(0) < \infty$ the limit in (5.4.11) is infinity for all $u > \frac{1}{\mathtt{d}}(\phi(0) + \bar{\mu}(0))$. Finally, regardless of the value of $\bar{\mu}(0)$, we have, as $b \to \infty$ and for any $a > 0$ such that $\mathtt{d}^{-1}\int_0^\infty e^{-ay}\bar{\mu}(y)\,dy < 1$,

$$\left|\frac{\Gamma(a+ib)}{W_\phi(a+ib)}\right| \lesssim e^{-\frac{\bar{\mu}\left(\frac{1}{b}\right)+\phi(0)}{\mathtt{d}}\ln b}. \tag{5.4.12}$$

Proof. Let $\phi \in \mathcal{B}$. Fix $a > 0$ and without loss of generality assume that $b > 0$. Applying (5.4.8) to $|\Gamma(a+ib)|$ and (5.3.19) to $|W_\phi(a+ib)|$ we get, as $b \to \infty$, that

$$\left|\frac{\Gamma(a+ib)}{W_\phi(a+ib)}\right| \asymp b^{a-\frac{1}{2}}\sqrt{|\phi(a+ib)|}e^{A_\phi(a+ib)-\frac{\pi}{2}b}. \tag{5.4.13}$$

It therefore remains to estimate $A_\phi(a+ib)$ in the different scenarios stated. Let us start with $\mathtt{d} = 0$ or equivalently $\phi \in \mathcal{B}_P^c$. Then from (5.3.3) we get that

$$\begin{aligned}\phi(a+ib) &= \phi(0) + \int_0^\infty \left(1 - e^{-ay}\cos(by)\right)\mu(dy) + i\int_0^\infty \sin(by)e^{-ay}\mu(dy)\\ &= \mathtt{Re}\left(\phi(a+ib)\right) + i\mathtt{Im}\left(\phi(a+ib)\right).\end{aligned}$$

Clearly, $\mathtt{Re}\left(\phi(a+ib)\right) \geq \phi(a) > 0$, see (5.3.32). Next,

$$\lim_{b\to\infty}\frac{|\mathtt{Im}\left(\phi(a+ib)\right)|}{b} = 0, \tag{5.4.14}$$

which follows from Proposition 5.3.13(3). These facts allow us to deduct that, for any $M > 0$ and any $u > u(M) > 0$,

$$\begin{aligned}|\arg\left(\phi(a+iu)\right)| &= \left|\arctan\left(\frac{\mathtt{Im}\left(\phi(a+iu)\right)}{\mathtt{Re}\left(\phi(a+iu)\right)}\right)\right|\\ &\leq \arctan\left(\frac{|\mathtt{Im}\left(\phi(a+iu)\right)|}{\phi(a)}\right) = \frac{\pi}{2} - \arctan\left(\frac{\phi(a)u}{u\,|\mathtt{Im}\left(\phi(a+iu)\right)|}\right)\\ &\leq \frac{\pi}{2} - \arctan\left(\frac{M\phi(a)}{u}\right),\end{aligned}$$

where the last inequality follows from (5.4.14). Therefore, from the definition of A_ϕ, see (5.3.11), we get that for any $b > u(M)$,

$$|A_\phi(a+ib)| \leq \int_0^b |\arg\phi(a+iu)|\,du \leq \frac{\pi}{2}b - \int_{u(M)}^b \arctan\left(\frac{M\phi(a)}{u}\right)du.$$

However, since $\arctan x \overset{0}{\sim} x$, we see that $\exists\, u'$ big enough such that for any $b > u'$

$$|A_\phi(a+ib)| \leq \frac{\pi}{2}b - \frac{M\phi(a)}{2}\left(\ln b - \ln u'\right).$$

Plugging this in (5.4.13) and using the fact that, for a fixed $a > 0$, $|\phi(a+ib)| \overset{\infty}{=} \mathrm{o}\left(|a+ib|\right)$, when $\mathtt{d} = 0$, see Proposition 5.3.13(3), we easily get that, as $b \to \infty$,

$$\left|\frac{\Gamma(a+ib)}{W_\phi(a+ib)}\right| \lesssim b^a e^{-\frac{M\phi(a)}{2}\ln b} = b^{a-\frac{M\phi(a)}{2}}.$$

Since M is arbitrary we conclude (5.4.11) when $\phi \in \mathcal{B}_P^c$. Assume next that $\phi \in \mathcal{B}_P$ and without loss of generality that $b > 0$. Then from (5.3.51) of Proposition 5.3.15 we get for the exponent of (5.4.13) that, as $b \to \infty$ and any fixed $a > 0$,

$$\begin{aligned} A_\phi(a+ib) - \frac{\pi}{2}b &= -b\arctan\left(\frac{a}{b}\right) - \left(a + \frac{\phi(0)}{\mathtt{d}}\right)\ln b \\ &\quad - \frac{\left(a+\frac{\phi(0)}{\mathtt{d}}\right)}{2}\left(\ln\left(1+\frac{b^2}{a^2}\right) - \ln b^2\right) - \int_0^\infty \frac{1-\cos(by)}{y}\mathfrak{L}_{\bar{\mu}_{a,\mathtt{d}}}(dy) + \bar{A}_\phi(a) \\ &= -\left(a+\frac{\phi(0)}{\mathtt{d}}\right)\ln b - \frac{\left(a+\frac{\phi(0)}{\mathtt{d}}\right)}{2}\ln\left(1+\frac{b^2}{a^2}\right) + \mathrm{O}(1), \end{aligned}$$

where we have used implicitly that $\arctan\frac{a}{b} + \arctan\frac{b}{a} = \frac{\pi}{2}$. Therefore, as $b \to \infty$, (5.4.13) is simplified to

$$\left|\frac{\Gamma(a+ib)}{W_\phi(a+ib)}\right| \asymp b^{a-\frac{1}{2}}\sqrt{|\phi(a+ib)|}e^{-\left(a+\frac{\phi(0)}{\mathtt{d}}\right)\ln b - \int_0^\infty \frac{1-\cos(by)}{y}\mathfrak{L}_{\bar{\mu}_{a,\mathtt{d}}}(dy)}.$$

We recall that $\mathfrak{L}_{\bar{\mu}_{a,\mathtt{d}}}(dy)$ is the measure associated to the measure $\bar{\mu}_{a,\mathtt{d}}(y)dy = \mathtt{d}^{-1}e^{-ay}\bar{\mu}(y)dy$ as defined in (5.3.46). When $\phi \in \mathcal{B}_P$ we have that $|\phi(a+ib)| \sim \mathtt{d}b$, as $b \to \infty$ and $a > 0$ fixed, see Proposition 5.3.13(3), and thus (5.4.13) is simplified further to

$$\left|\frac{\Gamma(a+ib)}{W_\phi(a+ib)}\right| \asymp e^{-\frac{\phi(0)}{\mathtt{d}}\ln b - \int_0^\infty \frac{1-\cos(by)}{y}\mathfrak{L}_{\bar{\mu}_{a,\mathtt{d}}}(dy)}. \tag{5.4.15}$$

Next, choose $a > a_0 > 0$ so as to have $||\bar{\mu}_{a,\mathtt{d}}||_{TV} = \mathtt{d}^{-1}\int_0^\infty e^{-ay}\bar{\mu}(y)dy < 1$. Then (5.3.54) of Proposition 5.3.15 applies and proves (5.4.12) regardless of the value of $\bar{\mu}(0)$. Moreover, (5.4.12) also settles (5.4.11) for those $a > a_0$ and any $u \geq 0$ whenever $\bar{\mu}(0) = \infty$. However, since

$$\left|\frac{\Gamma(1+a+ib)}{W_\phi(1+a+ib)}\right| = \frac{|a+ib|}{|\phi(a+ib)|}\left|\frac{\Gamma(a+ib)}{W_\phi(a+ib)}\right|$$

and $\lim_{b\to\infty}\frac{|a+ib|}{|\phi(a+ib)|} = \mathtt{d}^{-1}$, see Proposition 5.3.13(3), we trivially conclude (5.4.11), when $\bar{\mu}(0) = \infty$, for any $a > 0$, $u \geq 0$. Next, let $\bar{\mu}(0) < \infty$ and choose again $a > a_0$ so that $||\bar{\mu}_{a,\mathtt{d}}||_{TV} < 1$. Then (5.3.45) holds and thanks to Proposition 5.4.10 we have that $\bar{\mu}^{*n}_{a,\mathtt{d}}(y) = e^{-ay}\frac{\bar{\mu}^{*n}(y)}{\mathtt{d}^n}$. Therefore, on $(0,\infty)$,

$$\begin{aligned} \mathfrak{L}_{\bar{\mu}_{a,\mathtt{d}}}(dy) &= \sum_{n=1}^\infty (-1)^{n-1}\frac{\bar{\mu}^{*n}_{a,\mathtt{d}}(y)}{n}dy \\ &= e^{-ay}\sum_{n=1}^\infty (-1)^{n-1}\frac{\bar{\mu}^{*n}(y)}{\mathtt{d}^n n}dy = \frac{1}{\mathtt{d}}e^{-ay}\bar{\mu}(y)dy + h(y)dy. \end{aligned}$$

Now since $\bar{\mu}(0) < \infty$ and therefore $\bar{\mu} \in \mathrm{L}^\infty(\mathbb{R}^+)$ we have from Proposition 5.4.10(5.4.74) applied with $a = a' = 0$ that

$$|h(y)| \leq e^{-ay}\sum_{n=2}^\infty \frac{\bar{\mu}^{*n}(y)}{\mathtt{d}^n n}dy \leq ye^{-ay}\sum_{n=2}^\infty \frac{\bar{\mu}^n(0)y^{n-2}}{\mathtt{d}^n n!}.$$

Thus

$$\left|\int_0^\infty \frac{1-\cos(by)}{y}h(y)dy\right| \leq 2\sum_{n=2}^\infty \frac{\bar{\mu}^n(0)}{n(n-1)\mathtt{d}^n a^{n-1}} < \infty \iff a \geq \frac{\bar{\mu}(0)}{\mathtt{d}}.$$

Therefore, if we choose $a > a_0 \vee \frac{\bar{\mu}(0)}{\mathtt{d}}$ we see that the term above does not contribute to the asymptotic in (5.4.15). We are then left with the relation

$$\left|\frac{\Gamma(a+ib)}{W_\phi(a+ib)}\right| \asymp e^{-\frac{\phi(0)}{\mathtt{d}}\ln b - \frac{1}{\mathtt{d}}\int_0^\infty \frac{1-\cos(by)}{y}e^{-ay}\bar{\mu}(y)dy}. \tag{5.4.16}$$

First, since $y \mapsto e^{-ay}\frac{\bar{\mu}(y)}{y}$ is integrable on $(1,\infty)$, the Riemann-Lebesgue lemma yields that

$$\lim_{b\to\infty}\int_1^\infty \frac{1-\cos(by)}{y}e^{-ay}\bar{\mu}(y)dy = \int_1^\infty e^{-ay}\bar{\mu}(y)\frac{dy}{y}. \tag{5.4.17}$$

Next, recall that $\bar{\mu}(0) < \infty$. Henceforth, from the dominated convergence theorem

$$\lim_{b\to\infty}\int_0^{\frac{1}{b}} \frac{1-\cos(by)}{y}e^{-ay}\bar{\mu}(y)dy = \lim_{b\to\infty}\int_0^1 \frac{1-\cos y}{y}e^{-a\frac{y}{b}}\bar{\mu}\left(\frac{y}{b}\right)dy = \bar{\mu}(0)\int_0^1 \frac{1-\cos y}{y}dy. \tag{5.4.18}$$

Recall that $y \mapsto \bar{\mu}_a(y) = e^{-ay}\bar{\mu}(y)$ is decreasing on $\mathbb{R}^+$, thus defining a measure $\bar{\mu}_a(dv)$ on $(0,\infty)$. Therefore, the remaining portion of the integral in the exponent of (5.4.16) is written as

$$\begin{aligned}\int_{\frac{1}{b}}^1 \frac{1-\cos(by)}{y}e^{-ay}\bar{\mu}(y)dy &= \int_{\frac{1}{b}}^1 \frac{1-\cos(by)}{y}\left(\bar{\mu}_a(y) - \bar{\mu}_a\left(\frac{1}{b}\right) + \bar{\mu}_a\left(\frac{1}{b}\right)\right)dy \\ &= \bar{\mu}_a\left(\frac{1}{b}\right)\left(\ln b - \int_1^b \frac{\cos y}{y}dy\right) \\ &- \int_{\frac{1}{b}}^1 \frac{1}{y}\left(\bar{\mu}_a\left(\frac{1}{b}\right) - \bar{\mu}_a(y)\right)dy + \int_{\frac{1}{b}}^1 \frac{\cos(by)}{y}\int_{\frac{1}{b}}^y \bar{\mu}_a(dv)dy.\end{aligned} \tag{5.4.19}$$

Clearly, then

$$\overline{\lim_{b\to\infty}}\left|\int_{\frac{1}{b}}^1 \frac{\cos(by)}{y}\int_{\frac{1}{b}}^y \bar{\mu}_a(dv)dy\right| \leq \overline{\lim_{b\to\infty}}\int_{\frac{1}{b}}^1 \left|\int_{bv}^b \frac{\cos y}{y}dy\right|\bar{\mu}_a(dv) = 0,$$

since $\overline{\lim_{b\to\infty}}\left|\int_{bv}^b \frac{\cos y}{y}dy\right| = 0$, for all $v \in (0,1)$, and the validity of the dominated convergence theorem which is due to $\sup_{x\geq 1}\left|\int_x^\infty \frac{\cos y}{y}dy\right| < \infty$ and $\int_0^1 |\bar{\mu}_a(dv)| = \bar{\mu}(0) - e^{-1}\bar{\mu}(1) < \infty$. Henceforth, from (5.4.17), (5.4.18) and (5.4.19), we obtain, as $b \to \infty$, that

$$\left|\int_0^\infty \frac{1-\cos(by)}{y}e^{-ay}\bar{\mu}(y)dy - \bar{\mu}(0)\ln b\right| \leq \left(\bar{\mu}(0) - \bar{\mu}_a\left(\frac{1}{b}\right)\right)\ln b + \int_{\frac{1}{b}}^1 \left(\bar{\mu}(0) - \bar{\mu}_a(y)\right)\frac{dy}{y} + C + \mathrm{o}(1)$$

where $C > 0$. However, it can be seen easily that the first and the second term on the right-hand side are of order $\mathrm{o}(\ln b)$ and therefore, as $b \to \infty$,

$$\int_0^\infty \frac{1-\cos(by)}{y} e^{-ay}\bar{\mu}(y)dy = \bar{\mu}(0)\ln b + \mathrm{o}(\ln b).$$

This fed in (5.4.16) yields

$$\left|\frac{\Gamma(a+ib)}{W_\phi(a+ib)}\right| \asymp e^{-\frac{\phi(0)}{\mathtt{d}}\ln b - \frac{\bar{\mu}(0)}{\mathtt{d}}\ln b + \mathrm{o}(\ln b)}, \tag{5.4.20}$$

which proves (5.4.11) for $u < \frac{1}{\mathtt{d}}(\phi(0)+\bar{\mu}(0))$ and shows that the limit in (5.4.11) is infinity for $u > \frac{1}{\mathtt{d}}(\phi(0)+\bar{\mu}(0))$. This concludes the proof of Proposition 5.4.2. □

Proposition 5.4.2 essentially deals exhaustively with the proof of Theorem 5.2.5 via the term $\left|\frac{\Gamma(z)}{W_{\phi_+}(z)}\right|$ in (5.4.6). Since $\left|\frac{\Gamma(z)}{W_{\phi_+}(z)}\right|$ decays faster than any polynomial except when $\phi_+ \in \mathcal{B}_P$ and $\bar{\mu}_+(0) < \infty$, it remains to discuss this scenario. Before stating and proving Proposition 5.4.3 we introduce some more notation needed throughout below. We recall that with any $\phi \in \mathcal{B}$ we have an associated non-decreasing (possibly killed at independent exponential rate of parameter $\phi(0) > 0$) Lévy process ξ. Then the potential measure of ξ and therefore of ϕ is defined as

$$U(dy) = \int_0^\infty e^{-\phi(0)t}\mathbb{P}(\xi_t \in dy)\,dt,\; y > 0, \tag{5.4.21}$$

and from Proposition 5.3.13(5) we get that, for $z \in \mathbb{C}_{(0,\infty)}$,

$$\int_0^\infty e^{-zy}U(dy) = \frac{1}{\phi(z)}. \tag{5.4.22}$$

The renewal or potential function $U(y) = \int_0^y U(dx)$, $y > 0$, is subadditive on $(0,\infty)$. Recall that if $\Psi \in \overline{\mathcal{N}}$ then $\Psi(z) = -\phi_+(-z)\phi_-(z)$ and we have two potential measures $U_\pm$ related to $\phi_\pm$ respectively. If in addition $\phi \in \mathcal{B}_P$ then it is well known from [11, Chapter III] that the potential density $u(y) = \frac{U(dy)}{dy}$ exists. Moreover, it is continuous, strictly positive and bounded on $[0,\infty)$, that is $||u||_\infty < \infty$. Furthermore, [36, Proposition 1] establishes that in this case

$$u(y) = \sum_{j=0}^\infty \frac{(-1)^j}{\mathtt{d}^{j+1}}\left(\mathbf{1} * (\phi(0)+\bar{\mu})^{*j}\right)(y) = \frac{1}{\mathtt{d}} + \tilde{u}(y),\; y \ge 0, \tag{5.4.23}$$

where $\mathbf{1}(y) = \mathbb{I}_{\{y>0\}}$ stands for the Heavyside function and $f * g(x) = \int_0^x f(x-v)g(v)dv$ represents the convolution of two functions. We keep the last notation for convolutions of measures too. Then we have the result.

Proposition 5.4.3. *Let $\phi_+ \in \mathcal{B}_P$ and $\bar{\mu}_+(0) < \infty$. Then we have the following scenarios.*

1. *If $\phi_- \in \mathcal{B}_P$ then $\phi_- \in \mathcal{B}\left(\frac{\pi}{2}\right)$.*

2. *If* $\phi_- \in \mathcal{B}_P^c$ *then* $\phi_- \in \mathcal{B}_\infty \iff \overline{\Pi}_-(0) = \infty$.

3. *If* $\phi_- \in \mathcal{B}_P^c$ *and* $\overline{\Pi}_-(0) < \infty$ *then for any* $u < \frac{\int_0^\infty u_+(y)\Pi_-(dy)}{\phi_-(0)+\bar{\mu}_-(0)}$ *and any* $a > 0$ *we have that* $\lim_{|b|\to\infty} |b|^u |W_{\phi_-}(a+ib)| = 0$. *If* $u > \frac{\int_0^\infty u_+(y)\Pi_-(dy)}{\phi_-(0)+\bar{\mu}_-(0)}$ *then for any* $a > 0$ *the following* $\lim_{|b|\to\infty} |b|^u |W_{\phi_-}(a+ib)| = \infty$ *is valid.*

Remark 5.4.4. *We stress that the validity of item* (2)*, that is* $\phi_- \in \mathcal{B}_\infty$*, can be established for general* $\phi \in \mathcal{B}$ *as long as the following three conditions are satisfied. First, the Lévy measure of* ϕ *is absolutely continuous, that is* $\mu(dy) = \upsilon(y)dy$*,* $y > 0$*, and* $\upsilon(0) = \infty$*. Secondly,* $\upsilon(y) = \upsilon_1(y) + \upsilon_2(y)$ *such that* $\upsilon_1, \upsilon_2 \in \mathrm{L}^1(\mathbb{R}^+)$ *and* $\upsilon_1 \geq 0$ *is non-increasing on* $\mathbb{R}^+$*. Finally,* $\int_0^\infty \upsilon_2(y)dy \geq 0$ *and* $|\upsilon_2(x)| \leq \left(\int_x^\infty \upsilon_1(y)dy\right) \vee C$ *for some* $C > 0$ *on* $\mathbb{R}^+$*. Imposing* $\phi_+ \in \mathcal{B}_P$ *and* $\bar{\mu}_+(0) < \infty$ *ensures precisely those conditions for* ϕ_-*. In fact Lemma 5.4.5, Proposition 5.4.6 and Lemma 5.4.7 modulo to* (5.4.58),(5.4.59) *serve only the purpose to check the validity of those conditions. For item* (3) *for general* $\phi \in \mathcal{B}$ *it suffices to assume that* $\mu(dy) = \upsilon(y)dy$*,* $y > 0$*,* $\upsilon(0^+) = \lim_{y\to 0} \upsilon(y) < \infty$ *and* $\upsilon \in \mathrm{L}^\infty(\mathbb{R}^+)$*. Also note that* $\upsilon_-(0^+) = \int_0^\infty u_+(y)\Pi_-(dy)$*.*

Proof. When $\phi_- \in \mathcal{B}_P$ it follows immediately from (5.3.51) of Proposition 5.3.15 that $\phi_- \in \mathcal{B}\left(\frac{\pi}{2}\right)$ and item (1) is proved. Let us proceed with item (2). Since $\phi_+ \in \mathcal{B}_P$ from [34, Chapter V, (5.3.11)] or Proposition 5.7.2 we can obtain the differentiated version of (5.7.5), that is

$$\mu_-(dy) = \upsilon_-(y)\,dy = \int_0^\infty u_+(v)\Pi_-(y+dv)dy. \tag{5.4.24}$$

Note that

$$\bar{\mu}_-(0) = \int_0^\infty \upsilon_-(y)\,dy = \infty \iff \int_0^1 \overline{\Pi}_-(y)dy = \infty. \tag{5.4.25}$$

Indeed, from (5.4.24), for any $y < 1$,

$$\left(\inf_{0\leq v\leq 1} u_+(v)\right)\left(\overline{\Pi}_-(y) - \overline{\Pi}_-(1)\right) \leq \upsilon_-(y) \leq ||u_+||_\infty \overline{\Pi}_-(y).$$

Then $\inf_{0\leq v\leq 1} u_+(v) > 0$ since $u_+(0) = \frac{1}{\mathtt{d}_+} > 0$, $u_+ \in \mathtt{C}([0,\infty))$ and u_+ never touches 0 whenever $\mathtt{d}_+ > 0$, see [11, Chapter III], and (5.4.25) is established. Next $\phi_+ \in \mathcal{B}_P$ and $\bar{\mu}_+(0) < \infty$ trigger the simultaneous validity of Lemma 5.4.5, Lemma 5.4.7 and Lemma 5.4.8 provided $\bar{\mu}_-(0) = \infty$ or equivalently $\int_0^1 \overline{\Pi}_-(y)dy = \infty$, see (5.4.25) above. Assume the latter and note that $\bar{\mu}_-(0) = \infty$ is only needed for Lemma 5.4.8 so that $\phi_-^c(\infty) = \infty$ is valid when $\mathtt{d}_- = 0$, as it is the case in item (2) here. Then, we can always choose $c \in \mathbb{R}^+$ such that (5.4.63) holds for any $a \geq a_c > 0$, namely, for all b large enough

$$\arg(\phi_-(a+ib)) = \arg(\phi_-^c(a+ib))(1+\mathrm{o}(1)) + \arg\left(1 - \mathcal{F}_{\Xi_a^c}(-ib)\right)$$

with Ξ_a^c as in Lemma 5.4.8. Thus, from the definition of A_ϕ, see (5.3.11), we get that, as $b \to \infty$,

$$A_{\phi_-}(a+ib) = A_{\phi_-^c}(a+ib)\left(1+\mathrm{o}\left(1\right)\right) + \int_0^b \arg\left(1 - \mathcal{F}_{\Xi_a^c}(-iu)\right) du.$$

However, since in (5.3.19) of Theorem 5.3.2, G_ϕ does not depend on b, whereas E_ϕ and R_ϕ are uniformly bounded on $\mathbb{C}_a$ and $\phi_- \in \mathcal{B}$ and $A_\phi \geq 0$, we conclude that, for every $a > a_c$ fixed and any $\eta \in \left(0, \frac{1}{2}\right)$, as $b \to \infty$,

$$\begin{aligned} |W_{\phi_-}(a+ib)| &\lesssim \frac{\left|\phi_-^c\left(a+ib\right)\right|^{\frac{1}{2}+\eta}}{\sqrt{\left|\phi_-\left(a+ib\right)\right|}} e^{-\int_0^b \arg\left(1-\mathcal{F}_{\Xi_a^c}(-iu)\right)du} \left|W_{\phi_-^c}(a+ib)\right|^{1+2\eta} \\ |W_{\phi_-}(a+ib)| &\gtrsim \frac{\left|\phi_-^c\left(a+ib\right)\right|^{\frac{1}{2}-\eta}}{\sqrt{\left|\phi_-\left(a+ib\right)\right|}} e^{-\int_0^b \arg\left(1-\mathcal{F}_{\Xi_a^c}(-iu)\right)du} \left|W_{\phi_-^c}(a+ib)\right|^{1-2\eta}. \end{aligned} \tag{5.4.26}$$

However, the Lévy measure of ϕ_-^c is $\mu_-^c(dy) = \mathbb{I}_{\{y<c\}}\mathtt{d}_+^{-1}\overline{\Pi}_-^c(y)dy$, see (5.4.56), and then since $\overline{\Pi}_-^c(y)$ is non-increasing on $(0,\infty)$ we deduce via integration by parts of (5.4.56) that

$$\Psi^c(z) = z\phi_-^c(z) = \phi_-(0)z + \frac{1}{\mathtt{d}_+}\int_{-c}^0 \left(e^{zr} - 1 - zr\right)\Pi^c(-dr) \in \overline{\mathcal{N}}.$$

However, [83, Theorem 5.1 (5.3)] shows that $\phi_-^c \in \mathcal{B}_\infty \iff \overline{\Pi}_-^c(0) = \infty$ (the latter being equivalent to $\mathtt{N}_{\mathfrak{r}} = \infty$ in the notation of [83] and $W_{\phi_-^c} = \mathcal{M}_{V_\psi}$ therein). Moreover, since $\overline{\Pi}_-^c(y) = \left(\overline{\Pi}_-(y) - \overline{\Pi}_-(c)\right)\mathbb{I}_{\{y\leq c\}}$, see Lemma 5.4.5, we obtain that $\phi_-^c \in \mathcal{B}_\infty \iff \overline{\Pi}_-(0) = \infty$. It remains henceforth to understand the terms to the right-hand side of (5.4.26) and show that they cannot disrupt the subexponential decay brought in by $\left|W_{\phi_-^c}(a+ib)\right|^{1+2\eta}$. With the notation and the claim of Lemma 5.4.8 we have that $\lim_{a\to\infty} ||\Xi_a^c||_{TV} = 0$ and thus there exists $a_0 > a_c$ such that for all $a > a_0$, $||\Xi_a^c||_{TV} < 1$. Therefore, from (5.3.49) of Proposition 5.3.14 we get that for all $u \in \mathbb{R}$, $\log_0\left(1 - \mathcal{F}_{\Xi_a^c}(-iu)\right) = -\mathcal{F}_{\mathfrak{L}_{\Xi_a^c}}(-iu)$. Moreover, $||\Xi_a^c||_{TV} < 1$ implies that the first expression in (5.3.45) holds. Henceforth,

$$\begin{aligned} \arg\left(1 - \mathcal{F}_{\Xi_a^c}(-iu)\right) &= \mathtt{Im}\left(\log_0\left(1 - \mathcal{F}_{\Xi_a^c}(-iu)\right)\right) \\ &= -\mathtt{Im}\left(\mathcal{F}_{\Xi_a^c}(-iu)\right) - \sum_{n=2}^\infty \frac{\mathtt{Im}\left(\mathcal{F}^n_{\Xi_a^c}(-iu)\right)}{n} \\ &= -\mathtt{Im}\left(\mathcal{F}_{\Xi_a^c}(-iu)\right) - \sum_{n=2}^\infty \frac{\mathtt{Im}\left(\mathcal{F}_{(\Xi_a^c)^{*n}}(-iu)\right)}{n}. \end{aligned} \tag{5.4.27}$$

We work with the remainder term of (5.4.26), that is $\int_0^b \arg\left(1 - \mathcal{F}_{\Xi_a^c}(-iu)\right)du$ and $b > 0$. We start with the infinite sum in (5.4.27) clarifying that, for each $n \geq 1$,

$$\mathtt{Im}\left(\mathcal{F}_{(\Xi_a^c)^{*n}}(-iu)\right) = -\int_0^\infty \sin\left(uy\right)\left(\chi_a^c\right)^{*n}(y)dy, \tag{5.4.28}$$

where from Lemma 5.4.8 $\chi^c_a(y)dy = e^{-ay}\chi^c(y)dy$ is the density of $\Xi^c_a(dy)$ and for any $a \geq a_c$, $||\chi^c_a||_\infty < \infty$. By definition $a_0 = \max\{a_0, a_c\}$. We work from now on with $0 < a_0 < a$. Next, note that, for each $b > 0$, from Proposition 5.4.10(5.4.74), we have that

$$\begin{aligned}\int_0^b\int_0^\infty\left|\sin(uy)\sum_{n=2}^\infty\frac{(\chi^c_a)^{*n}(y)}{n}\right|dydu &\leq b\int_0^\infty\sum_{n=2}^\infty||\chi^c_{a_0}||^n_\infty\frac{y^{n-1}e^{-(a-a_0)y}}{n!}dy\\ &= b\sum_{n=2}^\infty\frac{||\chi^c_{a_0}||^n_\infty}{n\,(a-a_0)^n} < \infty \iff a-a_0 > ||\chi^c_{a_0}||_\infty.\end{aligned} \tag{5.4.29}$$

So since (5.4.26) is valid for any $a > a_0 > a_c$ we, from now on, fix $a > 2||\chi^c_{a_0}||_\infty + a_0$. Then (5.4.29) allows via integration by parts in (5.4.30) below to conclude using (5.4.27), (5.4.28) and Proposition 5.4.10(5.4.74) that, for any $b > 0$,

$$\begin{aligned}&\left|\int_0^b\left(\arg\left(1-\mathcal{F}_{\Xi^c_a}(-iu)\right)+\mathtt{Im}\left(\mathcal{F}_{\Xi^c_a}(-iu)\right)\right)du\right| = \left|\int_0^b\sum_{n=2}^\infty\frac{\mathtt{Im}\left(\mathcal{F}_{(\Xi^c_a)^{*n}}(-iu)\right)}{n}du\right|\\ &=\left|\int_0^b\sum_{n=2}^\infty\frac{\int_0^\infty\sin(uy)\,(\chi^c_a)^{*n}(y)dy}{n}du\right| = \left|\int_0^\infty\frac{1-\cos(by)}{y}\sum_{n=2}^\infty\frac{(\chi^c_a)^{*n}(y)}{n}dy\right|\\ &\leq 2\int_0^\infty\sum_{n=2}^\infty||\chi^c_{a_0}||^n_\infty\frac{y^{n-2}e^{-(a-a_0)y}}{n!}dy = 2\sum_{n=2}^\infty\frac{||\chi^c_{a_0}||^n_\infty}{n(n-1)\,(a-a_0)^{n-1}} < \infty.\end{aligned} \tag{5.4.30}$$

Since the right-hand side is independent of $b > 0$ we deduct that for $a > 2||\chi^c_{a_0}||_\infty + a_0$, (5.4.26) is simplified to

$$\begin{aligned}|W_{\phi_-}(a+ib)| &\lesssim \frac{|\phi^c_-(a+ib)|^{\frac{1}{2}+\eta}}{\sqrt{|\phi_-(a+ib)|}}e^{\int_0^b\mathtt{Im}\left(\mathcal{F}_{\Xi^c_a}(-iu)\right)du}\left|W_{\phi^c_-}(a+ib)\right|^{1+2\eta}\\ |W_{\phi_-}(a+ib)| &\gtrsim \frac{|\phi^c_-(a+ib)|^{\frac{1}{2}-\eta}}{\sqrt{|\phi_-(a+ib)|}}e^{\int_0^b\mathtt{Im}\left(\mathcal{F}_{\Xi^c_a}(-iu)\right)du}\left|W_{\phi^c_-}(a+ib)\right|^{1-2\eta}\end{aligned} \tag{5.4.31}$$

Next from (5.4.28)

$$\begin{aligned}\left|\int_0^b\mathtt{Im}\left(\mathcal{F}_{\Xi^c_a}(-iu)\right)du\right| &= \left|\int_0^\infty\frac{1-\cos(by)}{y}\chi^c_a(y)dy\right|\\ &\leq\left|\int_0^1\frac{1-\cos(by)}{y}\chi^c_a(y)dy\right|+\left|\int_1^\infty\frac{1-\cos(by)}{y}\chi^c_a(y)dy\right|.\end{aligned} \tag{5.4.32}$$

From the Riemann-Lebesgue lemma applied to the absolutely integrable function $\chi^c_a(y)y^{-1}\mathbb{I}_{\{y>1\}}$ we get that

$$\lim_{b\to\infty}\left|\int_1^\infty\frac{1-\cos(by)}{y}\chi^c_a(y)dy\right| = \left|\int_1^\infty\frac{\chi^c_a(y)}{y}dy\right| =: D_a.$$

Therefore, using the fact that $||\chi_a^c||_\infty < \infty$, see Lemma 5.4.8, we conclude, for all $b > 1$ big enough, that

$$\left|\int_0^b \mathtt{Im}\left(\mathcal{F}_{\Xi_a^c}(-iu)\right)du\right| \leq \int_0^b \frac{1-\cos y}{y}\left|\chi_a^c\left(\frac{y}{b}\right)\right|dy + 2D_a \tag{5.4.33}$$
$$\leq ||\chi_a^c||_\infty\left(\int_0^1 \frac{1-\cos y}{y}dy + \int_1^b \frac{1-\cos y}{y}dy\right) + 2D_a$$
$$\leq ||\chi_a^c||_\infty \int_0^1 \frac{1-\cos y}{y}dy + ||\chi_a^c||_\infty \ln b + \tilde{D}_a,$$

where $\tilde{D}_a = 2D_a + \sup_{b>1}\left|\int_1^b \frac{\cos y}{y}dy\right| < \infty$. This allows us to conclude in (5.4.31), as $b \to \infty$,

$$\begin{aligned}|W_{\phi_-}(a+ib)| &\lesssim b^{||\chi_a^c||_\infty + \frac{1}{2}+\eta}\left|W_{\phi_-^c}(a+ib)\right|^{1+2\eta}\\ |W_{\phi_-}(a+ib)| &\gtrsim b^{-||\chi_a^c||_\infty - \frac{1}{2}}\left|W_{\phi_-^c}(a+ib)\right|^{1-2\eta},\end{aligned} \tag{5.4.34}$$

where we have also used the standard relation $|\phi(a+ib)| \overset{\infty}{\sim} \mathtt{d}b + \mathrm{o}(b)$, see Proposition 5.3.13(3), and for any $a > 0$ and any $\phi \in \mathcal{B}$, $|\phi(a+ib)| \geq \mathtt{Re}(\phi(a+ib)) \geq \phi(a) > 0$, see (5.3.32). Hence, as mentioned below (5.4.26) from [83, Theorem 5.1 (5.3)] we have that $\phi_-^c \in \mathcal{B}_\infty \iff \overline{\Pi}_-^c(0) = \infty$ and since $\overline{\Pi}_-^c(y) = \left(\overline{\Pi}_-(y) - \overline{\Pi}_-(c)\right)\mathbb{I}_{\{y\leq c\}}$, see Lemma 5.4.5, we conclude that $\phi_-^c \in \mathcal{B}_\infty \iff \overline{\Pi}_-(0) = \infty$. This together with (5.4.34) and Lemma 5.4.1 shows that

$$\int_0^1 \overline{\Pi}_-(y)dy = \infty \implies \overline{\Pi}_-(0) = \infty \implies \phi_- \in \mathcal{B}_\infty.$$

Let next $\int_0^1 \overline{\Pi}_-(y)dy < \infty$ but $\overline{\Pi}_-(0) = \infty$. Unfortunately, we cannot easily use similar comparison as above despite that $\phi_-^c \in \mathcal{B}_\infty \iff \overline{\Pi}_-^c(0) = \infty$. In fact Lemma 5.4.8 fails to give a good and quick approximation of $\arg\phi_-$ with $\arg\phi_-^c$. We choose a different route. An easy computation involving (5.4.24), $u_+(0) = \frac{1}{\mathtt{d}_+} \in (0,\infty)$, see (5.4.23), and $u_+ \in \mathtt{C}([0,\infty))$, gives that

$$\lim_{y\to 0} \upsilon_-(y) = \infty, \tag{5.4.35}$$

since for any $\varepsilon > 0$, $\upsilon_-(y) \geq \left(\inf_{v\in[0,\varepsilon)} u_+(v)\right)\left(\overline{\Pi}_-(y) - \overline{\Pi}_-(y+\varepsilon)\right)$. Since we aim to show that $\phi_- \in \mathcal{B}_\infty$ from Lemma 5.4.1 we can work again with a single a which we will choose later. From $|W_{\phi_-}(a+ib)| = |W_{\phi_-}(a-ib)|$ we can focus on $b > 0$ only as well. From the alternative expression for A_{ϕ_-}, see (5.3.20) in the claim of Theorem 5.3.1(1), we get that

$$\begin{aligned}A_{\phi_-}(a+ib) = b\Theta_{\phi_-}(a+ib) &= \int_a^\infty \ln\left(\left|\frac{\phi_-(u+ib)}{\phi_-(u)}\right|\right)du\\ &\geq \int_a^\infty \ln\left(\left|1 + \frac{\mathtt{Re}\left(\phi_-(u+ib) - \phi_-(u)\right)}{\phi_-(u)}\right|\right)du.\end{aligned} \tag{5.4.36}$$

Next, we note that since $\mathtt{d}_- = 0$,

$$\frac{\mathtt{Re}\left(\phi_-(u+ib) - \phi_-(u)\right)}{\phi_-(u)} = \frac{\int_0^\infty (1-\cos(by))\,e^{-uy}\upsilon_-(y)\,dy}{\phi_-(u)} \geq 0.$$

Moreover, since $\bar{\mu}_{\text{-}}(0) < \infty$,

$$\lim_{u\to\infty} \sup_{b\in\mathbb{R}} \int_0^\infty (1-\cos(by))\, e^{-uy} \upsilon_{\text{-}}(y)\, dy \le \lim_{u\to\infty} 2\int_0^\infty e^{-uy}\upsilon_{\text{-}}(y)\, dy = 0$$

and $\lim\limits_{u\to\infty} \phi_{\text{-}}(u) = \phi_{\text{-}}(\infty) < \infty$, see (5.3.3). We choose $\widehat{a} > 0$ large enough so that $\phi_{\text{-}}(u) > \frac{\phi_{\text{-}}(\infty)}{2}$ and $\sup_{b\in\mathbb{R}} \int_0^\infty (1-\cos(by))\, e^{-uy}\upsilon_{\text{-}}(y)\, dy \le \frac{\phi_{\text{-}}(\infty)}{4}$, for all $u \ge \widehat{a}$. Therefore from (5.4.36) and $\ln(1+x) \ge Cx$, $\forall x < \frac{1}{2}$ with some $C > 0$, we deduce that for any $\varepsilon > 0$ and $b > \frac{1}{\varepsilon}$

$$\begin{aligned}
A_{\phi_{\text{-}}}(\widehat{a}+ib) &\ge C\int_{\widehat{a}}^\infty \frac{\int_0^\infty (1-\cos(by))\, e^{-uy}\upsilon_{\text{-}}(y)\, dy}{\phi_{\text{-}}(u)} du \\
&\ge \frac{C}{\phi_{\text{-}}(\infty)} \int_{\widehat{a}}^\infty \int_0^\infty (1-\cos(by))\, e^{-uy}\upsilon_{\text{-}}(y)\, dydu \\
&= \frac{C}{\phi_{\text{-}}(\infty)} \int_0^\infty (1-\cos(by))\, e^{-\widehat{a}y}\upsilon_{\text{-}}(y)\, \frac{dy}{y} \ge \frac{C}{\phi_{\text{-}}(\infty)} \int_{\frac{1}{b}}^\varepsilon (1-\cos(by))\, e^{-\widehat{a}y}\upsilon_{\text{-}}(y)\, \frac{dy}{y} \\
&\ge \frac{C}{\phi_{\text{-}}(\infty)} e^{-\widehat{a}\varepsilon} \int_1^{b\varepsilon} (1-\cos y)\, \upsilon_{\text{-}}\left(\frac{y}{b}\right) \frac{dy}{y} \\
&\ge \frac{C}{\phi_{\text{-}}(\infty)} e^{-\widehat{a}\varepsilon} \left(\inf_{v\in(0,\varepsilon)} \upsilon_{\text{-}}(v)\right) \int_1^{b\varepsilon} (1-\cos y)\, \frac{dy}{y}.
\end{aligned}$$

However, since $\int_1^\infty \frac{\cos y}{y} dy < \infty$ we conclude that for any $\varepsilon > 0$, as $b \to \infty$,

$$\lim_{b\to\infty} \frac{A_{\phi_{\text{-}}}(\widehat{a}+ib)}{\ln b} \ge \frac{C}{\phi_{\text{-}}(\infty)} e^{-\widehat{a}\varepsilon} \inf_{v\in(0,\varepsilon)} \upsilon_{\text{-}}(v). \tag{5.4.37}$$

Now, (5.4.35) and (5.4.37) prove the claim $\phi_{\text{-}} \in \mathcal{B}_\infty$ since for fixed $\widehat{a}$ and as $b \to \infty$

$$W_{\phi_{\text{-}}}(a+ib) \asymp \frac{1}{\sqrt{|\phi_{\text{-}}(\widehat{a}+ib)|}} e^{-A_{\phi_{\text{-}}}(\widehat{a}+ib)} \lesssim \frac{1}{\sqrt{|\phi_{\text{-}}(\widehat{a}+ib)|}} e^{-\frac{C}{\phi_{\text{-}}(\infty)} e^{-\widehat{a}\varepsilon} \left(\inf_{v\in(0,\varepsilon)} \upsilon_{\text{-}}(v)\right)\ln(b)},$$

see (5.3.19), and $|\phi_{\text{-}}(\widehat{a}+ib)| \ge \phi_{\text{-}}(\widehat{a}) > 0$, see (5.3.32). We conclude item (2). We proceed with the proof of item (3). Assume then that $\overline{\Pi}_{\text{-}}(0) < \infty$. In this case we study directly $A_{\phi_{\text{-}}}$. Since (5.4.24) holds in any situation when $\phi_{+} \in \mathcal{B}_P$ then $\mu_{\text{-}}(dy) = \upsilon_{\text{-}}(y)\, dy$, $y \in (0,\infty)$, and

$$||\upsilon_{\text{-}}||_\infty = \sup_{y\ge 0} \upsilon_{\text{-}}(y) \le ||u_{+}||_\infty \sup_{y>0} \overline{\Pi}_{\text{-}}(y) = ||u_{+}||_\infty \overline{\Pi}_{\text{-}}(0) = A < \infty.$$

Note that $\phi_{\text{-}}(\infty) = \phi_{\text{-}}(0) + \bar{\mu}_{\text{-}}(0)$ and put $\upsilon_a^\star(y) = \frac{e^{-ay}}{\phi_{\text{-}}(\infty)}\upsilon_{\text{-}}(y)$, $y > 0$. Then, clearly from the first expression in (5.3.3), for $z = a+ib \in \mathbb{C}_{(0,\infty)}$, $a > 0$,

$$\phi_{\text{-}}(z) = \phi_{\text{-}}(0) + \bar{\mu}_{\text{-}}(0) - \int_0^\infty e^{-iby} e^{-ay}\upsilon_{\text{-}}(y) dy = \phi_{\text{-}}(\infty)\left(1 - \mathcal{F}_{\upsilon_a^\star}(-ib)\right). \tag{5.4.38}$$

From $||\upsilon_{-}||_{\infty} \leq A$ then for all a big enough we have that $||\upsilon_a^{\star}||_{TV} < 1$. Fix such a. Then $\forall b \in \mathbb{R}$ we deduce from (5.3.49) of Proposition 5.3.14 and (5.3.45) that

$$\begin{aligned} \arg\left(\phi_{-}\left(a+ib\right)\right) = & \quad \mathtt{Im}\left(\log_0\left(1-\mathcal{F}_{\upsilon_a^{\star}}(-ib)\right)\right) \\ & = -\mathtt{Im}\left(\mathcal{F}_{\upsilon_a^{\star}}(-ib)\right) - \sum_{n=2}^{\infty} \frac{\mathtt{Im}\left(\mathcal{F}_{\upsilon_a^{\star}}(-ib)\right)^n}{n} = -\mathtt{Im}\left(\mathcal{F}_{\upsilon_a^{\star}}(-ib)\right) + g_a(b). \end{aligned} \tag{5.4.39}$$

Since $||\upsilon^{\star}||_{\infty} \leq \frac{A}{\phi_{-}(\infty)} < \infty$ with $\upsilon^{\star} = \upsilon_0^{\star}$ we can show repeating without modification (5.4.29) and (5.4.30) above and in the process estimating the convolutions $\left(\upsilon_a^{\star}\right)^{*n}$, $n \geq 2$, using Proposition 5.4.10(5.4.74) with $a' = 0$, that $\sup_{b>0}\left|\int_0^b g_a(v)dv\right| < \infty$ and thus it does not contribute more than a constant to $A_{\phi_{-}}$ at least for this fixed a big enough. Without loss of generality work with $b > 0$. Then, from the definition of $A_{\phi_{-}}$, see (5.3.11), (5.4.39) and the preceding discussion, we get that

$$\begin{aligned} A_{\phi_{-}}\left(a+ib\right) &= \int_0^b \arg\left(\phi_{-}\left(a+iu\right)\right)du = \int_0^b g_a(u)du - \int_0^b \mathtt{Im}\left(\mathcal{F}_{\upsilon_a^{\star}}(-iu)\right)du \\ &= \int_0^b g_a(u)du + \int_0^{\infty} \frac{1-\cos(by)}{y}\upsilon_a^{\star}(y)dy, \end{aligned}$$

where $\sup_{b>0}\left|\int_0^b g_a(u)du\right| < \infty$. Estimating precisely as in (5.4.32) and (5.4.33), since $y \mapsto \upsilon_a^{\star}(y)y^{-1}\mathbb{I}_{\{y>1\}} \in \mathrm{L}^1\left(\mathbb{R}^+\right)$ and $||\upsilon^{\star}||_{\infty} \leq \frac{A}{\phi_{-}(\infty)} < \infty$ one gets that for some positive constant C_a'

$$\left|A_{\phi_{-}}\left(a+ib\right) - \int_1^b \frac{1-\cos y}{y}\upsilon_a^{\star}\left(\frac{y}{b}\right)dy\right| \leq C_a'. \tag{5.4.40}$$

We investigate the contribution of the integral as $b \to \infty$. Fix $\rho \in (0,1)$. Then, clearly,

$$\sup_{b>1}\int_{b\rho}^b \frac{1-\cos y}{y}\upsilon_a^{\star}\left(\frac{y}{b}\right)dy \leq 2||\upsilon_a^{\star}||_{\infty}\left|\ln\rho\right|. \tag{5.4.41}$$

Next put

$$\upsilon_a^{\star}(0) = \lim_{y\to 0}\frac{\int_0^{\infty} u_{+}(v)\Pi_{-}(y+dv)}{\phi_{-}(\infty)} = \frac{\int_0^{\infty} u_{+}(v)\Pi_{-}(dv)}{\phi_{-}(0)+\bar{\mu}_{-}(0)} < \infty,$$

which follows from (5.4.24) and the continuity of u_{+}. Thus, $\upsilon_a^{\star}$ is continuous at zero. Set $\overline{\upsilon}_a(y) = \upsilon_a^{\star}(y) - \upsilon_a^{\star}(0)$, for $y \in (0,\infty)$. Then

$$\int_1^{b\rho}\frac{1-\cos y}{y}\upsilon_a^{\star}\left(\frac{y}{b}\right)dy = \int_1^{b\rho}\frac{1-\cos y}{y}\overline{\upsilon}_a\left(\frac{y}{b}\right)dy + \upsilon_a^{\star}(0)\left(\ln b - \int_1^b \frac{\cos y}{y}dy\right). \tag{5.4.42}$$

However, since $\upsilon_a^{\star}$ is continuous at zero we are able to immediately conclude that

$$\lim_{\rho\to 0}\sup_{y\leq b\rho}\left|\overline{\upsilon}_a\left(\frac{y}{b}\right)\right| = \lim_{\rho\to 0}\overline{\overline{\upsilon}}\left(\rho\right) = 0,$$

where $\overline{\overline{v}}(\rho) = \sup_{v\le\rho} |\overline{v}_a(v)|$, and thus

$$\left|\int_1^{b\rho} \frac{1-\cos y}{y}\overline{v}_a\left(\frac{y}{b}\right)dy\right| \le 2\left(\sup_{y\le\rho}|\overline{v}_a(y)|\right)\ln b = 2\overline{\overline{v}}(\rho)\ln b. \tag{5.4.43}$$

We then combine (5.4.41),(5.4.42) and (5.4.43) in (5.4.40) to get for any $\rho \in (0,1)$ and some constant $C_{a,\rho} > 0$ that

$$|A_{\phi_-}(a+ib) - v_a^\star(0)\ln b| \le C_{a,\rho} + 2\overline{\overline{v}}(\rho)\ln b. \tag{5.4.44}$$

Thus, for all a big enough and all $\rho \in (0,1)$ we have from (5.3.19) of Theorem 5.3.2 that

$$|W_{\phi_-}(a+ib)| \asymp \frac{1}{\sqrt{|\phi_-(a+ib)|}}e^{-v_a^\star(0)\ln b - 2\overline{\overline{v}}(\rho)\ln b - C_{a,\rho}}. \tag{5.4.45}$$

Since $\lim_{\rho\to 0}\overline{\overline{v}}(\rho) = 0$ and $v_a^\star(0) = v^\star(0) = \frac{v_-(0)}{\phi_-(\infty)}$ this settles the proof for item (3) at least for all a big enough. However, since μ_- is absolutely continuous, for any $a > 0$ fixed, we have from Proposition 5.3.13(4) that $\lim_{|b|\to\infty} \phi_-(a+ib) = \phi_-(\infty)$. From $W_{\phi_-}(1+a+ib) = \phi_-(a+ib)\,W_{\phi_-}(a+ib)$, see (5.3.4), we then get that (5.4.45) holds for any $a > 0$ up to a multiplicative constant. This concludes the proof of item (3) and therefore of Proposition 5.4.3. □

The first auxiliary result uses (5.4.23) and the celebrated *équation amicale inversée*, see [34, Chapter V, (5.3.11)], extended easily to killed Lévy processes in Proposition 5.7.2, to decompose and relate the Lévy measure of ϕ_+ to the Lévy measure associated to the Lévy process underlying $\Psi \in \overline{\mathcal{N}}$. This decomposition is used to relate ϕ_- and hence W_{ϕ_-} to $\phi_-^c \in \mathcal{B}$ and $W_{\phi_-^c}$ as in the proof of Proposition 5.4.3(2).

Lemma 5.4.5. *Let $\Psi \in \overline{\mathcal{N}}$ such that $\phi_+ \in \mathcal{B}_P$ and $\bar{\mu}_+(0) < \infty$. Then with the decomposition of the potential density $u_+(x) = \frac{1}{\mathtt{d}_+} + \tilde{u}_+(x)$, $x \ge 0$, see (5.4.23), $\exists\, c_0 = c_0(\Psi) \in (0,\infty)$ such that for any $c \in (0,c_0)$ the following identity of measures holds on $(0,c)$*

$$\begin{aligned}\mu_-(dy) &= \frac{1}{\mathtt{d}_+}\left(\overline{\Pi}_-^c(y) + \left(\overline{\Pi}_-(c) - \overline{\Pi}_-(y+c)\right)\right)dy \\ &+ \int_c^\infty u_+(v)\Pi_-(y+dv)dy + \int_0^c \tilde{u}_+(v)\Pi_-(y+dv)dy \\ &= \frac{1}{\mathtt{d}_+}\overline{\Pi}_-^c(y)dy + \tau_1^c(y)dy + \tau_2^c(y)dy,\end{aligned} \tag{5.4.46}$$

where $\overline{\Pi}_-^c(y) = \left(\overline{\Pi}_-(y) - \overline{\Pi}_-(c)\right)\mathbb{I}_{\{y\le c\}}$,

$$\tau_1^c(y) = \mathbb{I}_{\{y\le \frac{c}{2}\}}\int_0^{\frac{c}{2}} \tilde{u}_+(v)\Pi_-(y+dv) \tag{5.4.47}$$

and

$$\tau_2^c(y) = \left(\int_0^{\frac{c}{2}} \tilde{u}_+(v)\Pi_-(y+dv)\mathbb{I}_{\{y\in\left(\frac{c}{2},c\right)\}} + \int_{\frac{c}{2}}^c \tilde{u}_+(v)\Pi_-(y+dv)\mathbb{I}_{\{y<c\}}\right)dy + \\ + \left(\overline{\Pi}_-(c) - \overline{\Pi}_-(y+c)\right)\mathbb{I}_{\{y<c\}}dy + \left(\int_c^\infty u_+(v)\Pi_-(y+dv)\right)\mathbb{I}_{\{y<c\}}dy. \tag{5.4.48}$$

Moreover, $\sup_{y\in(0,c)} |\tau_2^c(y)| < \infty$ *and for some* $C^* > 0$

$$|\tau_1^c(y)| \leq C^* \left(\int_0^{\frac{c}{2}} \overline{\Pi}_-^c(\rho+y) - \overline{\Pi}_-^c\left(\frac{c}{2}+y\right)d\rho\right)\mathbb{I}_{\{y\leq\frac{c}{2}\}}. \tag{5.4.49}$$

Finally, both τ_1^c *and* τ_2^c *are absolutely integrable on* $(0,c)$*.*

Proof. Let $c > 0$. Recall (5.4.24) and decompose as follows

$$\mu_-(dy) = \upsilon_-(y)\,dy = \int_0^\infty u_+(v)\Pi_-(y+dv)dy \\ = \int_0^c u_+(v)\Pi_-(y+dv)dy + \int_c^\infty u_+(v)\Pi_-(y+dv)dy. \tag{5.4.50}$$

Recall also that the potential measure U_+ and its density u_+ have been discussed prior to the statement of Proposition 5.4.3. Most notably (5.4.23) gives that

$$u_+(x) = \sum_{j=0}^\infty \frac{(-1)^j}{\mathtt{d}_+^{j+1}}\left(\mathbf{1} * (\phi_+(0)+\bar{\mu}_+)^{*j}\right)(x) =: \frac{1}{\mathtt{d}_+} + \tilde{u}_+(x). \tag{5.4.51}$$

For any $c > 0$, plugging the right-hand side of (5.4.51) in the first term of the last identity of (5.4.50) we get upon trivial rearrangement the first identity in (5.4.46). The expressions for τ_1^c, τ_2^c, see (5.4.47) and (5.4.48), are up to a mere choice. Trivially, for the second term in (5.4.48), we get that $\left(\overline{\Pi}_-(c) - \overline{\Pi}_-(y+c)\right)\mathbb{I}_{\{y<c\}} \leq \overline{\Pi}_-(c)\mathbb{I}_{\{y\leq c\}}$ and it is integrable on $(0,c)$. Also since $||u_+||_\infty < \infty$, which is thanks to $\phi_+ \in \mathcal{B}_P$, we deduce that

$$\left(\int_c^\infty u_+(v)\Pi_-(y+dv)\right)\mathbb{I}_{\{y<c\}} \leq ||u_+||_\infty\overline{\Pi}_-(y+c)\mathbb{I}_{\{y<c\}} \leq ||u_+||_\infty\overline{\Pi}_-(c)\mathbb{I}_{\{y<c\}}.$$

Clearly, the upper bound is finite and integrable on $(0,c)$. Thus, the last term in (5.4.48) has been dealt with. Finally, using in an evident manner (5.4.51), we study the first term

$$\left|\int_0^c \tilde{u}_+(v)\Pi_-(y+dv)\right|\mathbb{I}_{\{y\in\left(\frac{c}{2},c\right)\}} + \left|\int_{\frac{c}{2}}^c \tilde{u}_+(v)\Pi_-(y+dv)\right|\mathbb{I}_{\{y<c\}} \\ \leq \left(\frac{1}{\mathtt{d}_+} + ||u_+||_\infty\right)\left(\overline{\Pi}_-(y)\,\mathbb{I}_{\{c>y>\frac{c}{2}\}} + \overline{\Pi}_-\left(\frac{c}{2}\right)\mathbb{I}_{\{y<c\}}\right). \tag{5.4.52}$$

However, the upper bound in (5.4.52) is clearly both bounded and integrable on $(0,c)$. Thus, we have proved that $\sup_{y\in(0,c)}|\tau_2^c(y)|<\infty$ and τ_2^c is absolutely integrable on $(0,c)$. It remains to investigate τ_1^c. Note that the term defining (5.4.51) has, for any $x\geq 0$, the form

$$\mathbf{1}*(\phi_+(0)+\bar{\mu}_+)(x)=\int_0^x(\phi_+(0)+\bar{\mu}_+(y))\,dy=\phi_+(0)x+\int_0^x\bar{\mu}_+(y)\,dy.$$

Since $\bar{\mu}_+(0)<\infty$ we conclude that

$$\mathbf{1}*(\phi_+(0)+\bar{\mu}_+)(x)\overset{0}{\sim}(\phi_+(0)+\bar{\mu}_+(0))\,x.$$

Then, since $1*(\phi_+(0)+\bar{\mu}_+)$ is non-decreasing on $\mathbb{R}^+$, [36, (4.2)] gives, for any $j\in\mathbb{N}$, that

$$\left(\mathbf{1}*(\phi_+(0)+\bar{\mu}_+)^{*j}\right)(x)\leq(\mathbf{1}*(\phi_+(0)+\bar{\mu}_+(x)))^j$$

and we conclude that for some $h>0$ and all $x\in(0,h)$,

$$\left(\mathbf{1}*(\phi_+(0)+\bar{\mu}_+)^{*j}\right)(x)\leq 2^j(\phi_+(0)+\bar{\mu}_+(0))^j x^j.$$

Therefore from (5.4.51), for $x<\frac{1}{4(\phi_+(0)+\bar{\mu}_+(0))}\wedge h$,

$$|\tilde{u}_+(x)|\leq C^*x, \tag{5.4.53}$$

where $C^*>0$ is some positive constant. Hence, from now on, we choose an arbitrary $c<c_0=\frac{1}{4(\phi_+(0)+\bar{\mu}_+(0))}\wedge h$. Using (5.4.53) in (5.4.47) we get that

$$|\tau_1^c(y)|\leq C^*\left(\int_0^{\frac{c}{2}}v\Pi_-(y+dv)\right)\mathbb{I}_{\{y\leq\frac{c}{2}\}}$$

and (5.4.49) follows by integration by parts. However, from (5.4.49) we get for $y\in\left(0,\frac{c}{2}\right)$ that

$$\begin{aligned}\frac{\tau_1^c(y)}{C^*}&\leq\int_0^{\frac{c}{2}}\left(\overline{\Pi}_-^c(\rho+y)-\overline{\Pi}_-^c\left(\frac{c}{2}+y\right)\right)d\rho\\&\leq\overline{\overline{\Pi}}_-^c(y):=\int_y^c\overline{\Pi}_-^c(v)dv\end{aligned}\tag{5.4.54}$$

and since

$$\int_0^c\overline{\overline{\Pi}}_-^c(y)dy=\int_0^c\int_y^c\overline{\Pi}_-^c(v)dvdy\leq\int_0^c\int_y^c\overline{\Pi}_-(v)dvdy=\int_0^c v\overline{\Pi}_-(v)dv<\infty,$$

we conclude that $|\tau_1^c|$ is integrable on $\left(0,\frac{c}{2}\right)$. □

We keep the notation of Lemma 5.4.5 and introduce the function

$$\tau^c(y)=\tau_1^c(y)+\tau_2^c(y)+\upsilon_-(y)\,\mathbb{I}_{\{y\geq c\}}. \tag{5.4.55}$$

The next result is technical.

Proposition 5.4.6. *Let $\Psi \in \overline{\mathcal{N}}$ such that $\phi_+ \in \mathcal{B}_P$ and $\bar{\mu}_+(0) < \infty$. Then for any $d \in [0, \bar{\mu}_-(0))$ there exists $c_1 = c_1(d, \Psi) \in (0, c_0)$ such that $\int_0^\infty \tau^c(y)dy \in (d, \bar{\mu}_-(0))$ for all $c \in (0, c_1)$.*

Proof. Note that from (5.4.55) and Lemma 5.4.5 for all $c \in (0, c_0)$

$$\int_c^\infty \tau^c(y)dy = \int_c^\infty \upsilon_-(y)\,dy = \bar{\mu}_-(c)\,.$$

By simple inspection of (5.4.47) and (5.4.48) we note that the only potential negative contribution to τ^c comes from the terms whose integrands are $\tilde{u}_+$. Since (5.4.53), that is $|\tilde{u}_+(x)| \leq C^* x$, holds for all $x \in (0, c_0)$ clearly an upper bound of those terms is the expression

$$\overline{\tau}(y) = C^* \int_0^c v\Pi_-(y + dv)\mathbb{I}_{\{y<c\}}.$$

Therefore, integrating by parts, we get that

$$\begin{aligned}\int_0^c \overline{\tau}(y)dy &= C^* \int_0^c \int_0^c \left(\overline{\Pi}_-(y+w) - \overline{\Pi}_-(y+c)\right) dwdy \\ &\leq C^* \int_0^c \int_0^c \overline{\Pi}_-(y+w)dwdy \leq C^* \int_0^c \overline{\overline{\Pi}}_-(y)dy.\end{aligned}$$

As the upper bound tends to 0 as $c \to 0$ we conclude the claim as the negative contribution of τ_1^c, τ_2^c cannot exceed in absolute value this quantity, whereas the positive contribution exceeds $\int_c^\infty \tau^c(y)dy = \bar{\mu}_-(c)$ which converges to $\bar{\mu}_-(0)$. $\square$

Lemma 5.4.5 allows us to prove the following result which transforms the decomposition of μ_- on $(0, c)$ to a decomposition of ϕ_-. We stress that although one of the terms in the aforementioned decomposition is a Bernstein function with better understood properties, the second term need not belong to $\mathcal{B}$.

Lemma 5.4.7. *Let $\Psi \in \overline{\mathcal{N}}$ such that $\phi_+ \in \mathcal{B}_P$ and $\bar{\mu}_+(0) < \infty$. Let $c \in (0, c_0)$ so that Lemma 5.4.5 is valid. Then, the function*

$$\phi_-^c(z) = \phi_-(0) + \mathtt{d}_- z + \int_0^c \left(1 - e^{-zy}\right) \frac{\overline{\Pi}_-^c(y)}{\mathtt{d}_+} dy \in \mathcal{B} \tag{5.4.56}$$

and with the definition of τ^c, see (5.4.55)*,*

$$\phi_-(z) = \phi_-^c(z) + \int_0^\infty \left(1 - e^{-zy}\right) \tau^c(y)dy = \phi_-^c(z) + \tilde{\phi}_-^c(z). \tag{5.4.57}$$

For any such choice and $a > 0$ fixed

$$\lim_{|b|\to\infty} \mathtt{Im}\left(\tilde{\phi}_-^c(a+ib)\right) = 0 \quad \text{and} \quad \lim_{|b|\to\infty} \mathtt{Re}\left(\tilde{\phi}_-^c(a+ib)\right) = \tilde{\phi}_-^c(\infty) = \int_0^\infty \tau^c(y)dy, \tag{5.4.58}$$

whereas

$$\lim_{|b|\to\infty} \mathtt{Re}\left(\phi_-^c(a+ib)\right) = \phi_-^c(\infty) \qquad \textit{and} \qquad \mathtt{Im}\left(\phi_-^c(a+ib)\right) \geq 0. \tag{5.4.59}$$

Finally, there exist $d \in [0, \bar{\mu}_-(0))$*,* $c_2 = c_2(d, \Psi) \in (0, c_0)$ *such that for any* $c \in (0, c_2)$ *we have that* $\tilde{\phi}_-^c(\infty) = \int_0^\infty \tau^c(y)dy > 0$*.*

Proof. Note that (5.4.56) can be defined for any $c > 0$ but we fix $c \in (0, c_0)$ ensuring the validity of Lemma 5.4.5. The validity of (5.4.57) is a simple rearrangement. Next, (5.4.58) follows from the application of the Riemann-Lebesgue lemma to the function $\tau^c \in \mathrm{L}^1(\mathbb{R}^+)$. The latter is a consequence from the absolute integrability of its constituting components τ_1^c, τ_2^c, see Lemma 5.4.5, and the fact that $\bar{\mu}_-(c) = \int_c^\infty \upsilon_-(y)\,dy < \infty$, see (5.3.3). Next, recall that (5.4.25) states

$$\bar{\mu}_-(0) = \int_0^\infty \upsilon_-(y)\,dy = \infty \iff \int_0^1 \overline{\Pi}_-(y)dy = \infty. \tag{5.4.60}$$

Thus, if $\bar{\mu}_-(0) = \infty$ (5.4.60) shows that the Lévy measure of ϕ_-^c, that is the quantity $\overline{\Pi}_-^c(y)dy = \left(\overline{\Pi}_-(y) - \overline{\Pi}_-(c)\right)\mathbb{I}_{\{y<c\}}dy$, assigns infinite mass on $(0, c)$ and is absolutely continuous therein. However, the latter facts trigger the validity of [90, Theorem 27.7] and thus the distribution of the non-increasing Lévy process underlying ϕ_-^c is absolutely continuous. This, in turn, thanks to the Riemann-Lebesgue lemma yields to

$$\lim_{|b|\to\infty} e^{-\phi_-^c(a+ib)} = \lim_{|b|\to\infty} \mathbb{E}\left[e^{-(a+ib)\xi_1^c}\right] = \lim_{|b|\to\infty} \int_0^\infty e^{-ax-ibx} h_c(x)dx = 0, \tag{5.4.61}$$

where ξ_1^c is the non-decreasing Lévy process associated to ϕ_-^c taken at time 1 and on $(0, \infty)$

$$h_c(x)dx = \mathbb{P}\left(\xi_1^c \in dx\right).$$

Thus, the first assertion of (5.4.59) is valid. It is clearly valid if $\mathtt{d}_- > 0$ as well. In both cases $\phi_-^c(\infty) = \infty$. It remains to settle the first statement of (5.4.59) when $\bar{\mu}_-(0) < \infty$ and $\mathtt{d}_- = 0$. It follows from the Riemann-Lebesgue lemma and (5.4.56) wherein by assumption $\mathtt{d}_- = 0$ and

$$\lim_{|b|\to\infty} \int_0^c e^{-ay-i|b|y} \frac{\overline{\Pi}_-^c(y)}{\mathtt{d}_+} dy = 0$$

which in turn comes from (5.4.60) which implies that $\int_0^c \overline{\Pi}_-^c(y)dy < \infty$. Regardless of $\bar{\mu}_-(0)$ being finite or not the second claim of (5.4.59) follows by integration by parts of $\overline{\Pi}_-^c(y) = \int_y^c \Pi_-(dr)$, $y \in (0, c)$, in (5.4.56) or the proof of [83, Lemma 4.6] since $\overline{\Pi}_-^c$ is non-increasing on $\mathbb{R}^+$. The final claim of the Lemma follows easily from the assertion of Proposition 5.4.6. □

The next result is the first step to the understanding of the quantity A_{ϕ_-} via studying the integrand $\arg\phi_-$. We always fix c such that $\tilde{\phi}_-^c(\infty) > 0$ in Lemma 5.4.7 and all claims of Lemma 5.4.5 hold, which from the final assertion of Lemma 5.4.7 is always possible as long as $\bar{\mu}_-(0) > 0$. We then decompose $\arg\phi_-$ as a sum of $\arg\phi_-^c$ and an error term and we simplify the latter. For this purpose we introduce some further notation. Let in the sequel (5.4.57) hold. Then we denote by $U^c(dy)$, $y > 0$, the potential measure of the subordinator associated to ϕ_-^c and by $U_a^c(dy) = e^{-ay}U^c(dy)$. Recall that $\tau_a^c(y) = e^{-ay}\tau^c(y)$, $y > 0$, where τ^c is defined in (5.4.55). Then, the following claim holds.

Lemma 5.4.8. *Let $\Psi \in \overline{\mathcal{N}}$ such that $\phi_+ \in \mathcal{B}_P$ and $\bar{\mu}_+(0) < \infty$. Assume furthermore that $\bar{\mu}_-(0) = \infty$ or equivalently $\int_0^1 \overline{\overline{\Pi}}_-(y)dy = \infty$, see* (5.4.60)*. Fix $a > 0$ and $c \in (0, c_0)$ so that Lemma 5.4.5 is valid. Then, modulo to $(-\pi, \pi]$ for all $b > 0$ and directly for all b large enough*

$$\begin{aligned}\arg\left(\phi_-(a+ib)\right) &= \arg\left(\phi_-^c(a+ib)\right) + \arg\left(1 + \frac{\tilde{\phi}_-^c(a+ib)}{\phi_-^c(a+ib)}\right)\\ &= \arg\left(\phi_-^c(a+ib)\right) + \arg\left(1 + \tilde{\phi}_-^c(\infty)\mathcal{F}_{U_a^c}(-ib) - \mathcal{F}_{U_a^c*\tau_a^c}(-ib)\right),\end{aligned} \tag{5.4.62}$$

where $\phi_-^c, \tilde{\phi}_-^c$ are as in the decomposition (5.4.57)*. For any $c \in (0, c_0)$ there exists $a_c > 0$ such that for any $a \geq a_c$ and as $b \to \infty$*

$$\arg\left(\phi_-(a+ib)\right) = \arg(\phi_-^c(a+ib))\left(1 + \mathrm{o}\left(1\right)\right) + \arg\left(1 - \mathcal{F}_{\Xi_a^c}(-ib)\right), \tag{5.4.63}$$

where Ξ_a^c is an absolutely continuous finite measure on $\mathbb{R}^+$. Moreover, its density χ_a^c is such that $\chi_a^c \in \mathrm{L}^1\left(\mathbb{R}^+\right) \cap \mathrm{L}^\infty\left(\mathbb{R}^+\right)$ and $\lim_{a\to\infty} ||\Xi_a^c||_{TV} = 0$.

Proof. Since the assumptions of Lemma 5.4.5 and Lemma 5.4.7 are satisfied we conclude that $\phi_-(z) = \phi_-^c(z) + \tilde{\phi}_-^c(z)$, see (5.4.57). Then, modulo to $(-\pi, \pi]$, the first identity of (5.4.62) is immediate whereas the second one follows from the fact that

$$\frac{\tilde{\phi}_-^c(a+ib)}{\phi_-^c(a+ib)} = \frac{1}{\phi_-^c(a+ib)}\left(\int_0^\infty \tau^c(y)dy - \mathcal{F}_{\tau_a^c}(-ib)\right) = \frac{\tilde{\phi}_-^c(\infty)}{\phi_-^c(a+ib)} - \frac{\mathcal{F}_{\tau_a^c}(-ib)}{\phi_-^c(a+ib)},$$

see (5.4.57), and (5.4.22). Note that (5.4.59) implies that $\arg\left(\phi_-^c(a+ib)\right) \in \left[0, \frac{\pi}{2}\right]$ at least for b large enough. Moreover, since $\bar{\mu}_-(0) = \infty$ we note that

$$\lim_{|b|\to\infty} \mathtt{Re}\left(\phi_-^c(a+ib)\right) = \phi_-^c(\infty) = \infty, \tag{5.4.64}$$

see (5.4.59). Also, (5.4.64) together with (5.4.58) yields that $\arg\left(1 + \frac{\tilde{\phi}_-^c(a+ib)}{\phi_-^c(a+ib)}\right) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ at least for b large enough. Henceforth, (5.4.62) holds directly for such b. From (5.4.22), (5.4.64) and $\tilde{\phi}_-^c(\infty) > 0$ we get that for all b large enough

$$0 < \mathtt{Re}\left(\tilde{\phi}_-^c(\infty)\mathcal{F}_{U_a^c}(-ib)\right) = \tilde{\phi}_-^c(\infty)\,\mathtt{Re}\left(\frac{1}{\phi_-^c(a+ib)}\right) = \tilde{\phi}_-^c(\infty)\frac{\mathtt{Re}\left(\phi_-^c(a-ib)\right)}{\left|\phi_-^c(a+ib)\right|^2}. \tag{5.4.65}$$

Therefore, from (5.4.65) we conclude that for all b large enough

$$\begin{aligned}&\arg\left(1+\tilde{\phi}^c_-(\infty)\mathcal{F}_{U^c_a}(-ib)-\mathcal{F}_{U^c_a*\tau^c_a}(-ib)\right)\\&=\arg\left(1+\tilde{\phi}^c_-(\infty)\mathcal{F}_{U^c_a}(-ib)\right)+\arg\left(1-\frac{\mathcal{F}_{U^c_a*\tau^c_a}(-ib)}{1+\tilde{\phi}^c_-(\infty)\mathcal{F}_{U^c_a}(-ib)}\right).\end{aligned}\tag{5.4.66}$$

From (5.4.22), as $b\to\infty$, we get with the help of the second fact in (5.4.59), that is $\mathtt{Im}\left(\phi^c_-(a+ib)\right)\geq 0$, and (5.4.64) that

$$\begin{aligned}H(b)=\left|\mathtt{Im}\left(\mathcal{F}_{U^c_a}(-ib)\right)\right|&=\frac{\left|\mathtt{Im}\left(\overline{\phi^c_-(a+ib)}\right)\right|}{\left|\phi^c_-(a+ib)\right|^2}\\&=\frac{\mathtt{Im}\left(\phi^c_-(a+ib)\right)}{\left(\mathtt{Re}\left(\phi^c_-(a+ib)\right)\right)^2+\left(\mathtt{Im}\left(\phi^c_-(a+ib)\right)\right)^2}=\mathrm{o}\left(1\right).\end{aligned}\tag{5.4.67}$$

From (5.4.59) and (5.4.64) we have again that $\arg\left(\phi^c_-(a+ib)\right)=\arctan\left(\frac{\mathtt{Im}(\phi^c_-(a+ib))}{\mathtt{Re}(\phi^c_-(a+ib))}\right)$ and we aim to show, as $b\to\infty$, that

$$H(b)=\left|\mathtt{Im}\left(\mathcal{F}_{U^c_a}(-ib)\right)\right|=\mathrm{o}\left(\arg\left(\phi^c_-(a+ib)\right)\right).\tag{5.4.68}$$

Fix $n\in\mathbb{N}$ and note from (5.4.67) that $\lim_{b\to\infty}nH(b)=0$. Therefore, from the second fact in (5.4.59) and (5.4.64), for all b large enough,

$$\tan(nH(b))\leq 2nH(b)\leq\frac{2n}{\mathtt{Re}\left(\phi^c_-(a+ib)\right)}\frac{\mathtt{Im}\left(\phi^c_-(a+ib)\right)}{\mathtt{Re}\left(\phi^c_-(a+ib)\right)}=\mathrm{o}\left(1\right)\frac{\mathtt{Im}\left(\phi^c_-(a+ib)\right)}{\mathtt{Re}\left(\phi^c_-(a+ib)\right)}.$$

Therefore, since from (5.4.59), $\mathtt{Im}\left(\phi^c_-(a+ib)\right)\geq 0$, as $b\to\infty$,

$$\begin{aligned}n\varlimsup_{b\to\infty}\frac{H(b)}{\arg\left(\phi^c_-(a+ib)\right)}&\leq\varlimsup_{b\to\infty}\frac{\arctan\left(\mathrm{o}\left(1\right)\frac{\mathtt{Im}(\phi^c_-(a+ib))}{\mathtt{Re}(\phi^c_-(a+ib))}\right)}{\arg\left(\phi^c_-(a+ib)\right)}\\&\leq\varlimsup_{b\to\infty}\frac{\arctan\left(\frac{\mathtt{Im}(\phi^c_-(a+ib))}{\mathtt{Re}(\phi^c_-(a+ib))}\right)}{\arg\left(\phi^c_-(a+ib)\right)}=1.\end{aligned}$$

Hence, since $n\in\mathbb{N}$ is arbitrary we conclude (5.4.68). However, from (5.4.65), that is the inequality $\mathtt{Re}\left(\tilde{\phi}^c_-(\infty)\,\mathcal{F}_{U^c_a}(-ib)\right)>0$, elementary geometry in the complex plane and (5.4.67) as $b\to\infty$

$$\begin{aligned}\left|\arg\left(1+\tilde{\phi}^c_-(\infty)\mathcal{F}_{U^c_a}(-ib)\right)\right|&=\left|\arg\left(1+\tilde{\phi}^c_-(\infty)\mathtt{Re}\left(\mathcal{F}_{U^c_a}(-ib)\right)+i\tilde{\phi}^c_-(\infty)\mathtt{Im}\left(\mathcal{F}_{U^c_a}(-ib)\right)\right)\right|\\&\leq\left|\arg\left(1+i\tilde{\phi}^c_-(\infty)\mathtt{Im}\left(\mathcal{F}_{U^c_a}(-ib)\right)\right)\right|=\left|\tan\left(\tilde{\phi}^c_-(\infty)\mathtt{Im}\left(\mathcal{F}_{U^c_a}(-ib)\right)\right)\right|\\&\lesssim\tilde{\phi}^c_-(\infty)\left|\mathtt{Im}\left(\mathcal{F}_{U^c_a}(-ib)\right)\right|\overset{5.4.68}{=}\mathrm{o}\left(\arg\left(\phi^c_-(a+ib)\right)\right)\end{aligned}$$

and therefore we deduce easily for the right-hand side of (5.4.66) that as $b \to \infty$

$$\begin{aligned}
&\arg\left(1 + \tilde{\phi}^c_-(\infty)\mathcal{F}_{U^c_a}(-ib) - \mathcal{F}_{U^c_a * \tau^c_a}(-ib)\right) \\
&= \arg\left(1 - \frac{\mathcal{F}_{U^c_a * \tau^c_a}(-ib)}{1 + \tilde{\phi}^c_-(\infty)\mathcal{F}_{U^c_a}(-ib)}\right) + \mathrm{o}\left(\arg\left(\phi^c_-\left(a+ib\right)\right)\right).
\end{aligned} \tag{5.4.69}$$

To confirm (5.4.63) we need to study the first in the second line of (5.4.69). From Lemma 5.4.9 below we know that for any $a > 0$, $G^c_a(dy) = U^c_a * \tau^c_a(dy) = g^c_a(y)dy$ with $g^c_a \in \mathrm{L}^1\left(\mathbb{R}^+\right) \cap \mathrm{L}^\infty\left(\mathbb{R}^+\right)$. Also, for fixed $c \in (0, c_0)$ from (5.4.22) we get that

$$U^c_a\left(\mathbb{R}^+\right) = \int_0^\infty e^{-ay}U^c(dy) = \frac{1}{\phi^c_-\left(a\right)}.$$

However, since $\int_0^1 \overline{\Pi}_-(y)dy = \infty$ then we obtain from $\overline{\Pi}^c_-(y) = \left(\overline{\Pi}_-(y) - \overline{\Pi}_-(c)\right)\mathbb{I}_{\{y<c\}}$ that $\int_0^1 \overline{\Pi}^c_-(y)dy = \infty$ and thus from (5.4.56) we conclude that $\lim_{a\to\infty} \phi^c_-(a) = \infty$ and hence $\lim_{a\to\infty} U^c_a\left(\mathbb{R}^+\right) = 0$. Thus, we choose $a_c > 0$ such that $U^c_a\left(\mathbb{R}^+\right) < \frac{1}{4\tilde{\phi}^c_-(\infty)}$ for all $a \geq a_c$ and work with arbitrary such a. Therefore $\sup_{b\in\mathbb{R}}\left|\tilde{\phi}^c_-(\infty)\mathcal{F}_{U^c_a}(-ib)\right| < \frac{1}{4}$ and then we can deduct that

$$\frac{\mathcal{F}_{G^c_a}(-ib)}{1 + \tilde{\phi}^c_-(\infty)\mathcal{F}_{U^c_a}(-ib)} = \mathcal{F}_{G^c_a}(-ib)\sum_{n=0}^\infty (-1)^n\left(\tilde{\phi}^c_-(\infty)\right)^n \mathcal{F}^n_{U^c_a}(-ib).$$

Since $G^c_a(dy) = g^c_a(y)dy$, formally, the right-hand side is the Fourier transform of a measure Ξ^c_a supported on $\mathbb{R}^+$ with density

$$\chi^c_a(y) = g^c_a(y) + \sum_{n=1}^\infty (-1)^n\left(\tilde{\phi}^c_-(\infty)\right)^n \int_0^y g^c_a(y-v)\left(U^c_a\right)^{*n}(dv). \tag{5.4.70}$$

However, it is immediate with the assumptions and observations above that

$$||\chi^c_a||_\infty \leq ||g^c_a||_\infty + ||g^c_a||_\infty \sum_{n=1}^\infty \left(\tilde{\phi}^c_-(\infty)U^c_a\left(\mathbb{R}^+\right)\right)^n < ||g^c_a||_\infty \sum_{n=0}^\infty \frac{1}{4^n} < \infty$$

and

$$||\Xi^c_a||_{TV} = \int_0^\infty \left|\chi^c_a(y)\right| dy \leq \int_0^\infty |g^c_a(y)|dy\left(1 + \sum_{n=1}^\infty\left(\tilde{\phi}^c_-(\infty)U^c_a\left(\mathbb{R}^+\right)\right)^n\right) \leq 2||G^c_a||_{TV} < \infty. \tag{5.4.71}$$

Therefore, Ξ^c_a is a well-defined finite measure with density $\chi^c_a \in \mathrm{L}^1\left(\mathbb{R}^+\right) \cap \mathrm{L}^\infty\left(\mathbb{R}^+\right)$ and from (5.4.69) we get for all $a \geq a_c$ and any $b \in \mathbb{R}$ that

$$\arg\left(1 - \frac{\mathcal{F}_{U^c_a * \tau^c_a}(-ib)}{1 + \tilde{\phi}^c_-(\infty)\mathcal{F}_{U^c_a}(-ib)}\right) = \arg\left(1 - \mathcal{F}_{\Xi^c_a}(-ib)\right).$$

Combining this, (5.4.69) and (5.4.62) we conclude (5.4.63) for any $a \geq a_c$ and as $b \to \infty$. The final claims are also immediate from the discussion above. We just note from (5.4.71) that $||\Xi_a^c||_{TV} \leq 2||G_a^c||_{TV}$ and Lemma 5.4.9 shows that $\lim_{a\to\infty} ||\Xi_a^c||_{TV} = 0$. $\square$

In the next result we discuss the properties of the measure $U_a^c * \tau_a^c$ used in the proof above.

Lemma 5.4.9. *Fix $a > 0$. The measure $G_a^c(dy) = U_a^c * \tau_a^c(dx)$ has a bounded on $(0, \infty)$ density*

$$g_a^c(x) = e^{-ax} \int_0^x \tau^c(x-v)U^c(dv) = e^{-ax} g^c x)$$

and $g_a^c \in \mathrm{L}^1(\mathbb{R}^+)$ with $\lim_{a\to\infty} ||G_a^c||_{TV} = 0$.

Proof. The existence and the form of χ_a is immediate from the definition of convolution and the fact that $\tau_a^c(y)dy = e^{-ay}\tau^c(y)dy$, $y > 0$. Recall from (5.4.55) that

$$\tau^c(y)dy = \left(\tau_1^c(y) + \tau_2^c(y) + \mathbb{I}_{\{y>c\}}\upsilon_{\text{-}}(y)\right) dy.$$

Then, $A_1 = ||\tau_2^c||_\infty + ||\upsilon_{\text{-}}\mathbb{I}_{\{y>c\}}||_\infty < \infty$ follows from Lemma 5.4.5 and (5.4.50). Therefore, we have with some constant $A_3 > 0$

$$\begin{aligned}\sup_{x>0} e^{-ax} \int_0^x \left|\tau_2^c(x-y) + \upsilon_{\text{-}}(x-y)\,\mathbb{I}_{\{x-y>c\}}\right| U^c(dy) &\leq \sup_{x>0} A_1 e^{-ax} U^c(x) \\ &\leq A_2 \sup_{x>0} x e^{-ax} \leq A_3,\end{aligned} \tag{5.4.72}$$

where we have used the fact that $U^c : \mathbb{R}^+ \mapsto \mathbb{R}^+$ is subadditive, see [11, p.74]. It remains to study the portion coming from $\tau_1^c(y)$, which according to Lemma 5.4.5(5.4.47) is supported on $\left(0, \frac{c}{2}\right)$ and is bounded by the expression in (5.4.49). Thus, recalling that $\overline{\Pi}_{\text{-}}^c(y) = \left(\overline{\Pi}_{\text{-}}(y) - \overline{\Pi}_{\text{-}}(c)\right)\mathbb{I}_{\{y<c\}}$, we get that

$$\begin{aligned}&\sup_{x>0} e^{-ax} \int_0^x |\tau_1^c(x-y)|\, U^c(dy) = \sup_{x>0} e^{-ax} \int_{\max\left\{0,\, x-\frac{c}{2}\right\}}^x |\tau_1^c(x-y)|\, U^c(dy) \\ &\leq C^* \sup_{x>0} e^{-ax} \int_{\max\left\{0, x-\frac{c}{2}\right\}}^x \left(\int_0^{\frac{c}{2}} \overline{\Pi}_{\text{-}}^c(\rho + x - y) - \overline{\Pi}_{\text{-}}^c\left(\frac{c}{2} + x - y\right) d\rho\right) U^c(dy) \\ &\leq C^* \sup_{x>0} e^{-ax} \int_{\max\left\{0, x-\frac{c}{2}\right\}}^x \left(\int_0^{\frac{c}{2}} \overline{\Pi}_{\text{-}}^c(\rho + x - y)\, d\rho\right) U^c(dy) \\ &\leq C^* \sup_{x>0} e^{-ax} \int_0^x \overline{\overline{\Pi}}_{\text{-}}^c(x-y)\, U^c(dy) \\ &\leq C^* \sup_{x>0} e^{-ax} = C^*,\end{aligned} \tag{5.4.73}$$

where for the first inequality we have used the bound (5.4.49) and for the very last one that, see [11, Chapter III, Proposition 2],

$$\int_0^x \overline{\overline{\Pi}}_{\text{-}}^c(x-y)\, U^c(dy) = \mathbb{P}\left(\mathbf{e}_{\phi_{\text{-}}^c(0)} > T^\sharp_{(x,\infty)}\right) \leq 1,$$

where $\mathbf{e}_{\phi^c_-(0)}$ is an exponential random variable with parameter $\phi^c_-(0) = \phi_-(0) \geq 0$, see (5.4.56), $T^\sharp_{(x,\infty)}$ is the first passage time above $x > 0$ of the unkilled subordinator related to $(\phi^c_-)^\sharp(z) = \phi^c_-(z) - \phi^c_-(0) \in \mathcal{B}$ and, from (5.4.56), $\overline{\overline{\Pi}}{}^c_-(y) = \int_y^\infty \overline{\Pi}{}^c_-(v)dv$ is the tail of the Lévy measure associated to $(\phi^c_-)^\sharp$. Summing (5.4.72) and (5.4.73) yields that $||g^c_a||_\infty \leq A_3 + C^* < \infty$. Finally, $g^c_a \in \mathrm{L}^1(\mathbb{R}^+)$ follows immediately from the estimate before the last estimate in (5.4.72) and (5.4.73). Clearly, from them we also get $\lim_{a\to\infty} ||G^c_a||_{TV} = 0$ which ends the proof. □

Recall that for any $a \geq 0$ and any function $f : \mathbb{R}^+ \mapsto \mathbb{R}$, $f_a(y) = e^{-ay}f(y)$, $y > 0$. The next proposition is trivial.

Proposition 5.4.10. *For any function f, $f^{*n}_a(y) = e^{-ay}f^{*n}(y)$, $y \in (0,\infty)$. If $f_{a'} \in \mathrm{L}^\infty(\mathbb{R}^+)$ for some $a' \geq 0$ then for any $a \geq a'$, all $n \geq 1$ and $y > 0$*

$$|f^{*n}_a(y)| \leq ||f_{a'}||^n_\infty \frac{y^{n-1}e^{-(a-a')y}}{(n-1)!}. \tag{5.4.74}$$

Proof. First $f^{*n}_a(y) = e^{-ay}f^{*n}(y)$, $y \in (0,\infty)$, is a triviality. Then, (5.4.74) is proved by elementary inductive hypothesis based on the immediate observation

$$\left|f^{*2}_a(y)\right| = e^{-ay}\left|\int_0^y f(y-v)f(v)dv\right| \leq ||f_{a'}||^2_\infty ye^{-(a-a')y},\ y > 0.$$

□

5.5 Proofs for Exponential functionals of Lévy processes

5.5.1 Regularity, analyticity and representations of the density: Proof of Theorem 5.2.7(1)

Recall that for any $\Psi \in \overline{\mathcal{N}}$ and we have that $\mathcal{M}_\Psi$ satisfies (5.1.1) that is

$$\mathcal{M}_\Psi(z+1) = \frac{-z}{\Psi(-z)}\mathcal{M}_\Psi(z) \tag{5.5.1}$$

at least for $z \in \mathbb{C}_{(0,-\overline{\mathfrak{a}}_-)}$, see Theorem 5.2.1, and recall the quantity $\overline{\mathfrak{a}}_-$, see (5.2.5). Therefore, for any $\Psi \in \mathcal{N} \subset \overline{\mathcal{N}}$ such that $\overline{\mathfrak{a}}_- < 0$ from Remark 5.2.11, $\mathcal{M}_{I_\Psi}$ solves (5.5.1) at least for $z \in \mathbb{C}_{(0,-\overline{\mathfrak{a}}_-)}$. Since by definition if $\Psi \in \mathcal{N} \iff \phi_-(0) > 0$, see (5.2.17), we proceed to show that $\mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z)$ or that the identities in (5.2.21) hold. First, thanks to [83, Proposition 6.8] we have that

$$\mathbb{E}\left[I^{z-1}_{\phi_+}\right] = \frac{\Gamma(z)}{W_{\phi_+}(z)},\ z \in \mathbb{C}_{(0,\infty)}.$$

Furthermore, it is immediate to verify that $\phi_{-}(0)W_{\phi_{-}}(1-z)$, $z\in\mathbb{C}_{(-\infty,1-\overline{\mathfrak{a}}_{-})}$, is the Mellin transform of the random variable $X_{\phi_{-}}$ defined via the identity

$$\mathbb{E}\left[g\left(X_{\phi_{-}}\right)\right]=\phi_{-}(0)\mathbb{E}\left[\frac{1}{Y_{\phi_{-}}}g\left(\frac{1}{Y_{\phi_{-}}}\right)\right], \tag{5.5.2}$$

where from Definition 5.3.1 $Y_{\phi_{-}}$ is the random variable associated to $W_{\phi_{-}}\in\mathcal{W}_{\mathcal{B}}$ and $g\in\mathtt{C}_b\left(\mathbb{R}^+\right)$. Therefore

$$\begin{aligned}\mathcal{M}^{\circ}_{I_\Psi}(z)&=\mathbb{E}\left[I_{\phi_{+}}^{z-1}\right]\mathbb{E}\left[X_{\phi_{-}}^{z-1}\right]=\phi_{-}(0)\mathcal{M}_{\Psi}(z)\\&=\phi_{-}(0)W_{\phi_{-}}(1-z)\frac{\Gamma(z)}{W_{\phi_{+}}(z)},\quad z\in\mathbb{C}_{(0,1-\overline{\mathfrak{a}}_{-})},\end{aligned} \tag{5.5.3}$$

is the Mellin transform of $I_{\phi_{+}}\times X_{\phi_{-}}$ and solves (5.5.1) with $\mathcal{M}^{\circ}_{I_\Psi}(1)=1$ on $\mathbb{C}_{(0,-\overline{\mathfrak{a}}_{-})}$. Therefore, both $\mathcal{M}^{\circ}_{I_\Psi},\mathcal{M}_{I_\Psi}$ solve (5.5.1) on $\mathbb{C}_{(0,-\overline{\mathfrak{a}}_{-})}$ with $\mathcal{M}^{\circ}_{I_\Psi}(1)=\mathcal{M}_{I_\Psi}(1)=1$ and are holomorphic on $\mathbb{C}_{(0,1-\overline{\mathfrak{a}}_{-})}$. However, note that $\mathcal{M}^{\circ}_{I_\Psi}$ is zero-free on $(0,1-\overline{\mathfrak{a}}_{-})$ since Γ is zero-free and according to Theorem 5.3.1 $W_{\phi_{-}}$ is zero-free on $(\overline{\mathfrak{a}}_{-},\infty)$. Thus, we conclude that $f(z)\mathcal{M}^{\circ}_{I_\Psi}(z)=\mathcal{M}_{I_\Psi}(z)$ with f some entire holomorphic periodic function of period one, that is $f(z+1)=f(z)$, $z\in\mathbb{C}_{(0,1]}$, and $f(1)=1$. Next, considering $z=a+ib\in\mathbb{C}_{(0,1-\overline{\mathfrak{a}}_{-})}$, a fixed and $|b|\to\infty$, we get that

$$\begin{aligned}|f(z)|&=\left|\frac{\mathcal{M}_{I_\Psi}(z)}{\mathcal{M}^{\circ}_{I_\Psi}(z)}\right|\leq\frac{\mathbb{E}\left[I_\Psi^{a-1}\right]}{\left|\mathcal{M}^{\circ}_{I_\Psi}(z)\right|}\\&=\frac{\mathbb{E}\left[I_\Psi^{a-1}\right]}{\phi_{-}(0)}\frac{|W_{\phi_{+}}(z)|}{|\Gamma(z)W_{\phi_{-}}(1-z)|}=\mathrm{O}\left(1\right)\frac{|W_{\phi_{+}}(z)|}{|\Gamma(z)W_{\phi_{-}}(1-z)|}.\end{aligned}$$

Since $a>0$ is fixed, we apply (5.3.19) to $|W_{\phi_{+}}(z)|$ and (5.3.25) and (5.3.19) to $|W_{\phi_{-}}(1-z)|$ to obtain the inequality

$$|f(z)|\leq\mathrm{O}\left(1\right)\frac{e^{-A_{\phi_{+}}(a+ib)+A_{\phi_{-}}(a+(1-a)^{\rightarrow}-ib)}}{\sqrt{|\phi_{+}(z)|}\,|\Gamma(z)|}\times\prod_{j=0}^{(1-a)^{\rightarrow}-1}|\phi_{-}\left(z+j\right)|, \tag{5.5.4}$$

where we recall that $c^{\rightarrow}=\left(\lfloor -c\rfloor+1\right)\mathbb{I}_{\{c\leq 0\}}$. Also, we have regarded any term in (5.3.19) and (5.3.25) depending on a solely as a constant included in $\mathrm{O}\left(1\right)$. From Proposition 5.3.13(3) we have, for $a>0$ fixed, that $|\phi(a+i|b|)|\overset{\infty}{\approx}|b|\left(\mathtt{d}+\mathrm{o}\left(1\right)\right)$. We recall, from (5.3.32), that, for any $a>0$,

$$|\phi\left(a+ib\right)|\geq\mathtt{Re}(\phi(a+ib))\geq\phi(0)+\mathtt{d}a+\int_0^\infty\left(1-e^{-ay}\right)\mu(dy)=\phi(a)>0.$$

Applying these observations to ϕ_{+},ϕ_{-}, in (5.5.4) and invoking (5.3.20), that is $A_\phi\geq 0$ and $A_\phi\left(a+ib\right)\leq\frac{\pi}{2}|b|$, we get, as $|b|\to\infty$,

$$|f(z)|\leq|b|^{(1-a)^{\rightarrow}}\frac{e^{|b|\frac{\pi}{2}}}{|\Gamma(z)|}\mathrm{O}\left(1\right). \tag{5.5.5}$$

However, from the well-known Stirling asymptotic for the gamma functions, see (5.4.8), (5.5.5) is further simplified, as $|b| \to \infty$, to

$$|f(z)| \leq |b|^{(1-a)^{\rightarrow} - a + \frac{1}{2}} e^{|b|\pi} \mathrm{O}(1) = \mathrm{o}\left(e^{2\pi|b|}\right).$$

However, the fact that f is entire periodic with period 1 and $|f(z)| = \mathrm{o}\left(e^{2\pi|b|}\right)$, $|b| \to \infty$, for $z \in \mathbb{C}_{(0,1-\overline{\mathfrak{a}}_-)}$ implies by a celebrated criterion for the maximal growth of periodic entire functions, see [61, p.96, (36)], that $f(z) = f(1) = 1$. Hence, $\mathcal{M}_{I_\Psi}(z) = \mathcal{M}^\circ_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z)$, which concludes the proof of Theorem 5.2.1 verifying (5.2.21) whenever $\overline{\mathfrak{a}}_- < 0$. Recall that $\Psi \in \overline{\mathcal{N}}$ is

$$\Psi(z) = \frac{\sigma^2}{2} z^2 + cz + \int_{-\infty}^{\infty} \left(e^{zr} - 1 - zr\mathbb{I}_{\{|r|<1\}}\right) \Pi(dr) - q, \tag{5.5.6}$$

see (5.2.1). Next, assume that $\overline{\mathfrak{a}}_- = 0$, see (5.2.5), and that either $-\Psi(0) = q > 0$ or $\Psi'(0^+) = \mathbb{E}[\xi_1] \in (0,\infty]$ with $q = 0$ hold in the definition of $\Psi \in \mathcal{N}$, i.e. (5.2.17). For any $\eta > 0$ modify the Lévy measure Π in (5.5.6) as follows

$$\Pi_\eta(dr) = \mathbb{I}_{\{r>0\}}\Pi(dr) + \mathbb{I}_{\{r<0\}} e^{\eta r}\Pi(dr). \tag{5.5.7}$$

Denote then by Ψ_η the Lévy-Khintchine exponent based on c, σ^2 taken from Ψ and Lévy measure Π_η. Then set $\Psi_\eta(z) = -\phi^\eta_+(-z)\phi^\eta_-(z)$, see (5.1.2). However, (5.5.7) and (5.5.6) imply that Ψ_η (resp. ϕ^η_-) extends holomorphically at least to $\mathbb{C}_{(-\eta,0)}$ (resp. $\mathbb{C}_{(-\eta,\infty)}$), which immediately triggers that $\overline{\mathfrak{a}}_{\phi^\eta_-} < 0$ if $\Psi(0) = -\Psi_\eta(0) = q > 0$ in (5.5.6). However, when $\Psi'(0^+) = \mathbb{E}[\xi_1] \in (0,\infty]$ and $q = 0$ are valid then we get that $\Psi'_\eta(0^+) = \mathbb{E}[\xi^\eta_1] \geq \Psi'(0^+) = \mathbb{E}[\xi_1] > 0$ since relation (5.5.7) shows that ξ^η is derived from ξ via a removal of negative jumps only. Henceforth, a.s. $\lim_{t\to\infty} \xi^\eta_t = \infty$ which shows that the downgoing ladder height process associated to ϕ^η_- is killed, that is $\phi^\eta_-(0) > 0$. This combined with $\phi^\eta_- \in \mathtt{A}_{(-\eta,\infty)}$ gives that $\overline{\mathfrak{a}}_{\phi^\eta_-} < 0$, see (5.2.5). Therefore, we have that (5.2.21) is valid for I_{Ψ_η} and the probabilistic interpretation of $\mathcal{M}_{I_{\Psi_\eta}}$ above gives that

$$I_{\Psi_\eta} \stackrel{d}{=} X_{\phi^\eta_-} \times I_{\phi^\eta_+}. \tag{5.5.8}$$

However, since (5.5.7) corresponds to the thinning of the negative jumps of ξ we conclude that $I_{\Psi_\eta} \leq I_\Psi$ and clearly a.s. $\lim_{\eta\to 0} I_{\Psi_\eta} = I_\Psi$. Moreover, from [68, Lemma 4.9] we have that $\lim_{\eta\to 0} \phi^\eta_\pm(u) = \phi_\pm(u)$, $u \geq 0$. Therefore, with the help of Lemma 5.3.12 we deduce that $\lim_{\eta\to 0} W_{\phi^\eta_\pm}(z) = W_{\phi_\pm}(z)$, $z \in \mathbb{C}_{(0,\infty)}$, and establish (5.2.21) via

$$\begin{aligned} \mathcal{M}_{I_\Psi}(z) &= \lim_{\eta\to 0} \mathcal{M}_{I_{\Psi_\eta}}(z) = \lim_{\eta\to 0} \phi^\eta_-(0) \frac{\Gamma(z)}{W_{\phi^\eta_+}(z)} W_{\phi^\eta_-}(1-z) \\ &= \phi_-(0) \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z), \; z \in \mathbb{C}_{(0,1)}. \end{aligned}$$

It remains to consider the case $\Psi \in \mathcal{N}$, $q = 0$, $\mathbb{E}\left[\xi_1 \mathbb{I}_{\{\xi_1>0\}}\right] = \mathbb{E}\left[-\xi_1 \mathbb{I}_{\{\xi_1<0\}}\right] = \infty$ and $\lim_{t\to\infty} \xi_t = \infty$ hold. We do so by killing the Lévy process ξ at rate $r > 0$. Therefore, with the obvious notation, $\Psi^r(z) = -\phi^r_+(-z)\phi^r_-(z)$ and $\lim_{r\to 0} \phi^r_\pm(z) = \phi_\pm(z)$, $z \in \mathbb{C}_{(0,\infty)}$, since $\phi^r_\pm$ are the exponents of the bivariate ladder height processes $\left(\tau^\pm, H^\pm\right)$ as introduced in Section 5.7.1 and Proposition 5.7.3 holds. Also a.s. $\lim_{r\to 0} I_{\Psi_r} = I_\Psi$. However, since (5.2.21) holds whenever $r > 0$ we conclude (5.2.21) in this case by virtue of Lemma 5.3.12. □

5.5.2 Proof of Theorem 5.2.7(2)

We use the identity (5.2.44) that is

$$I_\Psi \stackrel{d}{=} I_{\phi_+} \times X_{\phi_-}, \tag{5.5.9}$$

where I_{ϕ_+} is the exponential functional of the possibly killed negative subordinator associated to $\phi_+ \in \mathcal{B}$. It is well known from [43, Lemma 2.1] that $\operatorname{Supp} I_{\phi_+} = \left[0, \frac{1}{\mathtt{d}_+}\right]$ unless $\phi_+(z) = \mathtt{d}_+ z$ in which case $\operatorname{Supp} I_{\phi_+} = \left\{\frac{1}{\mathtt{d}_+}\right\}$. When $\mathtt{d}_+ = 0$ then $\operatorname{Supp} I_\Psi = \operatorname{Supp} I_{\phi_+} = [0,\infty]$ in any case. Assume that $\mathtt{d}_+ > 0$ and note from (5.3.21) that

$$\ln \mathbb{E}\left[Y^n_{\phi_-}\right] = \ln W_{\phi_-}(n+1) \stackrel{\infty}{\sim} n \ln \phi_-(n+1),$$

which clearly shows that $\operatorname{Supp} Y_{\phi_-} \subseteq [0, \ln(\phi_-(\infty))]$ and $\operatorname{Supp} Y_{\phi_-} \not\subseteq [0, \ln(\phi_-(\infty)) - \varepsilon]$, for all $\varepsilon > 0$, where Y_{ϕ_-} is the random variable associated to W_{ϕ_-}, see Definition 5.3.1. However, Y_{ϕ_-} is multiplicative infinitely divisible, see $\mathcal{R} \stackrel{d}{=} Y_{\phi_-}$ in the notation of [46, Section 3.2], or [8, Theorem 2.2]. Then $\ln Y_{\phi_-}$ is infinitely divisible and again according to [46, Section 3.2] its Lévy measure, Θ in their notation, is carried by $\mathbb{R}^-$ and $\Theta(-dx)$ is equivalent to $k(dx) = \int_0^x U_-(dx-y) y \mu_-(dy)$. However, when $\mathtt{d}_+ > 0$ from (5.4.24) we have

$$\mu_-(dy) = \upsilon_-(y)dy = \left(\int_0^\infty u_+(v)\Pi_-(y+dv)\right)dy$$

and as $u_+ > 0$ on $\mathbb{R}^+$, as mentioned in the proof of Lemma 5.4.7, then $\upsilon_- > 0$ on $\mathbb{R}^+$. Also, in this case $\upsilon_-(y) = \int_0^\infty u_+(v)\Pi_-(y+dv)$ is at least a càdlàg function, see [34, Chapter 5, Theorem 16] or in more generality the differentiated version of (5.7.5) below. Thus, for any $c > 0$,

$$\int_0^c k(dx) = \int_0^c \int_y^c (x-y)\upsilon_-(x-y)dx U_-(dy) > 0$$

and a celebrated criterion in [96] shows that $\operatorname{Supp} \ln Y_{\phi_-} = [-\infty, \ln \phi_-(\infty)]$. Finally, from (5.5.2) we deduct that $\operatorname{Supp} X_{\phi_-} = \left[\frac{1}{\phi_-(\infty)}, \infty\right]$. From (5.5.9) this concludes the proof. □

5.5.3 Proof of items (3) and (5) of Theorem 5.2.7

The smoothness and the analyticity in each of the cases follow by a simple Mellin inversion

$$f_\Psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x^{-a-ib}\mathcal{M}_{I_\Psi}(a+ib)\,db \tag{5.5.10}$$

due to the implied polynomial $\mathtt{N}_\Psi \in (0,\infty]$ or exponential $\Theta \in \left[0,\frac{\pi}{2}\right]$ decay of $|\mathcal{M}_{I_\Psi}|$, taking derivatives and utilizing the dominated convergence theorem. Thus, (5.2.22) follows in ordinary sense. To prove its validity in the L^2-sense it is sufficient according to [95] to have that $b:\mathbb{R}\mapsto \mathcal{M}_{I_\Psi}(a+ib) \in \mathrm{L}^2(\mathbb{R})$, which is the case when $\mathtt{N}_\Psi > 1/2$. □

5.5.4 Proof of Theorem 5.2.7(4)

We start with an auxiliary result. It shows that the decay of $|\mathcal{M}_\Psi|$ can be extended to the left provided $\mathfrak{a}_+ < 0$.

Proposition 5.5.1. *Let $\Psi \in \overline{\mathcal{N}}$. Then, the decay of $|\mathcal{M}_\Psi(z)|$ which is always of value $\mathtt{N}_\Psi > 0$, see* (5.2.16)*, is preserved along $\mathbb{C}_a$ for any $a \in (\mathfrak{a}_+, 0]$.*

Proof. Let $\mathfrak{a}_+ < 0$ and take any $a \in (\max\{\mathfrak{a}_+,-1\},0)$. Then we can meromorphically extend via (5.1.1) and (5.1.2) to derive that

$$\mathcal{M}_\Psi(a+ib) = \frac{\Psi(-a-ib)}{-a-ib}\mathcal{M}_\Psi(a+1+ib) = \frac{\phi_+(a+ib)\,\phi_-(-a-ib)}{a+ib}\mathcal{M}_\Psi(a+1+ib). \tag{5.5.11}$$

Then, Proposition 5.3.13(3) gives that $|\Psi(-a-ib)| = \mathrm{O}(b^2)$ and the result when $\Psi \in \overline{\mathcal{N}}_\infty$ follows. If $\Psi \in \overline{\mathcal{N}}_{\mathtt{N}_\Psi}$, $\mathtt{N}_\Psi < \infty$, then according to Theorem 5.2.5 we have that $\phi_+ \in \mathcal{B}_P$, $\phi_- \in \mathcal{B}_P^c$, $\overline{\Pi}(0) < \infty$ and $\upsilon_-(y) = \mu_-(dy)/dy$. Therefore, again from Proposition 5.3.13(3), $\lim_{|b|\to\infty}\left|\frac{\phi_+(a+ib)}{a+ib}\right| = \mathtt{d}_+ > 0$ and from Proposition 5.3.13(4), $\lim_{|b|\to\infty}\phi_-(-a-ib) = \phi_-(\infty)$. With these observations, $\mathtt{N}_\Psi$ is preserved in the decay through (5.5.11). We recur this argument for any $a \in (\mathfrak{a}_+,-1)$, if $\mathfrak{a}_+ < -1$. For the case $a=0$ taking $b \neq 0$ and then using the recurrent equation (5.3.4) applied to (5.2.6) we observe that

$$\mathcal{M}_\Psi(ib) = \frac{\phi_+(ib)}{ib}\frac{\Gamma(1+ib)}{W_{\phi_+}(1+ib)}W_{\phi_-}(1-ib).$$

However, if $\Psi \in \overline{\mathcal{N}}_\infty$ then either $\phi_- \in \mathcal{B}_\infty$ and/or $\left|\frac{\Gamma(1+ib)}{W_{\phi_+}(1+ib)}\right|$ decays subexponentially. Then Proposition 5.3.13(3) shows the same subexponential decay for $|\mathcal{M}_\Psi(ib)|$. If $\Psi \in \overline{\mathcal{N}}_{\mathtt{N}_\Psi}$, $\mathtt{N}_\Psi < \infty$, then as above $\phi_+ \in \mathcal{B}_P$ and thus $\lim_{|b|\to\infty}\left|\frac{\phi_+(ib)}{ib}\right| = \mathtt{d}_+ > 0$ and we conclude the proof. □

We are ready to start the proof of Theorem 5.2.7(4). A standard relation of Mellin transforms gives that the restriction of

$$\mathcal{M}^\star_{I_\Psi}(z) = -\frac{1}{z}\mathcal{M}_{I_\Psi}(z+1) = -\frac{\phi_-(0)}{z}\mathcal{M}_\Psi(z+1) \in \mathtt{A}_{(-1,0)} \cap \mathtt{M}_{(\mathfrak{a}_+-1,-\mathfrak{a}_-)}, \tag{5.5.12}$$

on $\mathbb{C}_{(-1,0)}$ is the Mellin transform in the distributional sense of $F_\Psi(x) = \mathbb{P}(I_\Psi \le x)$, where we recall (5.2.21) for the form and the analytical properties of $\mathcal{M}_{I_\Psi}$. Note next that in this assertion we only consider $\Psi \in \mathcal{N}_\dagger$, that is $q = -\Psi(0) > 0$. From Theorem 5.2.1 and (5.5.12) we get that if $\mathfrak{u}_+ = -\infty$ or $-\mathfrak{u}_+ \notin \mathbb{N}$ then $\mathcal{M}^\star_{I_\Psi}$ has simple poles at all points in the set $\{-1, \cdots, -\lceil 1 - \mathfrak{a}_+ \rceil + 1\}$ and otherwise if $-\mathfrak{u}_+ \in \mathbb{N}$ simple poles at all points of $\{-1, \cdots, \mathfrak{u}_+\}$. The decay of $\left|\mathcal{M}^\star_{I_\Psi}(z)\right|$ along $\mathbb{C}_a$, $a \in (-1,0)$ is $\mathtt{N}_\Psi + 1$ since the decay of $|\mathcal{M}_{I_\Psi}(z)|$ along $\mathbb{C}_a$, $a \in (0,1)$, is of order $\mathtt{N}_\Psi$, see Theorem 5.2.5(1). However, thanks to Proposition 5.5.1 the decay of $\left|\mathcal{M}^\star_{I_\Psi}(z)\right|$ is of order $\mathtt{N}_\Psi + 1 > 1$ along $\mathbb{C}_a$, $a \in (\mathfrak{a}_+ - 1, 0)$. Therefore, (5.5.12) is the Mellin transform of $F_\Psi(x)$ in ordinary sense. Moreover, with $\mathtt{N}_+ = |\mathfrak{u}_+| \mathbb{I}_{\{|\mathfrak{u}_+| \in \mathbb{N}\}} + (\lceil |\mathfrak{a}_+| + 1 \rceil) \mathbb{I}_{\{|\mathfrak{u}_+| \notin \mathbb{N}\}}$, $\mathbb{N} \ni M < \mathtt{N}_+$ and $a \in ((-M-1) \vee (\mathfrak{a}_+ - 1), -M)$ so that $a + M \in (-1,0)$ we apply the Cauchy theorem in the Mellin inversion for $F_\Psi(x)$ to get that

$$
\begin{aligned}
F_\Psi(x) &= -\frac{\phi_-(0)}{2\pi i} \int_{a+M-i\infty}^{a+M+i\infty} x^{-z} \frac{\mathcal{M}_\Psi(z+1)}{z} dz \\
&= q \sum_{k=1}^{M} \frac{\prod_{j=1}^{k-1} \Psi(j)}{k!} x^k - \frac{\phi_-(0)}{2\pi i} \int_{a-i\infty}^{a+i\infty} x^{-z} \frac{\mathcal{M}_\Psi(z+1)}{z} dz,
\end{aligned}
\tag{5.5.13}
$$

since the residues are of values $\frac{\phi_-(0)}{k} \phi_+(0) \frac{\prod_{j=1}^{k-1} \Psi(j)}{(k-1)!} = q \frac{\prod_{j=1}^{k-1} \Psi(j)}{k!}$ at each of those poles at $-k$ and are computed as the residues of $\mathcal{M}_\Psi$, see Theorem 5.2.1, and the contribution of the other terms of (5.5.12), that is $-\phi_-(0)/z$. We recall that by convention $\sum_{j=1}^0 = 0$. Thus, we prove (5.2.23) for $n = 0$. The derivative of order n is easily established via differentiating (5.5.13) as long as $0 \le n < \mathtt{N}_\Psi$.

Proof of Corollary 5.2.12. We sketch the proof of Corollary 5.2.12. If $|\mathfrak{a}_+| = \infty$ and $-\mathfrak{u}_+ \notin \mathbb{N}$ then the fact the sum in (5.2.23) taken to infinity, that is

$$
F(x) = q \sum_{k=1}^{\infty} \frac{\prod_{j=1}^{k-1} \Psi(j)}{k!} x^k
$$

is an asymptotic expansion is immediate. From the identity $\Psi(j) = -\phi_+(-j)\phi_-(j)$ and (5.3.3) we see that

$$
\phi_+(-j) = \phi_-(0) - \mathtt{d}_- j - \int_0^\infty \left(e^{jy} - 1\right) \mu_+(dy)
$$

and clearly $|\phi_+(-j)|$ has an exponential growth in j if μ_+ is not identically zero. In the latter case the asymptotic series cannot be a convergent series for any $x > 0$. If $\phi_+(z) = \phi_+(0) + \mathtt{d}_+ z$ then $\Psi(j) \overset{\infty}{\sim} \mathtt{d}_+ j \phi_-(j)$ and hence

$$
\frac{\prod_{j=1}^{k-1} |\Psi(j)|}{k!} \overset{\infty}{\sim} \frac{\mathtt{d}_+^{k-1}}{k} \prod_{j=1}^{k-1} \phi_-(j).
$$

Therefore, F is absolutely convergent if $x < \frac{1}{\mathtt{d}_+\phi_-(\infty)}$ and divergent if $x > \frac{1}{\mathtt{d}_+\phi_-(\infty)}$. Finally, if $\phi_+ \equiv 1$ then $\frac{\prod_{j=1}^{k-1}|\Psi(j)|}{k!} \stackrel{\infty}{=} \frac{\prod_{j=1}^{k-1}\phi_-(j)}{k!}$ and since from Proposition 5.3.13(3), $\phi_-(j) \stackrel{\infty}{\approx} j\left(\mathtt{d}_- + \mathrm{o}\left(1\right)\right)$, we deduct that F is absolutely convergent for $x < \frac{1}{\mathtt{d}_-}$ and divergent for $x > \frac{1}{\mathtt{d}_-}$. □ □

5.5.5 Proof of Theorem 5.2.14

Recall the definitions of the *lattice class* and the *weak non-lattice class*, see around (5.2.25). We start with a result which discusses when the decay of $|\mathcal{M}_\Psi(z)|$ can be extended to the line $\mathbb{C}_{1-\mathfrak{u}_-}$.

Proposition 5.5.2. *Let $\Psi \in \mathcal{N}$. If $\Psi \in \mathcal{N}_\infty \cap \mathcal{N}_{\mathcal{W}}$ then the subexponential decay of $|\mathcal{M}_{I_\Psi}(z)|$ along $\mathbb{C}_{1-\mathfrak{u}_-}$ is preserved. Otherwise, if $\Psi \in \mathcal{N}_{\mathtt{N}_\Psi}$, $\mathtt{N}_\Psi < \infty$, then the same polynomial decay is valid for $|\mathcal{M}_{I_\Psi}|$ along $\mathbb{C}_{1-\mathfrak{u}_-}$.*

Proof. Let $-\mathfrak{u}_- = -\overline{\mathfrak{a}}_- \in (0,\infty)$. Using (5.1.2) we write for $b \neq 0$

$$\mathcal{M}_\Psi\left(1-\mathfrak{u}_- + ib\right) = \frac{\mathfrak{u}_- - ib}{\Psi\left(\mathfrak{u}_- - ib\right)}\mathcal{M}_\Psi\left(-\mathfrak{u}_- + ib\right). \tag{5.5.14}$$

Assume first that $\Psi \in \mathcal{N}_\infty$ then we choose $k_0 \in \mathbb{N}$, whose existence is guaranteed since $\Psi \in \mathcal{N}_{\mathcal{W}}$, such that $\varliminf_{|b|\to\infty} |b|^{k_0}\left|\Psi\left(\mathfrak{u}_- + ib\right)\right| > 0$. Premultiplying (5.5.14) with $|b|^{-k_0}$ and taking absolute values we conclude that $|\mathcal{M}_\Psi\left(1-\mathfrak{u}_- + ib\right)|$ has subexponential decay. Recall from Theorem 5.2.5(1) that $\Psi \in \mathcal{N}_{\mathtt{N}_\Psi}$ with $\mathtt{N}_\Psi < \infty \iff \phi_+ \in \mathcal{B}_P$, $\phi_- \in \mathcal{B}_P^c$, $\overline{\overline{\Pi}}(0) < \infty$ and from (5.4.50) that the Lévy measure behind ϕ_- is absolutely continuous. Then, we conclude from Proposition 5.3.13(3) that $\lim_{|b|\to\infty} \frac{|\phi_+(-\mathfrak{u}_+ + ib)|}{|\mathfrak{u}_+ + ib|} = \mathtt{d}_+ > 0$ and from an easy extension of Proposition 5.3.13(4) that $\lim_{|b|\to\infty} \phi_-\left(\mathfrak{u}_- + ib\right) = \phi_-\left(\infty\right) > 0$. Therefore, we conclude that in (5.5.14),

$$\lim_{|b|\to\infty} \frac{|\mathfrak{u}_- - ib|}{|\Psi\left(\mathfrak{u}_- - ib\right)|} = \frac{1}{\mathtt{d}_+\phi_-\left(\infty\right)} > 0.$$

This shows that the speed of decay of $|\mathcal{M}_\Psi\left(-\mathfrak{u}_- + ib\right)|$ and therefore via (5.2.21) the speed of decay of $|\mathcal{M}_{I_\Psi}(z)|$ are preserved. □

We start with the proof of (5.2.29) of item (2). Note that an easy computation related to the Mellin transforms shows that the restriction of

$$\overline{\mathcal{M}^\star_{I_\Psi}}(z) = \frac{1}{z}\mathcal{M}_{I_\Psi}(z+1) \in \mathtt{A}_{(-1,0)} \cap \mathtt{M}_{(\mathfrak{a}_+-1,-\mathfrak{a}_-)}, \tag{5.5.15}$$

to $\mathbb{C}_{(0,-\overline{\mathfrak{a}}_-)}$, if $-\overline{\mathfrak{a}}_- > 0$, is the Mellin transform of $\overline{F}_\Psi(x) = \mathbb{P}\left(I_\Psi \geq x\right)$ in the distributional sense. Next, note from Theorem 5.2.5(1) that since $\mathtt{N}_\Psi > 0$ for every $\Psi \in \mathcal{N}$ then along $\mathbb{C}_a$, $a \in (0,-\overline{\mathfrak{a}}_-)$, $|\overline{\mathcal{M}^\star_{I_\Psi}}(z)|$ decays subexponentially if $\Psi \in \mathcal{N}_\infty$ or by speed of $\mathtt{N}_\Psi + 1 > 1$ otherwise. Therefore, $\overline{\mathcal{M}^\star_{I_\Psi}}$ is the Mellin transform of $\overline{F}_\Psi(x)$ in the ordinary sense. When

$\Psi \in \mathcal{N}_{\mathcal{Z}}$, (5.2.29) is known modulo to an unknown constant, see [88, Lemma 4]. The value of this constant, that is $\frac{\phi_-(0)\Gamma(-\mathfrak{u}_-)W_{\phi_-}(1+\mathfrak{u}_-)}{\phi_-'\left(\mathfrak{u}_-^+\right)W_{\phi_+}(1-\mathfrak{u}_-)}$, can be immediately computed as in (5.5.22) below. We proceed to establish (5.2.30). For this purpose we assume that either $\Psi \in \mathcal{N}_\infty \cap \mathcal{N}_{\mathcal{W}}$ or $\Psi \in \mathcal{N}_{\mathtt{N}_\Psi}$, $\mathtt{N}_\Psi < \infty$. In any case, whenever $\mathtt{N}_\Psi > 1$, the Mellin inversion theorem applies and yields that, for any $z \in \mathbb{C}_a$, $a \in (0, 1 - \overline{\mathfrak{a}}_-)$,

$$f_\Psi(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} x^{-a-ib}\mathcal{M}_{I_\Psi}(a+ib)\,db. \tag{5.5.16}$$

However, the assumptions $\overline{\mathfrak{a}}_- = \mathfrak{u}_- < 0$, $\Psi(\mathfrak{u}_-) = -\phi_+(-\mathfrak{u}_-)\,\phi_-(\mathfrak{u}_-) = 0$ and $|\Psi'(\mathfrak{u}_-^+)| < \infty$ of item (2) together with (5.1.2) lead to

$$\Psi'\left(\mathfrak{u}_-^+\right) = \phi_-'\left(\mathfrak{u}_-^+\right)\phi_+(-\mathfrak{u}_-)$$

and hence $|\Psi'(\mathfrak{u}_-^+)| < \infty$ implies that $|\phi_-'(\mathfrak{u}_-^+)| < \infty$. This observation, the form of $\mathcal{M}_\Psi$, see (5.4.4) and the fact that $\Psi \in \mathcal{N}_{\mathcal{Z}} \subset \mathcal{N}_{\mathcal{W}}$ permit us to write

$$\begin{aligned}\mathcal{M}_{I_\Psi}(z) &= \phi_-(0)\frac{\Gamma(z)}{W_{\phi_+}(z)}W_{\phi_-}(1-z)\\ &= \phi_-(0)\frac{\Gamma(z)}{W_{\phi_+}(z)}P(1-z) + \frac{\phi_-(0)W_{\phi_-}(1+\mathfrak{u}_-)}{\phi_-'(\mathfrak{u}_-^+)}\frac{1}{1-\mathfrak{u}_- - z}\frac{\Gamma(z)}{W_{\phi_+}(z)},\end{aligned} \tag{5.5.17}$$

with P from Theorem 5.3.1(3) having the form and the property that

$$P(z) = W_{\phi_-}(z) - \frac{W_{\phi_-}(1+\mathfrak{u}_-)}{\phi_-'(\mathfrak{u}_-^+)(z-\mathfrak{u}_-)} \in \mathtt{A}_{[\mathfrak{u}_-,\infty)}.$$

Therefore, (5.5.17) shows that $\mathcal{M}_{I_\Psi} \in \mathtt{A}_{(0,1-\mathfrak{u}_-)}$ extends continuously to $\mathbb{C}_{1-\mathfrak{u}_-} \setminus \{1-\mathfrak{u}_-\}$. Next, we show that the contour in (5.5.16) can be partly moved to the line $\mathbb{C}_{1-\mathfrak{u}_-}$ at least for $|b| > c > 0$ for any $c > 0$. For this purpose we observe from Proposition 5.5.2 that whenever $\Psi \in \mathcal{N}_{\mathcal{W}} \cap \mathcal{N}_\infty$(resp. $\Psi \in \mathcal{N}_{\mathtt{N}_\Psi} \cap \mathcal{N}$, $\mathtt{N}_\Psi < \infty$) the decay of $|\mathcal{M}_{I_\Psi}(z)|$ extends with the same subexponential (resp. polynomial) speed to the complex line $\mathbb{C}_{1-\mathfrak{u}_-}$. Then, for any $c > 0$, $a \in (0, 1-\mathfrak{u}_-)$ and $x > 0$, thanks to the Cauchy integral theorem valid because $\mathtt{N}_\Psi > 1$,

$$\begin{aligned} f_\Psi(x) &= \frac{1}{2\pi i}\int_{z=a+ib} x^{-z}\mathcal{M}_{I_\Psi}(z)\,dz\\ &= x^{\mathfrak{u}_- - 1}\frac{1}{2\pi}\int_{|b|>c} x^{-ib}\mathcal{M}_{I_\Psi}(1-\mathfrak{u}_- + ib)\,db + \frac{1}{2\pi i}\int_{B^{\downharpoonleft}(1-\mathfrak{u}_-,c)} x^{-z}\mathcal{M}_{I_\Psi}(z)dz,\\ &= f_\Psi^*(x,c) + f_\Psi^\star(x,c),\end{aligned} \tag{5.5.18}$$

where $B^{\downharpoonleft}(1-\mathfrak{u}_-, c) = \{z \in \mathbb{C} :\ |z-1+\mathfrak{u}_-| = c \text{ and } \mathtt{Re}(z-1+\mathfrak{u}_-) \le 0\}$ that is a semi-circle centered at $1-\mathfrak{u}_-$. We note that the Riemann-Lebesgue lemma applied to the absolutely integrable function $g(b) = \mathcal{M}_{I_\Psi}(1-\mathfrak{u}_- + ib)\,\mathbb{I}_{\{|b|>c\}}$ yields that

$$\lim_{x\to\infty} x^{1-\mathfrak{u}_-} f_\Psi^*(x,c) = \lim_{x\to\infty}\frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ib\ln x}\mathcal{M}_{I_\Psi}(1-\mathfrak{u}_- + ib)\,\mathbb{I}_{\{|b|>c\}}db = 0. \tag{5.5.19}$$

Using (5.5.17) we write that

$$\begin{aligned} f_{\Psi}^{\star}(x,c) &= \frac{\phi_-(0)}{2\pi i}\int_{B^{\downarrow}(1-\mathfrak{u}_-,c)} x^{-z}\frac{\Gamma(z)}{W_{\phi_+}(z)}P(1-z)dz \\ &+ \frac{1}{2\pi i}\frac{\phi_-(0)W_{\phi_-}(1+\mathfrak{u}_-)}{\phi_-'(\mathfrak{u}_-^+)}\int_{B^{\downarrow}(1-\mathfrak{u}_-,c)} x^{-z}\frac{1}{1-\mathfrak{u}_- - z}\frac{\Gamma(z)}{W_{\phi_+}(z)}dz \\ &= f_{\Psi}^{\star\star}(x,c) + f_{\Psi}^{\star\star\star}(x,c). \end{aligned} \tag{5.5.20}$$

Since $\frac{\Gamma(z)}{W_{\phi_+}(z)}P(1-z) \in \mathtt{A}_{(0,1-\mathfrak{u}_-]}$ we can use the Cauchy integral theorem to show that, for every $c>0$ fixed,

$$\begin{aligned} |f_{\Psi}^{\star\star}(x,c)| &= \left|\frac{\phi_-(0)}{2\pi}\int_{-c}^{c} x^{-1+\mathfrak{u}_- - ib}\frac{\Gamma(1-\mathfrak{u}_- + ib)}{W_{\phi_+}(1-\mathfrak{u}_- + ib)}P(\mathfrak{u}_- - ib)db\right| \\ &\leq x^{\mathfrak{u}_- -1}\frac{c\phi_-(0)}{\pi}\sup_{z=1-\mathfrak{u}_- + ib;\, b\in(-c,c)}\left|\frac{\Gamma(z)}{W_{\phi_+}(z)}P(1-z)\right|. \end{aligned} \tag{5.5.21}$$

Next, we consider $f_{\Psi}^{\star\star\star}(x,c)$. Since $\frac{\Gamma(z)}{W_{\phi_+}(z)} \in \mathtt{A}_{(0,\infty)}$, choosing c small enough we have by the Cauchy's residual theorem that

$$\begin{aligned} f_{\Psi}^{\star\star\star}(x,c) &= \frac{\phi_-(0)W_{\phi_-}(1+\mathfrak{u}_-)\,\Gamma(1-\mathfrak{u}_-)}{\phi_-'(\mathfrak{u}_-^+)\,W_{\phi_+}(1-\mathfrak{u}_-)}x^{\mathfrak{u}_- -1} \\ &+ \frac{1}{2\pi i}\frac{\phi_-(0)W_{\phi_-}(1+\mathfrak{u}_-)}{\phi_-'(\mathfrak{u}_-^+)}\int_{B^{\uparrow}(1-\mathfrak{u}_-,c)} x^{-z}\frac{1}{1-\mathfrak{u}_- - z}\frac{\Gamma(z)}{W_{\phi_+}(z)}dz, \end{aligned}$$

where $B^{\uparrow}(1-\mathfrak{u}_-,c) = \{z\in\mathbb{C}:\, |z-1+\mathfrak{u}_-| = c \text{ and } \mathtt{Re}(z-1+\mathfrak{u}_-)\geq 0\}$. However, on $B^{\uparrow}(1-\mathfrak{u}_-,c)$, we have that $z = 1-\mathfrak{u}_- + ce^{i\theta}$, $\theta\in\left(-\frac{\pi}{2},\frac{\pi}{2}\right)$, and thus for any such θ

$$\lim_{x\to\infty}\left|x^{1-\mathfrak{u}_-}x^{-z}\right| = \lim_{x\to\infty} x^{-c\cos\theta} = 0.$$

Therefore, since

$$\sup_{z\in B^{\uparrow}(1-\mathfrak{u}_-,c)}\left|\frac{1}{1-\mathfrak{u}_- - z}\frac{\Gamma(z)}{W_{\phi_+}(z)}\right| \leq \frac{1}{c}\sup_{z\in B^{\uparrow}(1-\mathfrak{u}_-,c)}\left|\frac{\Gamma(z)}{W_{\phi_+}(z)}\right| < \infty,$$

we can apply the dominated convergence theorem to the integral term in the representation of $f_{\Psi}^{\star\star\star}$ above to conclude that for all $c>0$ small enough

$$\lim_{x\to\infty} x^{1-\mathfrak{u}_-}f_{\Psi}^{\star\star\star}(x,c) = \frac{\phi_-(0)W_{\phi_-}(1+\mathfrak{u}_-)\,\Gamma(1-\mathfrak{u}_-)}{\phi_-'(\mathfrak{u}_-^+)\,W_{\phi_+}(1-\mathfrak{u}_-)}. \tag{5.5.22}$$

Combining (5.5.22) and (5.5.21), we get from (5.5.20) that

$$\lim_{c\to 0}\lim_{x\to\infty} x^{1-\mathfrak{u}_-}f_{\Psi}^{\star}(x,c) = \lim_{c\to 0}\lim_{x\to\infty} x^{1-\mathfrak{u}_-}f_{\Psi}^{\star\star\star}(x,c) = \frac{\phi_-(0)W_{\phi_-}(1+\mathfrak{u}_-)\,\Gamma(1-\mathfrak{u}_-)}{\phi_-'(\mathfrak{u}_-^+)\,W_{\phi_+}(1-\mathfrak{u}_-)}.$$

However, since (5.5.19) holds too, we get from (5.5.18) that

$$\lim_{x\to\infty} x^{1-\mathfrak{u}_-} f_\Psi(x) = \frac{\phi_-(0)W_{\phi_-}(1+\mathfrak{u}_-)\,\Gamma(1-\mathfrak{u}_-)}{\phi'_-(\mathfrak{u}_-^+)\,W_{\phi_+}(1-\mathfrak{u}_-)}.$$

This completes the proof of (5.2.30) of item (2) for $n=0$. Its claim for any $n\in\mathbb{N}$, $n\leq \lceil \mathtt{N}_\Psi\rceil -2$, follows by the same techniques as above after differentiating (5.5.18) and observing that we have thanks to Proposition 5.5.1 that for every $n\in\mathbb{N}$, $n\leq\lceil\mathtt{N}_\Psi\rceil-2$,

$$\lim_{b\to\infty}|b|^n\left|\mathcal{M}_{I_\Psi}(1-\mathfrak{u}_-+ib)\right|=0$$

and $|b|^n\left|\mathcal{M}_{I_\Psi}(1-\mathfrak{u}_-+ib)\right|$ is integrable.

Let us proceed with the proof of item (1). We note that in any case the Mellin transform of $\overline{F}_\Psi(x)$, that is $\overline{\mathcal{M}^\star_{I_\Psi}}$ in (5.5.15), has a decay of value $\mathtt{N}_\Psi+1>1$ along $\mathbb{C}_a$, $a\in(0,-\overline{\mathfrak{a}}_-)$, if $-\overline{\mathfrak{a}}_->0$. Consider first (5.2.26), that is $\overline{\lim_{x\to\infty}}\; x^{\underline{d}}\,\overline{F}_\Psi(x)=0$ for $\underline{d}<|\overline{\mathfrak{a}}_-|$. Let $-\overline{\mathfrak{a}}_->0$, as otherwise there is nothing to prove in (5.2.26), and choose $\underline{d}\in(0,-\overline{\mathfrak{a}}_-)$. By a Mellin inversion as in (5.5.18) and with $a=\underline{d}+\varepsilon<-\overline{\mathfrak{a}}_-$, $\varepsilon>0$, we get that on $\mathbb{R}^+$

$$\overline{F}_\Psi(x)=x^{-\underline{d}-\varepsilon}\frac{1}{2\pi}\left|\int_{-\infty}^{\infty}x^{-ib}\overline{\mathcal{M}^\star_{I_\Psi}}\left(\underline{d}+\varepsilon+ib\right)db\right|\leq C_{\underline{d}}x^{-\underline{d}-\varepsilon}$$

and (5.2.26) follows. This also settles the claim (5.2.27) when $-\Psi(-z)=\phi_+(z)\in\mathcal{B}$ since then $\overline{\mathcal{M}^\star_{I_\Psi}}(z)=\frac{1}{z}\frac{\Gamma(z)}{W_{\phi_+}(z)}\in\mathtt{A}_{(0,\infty)}$, $|\overline{\mathfrak{a}}_-|=\infty$ and we can choose $\underline{d}$ as big as we wish. It remains therefore to prove (5.2.27) assuming that $-\Psi(-z)\not\equiv\phi_+(z)$. If $\Pi(dy)\equiv 0\,dy$ on $(-\infty,0)$ then necessarily $\phi_-(z)=\phi_-(0)+\mathtt{d}_-z$, $\mathtt{d}_->0$, and even the stronger item (2) is applicable since it can be immediately shown that $\Psi\in\mathcal{N}_\mathcal{W}$ since $\lim_{|b|\to\infty}\left|\phi_-\left(-\frac{\phi_-(0)}{\mathtt{d}_-}+ib\right)\right|=\infty$, see (5.2.25). We can assume from now that $\Pi(dy)\not\equiv 0\,dy$ on $(-\infty,0)$ and $\Psi\in\mathcal{N}_\mathcal{Z}$. If the conditions of item (2) are violated we proceed by approximation to furnish them. Recall (5.5.6) and with the given Ψ define $\forall\eta>0$,

$$\Psi^\eta(z)=\frac{\sigma^2}{2}z^2+cz+\int_{\mathbb{R}}\left(e^{zr}-1-zr\mathbb{I}_{\{|r|<1\}}\right)e^{-\eta r^2\mathbb{I}_{\{r<-1\}}}\Pi(dr)-q. \tag{5.5.23}$$

Put $\Pi^\eta(dr)=e^{-\eta r^2\mathbb{I}_{\{r<-1\}}}\Pi(dr)$. Since Π^η is absolutely continuous with respect to Π it is clear that $\forall\eta>0$, $\Psi\in\mathcal{N}_\mathcal{Z}\implies\Psi^\eta\in\mathcal{N}_\mathcal{Z}$ and set from (5.1.2), $\Psi^\eta(z)=-\phi^\eta_+(-z)\phi^\eta_-(z)$. Let ξ^η be the Lévy process underlying Ψ^η and note that the transformation $\Psi\mapsto\Psi^\eta$ leaves q invariant and has the sole effect of truncating at the level of paths some of the negative jumps smaller than -1 of the underlying Lévy process ξ. Therefore, pathwise

$$I_\Psi=\int_0^{\mathbf{e}_q}e^{-\xi_s}ds\geq\int_0^{\mathbf{e}_q}e^{-\xi^\eta_s}ds=I_{\Psi^\eta}. \tag{5.5.24}$$

Next, it is clear from (5.5.23) that $\Psi^\eta\in\mathtt{A}_{(-\infty,0)}$ and then it is immediate that $(\Psi^\eta)''>0$ on $\mathbb{R}^-$, and thus Ψ^η is convex on $\mathbb{R}^-$. Moreover, clearly

$$\lim_{u\to-\infty}\frac{1}{|u|}\int_{-\infty}^0\left(e^{ur}-1-ru\mathbb{I}_{\{|r|<1\}}\right)\Pi^\eta(dr)=\infty$$

since $\Pi(dy)$ is not identically the zero measure on $\mathbb{R}^-$. Therefore, $\lim_{u\to-\infty}\Psi^\eta(u)=\infty$. Thus, immediately we conclude that $-\mathfrak{u}_{\phi_-^\eta}\in(0,\infty)$ if $\Psi(0)=-q<0$. Also, if $q=0$, then necessarily, regardless of whether $\mathbb{E}[\xi_1]\in(0,\infty]$ or $\mathbb{E}\left[\xi_1\mathbb{I}_{\{\xi_1\}}>0\right]=\mathbb{E}\left[-\xi_1\mathbb{I}_{\{\xi_1<0\}}\right]=\infty$ with $\lim_{t\to\infty}\xi_t=\infty$ hold for ξ, then $\mathbb{E}\left[\xi_1^\eta\right]\in(0,\infty]$, see (5.5.23), and thus $\phi_-^\eta(0)>0$ leads to $-\mathfrak{u}_{\phi_-^\eta}\in(0,\infty)$. However, from (5.5.24) we get that

$$\mathbb{P}\left(I_\Psi>x\right)\geq\mathbb{P}\left(I_{\Psi^\eta}>x\right)$$

and from (5.2.29) of item (2) we obtain, for any $\epsilon>0$ and fixed $\eta>0$, that with some $C>0$

$$\overline{\lim_{x\to\infty}}\,x^{-\mathfrak{u}_{\phi_-^\eta}+\epsilon}\mathbb{P}\left(I_\Psi>x\right)\geq\overline{\lim_{x\to\infty}}\,x^{-\mathfrak{u}_{\phi_-^\eta}+\epsilon}\mathbb{P}\left(I_{\Psi^\eta}>x\right)=C\,\overline{\lim_{x\to\infty}}\,x^\epsilon=\infty.$$

Thus, it remains to show that $\lim_{\eta\to0}\mathfrak{u}_{\phi_-^\eta}=\overline{\mathfrak{a}}_-\in(-\infty,0]$, where $\overline{\mathfrak{a}}_->-\infty$ is evident since $z\mapsto-\Psi(-z)\notin\mathcal{B}$. It is clear from the identity

$$\begin{aligned}\Psi^\eta(u)=&\frac{\sigma^2}{2}u^2+cu+\int_0^\infty\left(e^{ur}-1-ur\mathbb{I}_{\{r<1\}}\right)\Pi(dr)\\&+\int_{-\infty}^0\left(e^{ur}-1-ur\mathbb{I}_{\{r>-1\}}\right)e^{-\eta r^2\mathbb{I}_{\{r<-1\}}}\Pi(dr),\end{aligned}$$

that for any $u\leq0$, $\Psi^\eta(u)$ is increasing as $\eta\downarrow0$ and $\lim_{\eta\to0}\Psi^\eta(u)=\Psi(u)$. Fix $\eta^*>0$ small enough. Then, clearly,

$$\Psi\left(\mathfrak{u}_{\phi_-^{\eta^*}}\right)=\lim_{\eta\to0}\Psi^\eta\left(\mathfrak{u}_{\phi_-^{\eta^*}}\right)\geq\Psi^{\eta^*}\left(\mathfrak{u}_{\phi_-^{\eta^*}}\right)=0.$$

If $\Psi\left(\mathfrak{u}_{\phi_-^{\eta^*}}\right)=\infty$ then from (5.1.2) we get that $\phi_-\left(\mathfrak{u}_{\phi_-^{\eta^*}}\right)=-\infty$ and thus $\mathfrak{u}_{\phi_-^{\eta^*}}\leq\overline{\mathfrak{a}}_-$. Similarly, if $\Psi\left(\mathfrak{u}_{\phi_-^{\eta^*}}\right)\in[0,\infty)$ then $\phi_-\left(\mathfrak{u}_{\phi_-^{\eta^*}}\right)\leq0$ and again $\mathfrak{u}_{\phi_-^{\eta^*}}\leq\overline{\mathfrak{a}}_-$. Therefore,

$$\underline{\mathfrak{u}}=\overline{\lim_{\eta\to0}}\,\mathfrak{u}_{\phi_-^\eta}\leq\overline{\mathfrak{a}}_-.$$

Assume that $\underline{\mathfrak{u}}<\overline{\mathfrak{a}}_-$ and choose $u\in(\underline{\mathfrak{u}},\overline{\mathfrak{a}}_-)$. Then $\Psi^\eta(u)<0$ and thus $\Psi(u)=\lim_{\eta\to0}\Psi^\eta(u)\leq0$. However, $\Psi(u)\in(0,\infty]$, for $u<\overline{\mathfrak{a}}_-$, which triggers $\underline{\mathfrak{u}}=\overline{\mathfrak{a}}_-$. Moreover, the monotonicity of Ψ^η when $\eta\downarrow0$ shows that in fact

$$\lim_{\eta\to0}\mathfrak{u}_{\phi_-^\eta}=\underline{\mathfrak{u}}=\overline{\mathfrak{a}}_-$$

and we conclude the statement (5.2.27) when $\Psi\in\mathcal{N}_\mathcal{Z}$. Next, if $\Psi\notin\mathcal{N}_\mathcal{Z}$ then as in Theorem 5.3.1(2) one can check that $\Pi(dx)=\sum_{n=-\infty}^\infty c_n\delta_{\bar{h}n}(dx)$, $\sum_{n=-\infty}^\infty c_n<\infty$, $c_n\geq0$, $\bar{h}>0$, and $\sigma^2=c=0$ in (5.5.6). The underlying Lévy process, ξ, is a possibly killed compound Poisson process living on the lattice $\left\{\bar{h}n\right\}_{n=-\infty}^\infty$. We proceed by approximation. Set $\Psi_{\mathtt{d}}(z)=\Psi(z)+\mathtt{d}z$, $\mathtt{d}>0$, with underlying Lévy process $\xi^{\mathtt{d}}$. Clearly, $\xi_t^{\mathtt{d}}=\xi_t+\mathtt{d}t$, $t\geq0$, and hence $I_\Psi \geq I_{\Psi_{\mathtt{d}}}$ and, for any $x>0$, $\mathbb{P}\left(I_\Psi > x\right) \geq \mathbb{P}\left(I_{\Psi_{\mathtt{d}}} > x\right)$. However, $\Psi_{\mathtt{d}} \in \mathcal{N}_{\mathcal{Z}}$ and therefore from (5.2.27) for any $\overline{d} > \left|\overline{\mathfrak{a}}_{-}^{\mathtt{d}}\right|$, where from $\Psi_{\mathtt{d}}(z) = -\phi_{+}^{\mathtt{d}}(-z)\phi_{-}^{\mathtt{d}}(z)$, $\overline{\mathfrak{a}}_{-}^{\mathtt{d}} = \overline{\mathfrak{a}}_{\phi_{-}^{\mathtt{d}}}$, see (5.3.8), we deduct that

$$\varlimsup_{x\to\infty} x^{\overline{d}}\mathbb{P}\left(I_\Psi > x\right) \geq \varlimsup_{x\to\infty} x^{\overline{d}}\mathbb{P}\left(I_{\Psi_{\mathtt{d}}} > x\right) = \infty.$$

To establish (5.2.27) for Ψ it remains to confirm that $\lim_{\mathtt{d}\to 0} \overline{\mathfrak{a}}_{-}^{\mathtt{d}} = \overline{\mathfrak{a}}_{-}$. Note that adding $\mathtt{d}z$ to $\Psi(z)$ does not alter its range of analyticity and hence with the obvious notation $\mathfrak{a}_{-} = \mathfrak{a}_{\phi_{-}^{\mathtt{d}}}$. Set

$$T = \inf\left\{t>0 : \xi_t < 0\right\} \in (0,\infty] \text{ and } T^{\mathtt{d}} = \inf\left\{t>0 : \xi_t^{\mathtt{d}} < 0\right\} \in (0,\infty].$$

Clearly, from $\xi_t^{\mathtt{d}} = \xi_t + \mathtt{d}t$ we get that a.s. $\lim_{\mathtt{d}\to 0}\left(T^{\mathtt{d}}, \xi_{T^{\mathtt{d}}}\right) = \left(T, \xi_T\right)$. However, (T,ξ_T) and $(T^{\mathtt{d}}, \xi_{T^{\mathtt{d}}})$ define the distribution of the bivariate descending ladder time and height processes of Ψ and $\Psi_{\mathtt{d}}$, see Section 5.7.1. Therefore, since from Lemma 5.7.1 we can choose $\phi_{-}, \phi_{-}^{\mathtt{d}}$ to represent the descending ladder height process, that is $\phi_{-} = k_{-}$, $\phi_{-}^{\mathtt{d}} = k_{-}^{\mathtt{d}}$ in the notation therein, we conclude that $\lim_{\mathtt{d}\to 0}\phi_{-}^{\mathtt{d}}(z) = \phi_{-}(z)$ for any $z \in \mathbb{C}_{(\mathfrak{a}_{-},\infty)}$ and hence $\lim_{\mathtt{d}\to 0}\mathfrak{u}_{\phi_{-}^{\mathtt{d}}} = \mathfrak{u}_{-}$. Thus $\lim_{\mathtt{d}\to 0}\overline{\mathfrak{a}}_{-}^{\mathtt{d}} = \overline{\mathfrak{a}}_{-}$. This, concludes Theorem 5.2.14. □

5.5.6 Proof of Theorem 5.2.19

Recall from (5.5.12) that $\mathcal{M}^{\star}_{I_\Psi}(z)$ is the Mellin transform of $F_\Psi(x)$ at least on $\mathbb{C}_{(-1,0)}$. We record and re-express it with the help of (5.2.6) and (5.3.4) as

$$\begin{aligned}\mathcal{M}^{\star}_{I_\Psi}(z) &= -\frac{1}{z}\mathcal{M}_{I_\Psi}(z+1) = -\frac{\phi_{-}(0)}{z}\mathcal{M}_{\Psi}(z+1)\\ &= -\frac{\phi_{-}(0)}{z}\frac{\Gamma(z+1)}{W_{\phi_{+}}(z+1)}W_{\phi_{-}}(-z),\ z\in\mathbb{C}_{(-1,0)}.\end{aligned} \tag{5.5.25}$$

From Theorem 5.2.1 we deduct that $\mathcal{M}_{I_\Psi} \in \mathtt{A}_{(0,1-\mathfrak{a}_{-})}$ and that it extends continuously to $\mathbb{C}_0 \setminus \{0\}$. Moreover, Proposition 5.5.1 applied to $\mathcal{M}_{I_\Psi}(z)$ for $z \in i\mathbb{R}$ shows that the decay of $|\mathcal{M}_{I_\Psi}(z)|$ is preserved along $\mathbb{C}_0$. Therefore, either $\left|\mathcal{M}^{\star}_{I_\Psi}(z)\right|$ decays subexponentially along $\mathbb{C}_a$, $a\in[-1,0)$, or it decays polynomially with a speed of $\mathtt{N}_\Psi + 1 > 1$. Via a Mellin inversion, choosing a contour, based on the line $\mathbb{C}_{-1}$ and a semi-circle, as in the proof of Theorem 5.2.14, see (5.5.18) and (5.5.19), we get that, for any $c \in \left(0,\frac{1}{2}\right)$, as $x\to 0$,

$$F_\Psi(x) = \frac{1}{2\pi i}\int_{B^{\upharpoonright}(-1,c)} x^{-z}\mathcal{M}^{\star}_{I_\Psi}(z)dz + \mathrm{o}\left(x\right), \tag{5.5.26}$$

where only the contour is changed to $B^{\upharpoonright}(-1,c) = \{z\in\mathbb{C} : |z+1| = c \text{ and } \mathtt{Re}(z+1)\geq 0\}$. Apply (5.3.4) to write from (5.5.25)

$$\mathcal{M}^{\star}_{I_\Psi}(z) = -\frac{\phi_{-}(0)}{z}\frac{\phi_{+}(z+1)}{z+1}\frac{\Gamma(z+2)}{W_{\phi_{+}}(z+2)}W_{\phi_{-}}(-z) = \frac{\phi_{+}(z+1)}{z+1}\mathcal{M}^{\star\star}_{I_\Psi}(z). \tag{5.5.27}$$

Clearly, $\mathcal{M}^{\star\star}_{I_\Psi} \in \mathtt{A}_{(-2,0)}$. Recall that $\phi^\sharp_+(z) = \phi_+(z) - \phi_+(0) \in \mathcal{B}$. Then, we have, noting $\mathcal{M}^{\star\star}_{I_\Psi}(-1) = \phi_-(0)$, that

$$\begin{aligned}\mathcal{M}^{\star}_{I_\Psi}(z) &= \frac{\phi_+(0)\phi_-(0)}{z+1} + \frac{\phi^\sharp_+(z+1)}{z+1}\mathcal{M}^{\star\star}_{I_\Psi}(-1) + \phi_+(z+1)\frac{\mathcal{M}^{\star\star}_{I_\Psi}(z) - \mathcal{M}^{\star\star}_{I_\Psi}(-1)}{z+1}\\ &= \mathcal{M}^*_1(z) + \mathcal{M}^*_2(z) + \mathcal{M}^*_3(z).\end{aligned}$$

Plugging this in (5.5.26) we get and set

$$F_\Psi(x) = \frac{1}{2\pi i}\int_{B^\uparrow(-1,c)} x^{-z}\sum_{j=1}^{3}\mathcal{M}^*_j(z)dz + \mathrm{o}\,(x) = \sum_{j=1}^{3} F_j(x) + \mathrm{o}\,(x)\,. \tag{5.5.28}$$

Since $z \mapsto \frac{\mathcal{M}^*_3(z)}{\phi_+(z+1)} \in \mathtt{A}_{(-2,0)}$ and $\phi_+(0) \in [0,\infty)$ then precisely as in (5.5.21) we get that

$$\lim_{c\to 0}\lim_{x\to 0} x^{-1}F_3(x) = 0. \tag{5.5.29}$$

Next, from (5.3.3) and $\mathcal{M}^{\star\star}_{I_\Psi}(-1) = \phi_-(0)$ we get that

$$\mathcal{M}^*_2(z) = \frac{\phi^\sharp_+(z+1)}{z+1}\mathcal{M}^{\star\star}_{I_\Psi}(-1) = \phi_-(0)\int_0^\infty e^{-(z+1)y}\bar{\mu}_+(y)\,dy + \phi_-(0)\mathtt{d}_+.$$

Clearly, if $\left(\phi^\sharp_+\right)'(0) = \phi'_+(0) = \int_0^\infty \bar{\mu}_+(y)\,dy < \infty$, then the same arguments used above to prove (5.5.29) yield that

$$\lim_{c\to 0}\lim_{x\to 0} x^{-1}F_2(x) = \lim_{c\to 0}\lim_{x\to 0}\frac{1}{2\pi i}\int_{B^\uparrow(-1,c)} x^{-z}\mathcal{M}^*_2(z)dz = 0. \tag{5.5.30}$$

Also, similarly, we deduce that in any case the term $\phi_-(0)\mathtt{d}_+$ does not contribute to (5.5.30) and therefore assume that $\mathtt{d}_+ = 0$ in the sequel. However, if

$$\left(\phi^\sharp_+\right)'\left(0^+\right) = \phi'_+\left(0^+\right) = \infty$$

we could not apply this argument. We then split $\bar{\mu}_+(y) = \bar{\mu}_+(y)\,\mathbb{I}_{\{y>1\}} + \bar{\mu}_+(y)\,\mathbb{I}_{\{y\leq 1\}}$ and write accordingly

$$\mathcal{M}^*_2(z) = \phi_-(0)\int_1^\infty e^{-(z+1)y}\bar{\mu}_+(y)\,dy + \phi_-(0)\int_0^1 e^{-(z+1)y}\bar{\mu}_+(y)\,dy = \mathcal{M}^*_{2,1}(z) + \mathcal{M}^*_{2,2}(z).$$

However, $\left|\mathcal{M}^*_{2,2}(-1)\right| = \phi_-(0)\int_0^1 \bar{\mu}_+(y)\,dy < \infty$ and the portion of $\mathcal{M}^*_{2,2}$ in F_2 is negligible in the sense of (5.5.30). Then, we need discuss solely the contribution of $\mathcal{M}^*_{2,1}$ to F_2 that is

$$\begin{aligned}F^*_2(x) &= \frac{\phi_-(0)}{2\pi i}\int_{B^\uparrow(-1,c)} x^{-z}\int_1^\infty e^{-(z+1)y}\bar{\mu}_+(y)\,dydz\\ &= \frac{\phi_-(0)}{2\pi i}\int_1^\infty\int_{B^\uparrow(-1,c)} x^{-z}e^{-(z+1)y}dz\bar{\mu}_+(y)\,dy.\end{aligned}$$

The interchange is possible since evidently on $B^{\upharpoonleft}(-1,c)$ we have that

$$\sup_{z\in B^{\upharpoonleft}(-1,c)} \left|\mathcal{M}^*_{2,1}(z)\right| < \infty.$$

The latter in turn follows from

$$\sup_{z\in B^{\upharpoonleft}(-1,c)} \left|\mathcal{M}^*_{2,1}(z)\right| = \sup_{z\in B^{\upharpoonleft}(-1,c)} \frac{|\phi_+(z+1)|}{|z+1|} = \sup_{z\in B^{\upharpoonleft}(-1,c)} \frac{|\phi_+(z+1)|}{c} < \infty.$$

However, for any $x, y > 0$, $z \mapsto x^{-z}e^{-(z+1)y}$ is an entire function and an application of the Cauchy theorem to the closed contour $B^{\upharpoonleft}(-1,c) \cup \{-1+i\beta,\, \beta \in [-c,c]\}$ implies that

$$\int_{B^{\upharpoonleft}(-1,c)} x^{-z}e^{-(z+1)y}dz = ix\int_{-c}^{c} e^{-i\beta(\ln x+y)}d\beta = 2x\frac{\sin\left(c\left(\ln x+y\right)\right)}{\ln x+y}i.$$

Therefore, integrating by parts after representing $\bar{\mu}_+(y) = \int_y^\infty \mu_+(dv)$, we get that

$$\begin{aligned}
F^*_2(x) &= x\frac{\phi_-(0)}{\pi}\lim_{A\to\infty}\int_1^A \frac{\sin\left(c\left(\ln x+y\right)\right)}{\ln x+y}\bar{\mu}_+(y)\,dy \\
&= x\frac{\phi_-(0)}{\pi}\lim_{A\to\infty}\int_1^A\int_{c(1+\ln x)}^{c(v+\ln x)} \frac{\sin(y)}{y}dy\mu_+(dv) \\
&+ x\frac{\phi_-(0)}{\pi}\lim_{A\to\infty}\bar{\mu}_+(A)\int_{c(\ln x+1)}^{c(A+\ln x)}\frac{\sin(y)}{y}dy \\
&= x\frac{\phi_-(0)}{\pi}\int_1^\infty\int_{c(1+\ln x)}^{c(v+\ln x)} \frac{\sin(y)}{y}dy\mu_+(dv)
\end{aligned}$$

since

$$\sup_{a<b;a,b\in\mathbb{R}}\left|\int_a^b \frac{\sin(y)}{y}dy\right| < \infty,$$

the mass of $\mu_+(dv)$ on $(1,\infty)$ is simply $\bar{\mu}_+(1) < \infty$ and $\lim_{A\to\infty}\bar{\mu}_+(A) = 0$. Also, this allows via the dominated convergence theorem to deduce that

$$\lim_{x\to 0} x^{-1}F^*_2(x) = \frac{\phi_-(0)}{\pi}\lim_{x\to 0}\int_1^\infty\int_{c(1+\ln x)}^{c(v+\ln x)}\frac{\sin(y)}{y}dy\mu_+(dv) = 0$$

and thus with the reasoning above to verify that (5.5.30) holds, that is $\lim_{c\to 0}\lim_{x\to 0} x^{-1}F_2(x) = 0$. The latter together with (5.5.29) applied in (5.5.28) allows us to understand the asymptotic of $x^{-1}F(x)$, as $x \to 0$, in terms of F_1. However, from its very definition, (5.5.28) and $\Psi(0) = -\phi_-(0)\phi_+(0)$, see (5.1.2),

$$F_1(x) = -\frac{\Psi(0)}{2\pi i}\int_{B^{\upharpoonleft}(-1,c)}\frac{x^{-z}}{z+1}dz = -\Psi(0)x + \frac{\Psi(0)}{2\pi i}\int_{B^{\downharpoonleft}(-1,c)}\frac{x^{-z}}{z+1}dz,$$

where we recall that

$$B^{\downharpoonleft}(-1,c) = \{z \in \mathbb{C} : |z+1| = c \text{ and } \mathtt{Re}(z+1) \leq 0\}.$$

Clearly, if $z \in B^{\downharpoonleft}(-1,c) \setminus \{-1 \pm ic\}$ then $\lim_{x\to 0} |x^{-1}x^{-z}| = 0$. Therefore, for any $c \in \left(0, \frac{1}{2}\right)$,

$$\lim_{x\to 0} x^{-1}F_\Psi(x) = \lim_{x\to 0} x^{-1}F_1(x) = -\Psi(0) \in [0,\infty). \tag{5.5.31}$$

Thus (5.2.31) holds true. When $\Psi \in \mathcal{N}_{\mathtt{N}_\Psi}$, for some $\mathtt{N}_\Psi > 1$, all arguments above applied to $\mathcal{M}^\star_{I_\Psi}$ can be carried over directly to $\mathcal{M}_{I_\Psi}$ but at $z = 0$. When p_Ψ is continuous at zero then the result is immediate from (5.2.31). Thus, we obtain (5.2.32). □

5.5.7 Proof of Theorem 5.2.22

Let $\Psi \in \overline{\mathcal{N}} \setminus \mathcal{N}_\dagger$ that is $\Psi(0) = 0$, see (5.2.18). Recall that $I_\Psi(t) = \int_0^t e^{-\xi_s}ds$. Then if $\mathfrak{a}_+ < 0$ we get that for any $a \in (0, -\mathfrak{a}_+)$

$$\mathbb{E}\left[I_\Psi^{-a}(t)\right] \leq t^{-a}\mathbb{E}\left[e^{a \sup_{v\leq t}\xi_v}\right] < \infty, \tag{5.5.32}$$

where the finiteness of the exponential moments of $\sup_{v\leq t}\xi_v$ of order less than $-\mathfrak{a}_+$ follows from the definition of $\mathfrak{a}_+$, see (5.2.4), that is $\phi_+ \in \mathtt{A}_{(a,\infty)}$ for $a \in (0, -\mathfrak{a}_+)$, see e.g. [11, Chapter VI] or [34, Chapter 4]. This of course settles (5.2.34), that is $a \in (0, 1-\mathfrak{a}_+) \implies \mathbb{E}\left[I_\Psi^{-a}(t)\right] < \infty$, when $\mathfrak{a}_+ = -\infty$. We rewrite (5.5.6) as follows

$$\begin{aligned}\Psi(z) &= \tilde{\Psi}(z) + \Psi^*(z)\\ &= \left(\frac{\sigma^2}{2}z^2 + cz + \int_{-\infty}^1 \left(e^{zr} - 1 - zr\mathbb{I}_{\{|r|<1\}}\right)\Pi(dr)\right) + \int_1^\infty \left(e^{zr}-1\right)\Pi(dr),\end{aligned} \tag{5.5.33}$$

where $\tilde{\Psi}, \Psi^* \in \overline{\mathcal{N}} \setminus \mathcal{N}_\dagger$ and from (5.5.33) $(\xi_s)_{s\geq 0} = \left(\tilde{\xi}_s + \xi^*_s\right)_{s\geq 0}$ where $\tilde{\xi}, \xi^*$ are independent Lévy processes with Lévy-Khintchine exponents $\tilde{\Psi}, \Psi^*$ respectively. Set as usual $\tilde{\Psi}(z) = -\tilde{\phi}_+(-z)\tilde{\phi}_-(z)$ and note from (5.5.33) that $\tilde{\Psi} \in \mathtt{A}_{(0,\infty)}$ and hence $\tilde{\phi}_+ \in \mathtt{A}_{(-\infty,0)}$ or equivalently $\mathfrak{a}_{\tilde{\phi}_+} = -\infty$, see (5.2.4). Similarly to (5.5.32) we get that, for any $a \in (0, 1-\mathfrak{a}_+)$,

$$\mathbb{E}\left[I_\Psi^{-a}(t)\right] \leq \mathbb{E}\left[e^{a\sup_{v\leq t}\tilde{\xi}_v}\right]\mathbb{E}\left[I_{\Psi^*}^{-a}(t)\right]. \tag{5.5.34}$$

However, since $\mathfrak{a}_{\tilde{\phi}_+} = -\infty$ we conclude from (5.5.32) that in fact $\mathbb{E}\left[e^{a\sup_{v\leq t}\tilde{\xi}_v}\right] < \infty$ for any $a > 0$. If $\Psi^* \equiv 0$ there is nothing to prove. So let $h = \Pi\{(1,\infty)\} > 0$ and write $\xi^*_s = \sum_{j=1}^{N_s} X_j$ where $(N_s)_{s\geq 0}$ is a Poisson counting process with $N_s \sim \mathit{Poisson}(hs)$ and $(X_j)_{j\geq 1}$ are i.i.d. random variables with law

$$\mathbb{P}(X_1 \in dx) = \mathbb{I}_{\{x>1\}}\frac{\Pi(dx)}{h}.$$

It is a well-known fact that

$$\mathbb{E}\left[e^{\lambda X_1}\right] < \infty \iff \lambda < -\mathfrak{a}_+.$$

Set $S_n = \sum_{j=1}^{n} X_j$ with $S_0 = 0$. Then

$$I_{\Psi^*}(t) = t\mathbb{I}_{\{N_t=0\}} + \mathbb{I}_{\{N_t>0\}}\left(\sum_{j=1}^{N_t} e_j e^{-S_{j-1}} + \left(t - \sum_{j=1}^{N_t} e_j\right) e^{-S_{N_t}}\right)$$

with $(e_j)_{j\geq 1}$ a sequence of independent identically distributed random variables with exponential law of parameter h. Clearly then

$$\begin{aligned}\mathbb{E}\left[I_\Psi(t)^{-a}\right] &= t^{-a}\mathbb{P}\left(N_t = 0\right) + \sum_{n=1}^{\infty} \mathbb{E}\left[\frac{1}{\left(\sum_{j=1}^{n} e_j e^{-S_{j-1}} + \left(t - \sum_{j=1}^{n} e_j\right) e^{-S_n}\right)^a}; N_t = n\right] \\ &= t^{-a}\mathbb{P}\left(N_t = 0\right) + \sum_{n=1}^{\infty} A_n.\end{aligned} \tag{5.5.35}$$

Note that

$$\{N_t = n\} = \left\{\sum_{j=1}^{n} e_j \leq \frac{t}{2}; \sum_{j=1}^{n+1} e_j > t\right\} \bigcup \left\{\sum_{j=1}^{n} e_j \in \left(\frac{t}{2}, t\right); \sum_{j=1}^{n+1} e_j > t\right\}.$$

We observe that with some $C(t) < \infty$, for any $t > 0$,

$$\sup_{s\in(0,t]} \mathbb{P}\left(N_s \geq n-1\right) \leq C(t)\frac{t^{n-1}h^{n-1}}{(n-1)!}, \tag{5.5.36}$$

which is an elementary consequence of $N_s \sim Poisson(sh)$. We split the quantity $A_n = A_n^{(1)} + A_n^{(2)}$ by considering the two possible mutually exclusive cases for the event $\{N_t = n\}$

discussed above. In the first scenario we have the following sequence of relations

$$
\begin{aligned}
A_n^{(1)} &= \mathbb{E}\left[\frac{1}{\left(\sum_{j=1}^n e_j e^{-S_{j-1}} + \left(t - \sum_{j=1}^n e_j\right) e^{-S_n}\right)^a}; \sum_{j=1}^n e_j \le \frac{t}{2}; \sum_{j=1}^{n+1} e_j > t\right] \\
&\le \mathbb{E}\left[\frac{1}{\left(e_1 + \frac{t}{2} e^{-S_n}\right)^a}; \, e_1 \le \frac{t}{2}\right] \mathbb{P}\left(N_t \ge n-1\right) \\
&\overset{5.5.36}{\le} C(t)\frac{h^n t^{n-1}}{(n-1)!}\mathbb{E}\left[\int_0^{\frac{t}{2}} \frac{1}{\left(x + \frac{t}{2}e^{-S_n}\right)^a} e^{-hx} dx\right] \le C(t)\frac{h^n t^{n-1}}{(n-1)!}\mathbb{E}\left[\int_0^{\frac{t}{2}} \frac{1}{\left(x + \frac{t}{2}e^{-S_n}\right)^a} dx\right] \\
&\le C(t)\frac{h^n t^{n-1}}{(n-1)!}\left(\frac{2^{a-1}(a-1)}{t^{a-1}}\mathbb{E}\left[e^{(a-1)S_n}\right]\mathbb{I}_{\{a>1\}} + \left(\mathbb{E}\left[S_n\right] + \ln(4)\right)\mathbb{I}_{\{a=1\}} + \frac{t^{1-a}}{1-a}\mathbb{I}_{\{a\in(0,1)\}}\right) \\
&= C(t)\frac{h^n t^{n-1}}{(n-1)!}\left(\frac{2^{a-1}}{(a-1)\, t^{a-1}}\left(\mathbb{E}\left[e^{(a-1)X_1}\right]\right)^n \mathbb{I}_{\{a>1\}} + \left(n\mathbb{E}\left[X_1\right] + \ln(4)\right)\mathbb{I}_{\{a=1\}}\right) \\
&+ C(t)\frac{h^n t^{n-1}}{(n-1)!}\frac{t^{1-a}}{1-a}\mathbb{I}_{\{a\in(0,1)\}},
\end{aligned}
\tag{5.5.37}
$$

where for the terms containing $\mathbb{I}_{\{a=1\}}$ and $\mathbb{I}_{\{a\in(0,1)\}}$ in the derivation of the last inequality we have used that $S_n \ge n > 0$ since $X_1 \ge 1$. In the second scenario we observe that the following inclusion holds

$$
\left\{\sum_{j=1}^n e_j \in \left(\frac{t}{2}, t\right); \sum_{j=1}^{n+1} e_j > t\right\} \subseteq \bigcup_{j=1}^n \left\{e_j \ge \frac{t}{2n}\right\} \bigcap \left\{\sum_{1\le i\le n; i\ne j} e_i < t\right\}. \tag{5.5.38}
$$

Clearly then for any j, the events $\left\{e_j \ge \frac{t}{2n}\right\}$ and $\left\{\sum_{1\le i\le n; i\ne j} e_i < t\right\}$ are independent and moreover

$$
\mathbb{P}\left(\sum_{1\le i\le n; i\ne j} e_i < t\right) = \mathbb{P}\left(\sum_{1\le i\le n-1} e_i < t\right) = \mathbb{P}\left(N_t \ge n-1\right).
$$

We are therefore able to estimate $A_n^{(2)}$ using the relation between events in (5.5.38) in the following manner

$$
\begin{aligned}
A_n^{(2)} &= \mathbb{E}\left[\frac{1}{\left(\sum_{j=1}^n e_j e^{-S_{j-1}} + \left(t - \sum_{j=1}^n e_j\right) e^{-S_n}\right)^a}; \sum_{j=1}^n e_j \in \left(\frac{t}{2}, t\right); \sum_{j=1}^{n+1} e_j > t\right] \\
&\le \left(\left|\frac{1}{1-a}\right| (2n)^a t^{1-a}\mathbb{I}_{\{a\ne 1\}} + \ln(2n)\mathbb{I}_{\{a=1\}}\right)\mathbb{P}\left(N_t \ge n-1\right) \\
&+ \mathbb{I}_{\{n>1\}}\left(\sum_{j=2}^n h\mathbb{E}\left[\int_0^t \frac{1}{\left(x + \frac{t}{2n}e^{-S_{j-1}}\right)^a} dx\right]\right)\mathbb{P}\left(N_t \ge n-1\right),
\end{aligned}
$$

where the terms in the second line correspond to the case $\left\{e_1 \geq \frac{t}{2n}\right\}$ whereas the term in the third line corresponds to other scenarios, that is $\bigcup_{j=2}^{n}\left\{e_j \geq \frac{t}{2n}\right\}$, and a disintegration of e_1. However, performing the integration and estimating precisely as in (5.5.37) we get with the help of (5.5.36) that

$$\begin{aligned}
A_n^{(2)} \leq & \left(\left|\frac{1}{1-a}\right|(2n)^a\, t^{1-a}\mathbb{I}_{\{a\neq 1\}} + \ln(2n)\mathbb{I}_{\{a=1\}}\right)\mathbb{P}\left(N_t \geq n-1\right) \\
& + \mathbb{I}_{\{n>1\}}\mathbb{I}_{\{a>1\}}C(t)\frac{t^{n-1}h^n}{(n-1)!}\sum_{j=2}^{n}\frac{(2j)^{a-1}}{(a-1)\,t^{a-1}}\left(\mathbb{E}\left[e^{(a-1)X_1}\right]\right)^{j-1} \\
& + \mathbb{I}_{\{n>1\}}\mathbb{I}_{\{a=1\}}C(t)\frac{t^{n-1}h^n}{(n-1)!}\sum_{j=2}^{n}\left(j\mathbb{E}\left[X_1\right] + \ln\left(4j\right)\right) \\
& + \mathbb{I}_{\{n>1\}}\mathbb{I}_{\{a\in(0,1)\}}C(t)\frac{t^{n-1}h^n}{(n-1)!}\sum_{j=2}^{n}\frac{t^{1-a}}{1-a}\mathbb{I}_{\{a\in(0,1)\}} \\
\leq & \left(\left|\frac{1}{1-a}\right|(2n)^a\, t^{1-a}\mathbb{I}_{\{a\neq 1\}} + \ln(2n)\mathbb{I}_{\{a=1\}}\right)\mathbb{P}\left(N_t \geq n-1\right) \\
& + C(t)\frac{t^{n-a}h^n}{(n-1)!}\frac{(4n)^a}{a-1}\left(\mathbb{E}\left[e^{(a-1)X_1}\right]\right)^{n-1}\mathbb{I}_{\{a>1\}} + C(t)\frac{t^{n-1}h^n}{(n-1)!}\left(n^2 + \ln\left(4n\right)\right)\mathbb{E}\left[X_1\right]\mathbb{I}_{\{a=1\}} \\
& + C(t)\frac{t^{n-a}h^n}{(n-1)!}\frac{h}{1-a}n\mathbb{I}_{\{a\in(0,1)\}}.
\end{aligned} \tag{5.5.39}$$

Therefore from $A_n = A_n^{(1)} + A_n^{(2)}$, (5.5.37) and (5.5.39) applied in (5.5.35) we easily get that $\mathbb{E}\left[I_\Psi(t)^{-a}\right] < \infty$ whenever: $a \in (0,1)$; $a = 1$ and $\mathbb{E}\left[X_1\right] < \infty$ and $a > 1$ and $\mathbb{E}\left[e^{(a-1)X_1}\right] < \infty$. However, it is very well-known fact that

$$\mathbb{E}\left[X_1\right] < \infty \iff \mathbb{E}\left[\max\left\{\xi_1, 0\right\}\right] < \infty \iff \left|\Psi'(0^+)\right| < \infty$$

and if $\mathfrak{a}_+ < 0$ then

$$\mathbb{E}\left[e^{(a-1)X_1}\right] < \infty \impliedby a \in (0, 1-\mathfrak{a}_+)$$

and

$$\mathbb{E}\left[e^{(\mathfrak{a}_+-1)X_1}\right] < \infty \iff |\Psi(-\mathfrak{a}_+)| < \infty \iff |\phi_+(\mathfrak{a}_+)| < \infty.$$

Hence via (5.5.34) the relation (5.2.34) and the backward directions of (5.2.35) and (5.2.36) follow. Let us provide a lower bound for $\mathbb{E}\left[I_\Psi^{-a}(t)\right]$ whenever $a \geq 1$. We again utilize the processes $\tilde{\xi}, \xi^*$ in the manner

$$\begin{aligned}\mathbb{E}\left[I_\Psi(t)^{-a}\right] &\geq \mathbb{E}\left[I_\Psi(t)^{-a}; \sup_{v\leq t}|\tilde{\xi_s}| \leq 1; N_t = 1\right]\\ &\geq e^{-a}\mathbb{P}\left(\sup_{v\leq t}|\tilde{\xi_s}| \leq 1\right)\mathbb{E}\left[I_{\Psi^*}^{-a}(t); N_t = 1\right]\\ &\geq e^{-at}\mathbb{P}\left(\sup_{v\leq t}|\tilde{\xi_s}| \leq 1\right)\mathbb{P}\left(e_2 > t\right)he^{-ht}\mathbb{E}\left[\int_0^t \frac{1}{\left(x+te^{-X_1}\right)^a}dx\right]\\ &= e^{-at}\mathbb{P}\left(\sup_{v\leq t}|\tilde{\xi_s}| \leq 1\right)\mathbb{P}\left(e_2 > t\right)he^{-ht}\\ &\times\left(\left(\frac{t^{1-a}}{a-1}\mathbb{E}\left[e^{(a-1)X_1}\right]-1\right)\mathbb{I}_{\{a>1\}} + \mathbb{E}\left[X_1\right]\mathbb{I}_{\{a=1\}}\right).\end{aligned}$$

The very last term is infinity iff $a-1 > \mathfrak{a}_+$, $\mathbb{E}[X_1] = \infty$ and $a = 1$, or $a - 1 = \mathfrak{a}_+$ and $\mathbb{E}\left[e^{\mathfrak{a}_+ X_1}\right] = \infty$. This shows the forward directions of (5.2.35) and (5.2.36) and relation (5.2.35) and (5.2.37). It remains to show (5.2.38). From $A_n = A_n^{(1)} + A_n^{(2)}$ and (5.5.37) and (5.5.39) we get that $t^a A_n = \mathrm{O}(t)$ as t goes to zero and therefore from (5.5.35) we get that

$$\lim_{t\to 0} t^a \mathbb{E}\left[I_\Psi^{-a}(t)\right] = \lim_{t\to 0}\mathbb{P}\left(N_t = 0\right) = 1.$$

This establishes the validity of (5.2.38).

5.5.8 Proof of Theorem 5.2.24

Recall that $\Psi \in \mathcal{N}^c = \overline{\mathcal{N}} \setminus \mathcal{N} = \left\{\Psi \in \overline{\mathcal{N}} :\ \phi_-(0) = 0\right\}$ and as usual set

$$\Psi_r(z) = \Psi(z) - r = -\phi_+^r(-z)\phi_-^r(z) \in \mathcal{N},\ r \geq 0.$$

We also repeat that $I_\Psi(t) = \int_0^t e^{-\xi_s}ds$, $t \geq 0$ and $I_\Psi(t)$ is non-decreasing in t. The relation (5.2.40) then follows from the immediate bound

$$\mathbb{P}\left(I_{\Psi_r} \leq x\right) = r\int_0^\infty e^{-rt}\mathbb{P}\left(I_\Psi(t) \leq x\right)dt \geq \left(1-e^{-1}\right)\mathbb{P}\left(I_\Psi\left(\frac{1}{r}\right) \leq x\right),$$

combined with the representation (5.2.23) with $n = 0$ for $\mathbb{P}\left(I_{\Psi_r} \leq x\right)$, the identity $\kappa_-(r)\kappa_+(r) = \phi_-^r(0)\phi_+^r(0) = -\Psi(0) = r$ which is valid since

$$\Psi \in \mathcal{N}^c \implies \Psi_r(0) = \Psi(0) - r = -r$$

and the fact that (5.7.4) holds. Thus, Theorem 5.2.24(1) is settled and we proceed with Theorem 5.2.24(2). Denote by $\mathcal{M}_r^{\star,a}$, $a \in (0,1)$, the Mellin transform of the cumulative distribution function of the measure $y^{-a}\mathbb{P}\left(I_{\Psi_r} \in dy\right)\mathbb{I}_{\{y>0\}}$, say V_a^r. Following (5.5.25) we conclude that

$$\mathcal{M}_r^{\star,a}(z) = -\frac{1}{z}\mathcal{M}_{I_{\Psi_r}}(z+1-a) = -\frac{\kappa_-(r)}{z}\mathcal{M}_{\Psi_r}\left(z+1-a\right),\ z \in \mathbb{C}_{(a-1,0)} \qquad (5.5.40)$$

and at least $\mathcal{M}^{\star,a}_r \in \mathtt{A}_{(a-1,0)}$ since $\mathcal{M}_{\Psi_r} \in \mathtt{A}_{(0,1)}$, see Theorem 5.2.1(5.2.7). Similarly, as for any $\Psi \in \overline{\mathcal{N}}$ we have that $\mathtt{N}_\Psi > 0$, we deduct that $\lim\limits_{|b|\to\infty} |b|^\beta \left|\mathcal{M}^{\star,a}_r(c+ib)\right| = 0$ for some $\beta \in (1, 1+\mathtt{N}_\Psi)$ and any $c \in (a-1, 0)$. Therefore, by Mellin inversion, for any $x > 0$,

$$\begin{aligned} V^r_a(x) &= \int_0^x y^{-a}\mathbb{P}\left(I_{\Psi_r} \in dy\right) = \frac{x^{-c}}{2\pi}\int_{-\infty}^{\infty} x^{-ib}\mathcal{M}^{\star,a}_r(c+ib)db \\ &= -\kappa_-(r)\frac{x^{-c}}{2\pi}\int_{-\infty}^{\infty} x^{-ib}\frac{\mathcal{M}_{\Psi_r}(c+1-a+ib)}{c+ib}db. \end{aligned} \tag{5.5.41}$$

However, since

$$V^r_a(x) = r\int_0^\infty e^{-rt}\int_0^x y^{-a}\mathbb{P}\left(I_\Psi(t) \in dy\right)dt, \tag{5.5.42}$$

we have that

$$\lim_{r\to 0}\int_0^\infty e^{-rt}\int_0^x y^{-a}\mathbb{P}\left(I_\Psi(t)\in dy\right)dt = \lim_{r\to 0}\left(-\frac{\kappa_-(r)}{r}\frac{x^{-c}}{2\pi}\int_{-\infty}^{\infty} x^{-ib}\frac{\mathcal{M}_{\Psi_r}(c+1-a+ib)}{c+ib}db\right).$$

From (5.5.55),(5.5.56) and (5.5.57) of Lemma 5.5.3 with $\beta \in (0, \mathtt{N}_\Psi)$ we conclude that the dominated convergence theorem applies and yields that

$$\lim_{r\to 0}\frac{r}{\kappa_-(r)}\int_0^\infty e^{-rt}\int_0^x y^{-a}\mathbb{P}\left(I_\Psi(t)\in dy\right)dt = -\frac{x^{-c}}{2\pi}\int_{-\infty}^{\infty} x^{-ib}\frac{1}{c+ib}\mathcal{M}_\Psi(c+1-a+ib)\,db.$$

Let $-\varliminf\limits_{t\to\infty}\xi_t = \varlimsup\limits_{t\to\infty}\xi_t = \infty$ a.s. or alternatively $\kappa_+(0) = \phi_+(0) = \phi_-(0) = \kappa_-(0) = 0$. Assume also that $\lim\limits_{t\to\infty}\mathbb{P}(\xi_t < 0) = \rho \in [0,1)$. Then from the discussion succeeding [34, Chapter 7, (7.2.3)] (a chapter dedicated to the Spitzer's condition) we have that $\kappa_- \in RV_\rho$. Then, since $\frac{\kappa_-(r)}{r} = \frac{1}{\kappa_+(r)}$, $\kappa_+ \in RV_{1-\rho}$ with $1-\rho > 0$, and

$$\int_0^x y^{-a}\mathbb{P}\left(I_\Psi(t)\in dy\right) = \mathbb{E}\left[I_\Psi^{-a}(t)\mathbb{I}_{\{I_\Psi(t)\le x\}}\right] \tag{5.5.43}$$

is non-increasing in t for any fixed $x > 0$ and $a \in (0,1)$, from a classical Tauberian theorem and the monotone density theorem, see [11, Section 0.7], we conclude from (5.5.41) that, for any $x > 0$ and any $c \in (a-1, 0)$,

$$\lim_{t\to\infty} t\kappa_+\left(\frac{1}{t}\right)\int_0^x y^{-a}\mathbb{P}\left(I_\Psi(t)\in dy\right) = -\frac{x^{-c}}{2\pi\Gamma(1-\rho)}\int_{-\infty}^{\infty} x^{-ib}\frac{1}{c+ib}\mathcal{M}_\Psi(c+1-a+ib)\,db. \tag{5.5.44}$$

With $t\kappa_+\left(\frac{1}{t}\right) = \frac{1}{\kappa_-\left(\frac{1}{t}\right)}$ we deduce that

$$\frac{y^{-a}\mathbb{P}\left(I_\Psi(t)\in dy\right)}{\kappa_-\left(\frac{1}{t}\right)} \text{ converges vaguely to } \vartheta_a,$$

whose distribution function is simply the integral to the right-hand side of (5.5.44). To show that it converges weakly, and thus prove (5.2.41) and (5.2.42), we need only show that

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[I_{\Psi}^{-a}(t)\right]}{\kappa_{-}(\frac{1}{t})} < \infty.$$

However, this is immediate from the fact that $\mathcal{M}_{I_{\Psi_r}}(z)$, $r > 0$, is always well defined on $\mathbb{C}_{(0,1)}$, see (5.2.21) of Theorem 5.2.1, $\lim_{r\to 0} \mathcal{M}_{\Psi_r}(1-a) = \mathcal{M}_{\Psi}(1-a)$, $a \in (0,1)$, justified in (5.5.55) below, and utilizing again a Tauberian theorem and the monotone density theorem to

$$\mathcal{M}_{I_{\Psi_r}}(1-a) = \kappa_{-}(r)\mathcal{M}_{\Psi_r}(1-a) = r\int_0^\infty e^{-rt}\mathbb{E}\left[I_{\Psi}^{-a}(t)\right]dt. \tag{5.5.45}$$

By putting $f \equiv 1$ in (5.2.41) we also prove (5.2.43) for any $a \in (0,1)$. Next, assume that $\mathfrak{a}_{+} < 0$ and fix any $r > 0$. For any $a \in (0, 1-\mathfrak{a}_{+})$ set

$$H(r,a) = \int_1^\infty e^{-rt}\mathbb{E}\left[I_{\Psi}^{-a}(t)\right]dt. \tag{5.5.46}$$

From Theorem 5.2.22 we have that $\mathbb{E}\left[I_{\Psi}^{-a}(t)\right] < \infty$ for all $t \geq 0$ and any $a \in (0, 1-\mathfrak{a}_{+})$. Therefore from $\mathbb{E}\left[I_{\Psi}^{-a}(1)\right] \geq \mathbb{E}\left[I_{\Psi}^{-a}(t)\right]$ for $t \geq 1$ we conclude that

$$H(a,r) \leq \mathbb{E}\left[I_{\Psi}^{-a}(1)\right]\int_1^\infty e^{-rt}dt < \infty$$

and thus $H(r,z)$ can be extended analytically so that $H(r,\cdot) \in \mathtt{A}_{[0,1-\mathfrak{a}_{+})}$. From (5.5.43) we immediately see that, for any $x > 0$,

$$H(a,r) \geq \int_1^\infty e^{-rt}\int_0^x y^{-a}\mathbb{P}\left(I_{\Psi}(t) \in dy\right)dt$$

and we conclude, re-expressing (5.5.42) as

$$\begin{aligned}\frac{1}{r}V_a^r(x) &= \int_0^1 e^{-rt}\int_0^x y^{-a}\mathbb{P}\left(I_{\Psi}(t) \in dy\right)dt + \int_1^\infty e^{-rt}\int_0^x y^{-a}\mathbb{P}\left(I_{\Psi}(t) \in dy\right)dt \\ &= \int_0^1 e^{-rt}\int_0^x y^{-a}\mathbb{P}\left(I_{\Psi}(t) \in dy\right)dt + W_x(r,-a),\end{aligned} \tag{5.5.47}$$

that $W_x(r,\cdot) \in \mathtt{A}_{(\mathfrak{a}_{+}-1,0]}$. Next, since the analyticity of Ψ_r obviously coincides with that of Ψ we conclude that $\mathfrak{a}_{+} = a_{\phi_{+}^{r}}$ and $\mathfrak{a}_{-} = a_{\phi_{-}^{r}}$ for any $r \geq 0$. Moreover, from $\phi_{+}(0) = 0$ then $\phi_{+} < 0$ on $(\mathfrak{a}_{+}, 0)$ and $\mathfrak{u}_{+} = 0$, and since $\lim_{r\to 0}\phi_{+}^{r}(a) = \phi_{+}(a)$ for any $a > \mathfrak{a}_{+}$, see (5.7.7), we easily deduct that

$$\lim_{r\to 0}\mathfrak{u}_{\phi_{+}^{r}} = \mathfrak{u}_{+} = 0$$

and for all $r \leq r_0$, $\mathfrak{u}_{\phi_{+}^{r}} \in (\max\{-1, \mathfrak{a}_{+}\}, 0)$. Therefore since $-\mathfrak{u}_{\phi_{+}^{r}}$ is not an integer from Theorem 5.2.1 we conclude that for all $r \leq r_0$, $\mathcal{M}_{\Psi_r} \in \mathtt{M}_{(\mathfrak{a}_{+}, 1-\mathfrak{a}_{-})}$ and it has simple poles with residues $\kappa_-(r)\frac{\prod_{k=1}^{n}\Psi_r(k)}{n!}$, with $\prod_{k=1}^{0}=1$, at all non-positive integers $-n$ such that $-n>\mathfrak{a}_+$. Henceforth, for any $-n_0>\mathfrak{a}_+$, $n_0\in\mathbb{N}\cup\{0\}$, we have that

$$\mathcal{M}_{\Psi_r}(z)=\kappa_+(r)\sum_{n=0}^{n_0}\frac{\prod_{k=1}^{n}\Psi_r(k)}{n!}\frac{1}{z+n}+\mathcal{M}^{(n_0)}_{\Psi_r}(z) \tag{5.5.48}$$

with $\mathcal{M}^{(n_0)}_{\Psi_r}\in\mathtt{A}_{(\max\{-n_0-1,\mathfrak{a}_+\},1-\mathfrak{a}_-)}$. Also as for any $\Psi\in\overline{\mathcal{N}}$ we have that $\mathtt{N}_\Psi>0$, see Theorem 5.2.5, we conclude from (5.5.48) that at least for any $\beta\in(0,\min\{N_{\Psi_r},1\})$ and any $c\in(\max\{-n_0-1,\mathfrak{a}_+\},1-\mathfrak{a}_-)$

$$\lim_{|b|\to\infty}|b|^\beta\left|\mathcal{M}^{(n_0)}_{\Psi_r}(c+ib)\right|=0. \tag{5.5.49}$$

For $a\in(0,1)$, $c\in(a-1,0)$ and $-n_0>\mathfrak{a}_+, n_0\in\mathbb{N}\cup\{0\}$, (5.5.41) together with (5.5.47), (5.5.48), (5.5.49) and $\kappa_+(r)\kappa_-(r)=r$, allow us to re-express (5.5.41) as follows

$$\begin{aligned}\frac{1}{r}V^r_a(x)&\overset{5.5.47}{=}\int_0^1 e^{-rt}\int_0^x y^{-a}\mathbb{P}\left(I_\Psi(t)\in dy\right)dt+W_x(r,-a)\\&\overset{5.5.41}{=}-\frac{\kappa_-(r)}{r}\frac{1}{2\pi i}\int_{z\in\mathbb{C}_c}x^{-z}\frac{\mathcal{M}_{\Psi_r}\left(z+1-a\right)}{z}dz\\&\overset{5.5.48}{=}\sum_{n=0}^{n_0}\frac{\prod_{k=1}^{n}\Psi_r(k)}{n!}\frac{1}{1-a+n}\left(x^{1-a+n}\mathbb{I}_{\{x\le1\}}-\mathbb{I}_{\{x>1\}}\right)\\&-\frac{\kappa_-(r)}{r}\frac{1}{2\pi i}\int_{z\in\mathbb{C}_c}x^{-z}\frac{\mathcal{M}^{(n_0)}_{\Psi_r}\left(z+1-a\right)}{z}dz,\end{aligned} \tag{5.5.50}$$

where the first term in the very last identity stems from the fact that for $a\in(0,1)$ the function $-\frac{1}{z(z+1-a+n)}$ is the Mellin transform of the function $\frac{1}{1-a+n}\left(x^{1-a+n}\mathbb{I}_{\{x\le1\}}-\mathbb{I}_{\{x>1\}}\right)$. However, from $W_x(r,\cdot)\in\mathtt{A}_{(\mathfrak{a}_+-1,0]}$ as noted beneath (5.5.47), the fact that $\mathcal{M}^{(n_0)}_{\Psi_r}\in\mathtt{A}_{(\max\{-n_0-1,\mathfrak{a}_+\},1-\mathfrak{a}_-)}$ and since we can choose $c<0$ as close to zero as we wish, we deduct upon substitution $-a\mapsto\zeta$ in (5.5.50) and equating the second and forth terms in (5.5.50) that as a function of ζ

$$\begin{aligned}&W_x(r,\zeta)+\frac{\kappa_-(r)}{r}\frac{1}{2\pi i}\int_{z\in\mathbb{C}_c}x^{-z}\frac{\mathcal{M}^{(n_0)}_{\Psi_r}\left(z+1+\zeta\right)}{z}dz\\&=\sum_{n=0}^{n_0}\frac{\prod_{k=1}^{n}\Psi_r(k)}{n!}\frac{1}{1+\zeta+n}\left(x^{1+\zeta+n}\mathbb{I}_{\{x\le1\}}-\mathbb{I}_{\{x>1\}}\right)-\int_0^1 e^{-rt}\int_0^x y^{\zeta}\mathbb{P}\left(I_\Psi(t)\in dy\right)dt\\&:=\sum_{n=0}^{n_0}\frac{\prod_{k=1}^{n}\Psi_r(k)}{n!}\frac{1}{1+\zeta+n}\left(x^{1+\zeta+n}\mathbb{I}_{\{x\le1\}}-\mathbb{I}_{\{x>1\}}\right)-G_x(r,\zeta)\\&=\tilde{G}_x(r,\zeta)\in\mathtt{A}_{(\max\{-n_0-2,\mathfrak{a}_+-1\},0]},\end{aligned} \tag{5.5.51}$$

where

$$G_x(r,\zeta)=\int_0^1 e^{-rt}\int_0^x y^{\zeta}\mathbb{P}\left(I_\Psi(t)\in dy\right)dt$$

and $\tilde{G}_x(r,\cdot)\in \mathtt{A}_{(\max\{-n_0-2,\mathfrak{a}_+-1\},0]}$ is a consequence of $W_x(r,\zeta)\in \mathtt{A}_{(\mathfrak{a}_+-1,0]}$ and the fact that $\mathcal{M}^{(n_0)}_{\Psi_r}\in \mathtt{A}_{(\max\{-n_0-1,\mathfrak{a}_+\},1-\mathfrak{a}_-)}$. Note that for any $k\in\mathbb{N}$ using the Taylor formula for e^{-x}

$$\begin{aligned}G_x(r,\zeta)&=\sum_{j=0}^{k}(-1)^j\frac{r^j}{j!}\int_0^1 t^j\int_0^x y^{\zeta}\mathbb{P}\left(I_\Psi(t)\in dy\right)dt+\int_0^1 f_{k+1}(t)\int_0^x y^{\zeta}\mathbb{P}\left(I_\Psi(t)\in dy\right)dt\\&:=\sum_{j=0}^{k}(-1)^j\frac{r^j}{j!}H_j(x,\zeta)+\tilde{H}_{k+1}(x,\zeta).\end{aligned} \tag{5.5.52}$$

However, as $\left|\int_0^x y^{\zeta}\mathbb{P}\left(I_\Psi(t)\in dy\right)\right|\leq \mathbb{E}\left[I_\Psi(t)^{\mathtt{Re}(\zeta)}\right]$, $\overline{\lim_{t\to 0}}\, t^{-k-1}f_{k+1}(t)<\infty$ and

$$\lim_{t\to 0} t^{-\mathtt{Re}(\zeta)}\mathbb{E}\left[I_\Psi(t)^{\mathtt{Re}(\zeta)}\right]=1,$$

for any $\mathtt{Re}(\zeta)\in(\mathfrak{a}_+-1,0)$, see Theorem 5.2.22(5.2.38), we conclude that

$$H_j\left(x,\cdot\right)\in \mathtt{A}_{(\max\{-1-j,-1+\mathfrak{a}_+\},0)} \text{ and } \tilde{H}_{k+1}\left(x,\cdot\right)\in \mathtt{A}_{(\max\{-2-k,-1+\mathfrak{a}_+\},0)}.$$

Set n' the largest integer smaller than $1-\mathfrak{a}_+$. Then, if $k<n'$, from (5.5.51) applied with $n_0=n'$, we deduce that H_k is meromorphic on $(\mathfrak{a}_+-1,0)$ with simple poles at $\{-n',\cdots,-k-1\}$. Set in these instances

$$H_k(x,\zeta)=\sum_{j=k+1}^{n'}\frac{b(k,x)}{\zeta+j}+H'_k(x,\zeta)$$

with $H'_k(x,\cdot)\in \mathtt{A}_{(\mathfrak{a}_+-1,0)}$. Then

$$G_x(r,\zeta)=\sum_{j=1}^{n'}\frac{P_{j,k}(x,r)}{\zeta+j}+Q_{n'}(x,r)H^*_{n'}(x,\zeta)$$

with $P_{j,k}, j=1,\cdots,k$ and Q_k polynomials in r and $H^*_{n'}(x,\cdot)\in \mathtt{A}_{(\mathfrak{a}_+-1,0)}$. However, since $\Psi_r(z)=\Psi(z)-r$ we deduce that $\prod_{k=1}^n\Psi_r(k)$ are polynomials in r and since from the definition of n', (5.5.51) defines analytic function, that is $\tilde{G}_x(r,\cdot)\in \mathtt{A}_{(\mathfrak{a}_+-1,0]}$, we conclude that

$$\sum_{n=0}^{n'}\frac{\prod_{k=1}^n\Psi_r(k)}{n!}\frac{1}{1+\zeta+n}\left(x^{1+\zeta+n}\mathbb{I}_{\{x\leq 1\}}-\mathbb{I}_{\{x>1\}}\right)-G_x(r,\zeta)=-Q_{n'}(x,r)H^*_{n'}(x,\zeta).$$

Clearly then

$$\lim_{r\to 0}Q_{n'}(x,r)H^*_{n'}(x,\zeta)=Q_{n'}(x,0)H^*_{n'}(x,\zeta)$$

and from (5.5.51) we conclude that, for any $-a \in (\mathfrak{a}_+ - 1, 0)$,

$$\lim_{r\to 0}\left(W_x(r,-a) + \frac{\kappa_-(r)}{r}\frac{1}{2\pi i}\int_{z\in\mathbb{C}_c} x^{-z}\frac{\mathcal{M}^{(n')}_{\Psi_r}(z+1-a)}{z}dz\right) = -Q_{n'}(x,0)H^*_{n'}(x,-a). \tag{5.5.53}$$

Since $\kappa_+(r)\kappa_-(r) = r$ then from (5.5.48)

$$\frac{\kappa_-}{r}\left|\mathcal{M}_{\Psi_r}(z) - \mathcal{M}^{(n')}_{\Psi_r}(z)\right| \leq \kappa_+(r)\frac{\kappa_-(r)}{r}\sum_{n=0}^{n'}\left|\frac{\prod_{k=1}^n \Psi_r(k)}{n!}\frac{1}{z+n}\right| = \mathrm{O}(1)\sum_{n=0}^{n'}\left|\frac{\prod_{k=1}^n \Psi(k)}{n!}\frac{1}{z+n}\right|$$

and therefore by our freedom for fixed $-a \in (\mathfrak{a}_+ - 1, 0)$ to choose $|c+1-a|$ to be non-integer and $c \in (a - 1 + \mathfrak{a}_+, 0)$ we get that

$$\begin{aligned}
&\frac{\kappa_-(r)}{r}\left|\int_{z\in\mathbb{C}_c} x^{-z}\frac{\mathcal{M}_{\Psi_r}(z+1-a)}{z}dz - \int_{z\in\mathbb{C}_c} x^{-z}\frac{\mathcal{M}^{(n')}_{\Psi_r}(z+1-a)}{z}dz\right| \\
&\leq x^{-c}\mathrm{O}(1)\int_{-\infty}^{\infty}\sum_{n=0}^{n'}\left|\frac{\prod_{k=1}^n \Psi(k)}{n!}\frac{1}{1-a+c+ib+n}\right|\frac{db}{|c+ib|} = \mathrm{O}(1),
\end{aligned} \tag{5.5.54}$$

where in the last step we have invoked the dominated convergence theorem. Finally, from these observations (5.5.55),(5.5.56) and (5.5.57) of Lemma 5.5.3 we deduct from (5.5.53) and (5.5.54) that as $r \to 0$

$$\begin{aligned}
W_x(r,-a) &= \int_1^{\infty} e^{-rt}\int_0^x y^{-a}\mathbb{P}(I_\Psi(t)\in dx)\,dt \\
&= -\frac{\kappa_-(r)}{r}\frac{1}{2\pi i}\int_{z\in\mathbb{C}_c} x^{-z}\frac{\mathcal{M}^{(n')}_{\Psi_r}(z+1-a)}{z}dz - Q_{n'}(x,r)H^*_{n'}(x,-a) \\
&= -\frac{\kappa_-(r)}{r}\frac{1}{2\pi i}\int_{z\in\mathbb{C}_c} x^{-z}\frac{\mathcal{M}^{(n')}_{\Psi_r}(z+1-a)}{z}dz + \frac{\kappa_-(r)}{r}\frac{1}{2\pi i}\int_{z\in\mathbb{C}_c} x^{-z}\frac{\mathcal{M}_{\Psi_r}(z+1-a)}{z}dz \\
&- Q_{n'}(x,r)H^*_{n'}(x,-a) - \frac{\kappa_-(r)}{r}\frac{1}{2\pi i}\int_{z\in\mathbb{C}_c} x^{-z}\frac{\mathcal{M}_{\Psi_r}(z+1-a)}{z}dz \\
&\overset{0}{\sim} -\frac{\kappa_-(r)}{r}\frac{1}{2\pi i}\int_{z\in\mathbb{C}_c} x^{-z}\frac{\mathcal{M}_{\Psi}(z+1-a)}{z}dz
\end{aligned}$$

and we conclude (5.2.41) and(5.2.42) as in the case $a \in (0,1)$. We show, for $a \in (0, 1-\mathfrak{a}_+)$ that

$$\lim_{t\to\infty}\frac{\mathbb{E}\left[I_\Psi^{-a}(t)\right]}{\kappa_-\left(\frac{1}{t}\right)} = \vartheta_a\left(\mathbb{R}^+\right) < \infty$$

by decomposing (5.5.45) precisely as $V_a^r(x)$ in (5.5.50) and proceeding as there. This also verifies the expression for $\vartheta_a(\mathbb{R}^+)$ in (5.2.43).

The proof above is relies on the ensuing claims.

Lemma 5.5.3. *Let $\Psi \in \overline{\mathcal{N}}$ and for any $r \geq 0$, $\Psi_r(z) = \Psi(z) - r$. Fix $a \in (\mathfrak{a}_+, 1)$ such that $-a \notin \mathbb{N}$. Then for any $z \in \mathbb{C}_a$ we have that*

$$\lim_{r\to 0} \mathcal{M}_{\Psi_r}(z) = \mathcal{M}_\Psi(z). \tag{5.5.55}$$

Moreover, for any $\widehat{b} > 0$ and $\mathfrak{r} < \infty$,

$$\sup_{0\leq r\leq \mathfrak{r}} \sup_{|b|\leq \widehat{b}} |\mathcal{M}_{\Psi_r}(a+ib)| < \infty. \tag{5.5.56}$$

Finally, for any $\beta < \mathtt{N}_\Psi$, we have that

$$\overline{\lim_{|b|\to\infty}} |b|^\beta \sup_{0\leq r\leq \mathfrak{r}} |\mathcal{M}_{\Psi_r}(a+ib) - \mathcal{M}_\Psi(a+ib)| = 0. \tag{5.5.57}$$

Proof. Let $r \geq 0$. Set $\Psi_r(z) = \Psi(z) - r = -\phi_+^r(-z)\phi_-^r(z)$. Since the analyticity of Ψ_r obviously coincides with that of Ψ we conclude that $\mathfrak{a}_+ = a_{\phi_+^r}$ and $\mathfrak{a}_- = a_{\phi_-^r}$ for any $r \geq 0$. We start with some preparatory work by noting that for $a \in (\mathfrak{a}_+, 1)$ and any $r \geq 0$,

$$\sup_{b\in\mathbb{R}} \left|W_{\phi_-^r}(1-a-ib)\right| \leq W_{\phi_-^r}(1-a), \tag{5.5.58}$$

since, from Definition 5.3.1, $W_{\phi_-^r}$ is the moment transform of the random variable $Y_{\phi_-^r}$. Also, for any non-integer $a \in (\mathfrak{a}_+, 1)$, $z = a+ib \in \mathbb{C}_a$ and any $r \geq 0$ we get from (5.3.25) that

$$\frac{\Gamma(a+ib)}{W_{\phi_+^r}(a+ib)} = \left(\prod_{j=0}^{a^{\rightarrow}-1} \frac{\phi_+^r(a+j+ib)}{a+j+ib}\right) \frac{\Gamma(a+a^{\rightarrow}+ib)}{W_{\phi_+^r}(a+a^{\rightarrow}+ib)}, \tag{5.5.59}$$

where we recall that $c^{\rightarrow} = (\lfloor -c \rfloor + 1)\,\mathbb{I}_{\{c\leq 0\}}$. This leads to

$$\sup_{b\in\mathbb{R}} \left|\frac{\Gamma(a+ib)}{W_{\phi_+^r}(a+ib)}\right| \leq \sup_{b\in\mathbb{R}} \left(\prod_{j=0}^{a^{\rightarrow}-1} \frac{|\phi_+^r(a+j+ib)|}{|a+j+ib|}\right) \frac{\Gamma(a+a^{\rightarrow})}{W_{\phi_+^r}(a+a^{\rightarrow})} \tag{5.5.60}$$

since for $a > 0$, $\frac{\Gamma(a+ib)}{W_\phi(a+ib)}$ is the Mellin transform of I_ϕ, see the proof of Theorem 5.2.27 in section 5.5.1. Next, observe from (5.3.19) that for any $0 \leq r \leq \mathfrak{r}$, $z = a+ib$ and fixed $a > 0$

$$\sup_{0\leq r\leq \mathfrak{r}} \left|W_{\phi_\pm^r}(z)\right| = \sup_{0\leq r\leq \mathfrak{r}} \left(\frac{\sqrt{\phi_\pm^r(1)}}{\sqrt{\phi_\pm^r(a)\phi_\pm^r(1+a)|\phi_\pm^r(z)|}} e^{G_{\phi_\pm^r}(a) - A_{\phi_\pm^r}(z)} e^{-E_{\phi_\pm^r}(z) - R_{\phi_\pm^r}(a)}\right). \tag{5.5.61}$$

First, the error term, namely the product term above, is uniformly bounded over the whole class of Bernstein functions, see (5.3.18). Second, from Proposition 5.7.3, we have that

$$\lim_{r\to 0} \phi_\pm^r(1) = \phi_\pm(1); \ \lim_{r\to 0} \phi_\pm^r(a) = \phi_\pm(a) \text{ and } \lim_{r\to 0} \phi_\pm^r(1+a) = \phi_\pm(1+a).$$

Similarly, from (5.3.12) $\lim_{r\to 0} G_{\phi_+^r}(a) = G_{\phi_+}(a)$. Therefore, (5.5.61) is simplified to

$$\sup_{0\le r\le \mathfrak{r}} \left|W_{\phi_+^r}(z)\right| \asymp \sup_{0\le r\le \mathfrak{r}} \frac{1}{|\phi_+^r(z)|} e^{-A_{\phi_+^r}(z)}. \tag{5.5.62}$$

However, according to Theorem 5.3.2(1), $A_{\phi_+^r}(z)$ are non-increasing in r. Henceforth, (5.5.62) yields that, for any $z = a+ib$, $a>0$,

$$C_a' \inf_{0\le r\le \mathfrak{r}} \left(\frac{\sqrt{|\phi_+(z)|}}{\sqrt{\left|\phi_+^r(z)\right|}}\right) \left|W_{\phi_+}(z)\right| \le \sup_{0\le r\le \mathfrak{r}} \left|W_{\phi_+^r}(z)\right| \le C_a \sup_{0\le r\le \mathfrak{r}} \left(\frac{\sqrt{\left|\phi_+^{\mathfrak{r}}(z)\right|}}{\sqrt{\left|\phi_+^r(z)\right|}}\right) \left|W_{\phi_+^{\mathfrak{r}}}(z)\right|, \tag{5.5.63}$$

where C_a, C_a' are two absolute constants. Next, from Proposition 5.7.3(5.7.7) we have, for any $z\in\mathbb{C}_{(0,\infty)}$, that $\lim_{r\to 0}\phi_+^r(z) = \phi_+(z)$ and hence from Lemma 5.3.12 one obtains, for any $z\in\mathbb{C}_{(0,\infty)}$,

$$\lim_{r\to 0} W_{\phi_+^r}(z) = W_{\phi_+}(z).$$

Also from Proposition 5.7.3(5.7.7) and (5.5.59) we get that for any non-integer $a\in(\mathfrak{a}_+,1)$ and fixed $z\in\mathbb{C}_a$

$$\begin{aligned}
\lim_{r\to 0}\frac{\Gamma(z)}{W_{\phi_+^r}(z)} &= \lim_{r\to 0}\left(\prod_{j=0}^{a^{\rightarrow}-1}\frac{\phi_+^r(z+j)}{|z+j|}\frac{\Gamma(z+a^{\rightarrow})}{W_{\phi_+^r}(z+a^{\rightarrow})}\right)\\
&= \prod_{j=0}^{a^{\rightarrow}-1}\frac{\phi_+(z+j)}{|z+j|}\frac{\Gamma(z+a^{\rightarrow})}{W_{\phi_+}(z+a^{\rightarrow})} = \frac{\Gamma(z)}{W_{\phi_+}(z)}.
\end{aligned}$$

Recall (5.2.6)

$$\mathcal{M}_\Psi(z) = \phi_-(0)\frac{\Gamma(z)}{W_{\phi_+}(z)}W_{\phi_-}(1-z).$$

Therefore, have that

$$\begin{aligned}
\lim_{r\to 0}\mathcal{M}_{\Psi_r}(z) &= \lim_{r\to 0}\phi_-^r(0)\frac{\Gamma(z)}{W_{\phi_+^r}(z)}W_{\phi_-^r}(1-z)\\
&= \phi_-(0)\frac{\Gamma(z)}{W_{\phi_+}(z)}W_{\phi_-}(1-z)\\
&= \mathcal{M}_\Psi(z),\ z\in\mathbb{C}_a,
\end{aligned}$$

and (5.5.55) follows. Next, (5.5.59) and (5.5.63) give with the help of (5.2.6) that, for any

$\mathfrak{r} \in \mathbb{R}^{+}$,

$$
\begin{aligned}
\sup_{0\leq r\leq \mathfrak{r}} \left|\mathcal{M}_{\Psi_r}(a+ib)\right| &= \sup_{0\leq r\leq \mathfrak{r}} \prod_{j=0}^{a^{\to}-1} \frac{\left|\phi_+^r(a+j+ib)\right|}{|a+j+ib|} \frac{\left|\Gamma\left(a+a^{\to}+ib\right)\right|}{\left|W_{\phi_+^r}\left(a+a^{\to}+ib\right)\right|} \left|W_{\phi_-^r}(1-a-ib)\right| \\
&\leq C_a \left[\sup_{0\leq r\leq \mathfrak{r}} \frac{\sqrt{\left|\phi_+^r\left(a+a^{\to}+ib\right)\right|}}{\sqrt{\left|\phi_+\left(a+a^{\to}+ib\right)\right|}} \sup_{0\leq r<\mathfrak{r}} \frac{\sqrt{\left|\phi_-^{\mathfrak{r}}(1-a-ib)\right|}}{\sqrt{\left|\phi_-^r(1-a-ib)\right|}} \right. \\
&\quad \times \sup_{0\leq r\leq \mathfrak{r}} \prod_{j=0}^{a^{\to}-1} \frac{\left|\phi_+^r(a+j+ib)\right|}{|a+j+ib|} \\
&\quad \left. \times \frac{\left|\Gamma\left(a+a^{\to}+ib\right)\right|}{\left|W_{\phi_+}\left(a+a^{\to}+ib\right)\right|} \left|W_{\phi_-^{\mathfrak{r}}}(1-a-ib)\right| \right] \\
&= C_a \left(J_1(b) \times J_2(b) \times J_3(b)\right).
\end{aligned} \tag{5.5.64}
$$

However, Proposition 5.7.3(5.7.7) triggers that

$$
\begin{aligned}
\sup_{|b|\leq \widehat{b}} J_1(b) &= \sup_{|b|\leq \widehat{b}} \left(\sup_{0\leq r\leq \mathfrak{r}} \frac{\sqrt{\left|\phi_+^r\left(a+a^{\to}+ib\right)\right|}}{\sqrt{\left|\phi_+\left(a+a^{\to}+ib\right)\right|}} \sup_{0\leq r\leq \mathfrak{r}} \frac{\sqrt{\left|\phi_-^{\mathfrak{r}}(1-a-ib)\right|}}{\sqrt{\left|\phi_-^r(1-a-ib)\right|}} \right) \\
&\leq \sup_{|b|\leq \widehat{b}} \left(\sup_{0\leq r\leq \mathfrak{r}} \frac{\sqrt{\left|\phi_+^r\left(a+a^{\to}+ib\right)\right|}}{\sqrt{\phi_+\left(a+a^{\to}\right)}} \sup_{0\leq r\leq \mathfrak{r}} \frac{\sqrt{\left|\phi_-^{\mathfrak{r}}(1-a-ib)\right|}}{\sqrt{\phi_-^r(1-a)}} \right)
\end{aligned}
$$

since $1-a>0$, $a+a^{\to}>0$ and (5.3.32) holds, that is $\mathtt{Re}\left(\phi\left(a+ib\right)\right) \geq \phi(a) > 0$. The same is valid for $\sup_{|b|\leq \widehat{b}} J_3(b)$ (resp. $\sup_{|b|\leq \widehat{b}} J_2(b)$) thanks to (5.5.58), (5.5.60) and $\lim_{r\to 0} W_{\phi_+^r}(z) = W_{\phi_+}(z)$ (resp. (5.7.7)). Henceforth, (5.5.56) follows. It remains to show (5.5.57). Let $\mathtt{N}_\Psi = \infty$ first. We note that for any $r>0$,

$$
\Psi \in \mathcal{N}_\infty \iff \Psi_r \in \mathcal{N}_\infty, \text{ or equivalently,} \quad \mathtt{N}_\Psi = \infty \iff \mathtt{N}_{\Psi_r} = \infty.
$$

This is due to the fact that the decay of $|\mathcal{M}_\Psi|$ along complex lines is only determined by the Lévy triplet (c, σ^2, Π), see (5.5.6), which is unaffected in this case. Henceforth, we are ready to consider the terms in (5.5.64) and observe that for any $\beta > 0$,

$$
\overline{\lim_{|b|\to\infty}} |b|^\beta J_3(b) = \overline{\lim_{|b|\to\infty}} |b|^\beta \frac{\left|\Gamma\left(a+a^{\to}+ib\right)\right|}{\left|W_{\phi_+}\left(a+a^{\to}+ib\right)\right|} \left|W_{\phi_-^{\mathfrak{r}}}(1-a-ib)\right| = 0, \tag{5.5.65}
$$

because if

$$
\overline{\lim_{|b|\to\infty}} |b^\beta| \frac{\left|\Gamma\left(a+a^{\to}+ib\right)\right|}{\left|W_{\phi_+}\left(a+a^{\to}+ib\right)\right|} > 0
$$

then from Theorem 5.2.5(1) and Proposition 5.7.3 $\phi_+ \in \mathcal{B}_P$, $\bar{\mu}_+(0) < \infty$, which implies that $\phi_+^{\mathfrak{r}} \in \mathcal{B}_P$, $\bar{\mu}_+^{\mathfrak{r}}(0) < \infty$ and hence

$$
\lim_{|b|\to\infty} |b|^\beta \left|W_{\phi_-^{\mathfrak{r}}}(1-a-ib)\right| = 0.
$$

Next from (5.7.8) and Proposition 5.3.13(3) we deduct that

$$\begin{aligned}\overline{\lim_{|b|\to\infty}} J_2(b) &= \overline{\lim_{|b|\to\infty}} \sup_{0\leq r\leq\mathfrak{r}} \left(\prod_{j=0}^{a^{\rightarrow}-1} \frac{|\phi_+^r(a+j+ib)|}{|a+j+ib|}\right)\\ &\leq \overline{\lim_{|b|\to\infty}} \left(\prod_{j=0}^{a^{\rightarrow}-1} \frac{\sup_{0\leq r\leq\mathfrak{r}} \left(|\phi_+^r(a+j+ib)-\phi_+(a+j+ib)|\right)+|\phi_+(a+j+ib)|}{|b|}\right) < \infty.\end{aligned} \tag{5.5.66}$$

Finally, from (5.3.32), (5.7.7), (5.7.8) and Proposition 5.3.13(3) we conclude that

$$\begin{aligned}\overline{\lim_{|b|\to\infty}} \frac{J_1(b)}{|b|^2} &= \overline{\lim_{|b|\to\infty}} \frac{1}{|b^2|} \sup_{0\leq r\leq\mathfrak{r}} \frac{\sqrt{|\phi_+^r(a+a^{\rightarrow}+ib)|}}{\sqrt{|\phi_+(a+a^{\rightarrow}+ib)|}} \sup_{0\leq r<\mathfrak{r}} \frac{\sqrt{|\phi_-^{\mathfrak{r}}(1-a-ib)|}}{\sqrt{|\phi_-^r(1-a-ib)|}}\\ &\leq C_a \sup_{0\leq r\leq\mathfrak{r}} \frac{1}{\sqrt{|\phi_+(a+a^{\rightarrow})|}} \frac{1}{\sqrt{|\phi_-^r(a+a^{\rightarrow})|}} < \infty.\end{aligned} \tag{5.5.67}$$

Collecting the estimates (5.5.65), (5.5.66) and (5.5.67) we prove (5.5.57) when $\mathtt{N}_\Psi = \infty$ since from (5.5.64)

$$\lim_{|b|\to\infty} |b|^\beta \sup_{0\leq r\leq\mathfrak{r}} |\mathcal{M}_{\Psi_r}(a+ib)| \leq C_a \lim_{|b|\to\infty} \frac{J_1(b)}{|b|^2} J_2(b)|b|^{\beta+2} J_3(b) = 0.$$

Assume next that $\mathtt{N}_\Psi < \infty$ which triggers from Theorem 5.2.5(1) and Proposition 5.7.3 that $\phi_+, \phi_+^r \in \mathcal{B}_P$ with $\mathtt{d}_+ = \mathtt{d}_+^r, \forall\, r>0,\ \phi_-, \phi_-^r \in \mathcal{B}_P^c$ and $\overline{\overline{\Pi}}(0) < \infty$. The latter implies that $\bar{\mu}_\pm^r(0) < \infty$, $r \geq 0$. Thus, from (5.7.6) and (5.7.7) of Proposition 5.7.3, we conclude, from (5.2.16), that

$$\begin{aligned}\lim_{r\to 0} \mathtt{N}_{\Psi_r} &= \lim_{r\to 0}\left(\frac{\upsilon_-^r(0^+)}{\phi_-^r(0)+\bar{\mu}_-^r(0)}\right) + \lim_{r\to 0} \frac{\phi_+^r(0)+\bar{\mu}_+^r(0)}{\mathtt{d}_+}\\ &= \left(\frac{\upsilon_-(0^+)}{\phi_-(0)+\bar{\mu}_-(0)}\right) + \frac{\phi_+(0)+\bar{\mu}_+(0)}{\mathtt{d}_+} = \mathtt{N}_\Psi,\end{aligned} \tag{5.5.68}$$

wherein it has not only been checked that $\lim_{r\to 0} \upsilon_-^r(0^+) = \upsilon_-(0^+)$ yet. However, from Remark 5.2.6 we have that $\upsilon_-^r(0^+) = \int_0^\infty u_+^r(y)\Pi_-(dy)$, since $\Pi^r = \Pi$, whenever $\mathtt{N}_{\Psi_r} < \infty$. Also, in this case, from (5.4.23), we have since $\mathtt{d}_+ = \mathtt{d}_+^r$ that

$$u_+^r(y) = \frac{1}{\mathtt{d}_+} + \sum_{j=1}^{\infty} \frac{(-1)^j}{\mathtt{d}_+^{j+1}} \left(\mathbf{1} * (\phi_+^r(0)+\bar{\mu}_+^r)^{*j}\right)(y) = \frac{1}{\mathtt{d}_+} + \tilde{u}_+^r(y),\ y \geq 0.$$

The infinite sum above is locally uniformly convergent, see the proof of [36, Proposition 1], and therefore we can show using $\lim_{r\to 0} \phi_+^r(0) = \phi_+(0)$, $\lim_{r\to 0} \bar{\mu}_+^r = \bar{\mu}_+$ and (5.7.6) and (5.7.7) of Proposition 5.7.3, that $\lim_{r\to 0} \upsilon_-^r(0^+) = \upsilon_-(0^+)$. Thus, (5.5.68) holds true. Note that since

$\mathtt{d}_{+} > 0$ the Lévy process underlying Ψ^r is not a compound Poisson process and hence from Lemma 5.7.1 we have that that $\mu^r_{\pm}(dy) = \int_0^\infty e^{-rt-\Psi(0)t}\mu^\sharp_{\pm}(dt, dy)$, where $\mu^\sharp$ stands for the Lévy measure of the conservative Lévy process underlying $\Psi^\sharp(z) = \Psi(z) - \Psi(0) = \Psi^\sharp_r(z)$. Therefore, in the sense of measures on $(0, \infty)$, $\mu^r_{\pm}(dy) \leq \mu^\sharp_{\pm}(dy)$, for all $r \geq 0$. Since $\bar{\mu}^r_{-}(0) < \infty$, $r \geq 0$, and $\phi_{-}, \phi^r_{-} \in \mathcal{B}^c_P$ we conclude from Proposition 5.7.3(5.7.7) that for any $a > \mathfrak{a}_{+}$ and $\mathfrak{r} > 0$

$$\sup_{b\in\mathbb{R}} \sup_{0\leq r\leq \mathfrak{r}} |\phi^r_{-}(a + ib)| \leq \sup_{0\leq r\leq \mathfrak{r}} (\phi^r_{-}(0)) + \int_0^\infty (e^{ay} + 1)\, \mu^\sharp_{-}(dy) < \infty. \tag{5.5.69}$$

Also from $\mathtt{d}^r_{+} = \mathtt{d}_{+} > 0$ and $\bar{\mu}^r_{+}(0) < \infty$, $r \geq 0$, we obtain, for fixed $a > \mathfrak{a}_{+}$, $-a \notin \mathbb{N}$, that

$$\sup_{b\in\mathbb{R}} \sup_{0\leq r\leq \mathfrak{r}} \frac{|\phi^r_{+}(a + ib)|}{|a + ib|} \leq \sup_{0\leq r\leq \mathfrak{r}} \frac{\phi^r_{+}(0)}{|a|} + \mathtt{d}_{+} + \int_0^\infty (e^{ay} + 1)\, \mu^\sharp_{+}(dy) < \infty. \tag{5.5.70}$$

Therefore from (5.5.70)

$$\overline{\lim_{|b|\to\infty}}\, J_2(b) = \overline{\lim_{|b|\to\infty}} \sup_{0\leq r\leq \mathfrak{r}} \left(\prod_{j=0}^{a^{\rightarrow}-1} \frac{|\phi^r_{+}(a + j + ib)|}{|a + j + ib|} \right) < \infty \tag{5.5.71}$$

and from (5.5.69) and (5.3.32)

$$\begin{aligned} \overline{\lim_{|b|\to\infty}}\, J_1(b) &= \overline{\lim_{|b|\to\infty}} \sup_{0\leq r\leq \mathfrak{r}} \frac{\sqrt{|\phi^r_{+}(a + a^{\rightarrow} + ib)|}}{\sqrt{|\phi_{+}(a + a^{\rightarrow} + ib)|}} \sup_{0\leq r<\mathfrak{r}} \frac{\sqrt{|\phi^{\mathfrak{r}}_{-}(1 - a - ib)|}}{\sqrt{|\phi^r_{-}(1 - a - ib)|}} \\ &\leq C_a \sup_{0\leq r\leq \mathfrak{r}} \frac{1}{\sqrt{|\phi_{+}(a + a^{\rightarrow})|}} \frac{1}{\sqrt{|\phi^r_{-}(a + a^{\rightarrow})|}} < \infty. \end{aligned} \tag{5.5.72}$$

Relations (5.5.71) and (5.5.72) allow the usage of (5.5.64) to the effect that

$$\sup_{0\leq r\leq \mathfrak{r}} \sup_{|b|\leq \widehat{b}} |\mathcal{M}_{\Psi_r}(a + ib)| \leq C_a \frac{|\Gamma(a + a^{\rightarrow} + ib)|}{|W_{\phi_{+}}(a + a^{\rightarrow} + ib)|} \left|W_{\phi^{\mathfrak{r}}_{-}}(1 - a - ib)\right|. \tag{5.5.73}$$

Then (5.5.57) follows from (5.5.73) and (5.5.68) as $\mathfrak{r}$ can be chosen as small as we wish and thus $\mathtt{N}_{\Psi_{\mathfrak{r}}}$ as close as we need to $\mathtt{N}_{\Psi}$. This concludes the proof of this lemma. □

5.6 Intertwining between self-similar semigroups and factorization of laws

5.6.1 Proof of Theorem 5.2.27

As in the case $\bar{\mathfrak{a}}_{-} < 0$, see section 5.5.1, we recognize $\frac{\Gamma(z)}{W_{\phi_{+}}(z)}$ as the Mellin transform of the random variable $I_{\phi_{+}}$ and $\phi_{-}(0)W_{\phi_{-}}(1 - z)$ as the Mellin transform of $X_{\phi_{-}}$ as defined in (5.5.2). This leads to the first factorization (5.2.44) of Theorem 5.2.27. Next, we proceed with the proof of the second identity in law of Theorem 5.2.27. We simply express in (5.2.21) the product representations of the functions $W_{\phi_+}, W_{\phi_-}, \Gamma$, see (5.3.9), to obtain, for $z \in \mathbb{C}_{(-1,0)}$, that

$$\begin{aligned}
\mathcal{M}_{I_\Psi}(z+1) = \mathbb{E}\left[I_\Psi^z\right] &= \phi_-(0)\frac{\Gamma(z+1)}{W_{\phi_+}(z+1)}W_{\phi_-}(-z) \\
&= \frac{e^{z\left(\gamma_{\phi_+}+\gamma_{\phi_-}-\gamma+1-\frac{\phi_+'(1)}{\phi_+(1)}\right)}\phi_-(0)}{\phi_-(-z)}\frac{\phi_+(1+z)}{\phi_+(1)(1+z)}\prod_{k=2}^{\infty}\frac{\phi_-(k-1)}{\phi_-(k-1-z)}\frac{k\phi_+(k+z)}{\phi_+(k)(k+z)}C_\Psi^z(k),
\end{aligned} \tag{5.6.1}$$

where

$$C_\Psi(k) = e^{\left(\frac{\phi_-'(k-1)}{\phi_-(k-1)}+\frac{1}{k}-\frac{\phi_+'(k)}{\phi_+(k)}\right)}.$$

Performing a change of variable in Proposition 5.3.13(5) (resp. in the expression (5.3.3)), we get, recalling that $\Upsilon_-(dv) = U_-(\ln(v)),\ v > 1$, is the image of U_- via the mapping $y \mapsto \ln y$ that

$$\begin{aligned}
&\int_1^\infty y^z \Upsilon_-(dy) = \frac{1}{\phi_-(-z)} \\
&\int_0^1 y^z\left(\bar{\mu}_+(-\ln y)dy + \phi_+(0)dy + \mathtt{d}_+\delta_1(dy)\right) = \frac{\phi_+(1+z)}{(1+z)}.
\end{aligned}$$

Note that the last identities prove (5.2.45), that is

$$\begin{aligned}
\mathbb{P}\left(X_\Psi \in dx\right) &= \frac{1}{\phi_+(1)}\left(\bar{\mu}_+(-\ln x)dx + \phi_+(0)dx + \mathtt{d}_+\delta_1(dx)\right),\ x \in (0,1) \\
\mathbb{P}\left(Y_\Psi \in dx\right) &= \phi_-(0)\Upsilon_-(dx),\ x > 1
\end{aligned}$$

are probability measures on $\mathbb{R}^+$. The in it is trivial that, for any $z \in i\mathbb{R}$,

$$\int_0^\infty x^z \mathbb{P}(X_\Psi \times Y_\Psi \in dx) = \frac{\phi_-(0)}{\phi_+(1)}\frac{\phi_+(1+z)}{\phi_-(-z)(1+z)}.$$

Then it is clear that for $k = 1, \ldots,$ we have that

$$\mathbb{E}\left[f(\mathfrak{B}_k X_\Psi)\right]\mathbb{E}\left[f(\mathfrak{B}_{-k} Y_\Psi)\right] = \frac{\phi_-(k)}{\phi_-(k-z)}\frac{(k+1)}{\phi_+(k+1)}\frac{\phi_+(k+1+z)}{(k+1+z)},$$

where we recall that

$$\mathbb{E}\left[f(\mathfrak{B}_k X)\right] = \frac{\mathbb{E}\left[X^k f(X)\right]}{\mathbb{E}[X^k]}$$

and evidently $\mathbb{E}\left[X_\Psi^k\right] = \phi_+(1)\frac{\phi_+(k+1)}{k+1}$ and $\mathbb{E}\left[Y_\Psi^{-k}\right] = \frac{\phi_-(0)}{\phi_-(k)}$. This concludes the proof. □

5.6.2 Proof of Theorem 5.2.29

Let $\Psi \in \mathcal{N}_m$. If $\Psi(0^+) \in (0,\infty)$, that is, the underlying Lévy process drifts to infinity, (5.2.47) and hence (5.2.46) can be verified directly from [19] wherein it is shown that $\mathbb{E}\left[f\left(V_\Psi\right)\right] = \frac{1}{\mathbb{E}\left[I_\Psi^{-1}\right]}\mathbb{E}\left[\frac{1}{I_\Psi} f\left(\frac{1}{I_\Psi}\right)\right]$ for any $f \in \mathtt{C}_0\left([0,\infty)\right)$ and (5.2.21). Indeed from the latter we easily get that $\mathbb{E}\left[I_\Psi^{-1}\right] = \phi_-(0)\phi_+'(0^+)$. Then a substitution yields the result. If $\Psi(0^+) = 0$ that is the underlying process oscillates we proceed by approximation. Set $\Psi_{\mathfrak{r}}(z) = \Psi(z) + \mathfrak{r}z$ and note that $\Psi_{\mathfrak{r}}'(0^+) = \mathfrak{r} > 0$. Then (5.2.47) and hence (5.2.46) are valid for $\Psi_{\mathfrak{r}}(z) = -\phi_+^{\mathfrak{r}}(-z)\phi_-^{\mathfrak{r}}(z)$. From the celebrated Fristedt's formula, see (5.7.2) below, Lemma 5.7.1 and fact that the underlying process is conservative, that is $\Psi(0) = 0$, we get that

$$\phi_+^{\mathfrak{r}}(z) = h^{\mathfrak{r}}(0)e^{\int_0^\infty\int_0^\infty\left(e^{-t}-e^{-zx}\right)\frac{\mathbb{P}(\xi_t+\mathfrak{r}t\in dx)}{t}dt},\ z \in \mathbb{C}_{[0,\infty)},$$

where $h^{\mathfrak{r}}(0) = 1$ since the Lévy process $\xi^{\mathfrak{r}}$ corresponding to $\Psi_{\mathfrak{r}}$ is not a compound Poisson process, see (5.7.3). Then as $\lim_{\mathfrak{r}\to 0}\mathbb{P}\left(\xi_t + \mathfrak{r}t \in \pm dx\right) = \mathbb{P}\left(\xi_t \in \pm dx\right)$ weakly on $[0,\infty)$ we conclude that $\lim_{\mathfrak{r}\to 0}\phi_+^{\mathfrak{r}}(z) = \phi_+(z)$, $z \in \mathbb{C}_{[0,\infty)}$. This together with the obvious $\lim_{\mathfrak{r}\to 0}\Psi_{\mathfrak{r}}(z) = \Psi(z)$ gives that $\lim_{\mathfrak{r}\to 0}\phi_-^{\mathfrak{r}}(z) = \phi_-(z)$, $z \in \mathbb{C}_{[0,\infty)}$. Thus, from Lemma 5.3.12 we get that $\lim_{\mathfrak{r}\to 0} W_{\phi_\pm^{\mathfrak{r}}}(z) = W_{\phi_\pm}(z)$ on $\mathbb{C}_{(0,\infty)}$. Therefore, (5.2.47), that is

$$\lim_{\mathfrak{r}\to 0}\mathcal{M}_{V_{\Psi_{\mathfrak{r}}}}(z) = \lim_{\mathfrak{r}\to 0}\frac{1}{\left(\phi_+^{\mathfrak{r}}(0^+)\right)'}\frac{\Gamma(1-z)}{W_{\phi_+^{\mathfrak{r}}}(1-z)}W_{\phi_-^{\mathfrak{r}}}(z) = \frac{1}{\phi_+'(0^+)}\frac{\Gamma(1-z)}{W_{\phi_+}(1-z)}W_{\phi_-}(z), \quad z \in \mathbb{C}_{(\overline{\mathfrak{a}}_-,1)},$$

holds provided that $\lim_{\mathfrak{r}\to 0}\left(\phi_+^{\mathfrak{r}}(0^+)\right)' = \phi_+'(0^+)$. However, from $\lim_{\mathfrak{r}\to 0}\phi_+^{\mathfrak{r}}(z) = \phi_+(z)$ we deduct from the second expression in (5.3.3) with $\phi_+(0) = \phi_+^{\mathfrak{r}}(0) = 0$ that on $\mathbb{C}_{(0,\infty)}$

$$\lim_{\mathfrak{r}\to 0}\left(\mathtt{d}_+^{\mathfrak{r}} + \int_0^\infty e^{-zy}\bar{\mu}_+^{\mathtt{d}_+^{\mathfrak{r}}}(y)\,dy\right) = \mathtt{d}_+ + \int_0^\infty e^{-zy}\bar{\mu}_+(y)\,dy.$$

Since by assumption $\phi_+'(0^+) < \infty$ and hence $\left(\phi_+^{\mathfrak{r}}\right)'(0^+) < \infty$. Then, in an obvious manner from (5.3.27) we can get that

$$\lim_{\mathfrak{r}\to 0}\left(\phi_+^{\mathfrak{r}}\right)'(0^+) = \lim_{\mathfrak{r}\to 0}\left(\mathtt{d}_+^{\mathfrak{r}} + \int_0^\infty \bar{\mu}_+^{\mathtt{d}_+^{\mathfrak{r}}}(y)\,dy\right) = \mathtt{d}_+ + \int_0^\infty \bar{\mu}_+(y)\,dy = \phi_+'(0^+).$$

Thus, item (1) is settled. All the claims of item (2) follow from the following sequence of arguments. First that $\Pi(dx) = \pi_+(x)dx$, $x > 0$, π_+ non-increasing on $\mathbb{R}^+$ and [68] imply that (5.2.44) is precised to $I_\Psi \stackrel{d}{=} I_{\phi_+} \times I_\psi$, where $\psi(z) = z\phi_-(z) \in \mathcal{N}_m$. Secondly, this factorization is transferred to $V_\Psi \stackrel{d}{=} V_{\phi_+} \times V_\psi$ via (5.2.47). Finally the arguments in the proof of [83, Theorem 7.1] depend on the latter factorization of the entrance laws and the zero-free property of $\mathcal{M}_{V_\Psi}(z)$ for $z \in \mathbb{C}_{(0,1)}$ which via (5.2.47) is a consequence of Theorem 5.3.1 which yields that $W_\phi(z)$ is zero free on $\mathbb{C}_{(0,\infty)}$ for any $\phi \in \mathcal{B}$.

5.7 Appendix

5.7.1 Details on Lévy processes and their exponential functional

Recall that a Lévy process $\xi = (\xi_t)_{t\geq 0}$ is a real-valued stochastic process which possesses stationary and independent increments with a.s. right-continuous paths. We allow killing of the Lévy process by means of the following procedure. For $q \geq 0$ pick an exponential variable $\mathbf{e_q}$, of parameter $q \geq 0$, independent of ξ, such that $\xi_t = \infty$ for any $t \geq \mathbf{e_q}$. Note that $\mathbf{e}_0 = \infty$ a.s. and in this case the Lévy process is conservative that is unkilled. The law of a possibly killed Lévy process ξ is characterized via its characteristic exponent, i.e. $\log \mathbb{E}\left[e^{z\xi_t}\right] = \Psi(z)t$, where $\Psi : i\mathbb{R} \to \mathbb{C}$ admits the following Lévy-Khintchine representation

$$\Psi(z) = \frac{\sigma^2}{2}z^2 + cz + \int_{-\infty}^{\infty} \left(e^{zr} - 1 - zr\mathbb{I}_{\{|r|<1\}}\right) \Pi(dr) - q, \tag{5.7.1}$$

where $q \geq 0$ is the killing rate, $\sigma^2 \geq 0$, $c \in \mathbb{R}$, and, the Lévy measure Π satisfies the integrability condition $\int_{-\infty}^{\infty}(1 \wedge r^2)\,\Pi(dr) < +\infty$. With each Lévy process, say ξ, there are the bivariate ascending and descending ladder height and time processes $\left(\tau^{\pm}, H^{\pm}\right) = \left(\tau^{\pm}_t, H^{\pm}_t\right)_{t\geq 0}$ associated to ξ via $\left(H^{\pm}_t\right)_{t\geq 0} = \left(\xi_{\tau^{\pm}_t}\right)_{t\geq 0}$ and we refer to [11, Chapter VI] for more information on these processes. Since these processes are bivariate subordinators we denote by $k_{\pm}$ their Laplace exponents. The celebrated Fristedt's formula, see [11, Chapter VI, Corollary 10], then evaluates those on $z \in \mathbb{C}_{[0,\infty)}$, $q \geq 0$, as

$$k_{\pm}(q, z) = e^{-\int_0^\infty \int_0^\infty \left(e^{-t} - e^{-zx-qt}\right)\mathbb{P}\left(\xi^\sharp_t \in dx\right)\frac{dt}{t}}, \tag{5.7.2}$$

where $\xi^\sharp$ is a conservative Lévy process constructed from ξ by letting it to evolve on an infinite time horizon. Set

$$h(q) = e^{-\int_0^\infty \left(e^{-t} - e^{-qt}\right)\mathbb{P}\left(\xi^\sharp_t = 0\right)\frac{dt}{t}} \tag{5.7.3}$$

and note that $h : [0,\infty) \mapsto \mathbb{R}^+$ is an increasing, positive function. Then, the analytical form of the Wiener-Hopf factorization of $\Psi \in \overline{\mathcal{N}}$ is given by the expressions

$$\Psi(z) = -\phi_+(-z)\phi_-(z) = -h(q)k_+(q, -z)\,k_-(q, z)\,,\ z \in i\mathbb{R}, \tag{5.7.4}$$

where $\phi_{\pm} \in \mathcal{B}$ with $\phi_{\pm}(0) \geq 0$ and characteristics of $\phi_{\pm}$, that is $(\phi_{\pm}(0), \mathtt{d}_{\pm}, \mu_{\pm})$, depend on $q = -\Psi(0) \geq 0$. Then we have the result.

Lemma 5.7.1. *For any $\Psi \in \overline{\mathcal{N}}$ it is possible to choose $\phi_+ = h(q)k_+$ and $\phi_- = k_-$. The function $h : [0,\infty) \mapsto \mathbb{R}^+$ is not identical to 1 if and only if $\overline{\Pi}(0) < \infty$, $\sigma^2 = c = 0$, see* (5.7.1)*, that is ξ is a compound Poisson process. Then, on $\mathbb{R}^+$, $\mu_-(dy) = \int_0^\infty e^{-qy_1}\mu^\sharp_-(dy_1, dy)$ and $\mu_+(dy) = h(q)\int_0^\infty e^{-qy_1}\mu^\sharp_+(dy_1, dy)$, where $\mu^\sharp_{\pm}(dy_1, dy)$ are the Lévy measures of the bivariate ascending and descending ladder height and time processes associated to the conservative Lévy process $\xi^\sharp$.*

Proof. The proof is straightforward from [34, p.27] and the fact that for fixed $q \geq 0$, $k_{\pm} \in \mathcal{B}$. □

We refer to the excellent monographs [11] and [90] for background on the probabilistic and path properties of general Lévy processes and their associated Lévy-Khintchine exponent $\Psi \in \overline{\mathcal{N}}$.

5.7.2 A simple extension of the celebrated équation amicale inversée

When $\Psi \in \overline{\mathcal{N}}$ with $\Psi(0) = 0$ the Vigon's celebrated *équation amicale inversée*, see [34, 5.3.4] states that

$$\bar{\mu}_{-}(y) = \int_0^\infty \overline{\Pi}_{-}(y+v)U_{+}(dv),\ y > 0, \tag{5.7.5}$$

where μ_{-} is the Lévy measure of the descending ladder height process and U_{+} is the potential measure associated to the ascending ladder height process, see Section 5.7.1 and relation (5.4.22). We now extend (5.7.5) to all $\Psi \in \overline{\mathcal{N}}$.

Proposition 5.7.2. *Let $\Psi \in \overline{\mathcal{N}}$ and recall that $\Psi(z) = -\phi_{+}(-z)\phi_{-}(z)$, $z \in i\mathbb{R}$. Then* (5.7.5) *holds.*

Proof. Recall that $\Psi^\sharp(z) = \Psi(z) - \Psi(0) \in \overline{\mathcal{N}}$ and $\Psi^\sharp$ corresponds to a conservative Lévy process. From Lemma 5.7.1 we have that $\mu_{-}(dy) = \int_0^\infty e^{-qt}\mu_{-}^\sharp(dt, dy)$, $y > 0$. However, from [34, Corollary 6, Chapter 5] we have that

$$\mu_{-}^\sharp(dt, dy) = \int_0^\infty U_{+}^\sharp(dt, dv)\Pi_{-}(v + dy),$$

where $U_{-}^\sharp$ is the bivariate potential measure associated to $\left((\tau^{-})^\sharp, (H^{-})^\sharp\right)$, see [34, Chapter 5] for more details. Therefore,

$$\bar{\mu}_{-}(y) = \int_0^\infty \int_0^\infty e^{-qt}U_{+}^\sharp(dt, dv)\overline{\Pi}_{-}(v + y).$$

Assume first that the underlying Lévy process is not a compound Poisson process. Then from [34, p. 50] and Lemma 5.7.1, we have, for any $\eta > 0$,

$$\frac{1}{\phi_{+}(\eta)} = \frac{1}{k_{+}(q, \eta)} = \int_0^\infty e^{-\eta v} \int_0^\infty e^{-qt}U_{+}^\sharp(dt, dv)$$

and from Proposition 5.3.13(5) we conclude that $U_{+}(dv) = \int_0^\infty e^{-qt}U_{+}^\sharp(dt, dv)$ since

$$\frac{1}{\phi_{+}(\eta)} = \int_0^\infty e^{-\eta v}U_{+}(dv).$$

Thus (5.7.5) is established for any $\Psi \in \overline{\mathcal{N}}$ such that the underlying Lévy process is not a compound Poisson process. In the case of compound Poisson process the claim follows by a modification of the proof in [34, p. 50] accounting for the function $h(q)$ appearing in (5.7.4) which is missed therein since in this case $h(q)k_{+}(q, 0)k_{-}(q, 0) = q$. □

5.7.3 Some remarks on killed Lévy processes

The next claim is also a general fact that seems not to have been recorded in the literature at least in such a condensed form.

Proposition 5.7.3. *Let* $\Psi \in \overline{\mathcal{N}}$ *and for any* $r > 0$, $\Psi_r(z) = \Psi(z) - r = \phi^r_+(-z)\phi^r_-(z)$, $z \in i\mathbb{R}$, *with the notation* $\left(\phi^r_\pm(0), \mathtt{d}^r_\pm, \mu^r_\pm\right)$ *for the triplets defining the Bernstein functions* $\phi^r_\pm$. *Then, for any* $r > 0$, $\mathtt{d}^r_\pm = \mathtt{d}_\pm$ *and*

$$\bar{\mu}^r_+(0) = \infty \iff \bar{\mu}_+(0) = \infty \text{ and } \bar{\mu}^r_-(0) = \infty \iff \bar{\mu}_-(0) = \infty.$$

Moreover, with $\Psi^\sharp(z) = \Psi(z) - \Psi(0) = \Psi^\sharp_r(z) \in \overline{\mathcal{N}}$ *we get that weakly on* $(0, \infty)$

$$\lim_{r\to 0} \mu^r_\pm(dx) = \mu_\pm(dx) \tag{5.7.6}$$

and therefore for any $a > \mathfrak{a}_+$ *and* $[b_1, b_2] \subset \mathbb{R}$ *with* $-\infty < b_1 < 0 < b_2 < \infty$

$$\overline{\lim_{\mathfrak{r}\to 0}} \sup_{b\in[b_1,b_2]} \sup_{0\le r\le \mathfrak{r}} |\phi^r_+(a+ib) - \phi_+(a+ib)| = 0 \tag{5.7.7}$$

and

$$\overline{\lim_{\mathfrak{r}\to 0}} \sup_{b\in\mathbb{R}\setminus[b_1,b_2]} \sup_{0\le r\le \mathfrak{r}} \frac{|\phi^r_+(a+ib) - \phi_+(a+ib)|}{|b|} = 0. \tag{5.7.8}$$

Relations (5.7.7) *and* (5.7.8) *also hold with* ϕ_-, ϕ^r_- *for any fixed* $a > \mathfrak{a}_-$.

Proof. The Lévy process ξ^r underlying Ψ_r is killed at rate $\Psi(0) + r$ but otherwise possesses the same Lévy triplet (c, σ^2, Π) as ξ. Therefore, for any $r > 0$, $\mathtt{d}^r_\pm = \mathtt{d}_\pm$,

$$\bar{\mu}^r_+(0) = \infty \iff \bar{\mu}_+(0) = \infty \text{ and } \bar{\mu}^r_-(0) = \infty \iff \bar{\mu}_-(0) = \infty$$

since those are local properties unaffected by the additional killing rate. Moreover, even $\mathfrak{a}_{\phi^r_\pm} = \mathfrak{a}_{\phi_\pm}$, see (5.3.7), since the analyticity of Ψ and hence of $\phi_\pm$ is unaltered. Next, (5.7.6) follows immediately from Lemma 5.7.1 as it represents $\mu^r_\pm$ in terms of the Lévy measure of the ladder height processes of the conservative process underlying $\Psi^\sharp$, that is

$$\mu_-(dy) = \int_0^\infty e^{-(r+\Psi(0))y_1} \mu^\sharp_-(dy_1, dy),$$

and $\lim_{r\to 0} h\left(r + \Psi(0)\right) = h\left(\Psi(0)\right)$. It remains to prove (5.7.7) and (5.7.8). Fix $a > \mathfrak{a}_+$ and $[b_1, b_2]$ as in the statement. Then from the second expression of (5.3.3)

$$\sup_{b\in[b_1,b_2]} |\phi^r_+(a+ib) - \phi_+(a+ib)| \le |\phi^r_+(0) - \phi_+(0)| + 2\max\left\{|b_1| + |a|, b_2 + |a|\right\} \int_0^\infty e^{-ay} \left|\bar{\mu}_+(y) - \bar{\mu}^r_+(y)\right| dy. \tag{5.7.9}$$

Clearly, from the celebrated Fristedt's formula, see (5.7.2), Lemma 5.7.1 and the monotone convergence theorem

$$\lim_{r\to 0}\phi_{+}^{r}(0)=\lim_{r\to 0}h\left(\Psi(0)+r\right)k_{+}(r+\Psi(0),0)=\lim_{r\to 0}h\left(\Psi(0)+r\right)e^{\int_0^\infty\int_0^\infty\left(e^{-t}-e^{-(\Psi(0)+r)t}\right)\frac{\mathbb{P}\left(\xi_t^\sharp\in dx\right)}{t}dt}$$
$$=h\left(\Psi(0)\right)e^{\int_0^\infty\int_0^\infty\left(e^{-t}-e^{-\Psi(0)t}\right)\frac{\mathbb{P}\left(\xi_t^\sharp\in dx\right)}{t}dt}=h\left(\Psi(0)\right)k_{+}\left(\Psi(0),0\right)=\phi_{+}(0),$$

where $\xi^\sharp$ is the conservative Lévy process underlying $\Psi^\sharp(z)=\Psi(z)-\Psi(0)$. Next, from Lemma 5.7.1 it follows that for any $y>0$ and any $\mathfrak{r}>0$

$$\sup_{0\leq r\leq \mathfrak{r}}e^{-ay}\bar{\mu}_{+}^{r}(y)\leq h\left(\mathfrak{r}\right)e^{-ay}\bar{\mu}_{+}^{\sharp}(y) \tag{5.7.10}$$

with the latter being integrable on $(0,\infty)$ since $a>\mathfrak{a}_{+}$. Moreover, again from Lemma 5.7.1 we get that for any $y>0$

$$\sup_{0\leq r\leq\mathfrak{r}}\left|\bar{\mu}_{+}\left(y\right)-\bar{\mu}_{+}^{r}(y)\right|=\sup_{0\leq r\leq\mathfrak{r}}\left|\int_0^\infty\left(1-e^{-\mathfrak{r}t}\right)e^{-\Psi(0)t}\mu_{+}^{\sharp}\left(dt,(y,\infty)\right)\right|$$
$$\leq\mathfrak{r}\int_0^1 t\mu_{+}^{\sharp}\left(dt,(y,\infty)\right)+\int_1^\infty\left(1-e^{-\mathfrak{r}t}\right)\mu_{+}^{\sharp}\left(dt,(y,\infty)\right)$$

provided ξ underlying Ψ is not a compound Poisson process and

$$\sup_{0\leq r\leq\mathfrak{r}}\left|\bar{\mu}_{+}\left(y\right)-\bar{\mu}_{+}^{r}(y)\right|$$
$$\leq h\left(\Psi(0)\right)\left(\mathfrak{r}\int_0^1 te^{-\Psi(0)t}\mu_{+}^{\sharp}\left(dt,(y,\infty)\right)+\int_1^\infty\left(1-e^{-\mathfrak{r}t}\right)e^{-\Psi(0)t}\mu_{+}^{\sharp}\left(dt,(y,\infty)\right)\right)$$
$$+\left(h\left(\Psi(0)+\mathfrak{r}\right)-h(\Psi(0))\right)\int_0^\infty e^{-\Psi(0)t}\mu_{+}^{\sharp}\left(dt,\mathbb{R}^{+}\right)$$

otherwise. Evidently, in both cases, the right-hand side goes to zero as $\mathfrak{r}\to 0$ for any $y>0$ and this together with (5.7.10) and the dominated convergence theorem show from (5.7.9) that (5.7.7) holds true. In fact (5.7.8) follows in the same manner from (5.7.9) by first dividing by $2\max\{|b|+|a|\}$ for $b\in\mathbb{R}\setminus[b_1,b_2]$. □

Chapter 6

Scientific contributions in the dissertation

6.1 Short review of the main scientific contributions

The exponential functional of Lévy processes is a random variable, that plays a key role in a number of theoretical and applied studies. It is usually the case that the understanding of the probabilistic properties of this random variable brings additional information in those studies. For this purpose the exponential functional has been investigated intensively in many papers. However, it can be safely said that the results have been largely fragmented.

The main contribution of this dissertation is the development of a unified methodology, which allows for the global study of the exponential functional of Lévy processes. In Chapter 5 the Mellin transform of every exponential functional has been computed by the means of a new class of special functions, called the Bernstein-gamma functions. Thus, the detailed study of these Bernstein-gamma functions, conducted in Chapter 5, has led to the thorough understanding of those Mellin transforms. This in turn made possible the application of the Mellin inversion theorem in different contexts and via analytical and probabilistic techniques allowed for the understanding of the law of the exponential functional. Thus, we have obtained a number of distributional properties for the exponential functionals such as asymptotics, smoothness, factorizations, computations of the moments, etc. We emphasize that for most of the derived distributional properties we impose no assumptions on the Lévy process underlying the exponential functional, whereas for the others we have aimed to work with minimal necessary conditions. Thus, for example, the asymptotic behaviour of the logarithm of the tail of the exponential functional is understood in complete generality, whereas under the assumption that the underlying Lévy process does not belong to the *weak non-lattice* class the asymptotic behaviour of the density and its derivatives is deduced. The last result is general enough to encompass at least two papers, published in the last five years, that consider such asymptotic. It must also be highlighted that the dissertation makes a contribution to the theory of the special functions by developing of the Stirling asymptotics for the Bernstein-gamma functions. This, albeit not hard, is useful enough since it allows for the effortless derivation of the asymptotic properties of some well-known special functions.

The main scientific contribution of Chapters 2 and 3 is the factorization of the exponential functional of Lévy processes, which has been obtained under some assumptions. In light of Chapter 5, which includes general factorizations of the exponential functional, the achievements of Chapters 2 and 3 can be regarded as a subset of the attainments of Chapter 5. However, these papers played a key role for the understanding of the problem and for predicting the link between the Mellin transform of the exponential functional and the Bernstein-gamma functions. Therefore, they are at the heart of the development and the conclusion of this dissertation.

6.2 Some notation and basic quantities

In this chapter we will briefly discuss the main scientific contributions of the thesis. The aim is to present the key results and their implications for the development of the theory of the exponential functional of Lévy processes and the advancement of its applications. For this purpose we briefly recall some of the main quantities in the thesis.

We denote by ξ a real-valued Lévy process that is uniquely determined by its Lévy -Khintchine exponent via the link

$$\Psi(z) = \log \mathbb{E}\left[e^{z\xi_1}\right] = cz + \frac{\sigma^2}{2}z^2 + \int_{-\infty}^{\infty} \left(e^{zr} - 1 - zr\mathbb{I}_{\{|r|\leq 1\}}\right)\Pi(dr) - q,\ \ z \in i\mathbb{R}. \quad (6.2.1)$$

We recall that $q \geq 0$ is the killing rate of the Lévy process ξ, that is $\xi_s = \infty, s \geq \mathbf{e}_q$ and $\mathbf{e}_q \sim Exp(q)$ is independent of ξ. When $q = 0$ then $\mathbf{e}_0 = \infty$ almost surely and the process is conservative, that is unkilled.

Next, we recall that $\phi \in \mathcal{B}$ is a Bernstein function if and only if

$$\phi(z) = \log \mathbb{E}\left[e^{-z\xi_1}\right] = \phi(0) + dz + \int_0^{\infty} \left(1 - e^{-zy}\right)\mu(dy),\ \mathtt{Re}(z) \geq 0$$

and ξ is a non-decreasing Lévy process or a subordinator. Then, the celebrated Wiener-Hopf factorization is valid for any Ψ and takes the form

$$\Psi(z) = -\phi_+(-z)\phi_-(z),\ \ z \in i\mathbb{R}, \quad (6.2.2)$$

where $\phi_\pm \in \mathcal{B}$ are related to the ascending and the descending ladder height processes of ξ, that is $H^\pm$.

We use the notation

$$I_\Psi = \int_0^{\infty} e^{-\xi_s} ds$$

for the exponential functional of a Lévy process ξ with a Lévy-Khintchine exponent Ψ and assume implicitly that, whenever we discuss I_Ψ, we work with Ψ such that $I_\Psi < \infty$ almost surely.

6.3 Contributions pertaining to Chapters 2 and 3

The main contribution of these two chapters is the fact that, when the Lévy measure Π in (6.2.1) is such that $\Pi_-(dr) = \Pi(-dr)\mathbb{I}_{\{r>0\}} = \pi_-(r)dr\mathbb{I}_{\{r>0\}}$ and π_- is non-increasing on $(0,\infty)$, then

$$I_\Psi \stackrel{d}{=} I_{\phi_+} \times I_\psi, \tag{6.3.1}$$

where I_{ϕ_+}, I_ψ are independent of each other and

$$I_{\phi_+} = \int_0^\infty e^{-H_s^+} ds, \; I_\psi = \int_0^\infty e^{-\eta_s} ds,$$

where H_s^+ is the ascending ladder height process of ξ and

$$\psi(s) = s\phi_-(s)$$

is a Lévy-Khintchine exponent of the Lévy process η, which possesses no positive jumps. The random variables I_{ϕ_+}, I_ψ being related to simpler Lévy processes are moment determinate and are therefore easier to understand. This allows for the easy derivation of a number of properties for the law of I_Ψ. Perhaps the most notable is the fact that the work [76] published in Annals of Probability is an immediate consequence of Corollary 2.2.1 and is extended significantly by Corollary 3.1.3(iv). Also, with the help of (6.3.1), one gets some fine properties for the density of the supremum of stable Lévy processes with index $\alpha \in (0,1]$, see Corollary 3.1.5.

From theoretical standpoint (6.3.1) shows that the Wiener-Hopf factorization (6.2.2) of Ψ is transferred, under the requirement π_- is non-increasing on $(0,\infty)$, to a Wiener-Hopf factorization of I_Ψ, that is (6.3.1). This triggered further investigations, which culminated in the results of Chapter 5.

6.4 Contributions pertaining to Chapters 4 and 5

We provide the central contributions only.

6.4.1 Contributions to the Bernstein-gamma functions introduced in Section 5.3

The first main contribution is the establishment of the facts:

(1) for any $\phi \in \mathcal{B}$ the Bernstein-gamma function

$$W_\phi(z) = \frac{e^{-\gamma_\phi z}}{\phi(z)} \prod_{k=1}^{\infty} \frac{\phi(k)}{\phi(k+z)} e^{\frac{\phi'(k)}{\phi(k)} z}, \tag{6.4.1}$$

solves the recurrent equation

$$W_\phi(z+1) = \phi(z) W_\phi(z), \; \mathtt{Re}(z) > 0; \quad W_\phi(1) = 1; \tag{6.4.2}$$

(2) W_ϕ is the Mellin transform of a positive random variable;

(3) the understanding of the properties of W_ϕ as a meromorphic function.

These are contained in Theorem 5.3.1 and complement some results in the literature which have considered W_ϕ only as a function on $(0,\infty)$, see [2, 46, 62, 100]

The second main contribution is the development of Stirling type asymptotics for the quantity $|W_\phi(z)|$, $z \in \mathtt{Re}(z) > 0$. With $z = a+ib$ we then have, see Theorem 5.3.2(5.3.19),

$$|W_\phi(z)| = \frac{\sqrt{\phi(1)}}{\sqrt{\phi(a)\phi(1+a)|\phi(z)|}} e^{G_\phi(a)-A_\phi(z)} e^{-E_\phi(z)-R_\phi(a)}, \tag{6.4.3}$$

with the additional knowledge that:

(1) the function

$$A_\phi(z) = A_\phi(a+ib) = \int_0^{|b|} \arg\phi(a+iu)\, du$$

links the geometry of the set $\phi\left(\mathbb{C}_{(0,\infty)}\right)$ with the decay of $|W_\phi(z)|$ when $|\mathtt{Im}(z)| = |b|$ goes to infinity;

(2) the function

$$G_\phi(a) = \int_1^{1+a} \ln\phi(u) du$$

reflects the asymptotic of $|W_\phi(z)|$ when $\mathtt{Re}(z) = a$ goes to infinity;

(3) the expression $e^{-E_\phi(z)-R_\phi(a)}$ is uniformly bounded for the whole class of Bernstein functions and represents the error.

Relation (6.4.3) is universal for all $\phi \in \mathcal{B}$ and as such captures some well-known special functions like the Gamma function and the Barnes-gamma function, see [5, 6]. Thus, basic asymptotic quantities can be retrieved for such functions from (6.4.3) without looking at special cases.

6.4.2 Contributions to the Mellin transform of exponential functionals of Lévy processes

We recall that the Mellin transform of I_Ψ is defined formally as $\mathcal{M}_{I_\Psi}(z+1) = \mathbb{E}\left[I_\Psi^z\right]$ and that the class of all negative-definite functions Ψ or equivalently the class of all Lévy -Khintchine exponents of Lévy processes is denoted by $\overline{\mathcal{N}}$. Also $\mathcal{N} \subsetneq \overline{\mathcal{N}}$ stands for the set of all Ψ such that $I_\Psi < \infty$ almost surely.

The first achievement is the solution for any $\Psi \in \overline{\mathcal{N}}$ of the equation

$$\mathcal{M}_\Psi(z+1) = \frac{-z}{\Psi(-z)} \mathcal{M}_\Psi(z) \tag{6.4.4}$$

valid at least on the domain $i\mathbb{R} \setminus (\mathcal{Z}_0(\Psi) \cup \{0\})$, where we have put the set $\mathcal{Z}_0(\Psi) = \{z \in i\mathbb{R} : \Psi(-z) = 0\}$. Indeed, Theorem 5.2.1 gives that

$$\mathcal{M}_\Psi(z) = \frac{\Gamma(z)}{W_{\phi_+}(z)} W_{\phi_-}(1-z), \; z \in \mathbb{C}_{(0,1)}, \tag{6.4.5}$$

where $\phi_\pm \in \mathcal{B}$ are the functions of the Wiener-Hopf factorization of Ψ, see (6.2.2). Moreover, in Theorem 5.2.1 the main analytical properties of $\mathcal{M}_\Psi$ are established.

The second main contribution is the evaluation of the rate of polynomial decay of $|\mathcal{M}_\Psi(a+ib)|$ for a fixed and $|b| \to \infty$. Thus, Theorem 5.2.5 gives that

$$\begin{aligned} &\lim_{|b|\to\infty} |b|^\beta |\mathcal{M}_\Psi(a+ib)| = 0 \quad \text{if } \beta < \mathtt{N}_\Psi \\ &\lim_{|b|\to\infty} |b|^\beta |\mathcal{M}_\Psi(a+ib)| = \infty \text{ if } \beta > \mathtt{N}_\Psi \end{aligned} \tag{6.4.6}$$

with

$$\mathtt{N}_\Psi = \begin{cases} \frac{\upsilon_-(0^+)}{\phi_-(0)+\bar{\mu}_-0} + \frac{\phi_-(0)+\bar{\mu}_+0}{d_+} < \infty & \text{if } d_+ > 0, d_- = 0 \text{ and } \overline{\overline{\Pi}}(0) = \int_{-\infty}^{\infty} \Pi(dy) < \infty, \\ \infty & \text{otherwise.} \end{cases} \tag{6.4.7}$$

We note that $\mathtt{N}_\Psi \neq \infty$ if and only if the underlying Lévy process is a Compound Poisson process with positive drift.

The third main contribution is the fact that if $\Psi \in \mathcal{N}$ then

$$\mathbb{E}\left[I_\Psi^{z-1}\right] = \mathcal{M}_{I_\Psi}(z) = \phi_-(0)\mathcal{M}_\Psi(z), \; z \in \mathbb{C}_{(0,1)},$$

see Theorem 5.2.7. This allows the computation of the Mellin transform of any exponential functional of Lévy process. Prior to this work there have been only some special cases for which $\mathcal{M}_{I_\Psi}$ is computed, see [45, 50, 54]. In these latter works the decay of $|\mathcal{M}_{I_\Psi}(a+ib)|$ has also been determined thanks to the special form of $\mathcal{M}_{I_\Psi}$. We are not aware of any general result as the one contained in (6.4.6).

6.4.3 Contributions to the properties of I_Ψ

We start with contributions that seem to have the largest impact in the study of the distributional properties of I_Ψ.

6.4.3.1 Large asymptotic

Let $\overline{F}_\Psi(x) = \mathbb{P}(I_\Psi > x)$, $x > 0$, and $f_\Psi(x) = \mathbb{P}(I_\Psi \in dx)/dx$. Then, under the minimal requirement of Ψ not belonging to the weak non-lattice class, we obtain that, for every $n \leq \lceil \mathtt{N}_\Psi \rceil - 2$, see (6.4.7) and Theorem (5.2.14),

$$f_\Psi^{(n)}(x) \overset{\infty}{\approx} (-1)^n \frac{\phi_-(0)\Gamma(n+1-\mathfrak{u}_-) W_{\phi_-}(1+\mathfrak{u}_-)}{\phi'_-(\mathfrak{u}_-{}^+) W_{\phi_+}(1-\mathfrak{u}_-)} x^{-n-1+\mathfrak{u}_-}. \tag{6.4.8}$$

The asymptotic (6.4.8) retrieves the main contributions of two papers published in Annals of Probability [35, 49] and it recovers further asymptotic properties for the density f_Ψ and its derivatives that can be extracted from [44, 50, 54, 76]. Relation (6.4.8) in fact cuts deeper by showing that such asymptotic as (6.4.8) is a consequence of broad analytical properties of Ψ and does not require any special structure imposed on Ψ. It also confirms the expectations that the random variables I_Ψ are much more regular than the underlying Lévy processes.

Another contribution of Theorem (5.2.14) is the following asymptotic at the log-scale

$$\lim_{x\to\infty} \frac{\log \overline{F}_\Psi(x)}{\log x} = \mathfrak{a}_- \in [-\infty, 0)\,. \tag{6.4.9}$$

This result is completely general and refines [4, Lemma 2] which solely provides conditions when the limit in (6.4.9) is not $-\infty$.

6.4.3.2 Smoothness of f_Ψ

The density f_Ψ is known to exists from [16]. Also the works [27, 69] consider some further properties as to the continuity of f_Ψ. For special cases it can be established that f_Ψ is infinitely differentiable, see [44, 50, 54, 76].

Based on the decay as evaluated in (6.4.7), Theorem 5.2.7(3) shows that f_Ψ is at least $\lceil \mathtt{N}_\Psi \rceil - 2$ times continuously differentiable. Since almost invariably $\mathtt{N}_\Psi = \infty$ we deduce that in those cases f_Ψ is infinitely continuously differentiable. This computation of the number of available derivatives for f_Ψ resembles a similar work on the smoothness of self-decomposable laws by Sato and Yamazato, see [91]. However, here the setting is more general and somewhat harder.

6.4.3.3 Factorizations of I_Ψ

Theorem 5.2.27 shows that in absolute generality

$$I_\Psi \;\stackrel{d}{=}\; I_{\phi_+} \times X_{\phi_-} \stackrel{d}{=} \bigotimes_{k=0}^{\infty} \left(C_k \mathfrak{B}_k X_\Psi \times \mathfrak{B}_{-k} Y_\Psi \right), \tag{6.4.10}$$

where $\times$ stands for the product of independent random variables. The law of the positive variables X_Ψ, Y_Ψ are given by

$$\begin{aligned} \mathbb{P}\left(X_\Psi \in dx\right) &= \frac{1}{\phi_+(1)} \left(\bar{\mu}_+(-\ln x)dx + \phi_+(0)dx + d_+\delta_1(dx)\right),\, x \in (0,1) \\ \mathbb{P}\left(Y_\Psi \in dx\right) &= \phi_-(0)\Upsilon_-(dx),\, x > 1 \end{aligned} \tag{6.4.11}$$

where $\Upsilon_-(dv) = U_-\left(d\ln(v)\right),\, v > 1$ is the image of the potential measure U_- by the mapping $y \mapsto \ln y$,

$$C_0 = e^{\gamma_{\phi_+} + \gamma_{\phi_-} - \gamma + 1 - \frac{\phi'_+(1)}{\phi_+(1)}}, C_k = e^{\frac{1}{k+1} - \frac{\phi'_+(k+1)}{\phi_+(k+1)} - \frac{\phi'_-(k)}{\phi_-(k)}}, k = 1, 2, \ldots,$$

where for any integer $k, \mathfrak{B}_k X$ is the random variable defined by

$$\mathbb{E}\left[f(\mathfrak{B}_k X)\right] = \frac{\mathbb{E}\left[X^k f(X)\right]}{\mathbb{E}[X^k]}.$$

The factorization (6.4.10) is completely general and as such completes the story about finding factorizations of I_Ψ. In fact an infinite product factorization has been known in the literature only when Ψ is a Lévy-Khintchine exponent of a subordinator, see [2].

6.4.3.4 Asymptotic behaviour of $I_\Psi(t) = \int_0^t e^{-\xi_s} ds$ when $I_\Psi = \infty$

When $I_\Psi = \infty$ almost surely, that is $\Psi \in \overline{\mathcal{N}} \setminus \mathcal{N}$, then one can investigate the properties of $I_\Psi(t) = \int_0^t e^{-\xi_s} ds$ as t goes to infinity. In Theorem 5.2.24 we have contributed to the understanding the probability measures $\mathbb{P}\left(I_\Psi(t) \in dx\right)$ with the most notable contribution being item 2 which under the celebrated Spitzer's condition shows that

$$\lim_{t\to\infty} \frac{\mathbb{E}\left[I_\Psi^{-a}(t) f\left(I_\Psi(t)\right)\right]}{\kappa_-\left(\frac{1}{t}\right)} = \int_0^\infty f(x)\vartheta_a(dx), \tag{6.4.12}$$

for any $a \in (0, 1-\mathfrak{a}_+)$ and any continuous function f. Note that ϑ_a is a finite positive measure on $(0,\infty)$ and $\kappa_-(r) = \phi_-^r(0)$ with $\Psi^r(z) = \Psi(z) - r = -\phi_+^r(-z)\phi_-^r(z)$.

The result (6.4.12) has a similar counterpart in the discrete world of random walks. Indeed, with $I_S(n) = \sum_{j=0}^n e^{-S_j}$, $S = (S_j)_{j\geq 0}$ being an oscillating random walk, in relation to branching processes in random environments a result resembling (6.4.12) has been derived in [1, 48]. Using discretization the authors of [59, 67] have obtained some results for exponential functionals of Lévy processes into the direction of the limit (6.4.12). However, they consider only the case when $\mathbb{E}\left[\xi_1^2\right] < \infty$, ξ possesses exponential moments and work with a much narrower class of functions f. Here, we settle the question in generality.

6.4.4 Other contributions

The results presented in this dissertation have other applications. In future works they will play a key role in several spectral studies of non-self-adjoint Markov processes. The interested reader can have a look at the following preprint [83] for more details how the achievements of this work can be applied in quite a different area.

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