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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР

INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

On certain systems of generators of infinite symmetric and alternating groups

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Abstract

Let $S(N_0)$ be the symmetric group on the set N_0 of non-negative integers, consisting of all permutations σ with finite support. Let ω be the endomorphism of the group $S(N_0)$, defined by the rule: $\omega(\sigma)(p) = \omega(p-1)+1$ and $\omega(\sigma)(0) = 0$. Necessary and sufficient conditions for an ω-stable set $B \subset S(N_0)$ to generate $S(N_0)$, or its alternating subgroup $A(N_0)$, are given. The corresponding proof is valid in a more general framework. Given a natural d, let $S(d,0)$ (respectively, $A(d,0)$) be the direct product of the symmetric groups $S(j+dN_0)$ (respectively, the alternating groups $A(j+dN_0)$) where $0 \le j \le d-1$. Then a characterization of the ω-stable sets $B \subset S(N_0)$ which generate groups W between $A(d,0)$ and $S(d,0)$, is presented. When $d=1$ we obtain the result concerning the symmetric or the alternating group.

Introduction

Throughout the text below by $S(X)$ (respectively, by $A(X)$) we denote the symmetric (respectively, the alternating) group of permutations of a given set X, with finite support. By N_0 we denote the set of non-negative integers. The letter B with or without subscript, always denotes a subset of $S(N_0)$ with non-empty support.

The prototypes of the sets B of generators, which we consider in this note, are the standard ones: $(01),(12),(23),(34),\ldots$ and $(012),(123),(234),\ldots$, which generate $S(N_0)$ and $A(N_0)$, respectively. Both systems are ω-stable: $\omega(B) \subset B$, where ω is a fascinating injective endomorphism of $S(N_0)$, defined in [1]. It is clear that any ω-stable set B with non-empty support generates a non-trivial ω-stable subgroup $W = \langle B \rangle$ of $S(N_0)$. The radical of W is an (*a posteriori* ω-stable) group U which depends on W. The radicals U of all non-trivial ω-stable groups W are classified in [1, sect. 3]. It turns out that the classification depends upon two ingredients: a natural number $d = class(U)$ called class of the group U, with $class(W) = class(U)$ and a polynomial $f_U(x) \in F_2[x]$ which divides $x^d - 1$ (here F_2 is the field with two elements). In particular, $class(U) = 1$ if and only if $U = S(N_0)$ (case $f_U(x) = 1$), or $U = A(N_0)$ (case $f_U(x) = x-1$). In order to describe those ω-stable sets B which generate $S(N_0)$ or $A(N_0)$, we have to investigate

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the difference between a group W and its radical U. Moreover, we have to find a natural invariant $class(B)$ of the system B, such that $class(B) = class(W)$. Fortunately, both tasks are easy achievable. The difference between W and U is the simplest possible: $W = \omega^e(U)$, where $e = \min(suppB)$ (see (2.1.6)). On the other hand, we can define $class(B)$ with above property (see 3.1). Thus, we have

THEOREM A. *Let W be a non-trivial ω-stable subgroup of $S(N_0)$ and let B be an ω-stable system of generators for W. The following statements are equivalent:*

(i) The group W satisfies the inequalities $A(N_0) \leq W \leq S(N_0)$;

(ii) The set B satisfies the equalities $class(B) = 1$ and $\min(suppB) = 0$.

As usually happens, the proof of a theorem is more general than the theorem itself. The corresponding generalized statement is Theorem 3.1.2 below which describes the ω-stable sets of generators of the non-trivial ω-stable groups of any class $d \geq 1$.

1. PRELIMINARIES

1.1. In this subsection we will remind some definitions and notation from [1]. Given integers $d \geq 1$ and $e \geq 0$, we set

$$S(d,e) = S(e + dN_0) \times S(e + 1 + dN_0) \times \cdots \times S(e + d - 1 + dN_0),$$

$$A(d,e) = A(e + dN_0) \times A(e + 1 + dN_0) \times \cdots \times A(e + d - 1 + dN_0),$$

where $e + j + dN_0$ is the infinite arithmetical progression with first term $e + j$ and difference d, for $0 \leq j \leq d - 1$. Let ω be the endomorphism of the group $S(N_0)$, defined by the rule: $\omega(\sigma)(p) = \omega(p-1) + 1$ for $p \geq 1$ and $\omega(\sigma)(0) = 0$. Both direct products are ω-stable subgroups of $S(N_0)$. In particular, $S(d,e)$ and $A(d,e)$ have structures of ω-operator groups. On the other hand, the additive group of the ring

$$R_d = F_2[x]/(x^d - 1)$$

also has a structure of ω-operator group: $\omega g(x) = xg(x)$, for any $g(x) \in R_d$. We have a canonical epimorphism of ω-operator groups: for $\sigma = \sigma_0\sigma_1 \ldots \sigma_{d-1}$ with $\sigma_j \in S(j + dN_0)$, we define

$$f^{(d)}\colon S(d,0) \to R_d, \tag{1.1.1}$$

$$\sigma \to f_\sigma^{(d)}(x) = \sum_{j=0}^{d-1} \alpha_j(\sigma)x^j,$$

where the group homomorphism $\alpha_j: S(d,0) \to F_2$ is the additively written signature of the j-th component σ_j of σ. Further, we have $S(d,e) \geq S(d,e+1)$ and $A(d,e) \geq A(d,e+1)$ where $e \geq 0$, and the restriction $f^{(d)}|_{S(d,e)}$ induces an isomorphism of ω-operator groups $S(d,e)/A(d,e) \simeq R_d$ at any level e. Clearly, the ω-stable groups U with $A(d,e) \leq U \leq S(d,e)$ are in 1–1 correspondence with the ideals I of the ring R_d via $f^{(d)}|_{S(d,e)}$. Let

$$f_I(x) = \alpha_0 + \alpha_1 x + \ldots + \alpha_{d-k} x^{d-k} \in F_2[x], \; \alpha_{d-k} = 1,$$

be the divisor of $x^d - 1$, which generates the ideal I. In particular, $0 \leq k \leq d$. Given a non-negative integer s, we set

$$g_{I,s}(x) = x^s f_I(x), \tag{1.1.2}$$

$$G_{I,s} = \{g_{I,s}(x), g_{I,s+1}(x), \ldots, g_{I,s+k-1}(x)\}. \tag{1.1.3}$$

Then the k-element set $G_{I,s}$ generates I as a F_2-linear space, or, equivalently, as an Abelian 2-group of type $(2, 2, \ldots, 2)$.

1.2. We shall recall some definitions from [1, subsect. 3.3]. Let W be an non-trivial ω-stable subgroup of $S(N_0)$. The set

$$r_0(W) = \{\sigma \in S(N_0) \mid \omega^t \sigma \in W \text{ for some } t \geq 0\}$$

also is an ω-stable subgroup of $S(N_0)$, which we call *radical* of W. Clearly, $W \leq r_0(W)$.

According to [1, (3.3.1)], the group W contains at least one 3-cycle. The natural number

$$d = \gcd\{\zeta(i) - i \mid \zeta \text{ is a 3-cycle in } W \text{ and } i \in N_0\}.$$

is called *class* of the group W. We use notation $d = class(W)$. If W is an non-trivial ω-stable group of class d with radical U, then [1, (3.3.2)] yields that $A(d,0) \leq U \leq S(d,0)$. In particular, $W \leq S(d,0)$; hence we can take the image $f_W^{(d)}$ of the group W via the homomorphism (1.1.1).

LEMMA 1.2.1. *Let W be an non-trivial ω-stable subgroup of class d of $S(N_0)$ with radical U and let I be the ideal of the ring R_d, corresponding to the group U. Then one has:*

(i) The ideal I coincides with $f_W^{(d)}$;

(ii) If the set B generates the group W, then the set $f_B^{(d)}$ generates the ideal I.

PROOF: (i) Since $W \leq U$, then $f_W^{(d)} \subset I$. Let $h(x) \in I$ and let $\sigma \in U$ be such that $f_\sigma^{(d)}(x) = h(x)$. There exists a $t \geq 0$ with $\eta = \omega^t \sigma \in W$. We can assume t to be a multiple of d, because the group W is ω-stable. Since (1.1.1) is a homomorphism of ω-operator groups, then $f_\eta^{(d)} = x^t h(x) = h(x)$ in I. Thus, $I = f_W^{(d)}$.

(ii) The equality $f_W^{(d)} = (f_B^{(d)})$ is obvious.

1.3. Since the homomorphism (1.1.1) behaves as a "logarithm", then there should be something like "exponential" map. Indeed, let τ be the transposition $(0, d) \in S(d, 0)$. For any polynomial $g(x) = \sum_{i \geq 0} \beta_i x^i$ with coefficients in the field F_2 we set

$$\tau^{g(x)} = \prod_{i \geq 0} (\omega^i \tau)^{\beta_i} \in S(d, 0).$$

LEMMA 1.3.1. *Given polynomials $g(x)$, $h(x) \in F_2[x]$, one has:*

(i) If $\sigma = \tau^{g(x)}$, then $f_\sigma^{(d)}(\bar{x}) = g(x)$ in the ring R_d;

(ii) The equality $\omega^s \tau^{g(x)} = \tau^{x^s g(x)}$ holds for any $s \geq 0$;

(iii) If $deg(g(x))$, $deg(h(x)) \leq d-1$, then $\tau^{g(x)+h(x)} = \tau^{g(x)} \tau^{h(x)}$;

(iv) If $0 \leq deg(g(x)) \leq d-1$, then the permutation $\tau^{g(x)}$ has order 2.

PROOF: Parts (i) and (ii) follow by definition. Since every d consecutive powers $\omega^i \tau$ commute, then part (iii) holds. (iv). Since $g(x) \neq 0$, then $\tau^{g(x)} \neq (1)$. Using part (iii), we have $(\tau^{g(x)})^2 = \tau^{2g(x)} = \tau^0 = (1)$.

2. DESCRIPTION OF THE ω-STABLE GROUPS

2.1. For any $s \geq 0$ and for any ideal I of the ring R_d we set

$$\theta_{I,s} = \tau^{g_{I,s}(x)}, \tag{2.1.1}$$

$$T_{I,s} = \langle \theta_{I,s}, \theta_{I,s+1}, \ldots, \theta_{I,s+k-1} \rangle \leq S(d, 0), \tag{2.1.2}$$

where $g_{I,s}(x)$ are the polynomials from (1.1.2). Given $s \geq e$, we define an injective map

$$\varphi_{I,s}: G_{I,s} \to S(d, e),$$

$$g_{I,s+i}(x) \to \theta_{I,s+i} \text{ for } 0 \leq i \leq k-1,$$

where $G_{I,s}$ is the basis for I from (1.1.3). Note that $\varphi_{I,s}$ is the empty map if and only if $k = 0$, or, equivalently, $I = 0$.

LEMMA 2.1.3. *For any $s \geq e$ the map $\varphi_{I,s}$ can be extended to a monomorphism of groups $\psi_{I,s}: I \to S(d,e)$ such that:*

(i) The image of $\psi_{I,s}$ coincides with the group $T_{I,s}$ from (2.1.2);

(ii) The composition $f^{(d)}|_{S(d,e)} \circ \psi_{I,s}$ coincides with the identity map Id_I.

PROOF: Since the system $G_{I,s}$ is a basis for I, then the rule

$$\psi_{I,s}: I \to S(d,e),$$

$$(u(x) = \sum_{i=0}^{k-1} \beta_i g_{s+i}(x)) \to \tau^{u(x)},$$

defines a map. We have $u(x) = x^s g(x) f_I(x)$, where $g(x) = \sum_{i=0}^{k-1} \beta_i x^i$. Hence (1.3.1), (ii), implies that $\tau^{u(x)} = \omega^s \tau^{g(x) f_I(x)}$. Now, (1.3.1), (iii) and (iv), yield that $\psi_{I,s}$ is an injective homomorphism of groups. Since $\psi_{I,s}$ extends $\varphi_{I,s}$, then part (i) holds. Part (ii) follows from (1.3.1), (i).

The proposition below describes the non-trivial ω-stable subgroups of the symmetric group $S(N_0)$.

PROPOSITION 2.1.4. *Let W be a non-trivial ω-stable subgroup of $S(N_0)$. Given integers $d \geq 1$ and $e \geq 0$, and an ideal I of the ring R_d, one has the following equivalent statements:*

(i) The group W satisfies the inequalities $A(d,e) \leq W \leq S(d,e)$ and $f_W^{(d)} = I$;

(ii) The group $A(d,e)$ is a normal subgroup of W, the groups $T_{I,s}$ are subgroups of W for any $s \geq e$, and W is the semi-direct product of $A(d,e)$ with each $T_{I,s}$;

(iii) The group $A(d,e)$ is a normal subgroup of W, the group $T_{I,s}$ is a subgroup of W for some $s \geq e$, and W is the semi-direct product of $A(d,e)$ with $T_{I,s}$;

(iv) The group W satisfies the equalities $class(W) = d$, $\min(suppW) = e$ and $f_W^{(d)} = I$.

PROOF: *(i) ⇒ (ii)*. Lemma 2.1.3 yields that $T_{I,s}$ is a subgroup of W and that $\psi_{I,s}$ splits the homomorphism $f^{(d)}|_{S(d,e)}$ with kernel $A(d,e)$, for any $s \geq e$. In other words, W is the semi-direct product of $A(d,e)$ with any $T_{I,s}$, where $s \geq e$.

The implications *(ii) ⇒ (iii)*, *(iii) ⇒ (i)* and *(i) ⇒ (iv)* are trivial.

(iv) ⇒ (iii). Since $class(W) = d$, then, according to [1, (3.3.2)], the radical $U = r_0(W)$ satisfies $A(d,0) \leq U \leq S(d,0)$. In particular, $(0, d, 2d) \in U$. Hence there exists a $t \geq 0$ such that the 3-cycle $(t, t+d, t+2d)$ is contained in W. By [1, (3.2.1)], we obtain

$$A(d,t) \leq W. \tag{2.1.5}$$

On the other hand, (1.2.1), (i), implies that $I = f_U^{(d)}$. Now, the eqivalence of (i) and (ii), applied to the group U, yields that U is the semi-direct product of its normal subgroup $A(d,0)$ and its subgroup $T_{I,0}$. Since $T_{I,0}$ is finite, then there exists a $s \geq 0$ with $\omega^s T_{I,0} \leq W$, that is, $T_{I,s} \leq W$. We choose a minimal t satisfying (2.1.5). It can be supposed that $s \geq t$. We claim:

(1) The intersection $A(d,0) \cap W$ coincides with $A(d,t)$.

(2) The group W is the semi-direct product of $A(d,t)$ and $T_{I,s}$.

Proof of Claim (1): When $t = 0$, claim (1) is obvoius. In case $t \geq 1$ we suppose that there exists a permutation $\eta \in A(d,0) \cap W$, which does not belong to $A(d,t)$. Then $\min(supp(\eta)) \leq t-1$. After eventual applying of the endomorphism ω on η, we can suppose $\min(supp(\eta)) = t-1$; hence $\eta \in A(d,t-1)$. Our aim is to prove that the subgroup $\langle A(d,t), \eta \rangle$ of W coincides with $A(d,t-1)$, which would be a contradiction with the choice of t. We have $\eta = \eta_{t-1}\eta_t \dots \eta_{t+d-2}$, where $\eta_{t-1+i} \in A(t-1+i+dN_0)$ for $0 \leq i \leq d-1$, and $t-1 \in supp(\eta_{t-1})$. Therefore $\eta = \eta_{t-1}\eta' = \eta'\eta_{t-1}$, where $\eta' = \eta_t \dots \eta_{t+d-2} \in A(d,t)$. Thus,

$$\langle A(d,t), \eta \rangle = \langle A(d,t), \eta_{t-1} \rangle =$$

$$= \langle A(t-1+d+dN_0), \eta_{t-1} \rangle \times A(t+dN_0) \times \dots \times A(t+d-2+dN_0).$$

On the other hand, $\langle A(t-1+d+dN_0), \eta_{t-1} \rangle = \omega^{t-1} \langle A(d+dN_0), \zeta \rangle$, where $\omega^{t-1}\zeta = \eta_{t-1}$. It is enough to prove $\langle A(d+dN_0), \zeta \rangle = A(dN_0)$. The bijection $N_0 \to dN_0$, $n \to dn$, reduces the last statement to the following: $\langle A(1+N_0), \zeta \rangle = A(N_0)$ under conditions $\zeta \in A(N_0)$ and $0 \in supp(\zeta)$. Clearly, the group $G = \langle A(1+N_0), \zeta \rangle$ is ω-stable and transitive on N_0. Hence its subgroup $\omega^n G$ is transitive on the set $n + N_0$ for all $n \geq 0$. By an inductive argument, the group G is n-fold transitive on N_0 for all $n \geq 0$. Taking into account the inclusion $G \subset A(N_0)$, we obtain $G = A(N_0)$.

Proof of Claim (2): For if let $\sigma \in W$ and let $\theta \in T_{I,s}$ be such that $f_\theta^{(d)}(x) = f_\sigma^{(d)}(x) \in I$ (see (2.1.3), (ii)). Then $f_\eta^{(d)}(x) = 0$ for $\eta = \sigma\theta^{-1}$. Hence $\eta \in A(d,0) \cap W$ and this intersection is $A(d,t)$ by Claim (1). Thus, $A(d,t)$ is a normal subgroup of W and $W = A(d,t)T_{I,s}$. Moreover, (2.1.3), (ii), implies that $A(d,t) \cap T_{I,s} = \{(1)\}$.

Clearly, Claim (2) and the inequality $s \geq t$ yield that $t = \min(suppW) = e$ and part (iii) follows. The proof of Proposition 2.1.4 is done.

Corollary 2.1.6. *If W is a non-trivial ω-stable group, then $W = \omega^e U$ where $U = r_0(W)$ and $e = \min(suppW)$.*

PROOF: Let $d \geq 1$ be the class of W. Then (2.1.4) yields that $A(d,e) \leq W \leq S(d,e)$. Taking radicals, we obtain

$$A(d,0) \leq U \leq S(d,0) \tag{2.1.7}$$

Let I be the ideal of R_d corresponding to W. According to (1.2.1), (i), we have

$$I = f_U^{(d)}. \tag{2.1.8}$$

Applying the endomorphism ω^e to (2.1.7) and (2.1.8), we obtain $A(d,e) \leq V \leq S(d,e)$ and $x^e I = f_V^{(d)}$ for $V = \omega^e(U)$. Since the multiplication by x^e is a F_2-linear automorphism of I, then $I = f_V^{(d)}$. Thus, $A(d,e) \leq V$, $W \leq S(d,e)$ with $f_V^{(d)} = f_W^{(d)}$; hence $V = W$.

3. ω-STABLE SYSTEMS OF GENERATORS OF AN ω-STABLE GROUP

3.1. Given a subset B of the symmetric group $S(N_0)$, we define

$$class(B) = \gcd\{\beta(i) - i \mid \beta \in B \text{ and } i \in N_0\}.$$

The connection with the class of a non-trivial ω-stable group (see 1.2) is made by the following

LEMMA 3.1.1. *Let W be a non-trivial ω-stable subgroup of $S(N_0)$ and let B be an ω-stable system of generators for W. Then $class(B) = class(W)$.*

PROOF: We set $d = class(W)$. By Proposition (2.1.4) we obtain $A(d,e) \leq W \leq S(d,e)$ for $e = \min(suppW)$. The obvious equality $d = \gcd\{\sigma(i) - i \mid \sigma \in W \text{ and } i \in N_0\}$ implies that d divides $class(B)$. Because of the equalities

$$\beta^{-1}(i) - i = -(\beta\beta^{-1}(i) - \beta^{-1}(i)) \text{ with } i \in N_0 \text{ and } \beta \in B,$$

and

$$\beta\gamma(i) - i = (\beta\gamma(i) - \gamma(i)) + (\gamma(i) - i) \text{ with } i \in N_0 \text{ and } \beta,\ \gamma \in B \cup B^{-1},$$

it follows that $class(B)$ divides d. Therefore $class(B) = d$.

Using (2.1.4), (3.1.1), (1.2.1), (ii), as well as the equality $\min(suppB) = \min(suppW)$ for $W = \langle B\rangle$, we obtain immediately

THEOREM 3.1.2. *Let W be a non-trivial ω-stable subgroup of $S(N_0)$ and let B be an ω-stable system of generators for W. Given integers $d \geq 1$ and $e \geq 0$, and an ideal I of the ring R_d, one has the following equivalent statements:*

(i) The group W satisfies the inequalities $A(d,e) \leq W \leq S(d,e)$ and $f_W^{(d)} = I$;

(ii) The set B satisfies the equalities $class(B) = d$, $\min(suppB) = e$ and $(f_B^{(d)}) = I$.

REMARK 3.1.3. *In case $d = 1$ and $e = 0$ we obtain Theorem A from the Introduction.*

Examples. The foregoing theorem will be illustrated in the following examples. Given $\sigma \in S(N_0)$, with $\sigma \neq (1)$ we set $B_\sigma = \{\sigma, \omega\sigma, \omega^2\sigma, \ldots\}$. Obviously, $class(B_\sigma) = class(\sigma)$.

1) Let $\sigma = (04)(17)(2,11,14)$. Since $class(\sigma) = \gcd(4,6,9,12) = 1$ and $\min(suppB_\sigma) = 0$, it follows that $\langle B_\sigma \rangle = A(N_0)$.

2) Let $\sigma = (0,33,48)$ and $\xi = (265,337)(1864,1904,1994)$. We set $B = B_\sigma \cup B_\xi$. Then

$$class(B) = \gcd(class(\sigma), class(\xi)) = \gcd(3,2) = 1 \text{ and } \min(suppB) = 0.$$

Therefore we have $\langle B \rangle = S(N_0)$.

3) Let

$$\sigma = (0,9)(1,10,19)(2,83,92,110,911)(12,21,30,57)(5,14,23)(15,33,42,78,87,1995).$$

Then for $B = B_\sigma$ we have $class(B) = 9$ and $\min(suppB) = 0$. The ideal $I = (f_B^{(d)}) \subset R_9$ is generated by the irreducible factor $x^6 + x^3 + 1$ of the polynomial $x^9 - 1$. Hence the subgroup $W = \langle B \rangle$ of $S(N_0)$ satisfies the inequalities $A(9,0) < W < S(9,0)$ and, moreover, it is the inverse image of the ideal I via the homomorphism $f^{(9)}$.

4) Let

$$\sigma = (1,10,19)(2,83,92,110,911)(12,21,30)(5,14,23)(15,33,42,78,87,1995).$$

Then for $B = B_\sigma$ we have $class(B) = 9$ and $\min(suppB) = 1$. Moreover, the ideal $(f_B^{(d)})$ coincides with the ring R_9. Therefore $\langle B \rangle = S(9,1)$.

3.2. In the next proposition we show that any non-trivial ω-stable subgroup of $S(N_0)$ possesses a standard ω-stable system of generators.

PROPOSITION 3.2.1. *Let W be a non-trivial ω-stable subgroup of $S(N_0)$ of class d with $\min(suppW) = e$. Let I be the corresponding ideal of the ring R_d and let $\theta_{I,s}$ be the permutations given by (2.1.1). Then the ω-stable system*

$$B_{I,e} = \{\theta_{I,s} \mid s \geq e\}$$

generates the group W. When $I \neq 0$ each generator $\theta_{I,s}$ of W has order 2. When $I = 0$ the generators $\theta_{0,e}$ of $W = A(d,e)$ are 3-cycles.

PROOF: We have $\theta_{I,s} = \omega^s \theta_I$ where $\theta_I = \theta_{I,0}$. In case $I \neq 0$ Lemma 1.3.1, (iv), yields that every element of $B_{I,e}$ has order 2. When $I = 0$ we have $\theta_0 = \tau^{1+x^d} = (0,d)(d,2d) = (0,d,2d) \in A(d,0)$. Hence $B_{0,e}$ consists of 3-cycles. In both cases we have $class(B_{I,e}) = class(\theta_I) = d$ and it is obvious that $e = \min(suppB_{I,e})$. If we set $V = \langle B_{I,e} \rangle$, then (3.1.2) and (2.1.4), applied for V and W, respectively, yield that $A(d,e) \leq V$, $W \leq S(d,e)$. Moreover, we have $T_{I,s} \leq V$ for each $s \geq e$. Therefore $f_V^{(d)} = I = f_W^{(d)}$, that is, $V = W$.

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