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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР

INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

A Refinment of a Local Limit Theorem for a Branching Process Conditioned on the Total Progeny

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A Refinment of a Local Limit Theorem for a Branching Process Conditioned on the Total Progeny

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Abstract

Let N be the total progeny of a Bienaymé - Galton - Watson process $\{Z_t\}$, i.e. $N = \sum_{t=0}^{\infty} Z_t$. A local limit theorem for the distribution of Z_t , conditioned on the event $\{N = n\}$ as $n, t \to \infty$, $t^2 n^{-1} \to 0$ is proved . A corollary for a random rooted labeled tree is obtained .

branching process ; total progeny ; local limit theorem ; random trees ;

1 Introduction

Let us on the probability space $(\Omega, \mathcal{F}, P)$ define $\xi = \{\xi_i(t)\}$, $i, t = 0, 1, 2, \ldots$ - independent, identically distributed (i.i.d.) random variables taking non-negative integer values. The Bienaymé - Galton - Watson process (BGWP), starting with a single particle could be defined as follows

$$Z_{t+1} = \sum_{i=1}^{Z_t} \xi_i(t) \quad , \quad t = 0, 1, 2, \ldots \quad ; \quad Z_0 = 1 \text{ a.s.} \quad .$$

Usually Z_t is interpreted as the number of particles existing in the moment $t = 0, 1, 2, \ldots$ and the sum N of all Z_t - the total progeny of the process. The moment when Z_t becomes zero is called the moment of extinction and the distribution of ξ - offspring distribution of one particle .

In general three cases are considered, according to the mean of the offspring distribution - when it is less then one (the subcritical case), when it is one (the critical case) and when it is greater then one (the supercritical case). For the process, conditioned on non-extinction various limit theorems, depending on the criticality, are obtained .

The first proof of a local limit theorem for the BGWP, conditioned on non-extinction is attributed to Smirnov but it is not published. Chistyakov (1957) has given a proof for the continious time process. Later Kesten, Ney and Spitzer (1966) have proved the theorem under the assumption $E\xi^2 \log(1+\xi) < \infty$. To our regret a proof, based on finite second moment only has apparently not yet been found.

The asymptotic behavior of the BGWP, conditioned on the event $\{N = n\}$ as $n \to \infty$ has been thoroughly investigated by Kennedy (1975). This is analogous to the standard device in branching processes of conditioning on non-extinction. In the critical case the two methods of conditioning yield similar results, however, in the non-critical cases the obtained results are very different. It has been shown that conditioning on the total progeny for the suitably normalized BGWP yields limit results of the same form for all three cases .

Kolchin (1977) has considered a critical BGWP with finite variance σ^2. It has been proved that when the r-th ($r \geq 2$) moment of the offspring distribution exists and $x = \dfrac{2k}{\sigma^2 t}$ lies in a certain finite interval $[x_1, x_2]$

$$\frac{\sigma^2 t}{2} P(\frac{2}{\sigma^2 t} Z_t = x \mid N = n) = xe^{-x}(1 + o(1))$$

as $n, t \to \infty$ and $t^2 n^{-1+4/(2r+1)} \to 0$.

The author's interest to the topic was inspired by one problem, proposed by Kolchin (1986). The problem has the following formulation: the local limit theorem is still to be valid uder the same conditions for n and t as the integral one. Evidently the problem is interesting, especially in view of the connection between the random rooted labeled trees and the BGWP. (see Kolchin (1986;2.2); Vatutin (1993); Sevastyanov (1993)).

A tree could be defined as a connected non-ordered graph without cycles. When we choose one of the nodes of the tree for a root it becomes rooted. If it's nodes are numbered by $0, 1, 2, \ldots, n$ it is called labeled. Each node is connected to the root by an unique path. The length of this path (i.e. the number of the nodes in it) is called height of the node. It is well known that the number of all labeled rooted trees with n nodes is $(n+1)^{(n-1)}$, $n = 1, 2, \ldots$. If we define uniform distribution on the set of all labeled rooted trees with n nodes they become random, $n = 1, 2, 3, \ldots$.

The possibility of a relation between the random rooted labeled trees and the BGWP with a Poisson offspring distribution is first noted by Stepanov (1969). This connection has been found by Kolchin, who has used the studing of the BGWP with Poisson offspring distribution, conditioned on the total progeny as a part of a method to obtain asymptotic results for combinatorial objects such as random trees, random forests and random mapings. In Section 5 using this method we will obtain a corollary for a random rooted labeled tree.

2 Main results

Let $f(s) = \sum_{i=0}^{\infty} p_i s^i$, $| s | \leq 1$ denote the offspring probability generating function (pgf) of the process , $p_0 + p_1 < 1$ and $p_0 > 0$.

Further on we will suppose that

$$A)\begin{cases} f'(1)=1 \ , \quad 0<f''(1)=\sigma^2 \ , \\ E\xi^2 log(1+\xi)<\infty \ , \quad g.c.d.\{k: p_k>0\}=1 \ . \end{cases}$$

The main result of this paper is

Theorem 2.1 *If $t,n\to\infty$, $t^2n^{-1}\to 0$ in a way that* A) *holds, then*

$$\frac{\sigma^2 t}{2}P(\frac{2}{\sigma^2 t}Z_t = x \mid N=n) = xe^{-x}(1+o(1)) \quad .$$

uniformly for all k, $0<x_1<x=\frac{2k}{\sigma^2 t}<x_2<\infty$.

Now using Kennedy (1975), Lemma1 it is not difficult to extend the assertion of Theorem 1 to the subcritical and the supercritical cases.

We need the extra condition.

$$B)\begin{cases} \text{there exists } \alpha>0 \text{ with } f(\alpha)=\alpha f'(\alpha)<\infty \ , \\ f''(\alpha)<\infty \quad . \end{cases}$$

Theorem 2.2 *Suppose* A) *holds with $f'(1)=a<\infty$. Under the conditions $B)$ uniformly for all k,* $0<x_1<x=\frac{2k}{\beta t}<x_2<\infty$ *as* $t,n\to\infty$, $t^2n^{-1}\to 0$

$$\frac{\beta t}{2}P(\frac{2}{\beta t}Z_t = x \mid N=n) = xe^{-x}(1+o(1)) \quad ,$$

where $\beta=\alpha^2 f''(\alpha)/(2f(\alpha))$.

Finally, we will establish a corollary, exploiting the connection between the random trees and the BGWP.

Let $Z_t(T_n)$ be the number of the nodes with height t in the random labeled tree T_n. In the particular case when the process $\{Z_t\}$ has a Poison offspring distribution of one particle with parameter 1 we will obtain

Corollary 2.1 *If $n,t\to\infty$, $t^2n^{-1}\to\infty$, then*

$$\frac{t}{2}P(\frac{2}{t}Z_t(T_n)=x) = xe^{-x}+o(1) \ ,$$

uniformly for all k, $0<x_1\le x=\frac{2k}{t}\le x_2<\infty$

3 Preliminaries

Let $\{Z_t\}$ be a BGWP wth $Z_0 = m$, $m = 1, 2, 3, \ldots$. Let $N(m) = \sum_{t=0}^{\infty} Z_t(m)$ denote the total progeny and $N_t(m) = \sum_{i=0}^{t} Z_i(m)$, $m = 1, 2, 3, \ldots$; $N_t = N_t(1)$.

We will use the well known fact that as $t \to \infty$

$$(3.1) \qquad P(Z_t > 0) = \frac{2}{\sigma^2 t}(1 + o(1)) \quad .$$

H.Kesten, P.Ney, and F.Spitzer (1966) have proved that for $m, t = 1, 2, \ldots$

$$(3.2) \qquad sup_{k\geq 1} P(Z_t(m) = k) \leq \frac{C_1 m}{t^2} \quad , \quad C_1 > 0 \quad ,$$

$$(3.3) \qquad sup_{k\geq 1} P(Z_t(m) = k) \leq \frac{C_2}{mt} \quad , \quad C_2 > 0 \quad ,$$

and if $k, t \to \infty$ in such a way that k/t remains bounded, then, for fixed $i > 1$

$$(3.4) \qquad \lim \frac{t^2}{i}(\frac{\sigma^2}{2})^2 P(Z_t(i) = k) \exp\{\frac{2k}{\sigma^2 t}\} = 1$$

Dwass (1969) has shown that for $n \geq m \geq 0$, $n \geq 1$

$$(3.5) \qquad P(N(m) = n) = \frac{m}{n} P(\xi_1(t) + \ldots + \xi_n(t) = n - m) \quad .$$

The following results are used in the next Sections but are of independent interest also .

Lemma 3.1 *If $k \to \infty$ in a way that Λ) holds, then*

$$P(N_k = r) = O(\frac{1}{k^2}) \quad ,$$

uniformly for all $r > k$.

Proof. From (3.5) for each $k = 1, 2, 3, \ldots$ and each $r = k, k+1, \ldots$ we have

$$(3.6) \qquad P(N_k = r) = \frac{k}{r} P(\xi_1(t) + \ldots + \xi_r(t) = r - k) \quad .$$

Moreover

$$P(\xi_1(t) + \ldots + \xi_r(t) = r - k) \;=\; \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{-i\theta(r-k)} \varphi^r(\theta) d\theta \quad ,$$

where $\varphi(\theta) = f(e^{i\theta})$, $|\theta| \leq 1$ is the offspring characteristic function.

Put $z = \dfrac{k}{\sigma\sqrt{r}}$. Then

$$P(\xi_1(t) + \ldots + \xi_r(t) = r - k) = \frac{1}{2\pi}\int_{-\pi}^{\pi} e^{i\theta z\sigma\sqrt{r}}(\varphi^*(\theta))^r d\theta \quad ,$$

where $\varphi^*(\theta) = \varphi(\theta)e^{-i\theta}$.

Let $x = \theta\sigma\sqrt{r}$. Then

$$(3.7) \qquad \sigma\sqrt{r}P(\xi_1(t) + \ldots + \xi_r(t) = r - k) = \frac{1}{2\pi}\int_{-\pi\sigma\sqrt{r}}^{\pi\sigma\sqrt{r}} e^{ixz}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r dx \quad .$$

Consider a sequence of functions $G_1(z), G_2(z), G_3(z), \ldots$, defined as follows

$$G_r(z) = z^3 \int_{-\pi\sigma\sqrt{r}}^{\pi\sigma\sqrt{r}} e^{ixz}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r dx \; , \quad r = 1, 2, 3, \ldots \quad .$$

From (3.6) and (3.7) for each choice of k and r , $1 \leq k \leq r$

$$P(N_k = r) = \frac{1}{k^2}\frac{\sigma^2}{2\pi}G_r(\frac{k}{\sigma\sqrt{r}}) .$$

The lemma will be proved if we show that when $0 < \dfrac{\sigma z}{\sqrt{r}} \leq 1$ and $r \to \infty$

$$(3.8) \qquad G_r(z) = O(1) \quad .$$

Assume $\varepsilon > 0$. Since the process is aperiodic there exists $q = q(\varepsilon) < 1$, such that

$$(3.9) \qquad sup_{\varepsilon \leq |\theta| < \pi} \; |\varphi^*(\theta)| < q \quad , \quad 0 < \varepsilon < \pi \; .$$

Now we get

$$\begin{aligned}(3.10) \qquad |z^3 \int_{\varepsilon \leq \frac{|x|}{\sigma\sqrt{r}} \leq \pi} e^{ixz}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r dx| &\leq (\frac{r}{\sigma^2})^{\frac{3}{2}} \int_{\varepsilon \leq \frac{|x|}{\sigma\sqrt{r}} \leq \pi} |(\varphi^*(\frac{x}{\sigma\sqrt{r}}))|^r dx \\ &= \frac{r\sqrt{r}}{\sigma^2\sigma}\sigma\sqrt{r}\int_{\varepsilon \leq |\theta| \leq \pi} |\varphi^*(\theta)|^r d\theta \\ &\leq \frac{2\pi}{\sigma^2}r^2 q^r \quad \to 0 \qquad , r \to \infty \quad .\end{aligned}$$

We will prove that

$$(3.11)\qquad z^3\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ixz}(\varphi^*(\frac{x}{\sqrt{\sigma^2 r}}))^r\,dx = O(1)\ .$$

Since $\theta \to 0$

$$(3.12)\qquad \varphi^*(\theta) = 1 - \frac{\sigma^2\theta^2}{2}(1+o(1))\ ,$$

hence there exists $\varepsilon_1 > 0$ such that for $|\theta| < \varepsilon_1$

$$(3.13)\qquad |\varphi^*(\theta)| \leq 1 - \frac{\sigma^2\theta^2}{4} \leq e^{-\frac{\sigma^2\theta^2}{4}}\ .$$

The basic properties of the characteristic functions imply that there exists such ε_2 , that for $|\theta| < \varepsilon_2$

$$(3.14)\qquad |(\varphi^*(\theta))'| \leq 2\sigma^2|\theta|\ ,\quad |(\varphi^*(\theta))''| \leq \sigma^2\ .$$

Let $0 < \varepsilon \leq min(\varepsilon_1, \varepsilon_2)$ and without loss of generality assume $r \geq 4$. Using (3.13) and (3.14) we have for $|\frac{x}{\sigma\sqrt{r}}| < \varepsilon$

$$(3.15)\qquad \left|\frac{d(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r}{dx}\right| = r\left|(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1}\frac{d\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx}\right| \leq 2|x|e^{-\frac{3}{16}x^2}$$

and

$$(3.16)\qquad \left|r(r-1)(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-2}\left(\frac{d\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx}\right)^2\right| \leq 4x^2e^{-\frac{1}{8}x^2}\ ,$$

Under the same assumption

$$(3.17)\qquad \left|\frac{d^2(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r}{dx^2}\right| = \left|r(r-1)(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-2}\left(\frac{d(\varphi^*(\frac{x}{\sigma\sqrt{r}}))}{dx}\right)^2 + r(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1}\frac{d^2\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx^2}\right| \leq 4x^2e^{-\frac{1}{8}x^2} + e^{-\frac{3}{16}x^2}\ .$$

$$(3.18)\quad \left| r(r-1)\frac{d((\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-2}\,(\frac{d\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx})^2)}{dx}\right| \leq 8\mid x\mid^3 e^{-\frac{1}{16}x^2}+4\mid x\mid e^{-\frac{1}{8}x^2}$$

and

$$(3.19)\quad \left|(r-1)(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-2}\,\frac{d\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx}\right| \leq 2\mid x\mid e^{-\frac{1}{8}x^2}\quad .$$

Now integrating by parts the left-hand side of (3.11) we obtain

$$z^3\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ixz}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r\,dx = \frac{z^2}{i}\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r\,de^{ixz}$$

$$= \frac{z^2}{i}e^{ixz}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r\,\Big|_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} - \frac{z^2}{i}\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ixz}\frac{d(\varphi^*(\frac{x}{\sqrt{\sigma^2 r}}))^r}{dx}dx$$

$$= O(re^{-\frac{\sigma^2\varepsilon^2}{4}r}) - \frac{z}{i^2}\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}\frac{d(\varphi^*(\frac{x}{\sqrt{\sigma^2 r}}))^r}{dx}de^{ixz}$$

$$= o(1) - \frac{z}{i^2}e^{ixz}\frac{d(\varphi^*(\frac{x}{\sqrt{\sigma^2 r}}))^r}{dx}\,\Big|_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} + \frac{z}{i^2}\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ixz}\frac{d^2(\varphi^*(\frac{x}{\sqrt{\sigma^2 r}}))^2}{dx^2}dx$$

$$= o(1) - O(re^{-\frac{3\varepsilon^2}{16}r}) + \frac{1}{i^3}\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}\frac{d^2(\varphi^*(\frac{x}{\sqrt{\sigma^2 r}}))^r}{dx^2}de^{ixz}$$

$$= o(1) + i\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}(r(r-1)(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-2}\,(\frac{d\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx})^2 +$$

$$+r(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1}\,(\frac{d^2\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx^2})^2)de^{ixz}\quad ,$$

uniformly for all z, $0 < \frac{z\sigma}{\sqrt{r}} \leq 1$ as $r \to \infty$.

In the same way from (3.16)

$$z^3\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ixz}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r dx = o(1)+ir(r-1)(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-2}(\frac{d\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx})^2 e^{ixz}\Big|_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} -$$

$$-ir(r-1)\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ixz} d((\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-2}(\frac{d\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx})^2)+$$

$$+ir\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1}(\frac{d^2\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx^2})^2) de^{ixz}$$

$$= o(1)-ir(r-1)\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ixz}\frac{d((\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-2}(\frac{d\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx})^2)}{dx}dx+$$

$$+ir\int_{-\varepsilon\sigma\sqrt{r}}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1}(\frac{d^2\varphi^*(\theta)}{d\theta^2}\Big|_{\theta=\frac{x}{\sigma\sqrt{r}}})\frac{1}{\sigma^2 r}de^{ixz} \quad .$$

Finally using more integrating by parts , (3.18) and (3.19) we obtain

$$z^3\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ixz}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r dx = O(1)+\frac{i}{\sigma^2}\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1}(\sum_{m=0}^{\infty}(m-1)^2 i^2 p_m e^{(m-1)i\frac{x}{\sigma\sqrt{r}}})de^{ixz}$$

$$= O(1)-\frac{i}{\sigma^2}\sum_{m=0}^{\infty}p_m(m-1)^2(iz)\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1}e^{ix\frac{m-1}{\sigma\sqrt{r}}}e^{ixz}dx$$

$$= O(1)-\frac{i}{\sigma^2}\sum_{m=0}^{\infty}\frac{p_m(m-1)^2 iz}{i(z+\frac{m-1}{\sigma\sqrt{r}})}\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1} de^{ix(z+\frac{m-1}{\sigma\sqrt{r}})}$$

$$= O(1)-\frac{i}{\sigma^2}\sum_{m=0}^{\infty}\frac{p_m(m-1)^2 z}{(z+\frac{m-1}{\sigma\sqrt{r}})}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1} e^{ix(z+\frac{m-1}{\sigma\sqrt{r}})}\Big|_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}}+$$

$$+\frac{i}{\sigma^2}\sum_{m=0}^{\infty}\frac{p_m(m-1)^2 z}{(z+\frac{m-1}{\sigma\sqrt{r}})}\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ix(z+\frac{m-1}{\sigma\sqrt{r}})} d(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-1} \quad .$$

Hence

$$z^3\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ixz}(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^r dx = O(1)+\frac{i}{\sigma^2}\sum_{m=0}^{\infty}\frac{p_m(m-1)^2z}{(z+\frac{m-1}{\sigma\sqrt{r}})}\times$$

$$\times\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ix(z+\frac{m-1}{\sigma\sqrt{r}})}(r-1)(\varphi^*(\frac{x}{\sigma\sqrt{r}}))^{r-2}\frac{d\varphi^*(\frac{x}{\sigma\sqrt{r}})}{dx}dx$$

$$= O(1)+\frac{i}{\sigma^2}\sum_{m=0}^{\infty}\frac{p_m(m-1)^2z}{(z+\frac{m-1}{\sigma\sqrt{r}})}O(\int_{-\varepsilon\sigma\sqrt{r}}^{\varepsilon\sigma\sqrt{r}} e^{ix(z+\frac{m-1}{\sigma\sqrt{r}})}\mid x\mid e^{-\frac{1}{8}x^2}dx)\ .$$

Since the sum $\sum_{m=0}^{\infty} p_m(m-1)^2$ diverges (3.11) holds .
Now from (3.10) and (3.11) we get (3.8) , which complites the proof of the lemma .

Lemma 3.2 *If $t\to\infty$ in a way that* A) *holds, then*

$$EN_t\mathbf{I}(Z_t=k)=O(1)\quad ,$$

uniformly for all k, $0<c_1<\frac{k}{t}<c_2<\infty$.

Proof. Let the integers t,k and j satisfy the conditions

(3.20) $$c_1<\frac{k}{t}<c_2 \quad\text{and}\quad 0<j<t\quad .$$

From (3.2) and (3.3) for $j,m=1,2,3,\ldots$ and $k>\frac{c_1}{3}j$ we have

(3.21) $$P(Z_j(m)=k)<\frac{c_3}{j}\quad ,\quad c_3\geq 0\ .$$

Therefore

(3.22) $$sup_{i\in[\frac{k}{2},k]}P(Z_j(m)=i)<\frac{c_3}{j}.$$

Suppose $m>4k$. We will prove that in this case

(3.23) $$P(Z_j(m)=k)<\frac{4\sigma^2c_3}{c_1}\frac{1}{t}\quad .$$

Since $EZ_j(m)=m$ and $DZ_j(m)=m\sigma^2j$ using the Chebyshev's inequality we find

(3.24) $$P(Z_j(m)<k)\leq P(Z_j(m)<\frac{m}{4})\leq 4\sigma^2\frac{j}{k}\quad .$$

From (3.20) - (3.22) and (3.24)we obtain

$$
\begin{aligned}
P(Z_j(m)=k) &= \sum_{i=o}^{k} P(Z_j(m-2k)=i)P(Z_j(2k)=k-i) \\
&= \sum_{i=0}^{[\frac{k}{2}]} P(Z_j(m-2k)=i)P(Z_j(2k)=k-i)+ \\
&\quad + \sum_{[\frac{k}{2}]+1}^{k} P(Z_j(m-2k)=i)P(Z_j(2k)=k-i) \\
&< \frac{c_3}{j}\sum_{i=0}^{[\frac{k}{2}]} P(Z_j(m-2k)=i) + \frac{c_3}{j}\sum_{[\frac{k}{2}]+1}^{k} P(Z_j(2k)=k-i) \\
&= \frac{c_3}{j}\left(P(Z_j(m-2k)\le[\frac{k}{2}]) + P(Z_j(2k)\le k-[\frac{k}{2}]-1)\right) \\
&< \frac{c_3}{j}\left(P(Z_j(m-2k)<[\frac{k}{2}]) + P(Z_j(2k)<k-[\frac{k}{2}]-1)\right) \\
&< \frac{c_3}{j}\left(2\sigma^2\frac{j}{k}+2\sigma^2\frac{j}{k}\right) = \frac{4\sigma^2 c_3}{k} \quad ,
\end{aligned}
$$

i.e. (3.23) holds .

From (3.2) there exists a non-negative c_4 such that for each choice of k and t, $c_1 < \frac{k}{t} < c_2$

(3.25) $$P(Z_t = k) < \frac{c_4}{t^2} \quad .$$

Noting also (3.23) we find that for each $j \in [1,t]$ such that $c_1 < \frac{k}{t} < c_2$

(3.26)
$$
\begin{aligned}
EZ_j\mathbf{I}(Z_t=k) &= EZ_j\mathbf{I}(Z_t=k, Z_j<4k) + EZ_j\mathbf{I}(Z_t=k, Z_j\ge 4k)) \\
&< 4kP(Z_t=k) + \sum_{i\ge 4k} P(Z_j=i)P(Z_{t-j}(Z_j)=k) \\
&< 4kP(Z_t=k) + \frac{4\sigma^2 c_3}{c_1}\frac{1}{t}EZ_j\mathbf{I}(Z_j\ge 4k)
\end{aligned}
$$

$$
\begin{aligned}
&< \ 4kP(Z_t = k) + \frac{4\sigma^2 c_3}{c_1}\frac{1}{t} EZ_j \\
&< \ 4k\frac{c_4}{t^2} + \frac{4\sigma^2 c_3}{c_1 t} \\
&< \ (4c_2c_4 + 4\sigma^2\frac{c_3}{c_1}).\frac{1}{t} \quad .
\end{aligned}
$$

Then as $t \to \infty$ uniformly in all k which satisfy $c_1 < \frac{k}{t} < c_2$

$$
\begin{aligned}
EN_t\mathbf{I}(Z_t = k) &= E(1 + Z_1 + \ldots + Z_{t-1} + k)\mathbf{I}(Z_t = k) \\
&= (1+k)E\mathbf{I}(Z_t = k) + E(Z_1 + \ldots + Z_{t-1})\mathbf{I}(Z_t = k) \\
&= (1+k)E\mathbf{I}(Z_t = k) + EZ_1\mathbf{I}(Z_t = k) + \ldots + EZ_{t-1}\mathbf{I}(Z_t = k) \\
&< (1+k)\frac{c_4}{t^2} + (t-1)(4c_2c_4 + 4\sigma^2\frac{c_3}{c_1})\frac{1}{t} \quad ,
\end{aligned}
$$

which complites the proof of the Lemma.

Lemma 3.3 *If $t \to \infty$ in a way that* A) *holds, then*

$$
EN_t^2\mathbf{I}(Z_t = k) = O(t^2) \quad ,
$$

uniformly for all k, $0 < c_1 < \frac{k}{t} < c_2 < \infty$.

Proof. Under the conditions $c_1 < \frac{k}{t} < c_2$, $0 < j < t$ and $s > 4k$ the statements (3.21) - (3.23) hold.

Moreover

$$
\begin{aligned}
(3.27)\, EN_t^2\mathbf{I}(Z_t = k) &= E(1 + Z_1 + \ldots Z_{t-1} + k)^2\mathbf{I}(Z_t = k) \\
&= (1+k)^2E\mathbf{I}(Z_t = k) + 2(1+k)E(Z_1 + \ldots + Z_{t-1})\mathbf{I}(Z_t = k) + \\
&\quad + E(Z_1 + \ldots Z_{t-1})^2\mathbf{I}(Z_t = k) \quad .
\end{aligned}
$$

Using (3.23) and (3.25) we obtain

$$
\begin{aligned}
(3.28) \qquad E(Z_1 + \ldots + Z_{t-1})\mathbf{I}(Z_t = k) &= EZ_1\mathbf{I}(Z_t = k) + \ldots + EZ_{t-1}\mathbf{I}(Z_t = k) \\
&< (t-1)\frac{4\sigma^2 c_3}{c_1}\frac{1}{t} \quad .
\end{aligned}
$$

On the other hand

$$(3.29)\qquad E(Z_1+\ldots+Z_{t-1})^2\mathbf{I}(Z_t=k) \;=\; 2\sum_{1\le i\le j<t} EZ_iZ_j\mathbf{I}(Z_t=k) - {} - \sum_{1\le i<t} Z_i^2\mathbf{I}(Z_t=k) < 2\sum_{1\le i\le j<t} EZ_iZ_j\mathbf{I}(Z_t=k) \quad .$$

We shall use the decomposition

$$\begin{aligned}\mathbf{I}(Z_t=k) &= (\mathbf{I}(Z_t=k,Z_i\le 4k,Z_j\le 4k)+\mathbf{I}(Z_t=k,Z_i>4k,Z_j\le 4k)+\\ &\quad +\mathbf{I}(Z_t=k,Z_i\le 4k,Z_j>4k)+\mathbf{I}(Z_t=k,Z_i>4k,Z_j>4k)) \quad .\end{aligned}$$

Since $c_1<\frac{k}{t}<c_2$ we have

$$\begin{aligned}2\sum_{1\le i\le j<t} EZ_iZ_j\mathbf{I}(Z_t=k,Z_i\le 4k,Z_j\le 4k) &\le 2.16k^2\sum_{1\le i\le j<t} E\mathbf{I}(Z_t=k,Z_i\le 4k,Z_j\le 4k)\\ &< 32k^2\sum_{1\le i\le j<t} P(Z_t=k)\end{aligned}$$

and (3.25) implies

$$(3.30)\qquad 2\sum_{1\le i\le j<t} EZ_iZ_j\mathbf{I}(Z_t=k,Z_i\le 4k,Z_j\le 4k)=O(t^2) \quad .$$

From (3.27) for all k, $c_1<\frac{k}{t}<c_2$ we obtain

$$\begin{aligned}2\sum_{1\le i\le j<t} EZ_iZ_j\mathbf{I}(Z_t=k,Z_i>4k,Z_j\le 4k) &\le 8k\sum_{1\le i\le j<t} EZ_i\mathbf{I}(Z_t=k,Z_i>4k)\\ &< 8kt\sum_{i=1}^{t-1} EZ_i\mathbf{I}(Z_t=k) \quad .\end{aligned}$$

Therefore as $t\to\infty$

$$(3.31)\qquad 2\sum_{1\le i\le j<t} EZ_iZ_j\mathbf{I}(Z_t=k,Z_i>4k,Z_j\le 4k)=O(t^2) \quad .$$

Similarily as $t\to\infty$ one gets

$$(3.32)\qquad 2\sum_{1\le i\le j<t} EZ_iZ_j\mathbf{I}(Z_t=k)\mathbf{I}(Z_i\le 4k)\mathbf{I}(Z_j>4k)=O(t^2) \quad .$$

At the end using (3.23) we obtain

$$
\begin{aligned}
2\sum_{1\le i\le j<t} EZ_iZ_j\mathbf{I}(Z_t=k,Z_i>4k,Z_j>4k) &= 2\sum_{1\le i\le j<t}\sum_{m_1>4k}\sum_{m_2>4k} m_1m_2P(Z_i=m_1)\times \\
&\quad \times P(Z_{j-i}(m_1)=m_2)P(Z_{t-j}(m_2)=k) \\
&< \frac{8\sigma^2c_3}{c_1t}\sum_{1\le i\le j<t}\sum_{m_1=1}^{\infty} m_1P(Z_i=m_1)\times \\
&\quad \times\sum_{m_2=1}^{\infty} m_2P(Z_{j-i}(m_1)=m_2) \\
&= \frac{8\sigma^2c_3}{c_1t}\sum_{1\le i\le j<t}\sum_{m_1=1}^{\infty} m^2P(Z_i=m) \\
&< \frac{8\sigma^2c_3}{c_1}\sum_{i=1}^{t-1}\sum_{m=1}^{\infty} m^2P(Z_i=m) \quad .
\end{aligned}
$$

And since $EZ_i^2=O(i)$ as $t\to\infty$ then uniformly for all k , $c_1<\frac{k}{t}<c_2$

(3.33) $$2\sum_{1\le i\le j<t} EZ_iZ_j\mathbf{I}(Z_t=k,Z_i>4k,Z_j>4k)=O(t^2) \quad .$$

From (3.30) - (3.33) we find that uniformly for all k , $c_1<\frac{k}{t}<c_2$

(3.34) $$E(Z_1+\ldots+Z_{t-1})^2\mathbf{I}(Z_t=k)=O(t^2) \quad .$$

Then from (3.28) - (3.30) we get

$$EN_t^2\mathbf{I}(Z_t=k)=O(t^2)$$

and the Lemma is proved.

4 Proof of Theorem 1

Uniformly for all s and k , such that $\frac{k^2}{n-s-k}$ lies in a finite interval

(4.35) $$P(N_k=n-s)=\frac{k}{(n-s)\sqrt{2\pi\sigma^2(n-s)}}e^{-\frac{k^2}{2\sigma^2(n-s-k)}}(1+o(1)) \quad , \quad n\to\infty \quad .$$

(cf. Kolchin (1986;2.4)).

Moreover if $\frac{k^2}{n} \to 0$ and $\frac{s}{n} \to 0$ then

$$P(N_k = n - s) = \frac{k}{n\sqrt{2\pi\sigma^2 n}}(1 + o(1)) \ , \ n \to \infty \ . \tag{4.36}$$

In particular

$$P(N = n) = \frac{1}{n\sqrt{2\pi\sigma^2 n}}(1 + o(1)) \ , \ n \to \infty \ . \tag{4.37}$$

Puting $\gamma = \frac{t}{\sqrt{n}} \to 0$ and $0 < b < 1$ without loss of generality we assume

$$\gamma n < bn < n - k \ .$$

Now for all n , t and k

$$\begin{aligned} P(Z_t = k \mid N = n) &= \sum_{s=1}^{n-k} P(Z_t = k, N_t = s)\frac{P(N_k = n - s)}{P(N = n)} \\ &= S_1(n,t,k) + S_2(n,t,k) + S_3(n,t,k) \ , \text{ say } , \end{aligned} \tag{4.38}$$

where

$$\begin{aligned} S_1(n,t,k) &= \sum_{s \le \gamma n} P(Z_t = k, N_t = s)\frac{P(N_k = n - s)}{P(N = n)} \ , \\ S_2(n,t,k) &= \sum_{\gamma n < s \le bn} P(Z_t = k, N_t = s)\frac{P(N_k = n - s)}{P(N = n)} \ , \\ S_3(n,t,k) &= \sum_{bn < s \le n-k} P(Z_t = k, N_t = s)\frac{P(N_k = n - s)}{P(N = n)} \ . \end{aligned}$$

We are going to prove that uniformly for all $k, x_1 \le \frac{2k}{\sigma^2 t} \le x_2$

$$S_1(n,t,k) = kP(Z_t = k)(1 + o(1)) + O(\frac{1}{\sqrt{n}}) \ , \tag{4.39}$$

$$S_2(n,t,k) = O(\frac{1}{\sqrt{n}}) \ , \tag{4.40}$$

$$S_3(n,t,k) = O(\frac{1}{\sqrt{n}}) \tag{4.41}$$

as $t, n \to \infty$, $t^2 n^{-1} \to 0$.

From (4.36) and (4.37) uniformly for all $s \le \gamma n$ and all k, $x_1 \le x = \frac{2k}{\sigma^2 t} \le x_2$

$$\frac{P(N_k = n - s)}{P(N = n)} = k(1 + o(1)) .$$

as $t, n \to \infty$,$t^2 n^{-1} \to 0$

Therefore

$$(4.42) \qquad \begin{aligned} S_1(n, t, k) &= \sum_{s \le \gamma n} P(Z_t = k, N_t = s) k (1 + o(1)) \\ &= k(P(Z_t = k) - \sum_{s > \gamma n} P(Z_t = k, N_t = s))(1 + o(1)) \\ &= kP(Z_t = k)(1 + o(1)) - S_4(n, t, k)(1 + o(1)) \quad \text{, say ,} \end{aligned}$$

where

$$S_4(n, t, k) = k \sum_{s > \gamma n} P(Z_t = k, N_t = s).$$

Using Lemma 3.2 we obtain uniformly for all $k, x_1 \le \frac{2k}{\sigma^2 t} \le x_2$

$$\sum_{s > \gamma n} P(Z_t = k, N_t = s) = \frac{O(1)}{\gamma(t, n) n} = O(\frac{1}{t\sqrt{n}}) .$$

Now we have

$$S_4(n, t, k) = kO(\frac{1}{t\sqrt{n}}) = O(\frac{1}{\sqrt{n}}),$$

i.e. (4.39) holds .

Moreover uniformly for all $s, \gamma n < s < bn$ and all $k, x_1 \le \frac{2k}{\sigma^2 t} \le x_2$

$$P(N_k = n - s) = \frac{k}{(n - s)\sqrt{2\pi\sigma^2(n - s)}} e^{-\frac{k^2}{2\sigma^2(n - s - k)}} (1 + o(1)).$$

Since $(1 - b) \le \frac{n - s}{n} < 1$ we have

$$P(N_k = n - s) \le (1 - b)^{-\frac{3}{2}} \frac{k}{n\sqrt{2\pi\sigma^2 n}} (1 + o(1)).$$

Obviously from (4.37) there exists $a_1 > 0$ such that for $n = 1, 2, 3, \ldots$

(4.43) $$P(N = n) > a_1 \frac{1}{n\sqrt{2\pi\sigma^2 n}} \; .$$

Then uniformly for all $k, x_1 \leq \frac{2k}{\sigma^2 t} \leq x_2$

$$\begin{aligned} S_2(n,t,k) &< \sum_{\gamma n < s \leq bn} P(Z_t = k, N_t = s) \frac{(1-b)^{-\frac{3}{2}}}{a_1} k(1 + o(1)) \\ &< \frac{(1-b)^{-\frac{3}{2}}}{a_1} k P(Z_t = k, N_t > \gamma n)(1 + o(1)) \\ &< \frac{(1-b)^{-\frac{3}{2}}}{a_1} k O(\frac{1}{t\sqrt{n}}). \end{aligned}$$

Hence (4.40) holds .

Using Lemma 3.1 and (4.43) one can obtain uniformly for all $k, x_1 \leq \frac{2k}{\sigma^2 t} \leq x_2$

$$\begin{aligned} S_3(n,t,k) &< \sum_{bn < s \leq n-k} P(Z_t = k, N_t = s) O(\frac{1}{t^2}) \frac{n^{\frac{3}{2}}\sqrt{2\pi\sigma^2}}{a_1} \\ &< O(\frac{1}{t^2\sqrt{n}}) \sum_{bn < s \leq n-k} n^2 P(Z_t = k, N_t = s) \; . \end{aligned}$$

And since $bn < s$

$$\begin{aligned} S_3(n,t,k) &= O(\frac{1}{t^2\sqrt{n}}) \sum_{bn < s \leq n-k} s^2 P(Z_t = k, N_t = s) \\ &< O(\frac{1}{t^2\sqrt{n}}) \sum_{s=1}^{\infty} s^2 P(Z_t = k, N_t = s) \; . \end{aligned}$$

Now from Lemma 3.3 we get (4.41) .

From (4.39) - (4.41) uniformly for all $k, x_1 \leq \frac{2k}{\sigma^2 t} \leq x_2$ as $t, n \to \infty$,$t^2 n^{-1} \to 0$

$$\begin{aligned} P(Z_t = k \mid N = n) &= kP(Z_t = k)(1 + o(1)) + O(\frac{1}{\sqrt{n}}) \\ &= kP(Z_t = k \mid Z_t > 0)P(Z_t > 0)(1 + o(1)) + O(\frac{1}{\sqrt{n}}) \; . \end{aligned}$$

Finally using (3.4) and (3.1) it is not difficult to obtain uniformly for all $x = \frac{2k}{\sigma^2 t} \in [x_1, x_2]$ as $t, n \to \infty$,$t^2 n^{-1} \to 0$

$$\begin{aligned} P(Z_t = k, N = n) &= \frac{x\sigma^2 t}{2} \frac{2e^{-x}}{\sigma^2 t} \frac{2}{\sigma^2 t}(1 + O(1)) + O(\frac{1}{\sqrt{n}}) \\ &= \frac{2}{\sigma^2 t} x e^{-x}(1 + o(1)) , \end{aligned}$$

which complites the proof of the Theorem .

Kennedy (1975) has shown that if $a = \sum_{i=1}^{\infty} ip_i < \infty$ and the distribution $p_0, p_1, p_2, \ldots$ satisfies the condition A) we can define a new process $\overline{Z_0}, \overline{Z_1}, \overline{Z_2}, \ldots$ with offspring distribution $\overline{p_k} = \alpha^k p_k / f(\alpha)$, $k = 0, 1, 2, \ldots$ which is a critical one . For each $n \geq 1$, $j \geq 1$, $0 \leq k_1, \ldots, k_j \leq n$ and all choices of $r_1, \ldots, r_j \geq 1$

$$P(Z_{k_1} = r_1, \ldots, Z_{k_j} = r_j \mid N = n) = P(\overline{Z_{k_1}} = r_1, \ldots, \overline{Z_{k_j}} = r_j \mid \overline{N} = n) ,$$

where $\overline{N}$ denotes the sum $\sum_{r=0}^{\infty} \overline{Z_r}$.

From this construction and Theorem 2.1 it is clear that Theorem 2.2 holds.

5 An Application for Random Trees

In this Section we will apply Theorem 2.1 to obtain a limit theorem for random rooted labeled trees .

Consider a random labeled rooted tree T_n with n nodes . For each node the number of arcs , connected to it , excluding the one that belongs to the path , leading to the root , is called number of it's direct successors . Let $Z_t(r, T_n)$ be the number of the nodes in the tree with height t and exactly r direct successors, $n = 1, 2, \ldots$; $r = 0, 1, \ldots, n - t$.

Let G be a critical BGWP with a Poisson offspring distribution of one particle with parameter 1 . Let us denote the number of particles in the t-th generation of G with exactly r direct successors by $Z_t(r, G)$, $t, r = 0, 1, 2, \ldots$. Let $N(G)$ denote the total progeny of the process .

Using Theorem 2.1 and the arguments of Kolchin (1977 ; Th.7) one can prove Corollary 2.1 .

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