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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР

INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

Riesz summability of multiple Hermite series in L^p spaces

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Riesz summability of multiple Hermite series in L^p spaces *

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Abstract

Let $E_\lambda^\alpha f(y) = \sum(1-\mu_k/\lambda)^\alpha f_k\phi_k(y)$, $\mu_k < \lambda$, be the Riesz means of order α of the multiple Hermite series of the function f. Here it is proved that $E_\lambda^\alpha f \to f$ as $\lambda \to \infty$ in the following cases: (i) In the space $L^p(\mathbf{R}^n)$, $n \geq 2$, under the sharp condition on parameter $\alpha, \alpha > \alpha(p)$, where $\alpha(p) = \max(0, n|1/p - 1/2| - 1/2)$ and $1 \leq p \leq 2n/(n+2)$ or $2n/(n+2) \leq p < \infty$. (ii) In the space L_m^2 with a norm $(\int |f(x)|^2(1+|x|^2)^m dx)^{1/2}$ if $2\alpha > |m|$; or, equivalently, in the Sobolev space H^m with the usual norm $(\int |\hat{f}(\xi)|^2(1+|\xi|^2)^m d\xi)^{1/2}$.

1 Statement of the main results

Let $E_\lambda^\alpha f(y) = \sum(1-\mu_k/\lambda)^\alpha f_k\phi_k(y)$, $\mu_k < \lambda$, be the Riesz means of order α of the multiple Hermite series of the function f, where $f_k = \int f(x)\phi_k(x)dx$ and

$$\mu_k = 2|k| + n, \phi_k(x) = H_k(x)\exp(-x^2/2)$$

are the eigenvalues and orthonormalized eigenfunctions (Hermite functions) of the operator $-\Delta + x^2$ in $L^2(\mathbf{R}^n)$, $n \geq 2$. We are interested in the problem of the convergency $E_\lambda^\alpha f \to f$ as $\lambda \to \infty$. This convergence can be locally

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uniform [9],[18] or almost everywhere [10], [19] or on the Lebesgue set of the function f [10], [19] or in the L^p norm [10], [19]. In [10] the case of L^1_m spaces with a norm $(\int |f(x)|(1+|x|)^m dx)$ under sharp conditions on the parameters α and m, $\alpha > (n-1)/2$, $-2\alpha - 2/3 \le m \le 2\alpha + 1 - n$ was considered. See also [2], [4], [5], [7], [8], [14], [15] for the case of compact manifolds without boundary and the bibliography in [14], [19].

The results for L^p spaces [10], [19] are sharp only if $p = 1$. Using the analogy with the classical Bochner-Riesz conjecture, which corresponds to the case $A = -\Delta$ in $L^2(\mathbf{R^n})$, it is natural to expect that the convergence $E^\alpha_\lambda f \to f$ in L^p norm is valid under the following sharp condition on the parameter α,

$$\alpha > \alpha(p), \tag{1.1}$$

where

$$\alpha(p) = \max(0, n|1/p - 1/2| - 1/2),\ n \ge 2, \tag{1.2}$$

is the so-called critical index for the L^p spaces. This was confirmed in [19] for the radial functions in L^p.

Here we shall prove this conjecture for all p satisfying

$$|1/p - 1/2| \ge 1/n,\ n \ge 2, \tag{1.3}$$

using a variation of the main approach from [14], where the corresponding convergence was established for $|1/p - 1/2| \ge 1/(n+1)$ in the case of second order elliptic operators on a compact manifold without boundary (see also [6], [15]). Slightly changing the arguments, we investigate the convergence $E^\alpha_\lambda f \to f$ also in the Sobolev space H^m with the usual norm $\|f\|_{2,m} = (\int |\hat{f}(\xi)|^2(1+|\xi|^2)^m d\xi)^{1/2}$. Noticing that the operator A commutes with the Fourier transform, we see that the convergence $E^\alpha_\lambda f \to f$ as$\lambda \to \infty$ in the space H^m is equivalent to the convergence in the space L^2_m with a norm $\|f\|_{2,m} = (\int |f(x)|^2(1+|x|^2)^m dx)^{1/2}$.

By duality, it is sufficient to consider only the cases $1 \le p < 2$ and $m > 0$. Let $\|E^\alpha_\lambda\|_p$ denote the operator norm in L^p spaces and $\|E^\alpha_\lambda\|_{2,m}$ - the operator norm in H^m spaces or, equivalently, in L^2_m spaces. Then the desired convergence will follow from uniform estimates of these norms.

Theorem 1 *Let $\alpha > \alpha(p)$ and $1 \le p \le p_n$, where $p_n = 2n/(n+2)$. Then $\|E^\alpha_\lambda\|_p \le c$.*

Here and later on all estimates are uniform with respect to the parameter $\lambda > \lambda_0$ for some $\lambda_0 > 1$, where c is a positive constant.

Theorem 2 *If* $2\alpha > m$ *then* $\|E_\lambda^\alpha\|_{2,m} \le c$.

2 Proof of Theorem 1

Since the case $p = 1$ is already considered in [10],[19], we shall suppose here that $p > 1$. Note also that it is sufficient to prove the estimate $\|E_\lambda^\alpha\|_p \le c$ only for the case $p < p_n$. Indeed, we have $\alpha(p_n) = 1/2$. If $\alpha > 1/2$ then we can find p such that $p < p_n$ and $1/2 < \alpha(p) < \alpha$. Then $\|E_\lambda^\alpha\|_p \le c$ and $\|E_\lambda^\alpha\|_2 \le c$. Therefore the Riesz interpolation theorem [16] implies $\|E_\lambda^\alpha\|_{p_n} \le c$.

We intend to adapt the main arguments from [6],[14],[15]. To this end we need the following "restriction" theorem. Let $\chi_{(a,b)}(\mu)$ be the characteristic function of the interval (a, b) and let $\|f\|_p$ stand for the L^p norm. Denote

$$\beta(p) = \max(0, \frac{1}{2}n|1/p - 1/2| - 1/2), \tag{2.1}$$

and

$$|1/p - 1/2| \ne 1/n. \tag{2.2}$$

Theorem 3 *The uniform estimate*

$$\|\chi_{(0,1)}(A - \lambda)f\|_2 \le \lambda^{\beta(p)}\|f\|_p \tag{2.3}$$

is fulfilled if p *satisfies (2.2). In addition, if* $p = 1$ *then (2.3) is true for all* $n \ge 2$.

We postpone the proof of theorem 3 until section 3 and proceed further with the proof of theorem 1. Notice that

$$E_\lambda^\alpha = c_\alpha \lambda^{-\alpha} \int e^{i\lambda t}(t - i0)^{-\alpha-1}e^{-itA}\,dt. \tag{2.4}$$

and consider its average $E_{\lambda,\rho}^\alpha$,

$$E_{\lambda,\rho}^\alpha = c_\alpha \lambda^{-\alpha} \int e^{i\lambda t}\hat\rho(t)(t - i0)^{-\alpha-1}e^{-itA}\,dt, \tag{2.5}$$

where ρ is an even function from the Schwartz class such that the support of its Fourier transform $\hat{\rho}$ is close to zero and $\hat{\rho}(t) = 1$ if $|t| < \varepsilon$ for some $\varepsilon > 0$ small enough. Further the proof is devided into several steps.

Step 1. Here we prove

$$\|E_\lambda^\alpha - E_{\lambda,\rho}^\alpha\|_p \le c \text{ if } \alpha > \alpha(p),\ 1 < p < p_n. \tag{2.6}$$

To this end we need the following estimates for the kernels $I^\alpha e(\lambda, x, y)$ and $I^\alpha e_\rho(\lambda, x, y)$, of the operators (2.4), (2.5) respectively, which follow from theorem 4 below:

$$|I^\alpha e(\lambda, x, y)| + |I^\alpha e_\rho(\lambda, x, y)| \le C_N |y|^{-N} \tag{2.7}$$

if $y^2 > 2\lambda$ and analogously for $x^2 > 2\lambda$.

Theorem 4 *Let*

$$e(\lambda, x, y) = \sum_{\mu_k < \lambda} \phi_k(x)\phi_k(y)$$

be the spectral function of the operator A. Then

$$|e(\lambda + \mu, x, x) - e(\lambda, x, x)| \le c\lambda^{n/2-1} \textit{ if } n \ge 2,\ |\mu| \le 1, \tag{2.8}$$

$$e(\lambda, x, x) \le c\lambda^{n/2} \exp(-cx^2/\lambda), \tag{2.9}$$

$$e(\lambda, x, x) \le C_N |x|^{-N} \textit{ if } x^2 > (1+\delta)\lambda,\ \delta > 0,\ N > 0. \tag{2.10}$$

This theorem will be proved in section 5. Note that (2.8) is also proved in [19] by another method.

Further, according to (2.7) in estimating the integral $\int |E_\lambda^\alpha f - E_{\lambda,\rho}^\alpha|^p dy$ we can suppose that $y^2 < 2\lambda$. Then

$$\|(E_\lambda^\alpha - E_{\lambda,\rho}^\alpha)f\|_p \le c\lambda^{\frac{n}{2}(\frac{1}{p}-\frac{1}{2})} \|(E_\lambda^\alpha - E_{\lambda,\rho}^\alpha)f\|_2,\ 1 < p < 2. \tag{2.11}$$

On the other hand, $E_\lambda^\alpha - E_{\lambda,\rho}^\alpha = \lambda^{-\alpha} g(\lambda - A)$, where $g(\mu) = c_\alpha \int e^{i\mu t}(1 - \hat{\rho}(t))(t - i0)^{-\alpha-1} dt$. In particular, if $\alpha > 0$ it follows $|g(\mu)| \le C_N(1 + |\mu|)^{-N}$ for large N, hence the Plancherel theorem gives

$$\|(E_\lambda^\alpha - E_{\lambda,\rho}^\alpha)f\|_2^2 \le C_N \lambda^{-2\alpha} \sum (1 + |\lambda - \mu_j|)^{-2N} f_j^2.$$

Since the last sum is majorized by $\sum(1+|\lambda-k|)^{-2N}\|\chi_{(0,1)}(A-k)f\|_2^2$ we can apply theorem 3, therefore

$$\|(E_\lambda^\alpha - E_{\lambda,\rho}^\alpha)f\|_2 \leq c\lambda^{-\alpha+\beta(p)}\|f\|_p. \tag{2.12}$$

Evidently, if $1<p<p_n$ (2.6) follows from (2.11) and (2.12) since $\frac{1}{2}n(1/p-1/2)+\beta(p)=\alpha(p)$ and $\alpha>\alpha(p)$.

Step 2. In step 1 we reduced the problem to an estimate of the form

$$\|E_{\lambda,\rho}^\alpha\|_p \leq c \text{ if } \alpha>\alpha(p),\ 1<p<p_n. \tag{2.13}$$

For proving (2.13) we follow [14] and write

$$E_{\lambda,\rho} = \sum_{k\geq 1} E_{\lambda,k} + E_{\lambda,0}, \tag{2.14}$$

where

$$E_{\lambda,k} = c_\alpha\lambda^{-\alpha}\int e^{i\lambda t}\hat{\rho}(t)(t-i0)^{-\alpha-1}\hat{g_k}(\lambda t)e^{-itA}dt, \tag{2.15}$$

and $\hat{g_k}(s)=\hat{g}(2^{-k}s)$ if $k\geq 1$ and $\hat{g_0}(s)=1-\sum_{k=1}^\infty \hat{g}(2^{-k}s)$. The function $\hat{g}$ is $C_0^\infty(R)$ such that $\hat{g}(s)=0$ for $|s|<\epsilon/2$ and for $|s|>\epsilon$, and $\sum_{-\infty}^{+\infty}\hat{g}(2^{-k}s)=1$ for $s\neq 0$. Note that the sum in (2.14) is finite and $2^k\leq c\lambda$.

We have for large λ

$$\|E_{\lambda,0}\|_p \leq c,\ 1<p<2. \tag{2.16}$$

Indeed, $\hat{g_0}\in C_0^\infty(R)$ and $E_{\lambda,0}=m_\lambda(A)$, where $m_\lambda(\mu)$ is a convolution of the functions $(1-\mu/\lambda)_+^\alpha$ and $1/\lambda g_0(\mu/\lambda)$ for large λ (on the support of $\hat{g_0}(\lambda t)$ one has $\hat{\rho}(t)=1$ if $\lambda>\lambda_0$ and λ_0 is large enough). Consequently, for every $N>0, j\geq 0$,

$$|m_\lambda^{(j)}(\mu)| \leq C_N\lambda^{-j}(1+\mu/\lambda)^{-N}, \mu>0, \lambda>\lambda_0,$$

therefore $m_\lambda(A)$ is a pseudodifferential operator of order 0, uniformly on $\lambda>\lambda_0$, hence (2.16) is fulfilled.

The estimate of $E_{\lambda,k}$ is more complicated. As in [14] we consider first the case when the kernel of $E_{\lambda,k}$ is supported in the domain $\{(x,y):|x-y|<\lambda^{-1/2}2^k\}$. Then, if B is a ball of radius $\lambda^{-1/2}2^k$ and we want to estimate $E_{\lambda,k}f$

in $L^p(B)$ norm then it sufficies to take f supported in the ball $2B$. With this in mind one proceeds as follows. By the Hölder inequality,

$$\|E_{\lambda,k}f\|_{L^p(B)} \leq c(\lambda^{-1/2}2^k)^{n(1/p-1/2)}\|E_{\lambda,k}f\|_2. \tag{2.17}$$

On the other hand, the Plancherel theorem gives

$$\|E_{\lambda,k}f\|_2^2 = \sum m^2(\lambda - \mu_j)f_j^2,$$

where

$$m(\mu) = c_\alpha \lambda^{-\alpha} \int e^{it\mu}\hat{\rho}(t)(t-i0)^{-\alpha-1}\hat{g}(2^{-k}\lambda t)dt.$$

Since for every $N > 0$,

$$|m(\mu)| \leq C_N 2^{-k\alpha}(1 + |\mu|2^k\lambda^{-1})^{-N},$$

it follows that

$$\|E_{\lambda,k}f\|_2^2 \leq c2^{-2k\alpha}\sum(1 + |\lambda - \mu_j|2^k\lambda^{-1})^{-2N}f_j^2. \tag{2.18}$$

Since $2^k < c\lambda$ we obtain

$$\|E_{\lambda,k}f\|_2^2 \leq c2^{-2k\alpha}\sum(1 + |\lambda - s|2^k\lambda^{-1})^{-2N}\|\chi(0,1)(A-s)f\|_2^2$$

and theorem 3 implies

$$\|E_{\lambda,k}f\|_2 \leq c2^{-k\alpha-k/2}\lambda^{1/2+\beta(p)}\|f\|_p,\ 1 < p < p_n.$$

This and (2.17) show that

$$\|E_{\lambda,k}f\|_p \leq c2^{-k(\alpha-\alpha(p))}\|f\|_p \text{ if } |x-y| < \lambda^{-1/2}2^k,\ 1 < p < p_n. \tag{2.19}$$

In the case $|x-y| > \lambda^{-1/2}2^k$ we shall prove that

$$\|E_{\lambda,k}f\|_p \leq c2^{-k\alpha}\|f\|_p,\ p \geq 1. \tag{2.20}$$

To this end we consider the kernel $E_{\lambda,k}(x,y)$ of the operator $E_{\lambda,k}$. According to (2.15)

$$E_{\lambda,k}(x,y) = c_\alpha\lambda^{-\alpha}\int e^{i\lambda t}\hat{\rho}(t)(t-i0)^{-\alpha-1}\hat{g}(2^{-k}\lambda t)U(t,x,y)dt,$$

where

$$U(t,x,y) = (2\pi i \sin 2t)^{-n/2} \exp(i\frac{x^2+y^2}{2}\cot 2t - i\frac{xy}{\sin 2t}).$$

In particular, if $F_\lambda(x,y) = \lambda^\alpha E_{\lambda,k}(\sqrt{\lambda}x, \sqrt{\lambda}y)$, then

$$F_\lambda(x,y) = \int e^{i\lambda\psi} q(t,\lambda)dt, \tag{2.21}$$

where $\psi(t,x,y) = t + \frac{1}{2}(x^2+y^2)\cot 2t - xy/\sin 2t$ and $q(t,\lambda) = (t-i0)^{-\alpha-1-n/2}\hat{g}(2^{-k}\lambda t)h(t), h \in C_0^\infty$. We assert that for every $N > 0$

$$|F_\lambda(x,y)| \le C_N \lambda^{-2N+n/2+\alpha} 2^{k(N-\alpha-n/2)}|x-y|^{-2N}. \tag{2.22}$$

For proving this we shall integrate by parts in (2.21), using the operator L which transpose is $(i\partial_t\psi)^{-1}\partial_t$. Since

$$\partial_t\psi = \sin^{-2} 2t\ (\sin^2 2t - (x-y)^2 + 4xy\sin^2 t)$$

and $x^2, y^2 < 2, |x-y| > 2^k\lambda^{-1} > \frac{1}{\epsilon}|t|$ on the support of $\hat{g}(2^{-k}\lambda t)$ we obtain for small $\epsilon > 0$

$$|\partial_t\psi| \ge c|x-y|^2|t|^{-2} \ge c_1. \tag{2.23}$$

On the other hand

$$|\partial_t^{k+1}\psi| \le c_k|t|^{-k}(1+|\partial_t\psi|),\ k \ge 1 \tag{2.24}$$

and

$$|\partial_t^k q| \le c_k|t|^{-\alpha-1-n/2-k}. \tag{2.25}$$

Therefore (2.23)-(2.25) imply $|L^N q| \le C_N|t|^{-\alpha-1-n/2-N}|\partial_t\psi|^{-N}$, or

$$|L^N q| \le c_N|t|^{-\alpha-1-n/2+N}|x-y|^{-2N}. \tag{2.26}$$

Now (2.26) gives the estimate (2.22) for the integral (2.21). Consequently

$$|E_{\lambda,k}(x,y)| \le C_N\lambda^{-N+n/2}2^{k(N-\alpha-n/2)}|x-y|^{-2N}.$$

Using this for $N > n/2$, we obtain the bound

$$\int |E_{\lambda,k}(x,y)|dy \le c2^{-k\alpha} \text{ if } |x-y| > \lambda^{-1/2}2^k.$$

Thus the estimate (2.20) is proved.

Finally, (2.19) and (2.20) imply

$$\sum_{k\geq 1} \|E_{\lambda,k}f\|_p \leq c\|f\|_p \text{ if } 1 < p < p_n,\ \alpha > \alpha(p),$$

which together with (2.16) give (2.13). Theorem 1 is proved.

3 Proof of theorem 3

Step 1. $1 < p < p_n, n > 2$.

Let ρ be a real function so that $\rho > \chi_{(0,1)}$ and the support of $\hat{\rho}$ be close to zero. Then $\chi_{(0,1)}(A-\lambda) < \rho(A-\lambda)$, therefore

$$\|\chi_{(0,1)}(A-\lambda)f\|_2^2 \leq (B_\lambda f, f) \leq \|B_\lambda f\|_{p'}\|f\|_p, \tag{3.1}$$

where $1/p + 1/p' = 1$, $B_\lambda = g(A-\lambda)$ and $g = \rho^2$. Note that the support of $\hat{g}$ is also close to zero. Since

$$B_\lambda = \frac{1}{2\pi}\int e^{i\lambda t}\hat{g}(t)e^{-itA}dt$$

and we want to apply the Stein interpolation theorem [16], we consider the family

$$B_\lambda(z) = \frac{1}{2\pi}\int e^{i\lambda t}\hat{g}(t)(t-i0)^{-z}e^{-itA}dt$$

for $Re\, z$ running over a compact interval.

If $Re\, z = 1$, then $B_\lambda(1+is) = m(\lambda - A)$, where m is a convolution of the functions $g(\mu)$ and $\mu_+^{is}/\Gamma(1+is)$, in particular, $|m(\mu)| \leq ce^{c|s|}$ for some $c > 0$. Hence

$$\|B_\lambda(1+is)f\|_2 \leq ce^{c|s|}\|f\|_2. \tag{3.2}$$

Let now $Re\, z = -\gamma < 0$. Then for the kernel $B_\lambda(x,y)$ of the operator $B_\lambda(-\gamma + is)$ we have

$$B_\lambda(x,y) = c\int e^{i\lambda t}\hat{g}(t)(t-i0)^{\gamma - is}U(t,x,y)dt.$$

In estimating the size of $B_\lambda(x,y)$ we can suppose that $x=(x_1,0,...,0)$ and $y=(y_1,y_2,0,...,0)$ since the function $(x,y)\to U(t,x,y)$ is rotationally invariant. Then

$$\hat g(t)U(t,x,y)=\hat g(t)U_2(t,x,y)(2\pi i\frac{\sin 2t}{t})^{-n/2+1}(t-i0)^{-n/2+1},$$

where $U_2(t,x,y)=\int e^{-i\lambda t}de(\lambda,x,y)$ is the Fourier transform of the spectral function $e(\lambda,x,y)$ of the operator A, considered now in $\mathbf{R}^2$ space. If $\gamma<n/2-1$ we can write

$$B_\lambda(x,y)=c_s\int(\lambda-\mu)_+^{-\gamma+n/2-2+is}e_h(\mu,x,y)d\mu,$$

where $c_s=1/\Gamma(-\gamma+n/2-1+is)$, hence $|c_s|\le ce^{c|s|}$, and

$$e_h(\lambda,x,y)=\int h(\lambda-\mu)de(\mu,x,y)$$

for the corresponding function h. Since h is from the Schwartz class, we have the estimate

$$|e_h(\lambda,x,y)|\le c. \tag{3.3}$$

Indeed,

$$e_h(\lambda,x,y)=\sum h(\lambda-\mu_k)\phi_k(x)\phi_k(y),\ \mu_k=2|k|+2,$$

whence

$$e_h(\lambda,x,y)=\sum_{j=0}^{\infty}h(\lambda-2j-2)(e(j+1,x,y)-e(j,x,y)).$$

Now (3.3) follows from the bound (2.8). Therefore we have the estimate

$$|B_\lambda(x,y)|\le c_s\lambda^{-\gamma+n/2-1}\text{ if }0<\gamma<n/2-1,n>2,$$

which shows that

$$\|B_\lambda(-\gamma+is)f\|_\infty\le c_s\lambda^{-\gamma+n/2-1}\|f\|_1\text{ for }0<\gamma<n/2-1. \tag{3.4}$$

Now we can apply the Stein interpolation theorem, hence (3.2), (3.4) imply

$$\|B_\lambda f\|_{p'} \le c\lambda^{2\beta(p)}\|f\|_p \text{ if } 1 < p < p_n, n > 2. \tag{3.5}$$

Evidently (3.1) and (3.5) give theorem 3 in the considered case.

Step 2. $p = 1$

Since

$$\|\chi_{(0,1)}(A-\lambda)f\|_2 \le \int (e(\lambda+1,x,x) - e(\lambda,x,x))^{1/2}|f(x)|dx$$

the estimate (2.3) follows from (2.8).

Step 3. $p_n < p < 2, n > 2$

Starting with (3.1) we have to prove

$$\|B_\lambda f\|_{p'} \le c\|f\|_p, \; p_n < p < 2. \tag{3.6}$$

Evidently, $\|e^{-itA}f\|_2 \le \|f\|_2$ and $\|e^{-itA}f\|_\infty \le c|t|^{-n/2}\|f\|_1$ for small $|t|$. Hence the Riesz interpolation theorem implies

$$\|e^{-itA}f\|_{p'} \le c|t|^{-n(1/p-1/2)}\|f\|_p, \; 1 < p < 2,$$

whence we get (3.6). Theorem 3 is proved.

4 Proof of theorem 2

We argue as in the proof of theorem 1. Using (2.7) we see that it is sufficient to estimate $E^\alpha_{\lambda,\rho}f$ and $(E^\alpha_\lambda - E^\alpha_{\lambda,\rho})f$ in L^2_m norm only for $x^2, y^2 < 2\lambda$. Beginning with the first quantity, we notice that the operator $E_{\lambda,0} = m_\lambda(A)$ from (2.15) is L^2 bounded, uniformly in λ. Consequently, the same is true in the Sobolev space H^m :

$$\|E_{\lambda,0}\|_{2,m} \le c. \tag{4.1}$$

To estimate $\|E_{\lambda,k}\|_{2,m}$ we first consider the case $|x-y| < 2^{k/2}$, which means that the kernel of the operator $E_{\lambda,k}$ is supported in this domain. Let $\{B_j(y_j)\}_{j\ge1}$ be a sequence of balls with radius $2^{k/2}$ and center y_j such that the region $\{y : y^2 < 2\lambda\}$ be covered by the union of $\{B_j\}$. In estimating $E_{\lambda,k}f$ in $L^2_m(B_j)$ norm we can suppose that $f(x)$ is supported in the ball $2B_j$.

Step 1. $|y_j| < 2^{k/2+3}$.

Then, if $y \in B_j$ it follows $|y| < 9\ 2^{k/2}$, whence

$$\int_{B_j} |E_{\lambda,k}f(y)|^2(1+|y|^2)^m dy \le c_m 2^{km} \|E_{\lambda,k}f\|_2^2,$$

therefore (2.18) implies

$$\int_{B_j} |E_{\lambda,k}f(y)|^2(1+|y|^2)^m dy \le c_m 2^{km-2k\alpha} \int_{2B_j} |f(x)|^2 dx. \tag{4.2}$$

Step 2. $|y_j| \ge 2^{k/2+3}$.

Now, if $y \in B_j$ then $|y| \le |y_j| + 2^{k/2} \le 2(|y_j| - 2^{k/2+1})$ and for $x \in 2B_j$ we have $|x| \ge |y_j| - 2^{k/2+1}$. Thus, analogously to (4.2),

$$\int_{B_j} |E_{\lambda,k}f(y)|^2(1+|y|^2)^m dy \le c_m 2^{-2k\alpha} \int_{2B_j} |f(x)|(1+|x|^2)^m dx. \tag{4.3}$$

Evidently, (4.2) and (4.3) imply

$$\|E_{\lambda,k}f\|_{2,m} \le c_m 2^{k(m/2-\alpha)} \|f\|_{2,m} \text{ if } |x-y| < 2^{k/2}. \tag{4.4}$$

Now we turn to the case $|x-y| > 2^{k/2}$. Repeating the proof of (2.22), we can establish the estimate

$$|F_\lambda(x,y) \le C_N \lambda^{\alpha+1+n/2-N} 2^{-k}, \forall N > 0, \tag{4.5}$$

where $|x-y| \ge 2^{k/2}\lambda^{-1/2}$ and $x^2, y^2 < 2$. Indeed, starting with (2.21) and noticing that the hypotheses imply $|t| < c\epsilon|x-y|$ on the support of $t \to \hat{g}(2^{-k}\lambda t)$, we see that the bounds (2.23)-(2.25) are valid as before, thus (2.26) gives for $\alpha > 0, N > 0$,

$$|L^N q| \le c(2^k\lambda^{-1})^{-\alpha-1-n/2+N}(2^{k/2}\lambda^{-1/2})^{-2N}$$

or

$$|L^N q| \le c\lambda^{\alpha+1+n/2} 2^{-k}. \tag{4.6}$$

It is clear that (4.6) implies the estimate (4.5) for the integral (2.21). Further the bound (4.5) can be rewriten in the form

$$|E_{\lambda,k}(x,y)| \le C_N \lambda^{1+n/2-N} 2^{-k}, \forall N > 0,$$

where $|x-y| > 2^{k/2}$ and $x^2, y^2 < 2\lambda$. This estimate and the remarks at the beginning of the proof of theorem 2 are sufficient to assert that

$$\|E_{\lambda,k}f\|_{2,m} \leq c2^{-k}\|f\|_{2,m} \text{ if } |x-y| > 2^{k/2}. \tag{4.7}$$

Finally, (4.4) and (4.7) give

$$\sum_{k\geq 1} \|E_{\lambda,k}\|_{2,m} < c \text{ if } 2\alpha > m. \tag{4.8}$$

Thus (4.1), (4.8) and (2.14) imply

$$\|E^{\alpha}_{\lambda,\rho}\|_{2,m} < c \text{ if } 2\alpha > m. \tag{4.9}$$

It remains to estimate $E^{\alpha}_{\lambda} - E^{\alpha}_{\lambda,\rho}$. To this end we notice that analogously to (2.14)

$$E^{\alpha}_{\lambda} - E^{\alpha}_{\lambda,\rho} = \sum E_{\lambda,k}, 2^k > c\lambda, \lambda > \lambda_o, \tag{4.10}$$

where now

$$E_{\lambda,k} = c_{\alpha}\lambda^{-\alpha}\int e^{i\lambda t}(1-\hat{\rho}(t))(t-i0)^{-\alpha-1}\hat{g}(2^{-k}\lambda t)e^{-itA}dt.$$

Since

$$\int_{y^2<2\lambda} |E_{\lambda,k}f(y)|^2(1+|y|^2)^m dy \leq c\lambda^m\|E_{\lambda,k}f\|_2^2$$

we obtain as before

$$\|E_{\lambda,k}f\|_{2,m} \leq c\lambda^{m/2}2^{-k\alpha}\|f\|_{2,m}, \; m > 0, \; 2^k > c\lambda. \tag{4.11}$$

Consequently (4.10) and (4.11) imply

$$\|E^{\alpha}_{\lambda} - E^{\alpha}_{\lambda,\rho}\|_{2,m} \leq c\lambda^{m/2-\alpha}. \tag{4.12}$$

Evidently (4.9) and (4.12) finish the proof of theorem 2.

5 Proof of theorem 4

To prove (2.8) we use the relation

$$\int \rho(\lambda-\mu)de(\mu,x,x) = \frac{1}{2\pi}\int e^{i\lambda t}\hat{\rho}(t)U(t,x,x)dt,$$

where $U(t,x,x) = (2\pi i \sin 2t)^{-n/2}\exp(-ix^2\tan t)$.

Set

$$I(\lambda,x) = \int \rho(\lambda-\mu)de(\mu,\sqrt{\lambda}x,\sqrt{\lambda}x). \tag{5.1}$$

Evaluating the singularity at $t=0$ we obtain

$$I(\lambda,x) \sim \lambda^{n/2}\int e^{i\lambda\psi}q(t,\xi)dtd\xi, \tag{5.2}$$

where

$$\psi(t,\xi,x) = t - x^2\tan t - \frac{1}{2}\xi^2\sin 2t, \tag{5.3}$$

and $q(t,\xi) = \hat{\rho}(t)g(\xi), g \in C_0^\infty, g$ being an even cutoff function. Here the equivalence $a(\lambda,x)\sim b(\lambda,x)$ means that $|a-b| \le C_N(\lambda+x^2)^{-N}, N>0$.

Case $x^2 < 1-\delta, \delta > 0$.

To find the asymptotics of the integral (5.2) we apply the stationary phase method. In polar coordinates $\xi = \sigma\omega$ one obtains an integral with nondegenerate critical points $t=0, \sigma^2 = 1-x^2$. Therefore

$$I(\lambda,x) = O(\lambda^{n/2-1}),\ x^2 < 1-\delta. \tag{5.4}$$

Case $|x^2-1| < \delta$.

Now the critical points of the phase function $(t,\xi)\to\psi$, given by (5.3), degenerate if $x^2 = 1$, coinciding with $(0,0)$. Then there exists a smooth change of variables near $(0,0)$ such that in the new coordinates $\psi(t,\xi,x) = t\xi^2 + t^3/3$ if $x^2 = 1$. By the theory of the versal defformations [1], [13] there exists a smooth and odd change of variables near $(0,0)$ such that $(0,\xi)\to(0,\xi)$ and for some $\delta > 0$,

$$\psi(t,\xi,x) = -B(x)t + t\xi^2 + t^3/3 \text{ if } |x^2-1| < \delta.$$

In addition, $B(x) = 1 - x^2 + O((1-x^2)^2)$ as $x^2 \to 1$.

Using the principle of the stationary phase and polar coordinates $\xi = \sigma\omega$, we obtain

$$I(\lambda, x) \sim \lambda^{n/2} \int_0^\infty \int e^{i\lambda(-Bt+t\sigma^2+t^3/3)} \sigma^{n-1} g(t,\sigma) dt d\sigma, \qquad (5.5)$$

where

$$g(t,\sigma) = \int_{|\omega|=1} q(t,\sigma\omega) J(t,\sigma\omega) d\omega$$

and $J(t,\xi)$ is the corresponding jacobian. Since the function $\sigma \to g(t,\sigma)$ is even the Malgrange preparation theorem implies

$$g(t,\sigma) = a_0 + a_1 t + a_2\sigma^2 + (t^2 + \sigma^2 - B) g_1 + t\sigma^2 g_2. \qquad (5.6)$$

Integrating by parts in (5.5) and using (5.6) we get $I(\lambda, x) \sim$

$$\lambda^{n/2} \int_0^\infty \int e^{i\lambda(-Bt+t\sigma^2+t^3/3)} \sigma^{n-1} (a_0 + a_1 t + a_2\sigma^2) dt d\sigma + O(\lambda^{n/2-1}).$$

Consequently, $|I(\lambda, x)| \le$

$$c[\lambda^{n/6-1/3} f_{n-2}(B\lambda^{2/3}) + \lambda^{n/6-2/3} |f'_{n-2}(B\lambda^{2/3})| + \lambda^{n/6-1} f_n(B\lambda^{2/3}) + \lambda^{n/2-1}],$$

where $f_n(s) = \int_0^\infty \sigma^{n/2} Ai(\sigma - s) d\sigma$. Note the properties:

$$f_n(s) = -s f_{n-2}(s) + f''_{n-2}(s),\ n \ge 2;\ f_0(s) = \int_s^\infty Ai(\sigma) d\sigma,$$

$$f_1(s) = \pi 2^{1/3} [-4^{-1/3} s (Ai(4^{-1/3}s))^2 + (Ai'(4^{-1/3}s))^2].$$

Using $f'_n(s) = -\frac{n}{2} f_{n-2}(s), n \ge 2$ and the asymptotics of the Airy function we get $f_n(s), f'_n(s) = O(s^{n/2})$ as $s \to +\infty$, whence

$$|f_n(s)| + |f'_n(s)| \le c(1 + |s|)^{n/2},\ n \ge 0.$$

Thus we obtain the uniform estimate

$$I(\lambda, x) = O(\lambda^{n/2-1}),\ |x^2 - 1| < \delta,\ n \ge 2. \qquad (5.7)$$

Case $x^2 > 1 + \delta$.

Now $I(\lambda, x) = O(\lambda^{-\infty})$ since in the integral (5.2) the phase function ψ has no critical points on the support of q. Therefore (5.1), (5.4), (5.7) show that $\int \rho(\lambda - \mu) de(\mu, x, x) = O(\lambda^{n/2-1})$, $n \geq 2$, uniformly, whence (2.8) follows.

For proving (2.9) it sufficies to use the relation $\int_0^\infty e^{-\lambda t} de(\lambda, x, x) = (2\pi \sinh 2t)^{-n/2} \exp(-x^2 \tanh t)$.

Finally, (2.10) is a consequence of theorem 3 [11], taking into account the estimate $u\sqrt{u^2-1} - arccosh\ u \geq c\sqrt{u^2-1}$ if $u^2 - 1 > c$ and $0 < c < 1$. Theorem 4 is proved.

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