№. . . . . . .

ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

A Boundary Value Problem with a Finite Number of Impulses

R. P. Ivanov
P. M. Kitanov

БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ

1869

BULGARIAN ACADEMY OF SCIENCES

Preprint

September 1994
No 5

Department of Operations Research

A Boundary Value Problem with a Finite Number of Impulses *

R. P. Ivanov, P. M. Kitanov

Abstract

This work considers Impulsive differential systems with boundary conditions. The characteristic feature of the impulsive boundary value problems is the unknown impulse moment. The solvability of the differential systems without impulses does not imply solution for the corresponding impulsive problem. Another specialty of the impulsive boundary value problem is that the set of solutions may not be a closed set.

The problem is considered as a special logically controlled impulsive boundary value problem.

The control may choose to use an optional impulse on the surface S. It is proved that the set of solution of the above mentioned controlled problem with a fixed and finite number of impulses is a closed set. If the boundary conditions describe a compact set then the set of impulsive solutions is a compact set too. The sufficient conditions for the existence of solutions of single impulse linear boundary value problem are presented. An example is considered.

Key words: impulsive differential system, impulsive boundary value problem, Cauchy problem, Stieltjes integral, Caratheodory conditions.

AMS (MOS) subject classification. 34A60.

1 Introduction

In this paper we are going to consider an impulsive differential equation with boundary conditions. The boundary value problem for ordinary systems of differential equations appears in some physical problems, in optimal control theory, etc. There are many different results and solving methods for the boundary value problem (see f.e. [2], [4], [6], [7]).

The specificity of the impulsive boundary value problem dues to the unknown moments of impulses. The existence of solutions is the main problem. Namely, the existence of solutions of the corresponding non-impulsive problem does not, by any means, guarantee the existence of solutions of the impulsive problem.

Two formalizations of the problem are considered as follows:

*This work was partially supported by grants of the Bulgarian Ministry of Science and Education. Thanks to Operation Research Scientific Seminar of Mathematical Institute, Bulgarian Academy of Sciences.

1. Controlled impulsive boundary value problem where the control chooses to use an optional impulse on a surface.

2. Impulsive boundary value problem with not more then N^* impulses at the first $N \leq N^*$ moments when the trajectory is on a given surface.

In this paper we prove that the solution set for a controlled problem with a finite number of impulses (under some sufficient conditions), is a closed set. If the boundary conditions describe a compact set then the set of impulsive solutions is also compact. We consider the simplest case of a linear impulsive boundary value problem with only one impulse on a hyperplane and reduce it to the Cauchy problem with a fixed impulsive time moment. A theorem about uniqueness of the solution is proved. An example is presented.

2 Statement of the Problem

An impulsive differential system may be defined, as follows (see [5])

$$\dot{x} = f(t,x), \quad h(t,x) \neq 0, \quad t \in R, \quad x \in R^n, \tag{1}$$

$$\Delta x = I(t,x), \quad h(t,x) = 0, \quad t \in [t_0,T], \quad x \in R^n \quad 0 < \| I(t,x) \| \leq M < \infty. \tag{2}$$

Where $f : R \times R^n, \longmapsto R^n, I : R \times R^n \longmapsto R^n$. The equation $h(t,x) = 0$ defines a surface S in the space R^{n+1} and Δx is the impulsive function, i.e. the trajectory $x(t)$ jumps on the surface $h(t,x) = 0$. If $h(t,x(t)) = 0$ then

$$\Delta x(t) = x(t+0) - x(t) \quad \text{and} \quad x(t+0) = \lim_{h \to +0} x(t+h)$$

We are going to consider the boundary value problem for the impulsive differential system (1) - (2). Let us fix the time interval as $[t_0,T]$ and let the solutions of (1) - (2) satisfy the following linear boundary condition:

$$\int_{t_0}^{T} [\, d\,\Phi(t)\,]\, x(t) \;=\; a, \tag{3}$$

where $a \in R^m$, $\Phi(t)$ is a matrix $(n \times m)$ of functions with bounded variations on the interval $[t_0,T]$ and the left-hand side of (3) is the Stieltjes integral. In this paper we are going to consider the case when the measure $[d\Phi(t)]$ is supported at the finite number of moments $\tau_0, \tau_1, \ldots, \tau_k = T$. The condition (3) transforms to

$$\exists \;\; y(\tau_j) \in x(\tau_j), \;\; \sum_{j=1}^{k} \varphi_{ij} y(\tau_j) = a_i, \quad j = 1,\ldots,k, \quad i = 1,\ldots,m. \tag{4}$$

Additionally, we suppose that we have a finite number of possible jumps Δx of the trajectory on the surface $h(t,x) = 0$. We suppose that a subject can decide to be an impulse on the surface $h(t,x) = 0$ or does not happened. This special (logic) control is restricted by the minimal and maximal numbers of impulses. Denoting N the number of possible impulses we require:

$$0 \leq N_1 \leq N \leq N_2 < \infty, \tag{5}$$

where N_1 and N_2 are the minimal and maximal number of impulses respectively.

Definition 1 *A multi-function $x(t)$ is said to be a solution of the controlled impulsive boundary value problem if it satisfies (1) - (3), it is single-valued and absolutely continuous at every interval (t_i, t_{i+1}) and*

$$\begin{aligned} &h(t_i, x(t_i - 0)) = 0, \\ &h(t, x(t)) \neq 0, \quad t \in (t_i, t_{i+1}), \\ &h(t_{i+1}, x(t_{i+1} - 0)) = 0, \\ &t_0 \leq t_i < t_{i+1} \leq T, \\ &i = 1, 2, \ldots, N, \quad N_1 \leq N \leq N_2, \\ &x(t) = x(t-0) \cup x(t+0). \end{aligned}$$

Let us consider the following linear impulsive system:

$$\dot{x} = C(t)x + f(t) \quad t_0 \leq t \leq T, \tag{6}$$

$$x(t+0) = x(t-0) + d, \quad \langle c, x(t-0) \rangle = \alpha \tag{7}$$

where $S = \{x \in R^n | \langle c, x \rangle = \alpha\}$ and t is the first moment when $x(t) \in S$ and the trajectory obligatory undergoes a single impulse, i.e. $N_1 = N_2 = 1$. Let the boundary conditions have the following form:

$$Ax(t_0) = a \quad \text{and} \quad Bx(T) = b. \tag{8}$$

We suppose that A, B, C are matrices $(n \times n)$, x, f, a, b, c, d are n-dimensional vectors and α is a number, $C(\cdot)$ and $f(\cdot)$ belongs to $L_1[t_0, T]$.

As well as the single impulse is obligatory at the first moment t for which $x(t) \in S$ the statement of the above linear problem is different from the statement of the controlled impulsive boundary value problem.

In this paper we suppose that the following condition (i) is fulfilled

$$(i) \qquad x(t+0) \notin S \quad \text{if} \quad x(t-0) \in S.$$

It is easy to check that the condition (4) generalizes the condition (8) and that the condition (i) transforms to the condition $\langle c, d \rangle \neq 0$.

3 Main Result

Theorem 1 *Let us consider the controlled impulse boundary value problem (1) - (5) with the condition (i) for which the measure $[d\Phi(t)]$ is supported at the finite number k of fixed moments $\tau_i, i = 1, 2, \ldots, k$. Let the function $I(t, x)$, which describes impulses, be continuous. Suppose that $f(t, x) : R \times R^n \longmapsto R^n$ is the Caratheodory function (measurable in t for every x, continuous in x,*

$$\| f(t, x) \| \leq m(t)(1 + \| x \|), \quad \int_{t_0}^{T} m(t)\, dt < \infty).$$

Then, the intersection of the hyperplane $t = const.$ and the solutions set for the considered problem is a closed subset of R^n.

Proof. The proof is trivial for the case of an empty solution set.

We need the following

Lemma 1 ([1], p.10). *Let $x_i(t)\,(\alpha_i \le t \le \beta_i, i = 1, 2, \ldots)$ be solutions of the Caratheodory equation which graphs belong to the closed and bounded domain $D \subset R \times R^n$, and*

$$(\alpha_i, x_i(\alpha_i)) = p_i \to p = (\alpha, x_0), \quad (\beta_i, x_i(\beta_i)) = q_i \to q = (\beta, x^*).$$

Then there is a subsequence of solutions which converges to a solution for which:

1. *Its graph connects the two points $p = (\alpha, x_0)$ and $q = (\beta, x^*)$,*
2. *Its graph belongs to D when $\alpha \le t \le \beta$,*
3. *For every $\delta > 0$, the convergence is uniform on the interval $[\alpha + \delta, \beta - \delta]$.*

Under the Caratheodory conditions and lemma 1 (see f.e. [1], [2]) one can prove the following corollary.

Corollary 1 *1. Let the conditions of lemma 1 be fulfilled and $D = R^n$. Then the statement of lemma 1 (point 3) is valid with $\delta = 0$.*

2. Let the right-hand side of the differential equation satisfy the Caratheodory conditions and the set of initial positions be a compact set. Then the set of solutions of the Cauchy problem is a compact set in the space of continuous functions.

Let $x_i(t), i = 1, 2, \ldots$ be a sequence of solutions for which

$$x_i(t_i^j - 0) \in S, \quad j = 1, \ldots, N, \quad \sum_{j=1}^{k} \varphi_{sj} y_i(\tau_j) = a_s, \quad s = 1, \ldots, m, \quad i = 1, 2, \ldots.$$

Let us fix $t \in [t_0, T]$ and let $\lim_{i \to \infty} x_i(t) = x(t)$.

If the trajectories $x_i(t), i = 1, 2, \ldots$ do not have impulses on the interval $t_1 < t < t_2$ then by lemma 1 (corollary 1) the sequence $x_i(\cdot), i = 1, 2, \ldots$ is compact on this interval in the space of continuous functions $C[t_1, t_2]$. Now we can extend this interval up to the moment for which either $t_1 = t_0$ or the trajectories $x_i(\cdot)$ have impulses on the interval $(t_1 - \delta, t_1 + \delta)$, $\delta > 0$. We can extend the right bound t_2 up to the moment for which either $t_2 = T$ or the trajectories $x_i(\cdot)$ have impulses on the interval $(t_2 - \delta, t_2 + \delta)$, $\delta > 0$.

Let the trajectories $x_i(t), i = 1, 2, \ldots$ have impulses at moments t_i, $t = \lim_{i \to \infty} t_i$. As long as $x_i(t_i)$ is multi-valued we can choose a subsequence (we don't change the numeration) for which $\lim_{i \to \infty} x_i(t_i + 0) = x(t)$ if $t_i \le t$ or $\lim_{i \to \infty} x_i(t_i - 0) = x(t)$ if $t_i \ge t$. In the first case we put $x(t + 0) = x(t)$ and, respectively, $x(t - 0) = x(t)$ - in the second case.

If $\lim_{i \to \infty} x_i(t_i + 0) = x(t)$ then by (2) we have

$$\| x_i(t_i - 0) \| = \| x_i(t_i + 0) - I(t_i, x_i(t_i - 0)) \| \le \| x_i(t_i + 0) \| + M.$$

Thus we can choose a subsequence $x_i(t_i - 0), i = 1, 2, \ldots$ for which $\lim_{i \to \infty} x_i(t_i - 0) = x(t - 0)$. As long as $x_i(t_i - 0) \in S$ and $h(\cdot, \cdot)$ is a continuous function we obtain that $x(t - 0) \in S$ which implies that $x(\cdot)$ has an impulse at the point t.

If $\lim_{i\to\infty} x_i(t_i-0) = x(t)$ then $x(t-0)=x(t)\in S$ is well defined. By the continuity of $I(\cdot,\cdot)$ we obtain

$$x(t+0) = \lim_{i\to\infty} x_i(t_i+0).$$

Under corollary 1 we obtain that there exists a subsequence $x_i(\cdot), i = 1,2,\ldots$ which uniformly converges to $x(\cdot)$ on every interval $[t_1+\delta, t_2-\delta]$, where $\delta > 0$ is arbitrarily chosen and

$$\lim_{i\to\infty} x_i(t_i+0) = x(t_1+0), \quad \lim_{j\to\infty} x_j(t_j-0) = x(t_2-0), \quad \lim_{i\to\infty} t_i = t_1, \quad \lim_{j\to\infty} t_j = t_2.$$

Thus, we choose not more then $N+1$ subsequences and as long as every trajectory $x_i(\cdot)$ has exactly N impulses the same holds for $x(\cdot)$.

It is easy to check that the condition (4) is fulfilled for the obtained trajectory $x(\cdot)$.

Q.E.D.

Corollary 2 *If the boundary conditions (4) describe a compact subset of R^n then, under the conditions of the above theorem, the intersection of the hyperplane $t = const.$ and the solutions set for which the number of impulses is really N, is a compact subset of R^n.*

We are going to consider the boundary value problem (6) - (8) with the number of possible impulses equal to one.

Under the above proved theorem we have

Corollary 3 *The intersection of the hyperplane $t = const.$ and the solutions set of the controlled impulse boundary problem with single impulse ($N_1 = N_2 = 1$) for the linear system (6) - (8) is a closed subset of R^n.*

If $X(t,s)$ is the fundamental matrix of the solution for the system

$$\dot{x} = C(t)x$$

then the solution of system (6) can be represented using the Cauchy formula

$$x(t) = y + \int_{t_0}^{t} X(t,s)f(s)\,ds, \quad t_0 \le t \le T$$

where y is some initial position (see f.e. [2]). We are going to modify the transformation of the boundary value problem (6), (8) (without impulses) to a Cauchy problem. We denote the determinant of the matrix A by $det A$.

Theorem 2 *Let us consider the impulse boundary problem with single impulse for the linear system (6) - (8). Let the single impulse be obligatory at the first moment t^* for which $\langle c, x(t^*)\rangle = \alpha$. If $det(A+B)\neq 0$ and the following equation:*

$$\langle c, y(t^*) + \int_{t_0}^{t^*} X(t^*,s)f(s)\,ds\rangle = \alpha,$$

where

$$y(t^*) = (A+B)^{-1}\Big[a+b-B\,d - \int_{t_0}^{t^*} B\,X(t^*,s)f(s)\,ds - \int_{t^*}^{T} B\,X(T,s)f(s)\,ds\Big],$$

has a solution t^* on the interval $[t_0, T]$ then the impulse boundary value problem (6) - (8) has an unique solution. This solution can be obtained as a solution of the impulse Cauchy problem with the impulse at t^* and the initial position

$$y = (A+B)^{-1}\Big[a + b - B\,d - \int_{t_0}^{t^*} B\,X(t^*,s)f(s)\,ds - \int_{t^*}^{T} B\,X(T,s)f(s)\,ds\Big].$$

Proof. Consider the following system of equations:

$$\begin{aligned}
&A\,y = a\\
&\Big\langle c, x(t^*-0)\Big\rangle = \Big\langle c, y + \textstyle\int_{t_0}^{t^*} X(t^*,s)\,f(s)\,ds\Big\rangle = \alpha\\
&B\left[x(t^*+0) + \textstyle\int_{t^*}^{T} X(T,s)f(s)\,ds\right] = B\left[x(t^*-0) + d + \textstyle\int_{t^*}^{T} X(T,s)f(s)\,ds\right] = b
\end{aligned}$$

This system linearly depends on the unknown variable y and it is easy to obtain

$$(A+B)\,y = a + b - B\,d - \int_{t_0}^{t^*} B\,X(t^*,s)f(s)\,ds - \int_{t^*}^{T} B\,X(T,s)f(s)\,ds.$$

As long as $det(A+B) \neq 0$ there exists an unique solution $y(t^*)$ of the above linear system. By the conditions of the theorem there exists a minimal $t^* \in [t_0, T]$ for which

$$\langle c, y(t^*) + \int_{t_0}^{t^*} X(t^*,s)f(s)\,ds\rangle = \alpha.$$

Q.E.D.

4 Example

The following example shows that, in general, the set of the impulsive boundary value problem solutions should not be a closed set. We are going to consider the case where the impulses are obligatory at the moment t for which the trajectory $x(t)$ belongs to the surface $S = \{(t,x) \in R \times R^n \mid \quad h(t,x) = 0\}$. The following problem with not more then two impulses is in a plane.

$$\begin{aligned}
&\dot{x}_1 = 1\\
&\dot{x}_2 = \cos(t)\\
&h(t, x_1, x_2) = x_2 - \tfrac{1}{2}\\
&I(t, x_1, x_2) = I(t, x_1, \tfrac{1}{2}) =\\
&(*) \qquad (-x_1 + t, \quad -|x_1 - \tfrac{\pi}{6}| - \tfrac{1}{2}) \quad \text{or}\\
&(**) \quad (-x_1 + \tfrac{\pi}{6}, \quad -|x_1 - \tfrac{\pi}{6}| - \tfrac{1}{2})\\
&x_2(0) = 0, \quad x_1(\pi) = \pi, \quad t \in [0, \pi] \quad N = 2.
\end{aligned}$$

Supposing that $y_1 \in R$ is an arbitrarily chosen initial position for $x_1(0)$, we obtain the following solution:

$$x_1(t) = \begin{cases} y_1 + t, & \text{if} \quad y_1 \in R \setminus 0, \quad 0 \le t \le \frac{\pi}{6},\\ t \;, & \text{if} \quad \frac{\pi}{6} \le t \le \pi, \end{cases}$$

$$x_2(t) = \begin{cases} \sin(t) \quad , & \text{if} \quad 0 \le t \le \frac{\pi}{6}, \\ -|y_1| + \sin(t) - \frac{1}{2}, & \text{if} \quad \frac{\pi}{6} \le t \le \pi. \end{cases}$$

These solutions have only one impulse and if $y_1 = 0$ we obtain the following solution with two impulses in the case $(*)$:

$$x_1(t) = t, \quad \text{if} \quad 0 \le t \le \pi,$$

$$x_2(t) = \begin{cases} \sin(t) \quad , & \text{if} \quad 0 \le t \le \frac{\pi}{6}, \\ \sin(t) - \frac{1}{2}, & \text{if} \quad \frac{\pi}{6} \le t \le \frac{\pi}{2}, \\ \sin(t) - 1 - \frac{\pi}{3}, & \text{if} \quad \frac{\pi}{2} \le t \le \pi. \end{cases}$$

Thus, the point-wise limit of the solution with one impulse can be a solution with two impulses. It easy to check that in the case $(**)$ the trajectory with an initial position $x_1(0) = 0$ has to have two impulses. This trajectory does not satisfy the boundary conditions, i.e. the solutions set with really two impulses is empty. Thus, in the case where the impulses are obligatory, the set of all solutions may not be a closed set.

References

[1] A.F. Filippov, *Differential equations with discontinues right hand side*, Moscow, Nauka, (1985) (in Russian).

[2] Hartman F., *Ordinary Differential Equations*, Moscow, Mir, (1970) (in Russian).

[3] Kamke E., Zur Theorie der Systeme Gewönlicher Differentialgleichungen, *Acta Math.*, v. 58, 1, p. 57–85 (1932).

[4] Krasovsky N. N., *Theory of Motion Control*, Moscow, Nauka, (1968) (in Russian).

[5] Lakshmikantham V., Bainov D., Simeonov P., *Theory of Impulsive Differential Equations*, World Scientific, Singapore, New Jersey, London, Hong Kong, (1989).

[6] Pontryagin L. S., Boltyansky V. G., Gamkrelidze R. V., Mishtenko E. F., *Mathematical Theory of Optimal Processes*, Moscow, Nauka, (1969) (in Russian).

[7] Ralston, A. A First Course in Numerical Analysis. McGraw-Hill Book Company, New York, 1965.