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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР

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Differential Inclusions with Upper Semicontinuous Right-Hand Side

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Differential Inclusions with Upper Semicontinuous Right-Hand Side *

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Abstract

An existence theorem is proved for solutions of differential inclusions with an upper semicontinuous and nonconvex right-hand side. The proof is based on an inner and directional continuous parameterization. This parameterization leads to a familie of disturbed differential inclusions. The solution of the starting differential inclusion is obtained as an uniformly limit of the solutions of disturbed systems. Some aspects of the existence of the above mentioned inner parameterization are discussed. A few examples are presented.

Key words: multi-function, differential inclusion, directional continuity, upper and low semi-continuity, measurability.

AMS (MOS) subject classification. 34A60.

1 Introduction

This work considers differential inclusions with an upper semicontinuous and nonconvex right-hand side. A set of conditions, named the **Z condition,** is presented in the first section. These conditions guarantee the existence of solutions for the differential inclusions with an upper semi-continuous (u.s.c.) right-hand side. This part is related to the ideas of many authors, see f.e. [1–2], [4], [6–9], [10–11], [14–16], [18], [20]. The second section consideres the fulfilment of the **Z condition** for the different multi-functions. The first part of the **Z condition** is equivalent to the existence of the fixed point of some map. In the differential equation system case it is a solution of the Euler's implicit scheme. The same holds for the Yosida approximation of maximal monotone operators [2]. The second part of the **Z condition** is an existence of a measurable selection of some map and this selection should be directionally continuous at the point where the scalar parameter vanish, see f.e. [6–8].

The last section contains examples which particulary describe the clearance between the sufficient and necessary conditions for the existence of the solutions of differential inclusions.

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2 Existence Theorem

Let $F(t,x)$ be a multi-function with compact values which is measurable in t and u.s.c. in x:

$$F(t,x) \quad : \quad D \longrightarrow K(\mathbb{R}^{n+1}), \tag{1}$$

where $\mathbb{R}^n$ is the Euclidean space, $D \subset \mathbb{R}^n$ is a domain (a connected set with nonempty interior) and $K(\mathbb{R}^n)$ is a metric space of nonempty compact subsets of $\mathbb{R}^n$. The metric of this space is the Hausdorff distance $h(F,G)$ between the compact sets F and G:

$$h(F,G) = \max\{\max_{u\in F}\min_{v\in G} \| u-v \|, \quad \max_{v\in G}\min_{u\in F} \| u-v \|\},$$

where $\| \cdot \|$ is the Euclidean norm.

We are going to consider the following differential inclusion:

$$\dot{x} \in F(t,x), \quad x(t_0) = x_0, \quad t \in [t_0, t_1]. \tag{2}$$

Definition 1 *Every absolutely continuous function $x(t)$ which almost everywhere in $[0,1]$ satisfies the differential inclusion (2) is said to be a solution.*

Definition 2 (see [7]) *Let Γ be a cone in $\mathbb{R}^m$ and let Y be a metric space. A map $f : \mathbb{R}^m \longrightarrow Y$ is Γ-continuous at a point $\bar{x} \in \mathbb{R}^m$ iff for every $\varepsilon > 0$ there exists $\delta > 0$ such that $d(f(x), f(\bar{x})) < \varepsilon$ for all $x \in B(\bar{x}, \delta) \cap (\bar{x} + \Gamma)$. We say that f is Γ-continuous on a set A if f is Γ-continuous at every point $\bar{x} \in A$.*

In this paper we are going to use a modification of the above definition (see [6], [7], [8]):

Definition 3 *The function $f(\cdot,\cdot,\cdot)$: $\mathbb{R}^2 \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ is said to be directionally continuous at $(0,t,x)$ with a constant M if for every $\varepsilon > 0$ there exists a positive number $\delta(\varepsilon, x) > 0$ for which*

$$\| f(0,t,x) - f(s,t,y) \| < \varepsilon \quad if \quad \| x - y \| < M\, s, \quad 0 \le s < \delta(\varepsilon, x).$$

Z condition:

1. For every $t \in [t_0, t_1]$ and $x \in intD$ there exists a positive number $s(t,x)$ and vectors $z(s,t,x)$, $s \in [0, s(t,x))$ for which the following inclusion

$$z(s,t,x) \in F(t, x + sz(s,t,x)) + \omega(s,t,x)B \tag{3}$$

holds, where $s(t,x) \ge \bar{s} > 0$, $t \in [t_0, t_1]$, $x \in D$, B is the unit ball centered at the origin, $\lim_{s\to 0} \omega(s,t,x) = 0$, and $\omega(s,t,x)$ is a nonnegative scalar function which is measurable in s and continuous in (t,x).

2. The multi-function $\{z| \quad z \in F(t, x+sz) + \omega(s,t,x)B\}$ has a selection $z(s,t,x)$ which is jointly measurable in (t,x) and directionally continuous at $(0,t,x)$ with a constant M.

3. $\| F(t,x) \| \le L < M$.

Note, that the first part of the **Z condition** is an implicit and approximate inclusion which guarantees the existence of the selection $z(s,t,x)$, defined at least on $[0,\bar{s}] \times D$. The second part of the **Z condition** is related to some natural properties like measurability and directional continuity. The last part of the **Z condition** guarantees the extension of the solutions of the differential inclusions closely to the bound of the domain $[t_0, t_1] \times D$.

First, under the **Z condition** we are going to prove an existence theorem for the differential inclusions with u.s.c. and compact right-hand side. Let us denote

$$Z(s,t,x) = \operatorname*{ess\,lim}_{y \to x} z(s,t,y),$$

where $u \in \operatorname*{ess\,lim}_{y\to x} z(s,t,y)$ if for every set $N \subset \mathbb{R}^n$ with Lebesgue's measure $\mu(N) = 0$ there exists a sequence $\{y_k\}_{k=1}^{\infty} \notin N$ for which $\lim_{k\to\infty} y_k = x$ and $\lim_{k\to\infty} z(s,t,y_k) = u$. Note, the multi-function $coZ(s,t,x)$ (co means convex hull) is the Filippov's extension of the right-hand side for the ordinary differential equations with a measurable right-hand side ([11], [12]). For any fixed s this function is u.s.c. in x, jointly measurable in (t,x), measurable in t and for any continuous function $x(t)$ the multi-function $Z(s,t,x(t))$ is measurable (see [11], [12]).

Theorem 1 *Let the measurable in t and upper semi-continuous in x multi-function $F(\cdot,\cdot)$ with compact values satisfy the* **Z condition**, *i.e.:*

For all sufficiently small positive numbers s there exists a measurable in (t,x) function

$$z(s,t,x) \in F(t, x + sz(s,t,x)) + \omega(s,t,x)B \tag{3}$$

where $\lim_{s\to 0} \omega(s,t,x) = +0$, B is the unit ball;

$z(s,t,x)$ is directionally continuous at $(0,t,x)$ with a constant M;

$\| F(t,x) \| \leq L < M$.

Let the following equality

$$z(s+\tau, t, x - \tau\, z(s,t,x)) = z(s,t,x), \quad s,\tau \geq 0 \tag{4}$$

be fulfilled for the selection $z(s,x)$ of (3)).

Then the differential inclusion

$$\dot{x} \in F(t,x), \quad x(t_0) = x_0, \quad t \in [t_0, t_1]. \tag{2}$$

has a solution which can be extended closely to the bound of the domain $[t_0, t_1] \times D$.

If the differential inclusion (2) locally has a solution then, by traditional methods, it can be continued up to the bound of the domain $[t_0,t_1] \times D$. As long as $z(0,t,x) \in F(t,x)$, the above theorem 1 immediately follows from:

Theorem 2 *Let the bounded function $z(s,t,x)$ ($\| z(s,t,x) \| \leq L$) be jointly measurable in (t,x) and directionally continuous at $(0,t,x)$ with a constant M (see (3)). Let $L < M$ and the following equality be fulfilled*

$$z(s+\tau, t, x - \tau\, z(s,t,x)) = z(s,t,x), \quad s,\tau \geq 0. \tag{4}$$

Then the Cauchy problem

$$\dot{x} = z(0,t,x), \quad x(t_0) = x_0 \tag{5}$$

locally has a solution.

Proof. For a fixed $s > 0$, let us denote

$$Z(s,t,x) = \operatorname*{ess\,lim}_{y \to x} z(s,t,y), \tag{7}$$

where u belongs to $\operatorname*{ess\,lim}_{y \to x} z(s,t,y)$ if for every set $N \subset \mathbb{R}^n$ with Lebesgue's measure $\mu(N) = 0$ there exists a sequence $\{y_k\}_{k=1}^{\infty} \notin N$ for which $\lim_{k\to\infty} y_k = x$ and $\lim_{k\to\infty} z(s,t,y_k) = u$. If the function $z(s,t,x)$ is jointly measurable in (s,t,x) then the multi-function $coZ(s,t,x)$ is the Filippov's extension of the right-hand side for the ordinary differential equations with a measurable right-hand side ([11], [12]). This function is u.s.c. in x, jointly measurable in (t,x), measurable in t and for any continuous function $x(t)$ the multi-function $Z(s,t,x(t))$ is measurable (see [11], [12]).

Consider the following differential inclusion:

$$\dot{x}(s,t) \in coZ(s,t,x(s,t)), \quad x(s,0) = x_0, \quad s > 0, \tag{8}$$

where $s > 0$ is a constant, $Z(s,t,x)$ is defined by (7) and co means the convex hull. It is wellknown that the differential inclusion (8) has a solution (see f.e. [11], [12]) which can be extended closely to the bound of the domain $[t_0,t_1] \times D$ (see [11]). As long as $x_0 \in D$ there exists $T > t_0$ for which the solutions $x(s,t)$, $s \geq 0$ of (8) are well defined on the interval $[t_0,T]$.

The derivative $\dot{x}(s,t)$ of the solutions (8) a.e. can be represented as follows (see [12]):

$$\dot{x}(s,t) = \sum_{k=1}^{n+1} \alpha_k(s,t) z_k(s,t), \quad s > 0, \tag{9}$$

where $z_k(s,t)$ and $\alpha_k(s,t)$ are measurable functions on $[0,T]$, and a.e. in t

$$z_k(s,t) \in Z(s,t,x(s,t)), \quad \alpha_k(s,t) \geq 0, \quad \sum_{k=1}^{n+1} \alpha_k(s,t) = 1, \quad k = 1,2,\ldots,(n+1). \tag{10}$$

For fixed t which satisfies (9) we are going to estimate $\| \dot{x}(s,t) - z(0,t,x(0,t)) \|$, where $x(s,\cdot)$ uniformly converges to $x(0,\cdot)$ on the interval $[t_0,T]$ when $s \to +0$. As far as $z(s,t,x)$ is a bounded function with a constant L, the set of solutions of (8) is conditionally compact in the space of continuous functions $C[t_0,T]$. Thus, we can choose subsequence which uniformly converges to some function $x(0,t)$.

Let $\varepsilon > 0$ be sufficiently small, for example $\varepsilon < \dfrac{M-L}{4}$. Let $\tau > 0$ be chosen under the **Z condition**, i.e.

$$\| z(\tau,t,y) - z(0,t,x(0,t)) \| \leq \varepsilon \quad \text{if} \quad \| y - x(0,t) \| < \tau M$$

As well as $z_k(s,t) \in Z(s,t,x(s,t))$, one can choose $y_k(s,t)$ which are sufficiently close to $x(s,t)$ such that $(i = k, j, i = 1,2,\ldots,(n+1))$

$$\| z_k(s,t) - z(s,t,y_k(s,t)) \| \leq \varepsilon. \tag{11}$$

If $s > 0$ is sufficiently small we have

$$\| y_k(s,t) - \tau\, z(s,t,y_k(s,t)) - y_j(0,t) \| < \tau\,(L+\varepsilon) < \tau\, M.$$

Under the directional continuity of $z(s,t,x)$ and (4) we obtain

$$\| z(s,t,y_k(s,t)) - z(0,t,x(0,t)) \| = \| z(s+\tau,t,y_k(s,t) - \tau\, z(s,t,y_k(s,t))) - z(0,t,x(0,t)) \| < \varepsilon \tag{12}$$

We can write

$$\dot{x}(s,t) - z(0,t,x(0,t)) = \sum_{k=1}^{n+1} \alpha_k(s,t)\big(z_k(s,t) - z(0,t,x(0,t))\big) =$$

$$\sum_{k=1}^{n+1} \alpha_k(s,t)\big(z_k(s,t) - z(s,t,y_k(s,t))\big) + \sum_{k=1}^{n+1} \alpha_k(s,t)\,\big(z(s,t,y_k(s,t)) - z(0,t,x(0,t))\big)$$

By (10), (11) and (12) we have:

$$\| \dot{x}(s,t) - z(0,t,x(0,t)) \| < 2\varepsilon. \tag{13}$$

By (13), on the contrary, we obtain that a.e. in $t \in [0,T]$

$$\lim_{s\to+0} \dot{x}(s,t) = z(0,x(0,t)).$$

According Lebesgue's theorem, limiting s to $+0$, we obtain

$$x(0,t) = \lim_{s\to+0} x(s,t) = x_0 + \int_0^t \lim_{s\to+0} \dot{x}(s,\xi)\, d\xi = x_0 + \int_0^t z(0,\xi,x(0,\xi))\, d\xi$$

which is equivalent to

$$\dot{x}(0,t) = z(0,t,x(0,t)), \quad x(t_0) = x_0, \quad t \in [t_0,T].$$

Thus, the function $x(0,t)$ a.e. satisfies (5). **Q.E.D.**

3 Inner Parameterization

In this section we are going to consider the **Z condition** for u.s.c. multi-functions with respect to some properties as continuity, convex valued and monotonisity.

Note, that the inclusion (3) generalizes the well-known implicit Euler's scheme for the numerical solving of the ordinary system of differential equations. The implicit Euler's scheme (f.e., for autonomous systems) is the following:

$$x((i+1)h) = x(ih) + hf(x((i+1)h)), \quad i = 0,1,2,\dots, \quad h > 0.$$

We are going to substitute $(x((i+1)h) - x(ih))/h$ by $z(h,x(ih))$ and rewrite the above equalities as

$$z(h,x(ih)) = f(x(ih) + hz(h,x(ih))), \quad i = 0,1,2,\dots.$$

These equalities are equivalent to the inclusion (3) if $F(x)$ is single-valued and $\omega(s,x) \equiv 0$.

Lemma 1 *Let $F(x)$ be u.s.c. multi-function with compact and convex values. Then for every sufficiently small $s \geq 0$ it satisfies the first part of the* **Z condition** *with $\omega(s,x) \equiv 0$, i.e. there exists a function*

$$z(s,x) \in F(x + sz(s,x)).$$

Proof. As long as $F(x)$ is u.s.c. with compact values, the restriction of $F(x)$ on every compact set is bounded. Without loss of generality, we can suppose that there exists a constant M and its respective ball S_M with a radius M centered in the origin for which $F(x) \subset S_M$ ($\| F(x) \| \leq M$) . Let us fix x and a neighborhood

$$U(x) = \{y \in \mathbb{R}^n | \; \| x - y \| < \varepsilon\},$$

where $0 < \varepsilon \leq 1$ is chosen arbitrarily.

For every $s \in [0, \varepsilon \backslash M]$, $y \in U(x)$ and $z \in S_1 = \{u \in R^n \; \| u \| \leq 1\}$ we have

$$sF(y+z) \subset sS_M \subset S_1$$

From the Kukutani fix-point theorem, f.e. [2], there exists $v(s,y) \in S_1$ for which $v(s,y) \in sF(y + v(s,y))$. Denoting $z(s,x) = v(s,y) \backslash s$ we obtain $z(s,x) \in F(x + sz(s,x))$. The statement of the lemma for $s = 0$ is trivial. **Q.E.D.**

Now, we are going to show that $z(s,x) = -F_s(x)$, where $F_s(x)$ is the Yosida approximation for the maximal monotone operator $F(x)$ satisfies (3) with $\omega(s,x) \equiv 0$ if $F(x)$ is changed to $-F(x)$.

Definition 4 [2]. *A multi-function $F(x)$ is said to be monotone, if for any arbitrarily chosen sequence of points $x_i, i = 1,2$ and $y_i \in F(x_i), i = 1,2$, the following inequality*

$$\langle x_1 - x_2, y_1 - y_2 \rangle \geq 0$$

holds.

Let the multi-function $F(\cdot)$ be defined on the set D. Then, the following set is the graph of this function:

$$graph F \quad = \quad \{(x,y) | \quad y \in F(x), \quad x \in D\}.$$

Definition 5 ([2]). *A multi-function $F(x)$ is said to be maximal monotone if there is no other monotone function $G(x)$ for which*

$$graph F \subset graph G$$

It is well-known that the maximal monotone maps are u.s.c. with convex values. For more details see [2]. From lemma 1 we obtain:

Corollary 1 *Let $F(x)$ be maximal monotone multi-function with compact values. Then the functions $-F(x)$ and $F(x)$ satisfy the first part of the* **Z condition** *with $\omega(s,x) \equiv 0$.*

Definition 6 [9]. *A multi-function $F(x)$ is said to be cyclically monotone, if for every finite number of points $x_i, i = 1,2,\ldots,k, x_1 = x_k$ and arbitrarily chosen $y_i \in F(x_i), i = 1,2,\ldots,k$, the following inequality*

$$\sum_{i=1}^{k-1} (x_{i+1} - x_i, y_i) \geq 0$$

holds.

Definition 7 f.c. [2]. *Let $F(x)$ be a maximal monotone multi-function then the maps*

$$J_s = (1+sF)^{-1} \quad \text{and} \quad F_s = (1-J_s)\backslash s$$

are said to be the resolvent and the Yosida approximation, respectively.

We need the following theorem:

Theorem 3 [2]. *Let F be a maximal monotone map. Then for all $s > 0$*

1. The resolvent $J_s = (1+F)^{-1}$ is a nonexpansive single-valued map

2. The Yosida approximation $F_s = (1-J_s)\backslash s$ satisfies the following conditions:

(i) $F_s \in F(J_s)$,

(ii) F_s is maximal monotone and it satisfies the Lipschitz condition with a constant $1\backslash s$.

3. Let $m(F(x)) = \operatorname{Arg\,min}_{y \in F(x)} \| y \|$.

Then

$$\| F_s(x) - m(F(x)) \|^2 \leq \| m(F(x)) \|^2 - \| F_s(x) \|^2$$

and

(i) $J_s(x)$ converges to x,

(ii) $F_s(x)$ converges to $m(F(x))$.

Setting $z(s,x) = -F_s(x)$, by the above theorem (part 2) we obtain

$$z(s,x) \in -F(J_s) = -F(x - sF_s(x)) = -F(x + sz(s,x)).$$

If the above inclusion has an unique solution $z(s,x)$ then it coincides with the Yosida approximation $F_s(x)$.

Lemma 2 *Let $F(x)$ u.s.c., bounded and cyclically monotone multi-function with compact and convex values, defined on $\mathbb{R}^n$. Then it satisfies the first part of the* **Z condition** *with $\omega(s,x) \equiv 0$.*

Proof. According to Bresis [5] a multi-function $F(x)$ is cyclically monotone if and only if there exists a proper real, convex and lower semicontinuous function $V(x) : R^n \to R$ such that $F(x) \subset \partial V(x)$, where ∂V is a sub differential of V. Since $F(x)$ is bounded the same holds for $\partial V(x)$ ([1], [9]). According to Zarantonello [19] $\partial V(x)$ is single-valued almost everywhere.

Let us define the following scalar function:

$$V_s(x) = \min_{z \in \mathbb{R}^n} \left[\frac{s}{2} \| z \|^2 - V(x+sz) \right].$$

Under the well-known Weierstrass theorem, the above minimum exists. In fact, suppose that M is the constant for which $\| \partial V(x) \| \leq M$ we obtain:

$$V(x) - V(x+sz) \geq \max_{y \in \partial V(x+sz)} \langle y, sz \rangle > -sM \| z \|$$

$$\text{or} \quad \frac{s}{2} \| z \|^2 - V(x+sz) \geq \frac{s}{2} \| z \|^2 - sM \| z \| - V(x).$$

Obviously, for any fixed s and x, we have

$$\frac{s}{2} \parallel z \parallel^2 - sM \parallel z \parallel - V(x) \geq \frac{s}{2} M^2 - V(x) > -\infty \quad \text{and}$$

$$\lim_{\|z\| \to \infty} \frac{s}{2} \parallel z \parallel^2 - sM \parallel z \parallel - V(x) = \infty.$$

These relations imply that the Lebesgue sets are compact and the minimum exists. Let us fix as $z(s,x)$ any point which minimizes the function $\frac{s}{2} \parallel z \parallel^2 - V(x+sz)$. As well as $V(x+sz)$ is a convex function in z, for $z(s,x)$ we obtain

$$0 \quad \in \quad sz(s,x) - s\partial V(x + sz(s,x)) \quad \text{or} \quad z(s,x) \quad \in \quad \partial V(x + sz(s,x)).$$

Thus, the subdifferential of the convex function satisfies the first part of the **Z condition.**

Note, that for the directional derivative of the convex function we have:

$$\frac{d}{dy} V(x) = \lim_{t \to} \frac{V(x+ty) - V(x)}{t} = \max_{v \in \partial V(x)} \langle v, y \rangle.$$

Now we obtain:

$$0 \leq \frac{d}{d \pm . z(s,x)} [\frac{s}{2} \parallel z(s,x) \parallel^2 - V(x + sz(s,x))] =$$

$$\pm s \parallel z(s,x) \parallel^2 - s \max_{v \in \partial V(x+z(s,x))} \langle v, \pm z(s,x) \rangle.$$

Considering that $z(s,x) \in \partial V(x + sz(s,x))$, the above inequalities imply that

$$\parallel z(s,x) \parallel^2 = \max_{v \in \partial V(x+z(s,x))} \langle v, z(s,x) \rangle = \min_{v \in \partial V(x+z(s,x))} \langle v, z(s,x) \rangle.$$

Suppose that $\partial V(x + z(s,x))$ consisting of a point z_1 which is different from $z(s,x)$, i.e. $z_1 = z(s,x) + z^\perp$, where the scalar product $\langle z(s,x), z^\perp \rangle$ is equal to zero, $z^\perp \neq 0$. From the directional derivative we obtain:

$$0 \leq \frac{d}{dz^\perp} V(x + sz(s,x)) = s\langle z(s,x), z^\perp \rangle - s \max_{v \in \partial V(x+z(s,x))} \langle v, z^\perp \rangle \leq$$

$$-s\langle z_1, z^\perp \rangle = -s \parallel z^\perp \parallel^2 < 0.$$

This contradiction implies that $\partial V(x + z(s,x))$ consists of only one point and obviously the first part of the **Z condition** is satisfied. **Q.E.D.**

Without any proof we are going to formulate the following characterization of the cyclically monotone and u.s.c. maps $F(x) : \mathbb{R}^n \to K(\mathbb{R}^n)$. Let $A \subset \mathbb{R}^n$ be a compact and convex set. The point v is said to be an extreme point of A if it does not belong to the interior of some interval which is consisted in A.

Proposition 1 *Every cyclically monotone and u.s.c. maps $F(x) : \mathbb{R}^n \to K(\mathbb{R}^n)$ consists of the extreme points of the values of the subdifferential of the some real convex function, defined on $\mathbb{R}^n$.*

We need the following theorem:

Theorem 4 [7]. *Let F be a lower semi-continuous map with nonempty closed values, from $\mathbb{R}^m$ into a complete metric space Y. Then, for every cone $\Gamma \subseteq \mathbb{R}^m$, F admits a Γ-continuous selection.*

Lemma 3 *Let the continuous and compact valued multi-function $F(x)$ be defined on the bounded domain D. Then it satisfies the first and the second parts of the* **Z condition**.

Proof. Let us denote by $\omega(s,x)$ the following modulus of continuity of $F(x)$ at the point x:

$$\omega(s,x) = \sup_{\|x-y\|<sM} h(F(x), F(y)),$$

where M is a constant which bounds the function $F(\cdot)$ in the domain D, $h(F(x), F(y))$ is the Hausdorff distance between the compact sets $F(x)$ and $F(y)$.

As long as $F(\cdot)$ is continuous, the modulus of continuity is a continuous function in (s,x) and $\omega(0,x) = 0$.

Let $f(s,x)$ be a directionally continuous selection of $F(x)$. Therefore, this selection exists under the theorem 3 [7].

According to [7] $f(s,x)$ is jointly measurable in (s,x).

As well as $f(s,x) \in F(x)$ and $\rho(f(s,x), F(x + sf(s,x))) \leq h(F(x), F(x + sf(s,x))) \leq \omega(s,x)$, setting $z(s,x) = f(s,x)$ we obtain

$$z(s,x) \in F(x + sz(s,x)) + \omega(s,x)B.$$

Q.E.D.

4 Examples

In this section we are going to consider examples which show that the **Z condition** is sufficiently near to the necessary conditions for the existence of the solutions of the differential inclusion (2).

Note that the following well-known differential inclusion

$$\dot{x} \in -F(x) = -\begin{cases} -1, & \text{if } x \geq 0, \\ 1, & \text{if } x \leq 0, \end{cases}$$

does not have a solution for the initial position $x(0) = 0$. In this case the first part of the **Z condition** is not fulfilled.

Let us consider the convex function $V(x) = |x|$. For the subdifferential $\partial V(x)$ which is a maximal monotone operator, we have $F(x) \subset \partial V(x)$. $F(x)$ is a monotone u.s.c. multi-function and, under the lemma 2, $F(x)$ satisfies the first part of the **Z condition**. The following three differential inclusions with $x(0) = 0$

$$\dot{x} \in \pm\partial V(x)$$

$$\dot{x} \in F(x) = \begin{cases} -1, & \text{if } x \geq 0, \\ 1, & \text{if } x \leq 0, \end{cases}$$

admit solutions (f.e. see [1], [2] and [9]).

Example 1.

The following example shows that the inclusions (3) could not be valid for the continuous multi-functions if $\omega(s,x)\equiv 0$:

Let the domain D be the unit ball in $\mathbb{R}^2$. Define the continuous multi-function $F(x)$ with values which are subsets of the unit circle in the following way:

$$F(\rho\cos\alpha,\rho\sin\alpha)=\{(\cos\beta,\sin\beta)|\beta\in[0,2\pi]\setminus(\alpha-\rho,\alpha+\rho)\}.$$

Obviously, $F(\cdot,\cdot)=F(x)$ is a continuous multi-function but the beam $\{\lambda x|\lambda>0\}$ does not meet $F(x)$.

Example 2. [13].

The following example of an u.s.c. multi-function with compact and nonconvex values satisfies the first part of the **Z condition**, but not the second one. The respective differential inclusion does not admit a solution.

Let us define the following multi-function with a finite number of values:

$$F(x)=\begin{cases} -1, & \text{if } x\in[2^{-2k-1},2^{-2k}],\\ \{-1;1\}, & \text{if } x\in[2^{-2k-2},2^{-2k-1}]\cup[-2^{-2k},-2^{-2k-1}],\\ 1, & \text{if } x\in[-2^{-2k-1},-2^{-2k-2}],\end{cases}$$

$F(0)=\{-1;1\}, k=1,2,\ldots$. For every sufficiently small $s\geq 0$ we have:

$$Z(s,x)=\begin{cases} -1, & \text{if } x\in(2^{-2k-2},2^{-2k}]\cup(-2^{-2k},-2^{-2k-1}],\\ \{-1;1\}, & \text{if } x=0,\quad s=0,\\ 1, & \text{if } x\in[-2^{-2k},-2^{-2k-2})\cup[2^{-2k-2},2^{-2k-1}).\end{cases}$$

and

$$Z(s,0)=\begin{cases} 1, & \text{if } s\in[2^{-2k-2},2^{-2k-1}),\\ \{-1;1\}, & \text{if } s=0,\\ -1, & \text{if } s\in[2^{-2k-1},2^{-2k}).\end{cases}$$

There is no selection $z(s,0)\in Z(s,0)$ continuous in $s=+0$.

Consider the following differential inclusion

$$\dot{x}\in F(x),\quad x(0)=0.$$

The existence of a solution $x(t)$ implies the existence of a positive moment of the time $T>0$ for which we have $x(T)>0$ or $x(T)<0$. The solution $x(t)$ is an absolutely continuous function. Thus, for all sufficiently large numbers k there exists some t for which

$$x(T)>x(t)=2^{-2k-2}\quad\text{or}\quad\text{respectively}\quad x(T)<x(t)=-2^{-2k-2}.$$

We have

$$F(x)=-1\quad if\quad x\in(2^{-2k-1},2^{-2k})\quad or$$

$$F(x)=1\quad if\quad x\in(-2^{-2k-1},-2^{-2k-2}).$$

Thus, the trajectory $x(t)$ cannot run across points 2^{-2k-1} and -2^{-2k-2}. This implies that $x(t)$ cannot leave 0 and the above differential inclusion has no solutions.

Example 3.

We are going to consider an example for which the function $z(\cdot, x)$ is discontinuous for u.s.c. multi-functions with compact values.

Let us denote *conv* $\Omega(\mathbb{R}^n)$ all nonempty, compact and convex subsets of $\mathbb{R}^n$.

Theorem 5 ([17]) *There exist upper semi-continuous and almost everywhere in $t \in [0,1]$ discontinuous multi-valued maps $F : [0,1] \to$ conv $\Omega(\mathbb{R}^n)$, for which every single-valued selection is discontinuous on a set with a full measure.*

Theorem 6 ([17]) *For every set $E \subset [0,1]$ of the type F_σ and the first Bare category there exists an upper semi-continuous multi-valued map $F : [0,1] \to$ conv $\Omega(\mathbb{R}^n)$ for which the points of a discontinuity coincide with E. All single-valued selections of the map $F(t)$ are discontinuous on the set E.*

Under the theorem 4 there exists a multi-function $G(\cdot) : [0,1] \to$ *conv* $\Omega(\mathbb{R}^1)$ for which every single-valued selection is a.e. discontinuous. Denote

$$F(x) = F(x_1, x_2) : \mathbb{R}^2 \to (x_1, G(x_1)).$$

For the function $z(s,x) = (z_1(s,x), z_2(s,x))$ we have $z_1(s,x) = x_1 + s z_1(s,x)$ and $z_2(s,x) \in G(x_1 + s z_1(s,x))$. Thus,

$$z_1(s,x) = \frac{x_1}{1-s} \quad \text{and} \quad z_2(s,x) \in G(\frac{x_1}{1-s})$$

and for all fixed x_1 the function $z_2(s,x)$ is a.e. in s discontinuous.

If the map $G(\cdot)$ is chosen under the theorem 5 and x_2 belongs to its set of the discontinuity then the function $z(s,x)$ is discontinuous at the point $(0,x)$.

Example 4.

Let $f_1(x)$ and $f_2(x)$ be two continuous scalar functions which are defined on the real line $(-\infty, +\infty)$. Using these two functions one can construct some lower semicontinuous single-valued and left-hand side continuous function $f(x)$ for which

$$f(x) \in f_1(x) \cup f_2(x).$$

Suppose

$$|f_i(x)| \le L < 1, \quad i = 1, 2. \tag{13}$$

Finally, denote

$$F(x) = \operatorname{Lim\,sup}_{y \to x} f(y),$$

where Lim sup is the Kuratowski upper limit (see f.e. [3]). Applying theorem 1 (without presenting the proof) the following differential inclusion

$$\dot{x} \in F(x), \quad x(0) = x_0$$

has a solution.

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