ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

The asymptotic covariance matrix of multivariate serial correlations

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БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ

1869

BULGARIAN ACADEMY OF SCIENCES

PREPRINT
No 7

Computer Stochastics Laboratory

Sofia, April 1995

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ABSTRACT

We show that the entries of the asymptotic covariance matrix of the serial covariances and serial correlations of a multivariate stationary process can be expressed in terms of the autocovariances corresponding to the tensor square of its spectral density. The tensor convolution introduced in the paper may be of some interest on its own.

Keywords: Asymptotic distribution, Multivariate ARMA, Serial covariances, Serial correlations, Bartlett's formula, Tensor convolution.

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1 Introduction

The serial covariances and serial correlations are important tools in the analysis of time series. They are used in model identification, estimation of parameters, goodness-of-fit and other hypotheses tests (see e.g., [1, Chapter 6], [3], [17] and the references therein).

The statistical analysis of the procedures based on serial correlations is often based on their asymptotic distribution. It is normal under mild conditions in the univariate case (see [1, Chapter 8], Hannan and Heyde [13], Anderson [2]) and under more stringent conditions in the multivariate case (Hannan [12], Roy [16]).

The expressions for the asymptotic covariances of the serial covariances (correlations) contain infinite sums. For an univariate process these sums can be interpreted (up to a constant factor) as the autocovariances corresponding to the square of its spectral density (see [7], [3]). In particular, the asymptotic distribution of the serial covariances (correlations) of an univariate ARMA process can be expressed in terms of the autocovariance function of another ARMA process, whose parameters are obtained from the parameters of the initial process by squaring its autoregressive operator, moving average operator and the residual variance. Details are given in [8].

Our aim is to show that these statements remain essentially the same in the multivariate case (i.e. for the distribution of the serial crosscorrelations), provided that the square of a matrix a is interpreted as $a \otimes a$ (the Kronecker, or tensor, square of a). The interpretation of a number of individual infinite sums as the entries of the autocovariance matrices of the Kronecker square of a spectral density gives not only a nice mathematical way to write compactly these sums, but it provides also an efficient method for computation of the asymptotic distribution of the serial correlations (see Section 4).

The tensor convolution which is introduced in Section 3 seems to be of some interest by its own, beyond the context of the paper.

2 Notations

We consider a d-variate weakly stationary process $\{X_t\}$ with (vector) mean μ, autocovariance function $R(k) = E(X_t - \mu)(X_{t-k} - \mu)'$, autocorrelation function $r(k) = D_0^{-1/2} R(k) D_0^{-1/2}$ (D_0 is the diagonal matrix formed by taking the diagonal of $R(0)$), and spectral density $f(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} R(k) e^{-iwk}$.

The components of the column vectors X_t and μ are referred to by an additional index i, $i = 1, \ldots, d$. In scalar notations the i,j-th element $R_{ij}(k)$ of $R(k)$ is the covariance $E(X_{t,i} - \mu_i)(X_{t-k,j} - \mu_j)$. Similarly, $r_{ij}(k) = R_{ij}(k)/(R_{ii}0R_{jj}0)^{1/2}$, and $f_{ij}(\omega)$ is the Fourier transform of R_{ij}. We say the $\{X_t\}$ is white noise if $EX_t = 0$ and $R(k)$ is the zero matrix for $k \neq 0$.

3 Tensor convolution

Although weaker assumptions are possible, we suppose below that the sequences $\{a_k\}$, $\{b_k\}$, etc., are absolutely summable. This condition ensures the validity of the following formulas and the manipulations on them (see e.g. [10, Chapter 3]). It is fulfilled for the autocovariance functions of the ARMA processes.

The Fourier transform $\mathcal{F}\{a\}$ of the sequence $\{a_k\}$ is defined by

$$\mathcal{F}\{a\}(\omega) = \frac{1}{2\pi} \sum_{k=-\infty}^{\infty} a_k e^{-i\omega k}.$$

The inverse Fourier transform restores the sequence $\{a_k\}$,

$$a_k = \int_{-\pi}^{\pi} e^{i\omega k} \mathcal{F}\{a\}(\omega)\, d\omega.$$

The convolution of two sequences $\{a_k\}$ and $\{b_k\}$ is given by

$$(a * b)_k = \sum_{u=-\infty}^{\infty} a_{k-u} b_u.$$

The Fourier transform of the convolution multiplies the Fourier transforms of the arguments ([10, Corollary 3.4.1.1]),

$$\mathcal{F}\{a * b\} = 2\pi \mathcal{F}\{a\}\mathcal{F}\{b\}. \tag{1}$$

The tensor (Kronecker) product $A \otimes B$ of two matrices A and B is defined as the block matrix obtained by replacing every element a_{ij} of A by $a_{ij}B$.

To write compactly the inverse Fourier transform of a tensor product we need the notion for tensor convolution. The convolution of a single (scalar) sequence $\{a_k\}$ with the matrix sequence $\{B_k\} = (b_{ij}(k))_{ij}$ is defined to be the matrix sequence $(a * b_{ij})_{ij}$ of the elementwise convolutions of $\{a_k\}$ with the elements of $\{B_k\}$.

Definition 1 *The tensor convolution $A \boxplus B$ of two matrix sequences $\{A_k\}$ and $\{B_k\}$ is the block matrix*

$$A \boxplus B = \begin{pmatrix} a_{11} * B & \dots & a_{1n} * B \\ \vdots & & \vdots \\ a_{m1} * B & \dots & a_{mn} * B \end{pmatrix}$$

obtained by replacing every element a_{ij} of A by $a_{ij} * B$.

The properties of the tensor square $f \otimes f$ of a spectral density are summarized in the following proposition.

Proposition 1 *Let $\{R_k\}$ be an absolutely summable autocovariance (matrix) sequence and $f(\omega)$—its spectral density matrix. Suppose that the factorization of $f(\omega)$ is ([11, Chapter 3.2, Theorem 1; Chapter 3.5, Theorem 1″])*

$$f(\omega) = \frac{1}{2\pi}\Phi(e^{i\omega})G\Phi(e^{i\omega})^*.$$

Let $g(\omega) = 2\pi(f(\omega) \otimes f(\omega))$. Then

1. *$g(\omega)$ is a spectral density matrix;*
2. *the factorization of $g(\omega)$ is*

$$g(\omega) = \frac{1}{2\pi}\Psi(e^{i\omega})C\Psi(e^{i\omega})^*,$$

where $\Psi(e^{i\omega}) = (\Phi(e^{i\omega}) \otimes \Phi(e^{i\omega}))$, $C = (G \otimes G)$;

3. *the autocovariance function of $g(\omega)$ is $R \boxplus R$, i.e.*

$$(R \boxplus R)_k = \int_{-\pi}^{\pi} e^{ik\omega} g(\omega)\, d\omega \tag{2}$$

Proof. Since f is a spectral matrix it is Hermitian and nonnegative definite for each ω. It is straightforward to check that $f \otimes f$ inherits these properties and therefore is also a spectral density matrix. Indeed, let $a = (i-1)d + (r-1)$, $b = (j-1)d + (s-1)$, where r and s are in the range $[1, \dots, d]$. Then $(f \otimes f)_{a,b} = f_{ij}f_{rs}$ and $(f \otimes f)_{b,a} = f_{ji}f_{sr}$. The Hermitian property of f implies that $f_{ji} = f_{ij}^*$. Hence $(f \otimes f)_{b,a} = (f \otimes f)_{a,b}^*$ and $f \otimes f$ is also

Hermitian. The eigenvalues of $f \otimes f$ are of the form $\mu_{ij} = \lambda_i \lambda_j$, where λ_i are the eigenvalues of f and therefore are nonnegative (f is nonnegative definite). Therefore $\mu_{ij} \geq 0$. Hence $f \otimes f$ is also nonnegative definite. Furthermore, if the rank of $f(\omega)$ is constant for almost all ω then the same is true for $f \otimes f$.

The factorization of $g(\omega)$ can be obtained as follows:

$$\begin{aligned} 2\pi g(\omega) &= 2\pi(f(\omega) \otimes f(\omega)) \\ &= \frac{2\pi}{4\pi^2}(\Phi(e^{i\omega})G\Phi(e^{i\omega})^*) \otimes (\Phi(e^{i\omega})G\Phi(e^{i\omega})^*) \\ &= \frac{1}{2\pi}(\Phi(e^{i\omega}) \otimes \Phi(e^{i\omega}))(G \otimes G)(\Phi(e^{i\omega}) \otimes \Phi(e^{i\omega}))^*, \end{aligned}$$

since $(AB) \otimes (CD) = (A \otimes C)(B \otimes D)$ ([14, p. 408]).

The convolution theorem (1) implies that

$$\mathcal{F}\{R_{ij} * R_{lm}\} = 2\pi \mathcal{F}\{R_{ij}\}\mathcal{F}\{R_{lm}\} = 2\pi f_{ij}(\omega) f_{lm}(\omega).$$

Therefore, for $k = 0, \pm 1, \pm 2, \ldots$ we have

$$\begin{aligned} (R_{ji} * R_{lm})_k &= \mathcal{F}^{-1}\{\mathcal{F}\{R_{ji} * R_{lm}\}\}(k) \\ &= \int_{-\pi}^{\pi} e^{ik\omega}(2\pi f_{ji}(\omega) f_{lm}(\omega))\, d\omega \end{aligned} \quad (3)$$

With the help of the tensor convolution the last equation can be written in the form

$$(R \boxplus R)_k = \int_{-\pi}^{\pi} e^{ik\omega}(2\pi f(\omega) \otimes f(\omega))\, d\omega.$$

Elementwise comparison establishes (2). Hence, $(R \boxplus R)_k$ is the autocovariance sequence corresponding to the spectral density $g(\omega)$. □

In the ARMA case we have the following corollary.

Corollary 1 *Let $\{X_t\}$ be multivariate ARMA process with autocovariance sequence R_k and (causal) representation*

$$\Phi(B)X_t = \Theta(B)\epsilon_t,$$

where $\{\epsilon_t\}$ is white noise with covariance matrix $\Sigma = E\epsilon_t\epsilon_t'$. Then $R \boxplus R$ is the autocovariance function of another ARMA process, specified by the model

$$\alpha(B)\xi_t = \beta(B)\eta_t,$$

where η_t is white noise, $E\eta\eta' = \Sigma \otimes \Sigma$, $\alpha(B) = \Phi \otimes \Phi$, $\beta(B) = \Theta \otimes \Theta$.

Proof. The spectral density of $\{X_t\}$ is ([11, Chapter 2.5, (iv)])

$$\begin{aligned} f(\omega) &= \frac{1}{2\pi}\Phi^{-1}(e^{i\omega})\Theta(e^{i\omega})\Sigma(\Phi^{-1}(e^{i\omega})\Theta(e^{i\omega}))^* \\ &= \frac{1}{2\pi}\Phi^{-1}\Theta\Sigma\Theta^*\Phi^{-*}, \end{aligned}$$

where Φ^{-*} is the conjugate transpose of Φ^{-1}. Hence

$$\begin{aligned} 2\pi(f\otimes f) &= 2\pi(\frac{1}{2\pi}\Phi^{-1}\Theta\Sigma(\Phi^{-1}\Theta)^*)\otimes(\frac{1}{2\pi}\Phi^{-1}\Theta\Sigma(\Phi^{-1}\Theta)^*) \\ &= \frac{1}{2\pi}(\Phi^{-1}\Theta\Sigma\Theta^*\Phi^{-*})\otimes(\Phi^{-1}\Theta\Sigma\Theta^*\Phi^{-*}) \\ &= \frac{1}{2\pi}(\Phi^{-1}\otimes\Phi^{-1})(\Theta\otimes\Theta)(\Sigma\otimes\Sigma)(\Theta^*\otimes\Theta^*)(\Phi^{-*}\otimes\Phi^{-*}) \\ &= \frac{1}{2\pi}(\Phi\otimes\Phi)^{-1}(\Theta\otimes\Theta)(\Sigma\otimes\Sigma)(\Theta\otimes\Theta)^*(\Phi\otimes\Phi)^{-*}, \end{aligned}$$

which proves the corollary since the last line is exactly the spectral density of $\{\xi_t\}$. □

In the univariate case $\{X_t\}$ $\{\varepsilon_t\}$, ξ_t, and η_t are univariate processes, Σ is scalar, say $\sigma^2 = E\varepsilon_t^2$, $\alpha(B) = \Phi^2(B)$, $\beta(B) = \Theta^2(B)$, and $E\eta\eta' = E\eta^2 = \sigma^4$ (see [7] and [8]).

4 Covariances of the serial correlations

The serial covariances $\widehat{R}(k)$, $k = 0, 1, \ldots$, and the serial correlations $\widehat{r}(k)$, $k = 1, 2, \ldots$, from a stretch $(X_1, \ldots, X_N)$ from $\{X_t\}$ of length N are defined as

$$\widehat{R}(k) = \frac{1}{N}\sum_{i=k+1}^{N}(X_i - \bar{X})(X_{i-k} - \bar{X})', \quad \widehat{r}(k) = \hat{D}_0^{-1/2}\widehat{R}(k)\hat{D}_0^{-1/2}.$$

The asymptotic covariances between the individual entries of $\widehat{R}(k)$, and $\widehat{r}(k)$ are defined by the limits:

$$\begin{aligned} \Gamma_{k,h}(i,j,l,m) &= \lim_{N\to\infty} N\mathrm{Cov}(\widehat{R}_{ij}(k), \widehat{R}_{lm}(h)) \\ \gamma_{k,h}(i,j,l,m) &= \lim_{N\to\infty} N\mathrm{Cov}(\widehat{r}_{ij}(k), \widehat{r}_{lm}(h)). \end{aligned}$$

The formulae for $\Gamma_{k,h}(i,j,l,m)$ and $\gamma_{k,h}(i,j,l,m)$ are traditionally called Bartlett's formulae and look as follows ([16])

$$
\begin{aligned}
\Gamma_{k,h} &= \Theta_{h-k}(i,l,j,m) + \Theta_{h+k}(j,l,i,m) \\
\gamma_{k,h} &= \frac{1}{2} r_{ab}(k) r_{de}(h) \{\Delta_0(a,d,a,d) + \Delta_0(a,e,a,e) \\
&\qquad + \Delta_0(b,d,b,d) + \Delta_0(b,e,b,e)\} \\
&\quad - r_{ab}(k)\{\Delta_h(a,d,a,e) + \Delta_h(b,d,b,e)\} \\
&\quad - r_{de}(h)\{\Delta_k(b,d,a,d) + \Delta_h(b,e,a,e)\} \\
&\quad + \Delta_{h-k}(a,d,b,e) + \Delta_{h+k}(b,d,a,e),
\end{aligned}
$$

where

$$
\begin{aligned}
\Theta_k(i,j,l,m) &= \sum_{u=-\infty}^{\infty} R_{ij}(u) R_{lm}(u+k), \\
\Delta_k(i,j,l,m) &= \sum_{u=-\infty}^{\infty} r_{ij}(u) r_{lm}(u+k), \\
&= \Theta_k(i,j,l,m)\{R_{ii}(0)R_{jj}(0)R_{ll}(0)R_{mm}(0)\}^{-1/2}.
\end{aligned}
$$

The quantities $\Theta_k(i,j,l,m)$ and $\Delta_k(i,j,l,m)$ can be viewed as convolutions since

$$
\begin{aligned}
\sum_{u=-\infty}^{\infty} R_{ij}(u) R_{lm}(u+k) &= \sum_{u=-\infty}^{\infty} R_{ji}(-u) R_{lm}(u+k) \\
&= \sum_{v=-\infty}^{\infty} R_{ji}(k-v) R_{lm}(v) \qquad \text{(v=u+k)} \\
&= R_{ji} * R_{lm}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
\sum_{u=-\infty}^{\infty} R_{ij}(u) r_{lm}(u+k) &= r_{ji} * r_{lm}, \\
&= (R_{ii}(0)R_{jj}(0)R_{ll}(0)R_{mm}(0))^{-1/2} R_{ji} * R_{lm},
\end{aligned}
$$

Comparing the last expressions and (3) we can see that $\Theta_k(i,j,l,m)$ are elements of the tensor convolution $R \boxplus R$. More specifically, $\Theta_k(i,j,l,m)$ is the $((i-1)d+l, (j-1)d+m)$-th entry of the matrix $(R \boxplus R)_k$. On the other hand Corollary 1 shows that in the ARMA case $R \boxplus R$ is the autocovariance function of an ARMA process and therefore can be computed effectively (see e.g., [4]). This approach should be compared with the individual computation of $\Theta_k(i,j,l,m)$ for every necessary combination of k, i, j, l and m.

5 Acknowledgements

This research has been partly supported by a contract with the Bulgarian Ministry of Science and Education.

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