

ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР
INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

ZEROS OF PADÉ ERROR FUNCTIONS
FOR FUNCTIONS WITH
SMOOTH MACLAURIN COEFFICIENTS

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BULGARIAN
ACADEMY
OF SCIENCES

Preprint

N 8

April 1995

Department of Complex Analysis

ZEROS OF PADÉ ERROR FUNCTIONS FOR FUNCTIONS WITH SMOOTH MACLAURIN COEFFICIENTS

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Abstract: We deal with functions $f(z) := \sum_{n=0}^{\infty} a_n z^n$ whose coefficients satisfy Lubinsky's smoothness condition, namely: $a_{j+1} \cdot a_{j-1}/a_j^2 \rightarrow \eta$ as $j \rightarrow \infty$, $\eta \neq \infty$. In the present paper, theorems concerning the asymptotic behaviour of the normalized (in an appropriate way) Padé error functions $(f - \pi_{n,m})$ as $n \rightarrow \infty$, m - fixed, are provided. As a consequence, results concerning the number of the zeros and of their limiting distribution are deduced.

1. Introduction and main results

Let

$$f(z) := \sum_{j=0}^{\infty} a_j z^j \tag{1.1}$$

be a function with $a_j \neq 0$ for all nonnegative integers j , ($j \in \mathbb{N}$) large. We set

$$\eta_j := a_{j+1} \cdot a_{j-1}/a_j^2, \quad j = j_0, j_1, \dots$$

The basic assumption throughout the present work is that

$$\eta_j \rightarrow \eta, \text{ as } j \rightarrow \infty. \tag{1.2}$$

This kind of convergence has been introduced and studied by D. Lubinsky in [7], where important theorems resulting from (1.2) with respect to the asymptotic of Toeplitz determinants and uniform convergence of the m -th row of the table of classical Padé approximants to f are proved. Therefore, in what follows, condition (1.2) will be called "Lubinsky's smoothness condition".

Let $\rho(f)$ be the radius of convergence of the power series (1.1). We notice that under Lubinsky's smoothness condition (1.1) represents an entire function, if $|\eta| < 1$; the radius of convergence $\rho(f)$ is zero, if $|\eta| > 1$. If $|\eta| = 1$, then (1.1) may have a positive or a zero radius of convergence.

Further, we assume that the numbers η_n tend to η smoothly enough, namely: there exist complex numbers $\{c_i\}_{i=1}^{\infty}$ with $c_1 \neq 0$ such that for each $N, N \in \mathbb{N}, N > 1$, we have the representation

$$\eta_n = \eta \cdot \left\{ 1 + c_1/n + \sum_{i=2}^N c_i/n^i + o(n^{-N}) \right\}. \tag{1.3}$$

* This paper was supported in part under grant No 436 of the Bulgarian National Science Foundation

Introduce the function $H_\eta(z)$ with

$$H_\eta(z) := \sum_{j=0}^{\infty} \eta^{j(j+1)/2} z^j.$$

It is clear that for $|\eta| < 1$, $H_\eta(z)$ is an entire function. If $|\eta| \leq 1$, then $H_\eta(z)$ is holomorphic in the unit disk; in the case when $|\eta| > 1$, the radius of convergence is zero.

Notice that $H_\eta(z) = h(\eta z)$, where $h(z)$ is the partial theta function. Its properties (natural boundary, domains omitting zeros etc) have been studied in [8].

Let now m be a nonnegative integer. In our further considerations, we will assume that m is fixed.

Further, we assume that the power series (1.1) does not represent a rational function with a number of finite poles, not more than m (we write $f \notin \mathcal{R}_m$).

For each $n, n \in \mathbb{N}$, let $\pi_{n,m} (= \pi_{n,m}(f))$ be the Padé approximant to the function f of order (n, m) . We set

$$\pi_{n,m} = P_{n,m}/Q_{n,m},$$

where $Q_{n,m}(0) = 1$ and both polynomials $P_{n,m}$ and $Q_{n,m}$ do not have a common divisor.

Let $D(n, m) = \det \{a_{n-j+k}\}_{j,k=1}^m$ be Toeplitz's determinant formed from the Maclaurin coefficients of (1.1). Under our basic assumption concerning nonrationality of f , it is true that the inequality $D(n, m) \neq 0$ holds for an infinite sequence of positive integers n (see, for instance, [2] and [3]). Denote by Λ the sequence of those positive integers for which $\deg Q_{n,m} = m$; as it is known (see [2], [3]) Λ is infinite (recall that $f \notin \mathcal{R}_m$). For any $n \in \mathbb{N}$ the equality $\pi_{n,m} \equiv \pi_{k(n),m}$, where $k(n) := \max \{k, k \leq n, k \in \Lambda\}$ is valid. For $n \in \Lambda$ there holds (see [9])

$$(f \cdot Q_{n,m} - P_{n,m})(z) = z^{n+m+1} \cdot (-1)^m \cdot D(n+1, m+1)/D(n, m) + \dots,$$

and

$$Q_{n,m}(z) = 1 + \dots + z^m \cdot (-1)^m D(n+1, m)/D(n, m).$$

Recalling now the structure of Padé's table corresponding to f , we assume without losing the generality that the last formulas hold for every $n \in \mathbb{N}$ starting with number n_0 .

In what follows, we shall call the difference $f - \pi_{n,m}$ the Padé error function to f of order (n, m) .

Recently, the limiting distribution of the zeros of the m -th row in Padé's table for entire functions with "smooth" Maclaurin coefficients was considered (see [5]). As a consequence, the limiting distribution of the zeros of the sequence of Padé approximants $\pi_{n,m}$ as $n \rightarrow \infty$ was characterized. The goal of the present paper is to explore analogous problems with respect to the Padé error functions.

Denote by $S_n(z) = S_n(f, z)$ the n -th partial sum of the function $f(z)$:

$$S_n(z) := \sum_{j=0}^n a_j z^j.$$

We notice that $S_n(z) = \pi_{n,0}(z)$ for every $N \in \mathbf{N}$.

The starting point for the investigations is the following unpublished result by E. B. Saff which describes the limiting behaviour of the differences $f - S_n$ as $n \rightarrow \infty$ normalized in an appropriate way.

Theorem 1: Set

$$W_n(u) := (f - S_n)(ua_n/a_{n+1})/a_{n+1}(ua_n/a_{n+1})^{n+1}.$$

Assume that (1.2) holds with (i): $|\eta| < 1$ and (ii): with $|\eta| = 1$ in a way that $|\eta_n| \leq 1$ for all n large enough. Then, respectively

i:)

$$W_n(u) \rightarrow H_\eta(u) \tag{1.4i}$$

uniformly inside in \mathbf{C} and

ii:)

$$W_n(u) \rightarrow H_\eta(u) \tag{1.4ii}$$

uniformly inside $\{u : |u| < 1\}$.

As usual, "uniformly inside" a given set $M, M \in \mathbf{C}$, means uniform convergence on compact subsets of M in the uniform norm.

Let now m be a fixed positive integer. The first result in the present paper refers to functions (1.1) for those $\eta \neq 1$. Following [8], we introduce the polynomials $B_m(u) := B_m(u, q), m \in \mathbf{N}$ fixed, as follows:

$B_0(u) := 1$ and for $m = 1, 2, \dots$

$$B_m(u) := B_{m-1}(u) - u \cdot q^{m-1} \cdot B_{m-1}(u/q).$$

When q is not a root of unity, then

$$B_m(-u) = \sum_{j=0}^m \frac{u^j \prod_{k=1}^j (1 - q^{m+1-k})}{\prod_{k=1}^j (1 - q^k)};$$

furthermore, $B_m(u) = (1 - u)^m$, when $q = 1$.

These polynomials are of importance in the investigation of the distribution of the zeros of Padé error functions $f - \pi_{n,m}$ in the case when the number η in (1.2) is not a root of unity.

For $0 < q < 1$, the polynomials B_m (suitably normalized) are orthogonal with respect to a nonnegative weight on the unit circle (see [1]), so that all their zeros lie in $\{z, |z| \leq 1\}$. For $q = 1$ results concerning the distribution of the zeros of the polynomials $B_m(u), m = 0, 1, \dots$ can be found in [8].

For our goal, we introduce an appropriate normalization of the error functions. Set

$$e_{n,m} := \frac{(f - \pi_{n,m})(ua_n/a_{n+1})}{a_{n+1}(ua_n/a_{n+1})^{n+1}}.$$

The following theorem describes the limiting behaviour of the sequence $e_{n,m}$ for m fixed and $n \rightarrow \infty$.

Theorem 2: Assume that (1.2) holds for a number η with $\eta \neq \infty$ in the way that i): η is not a root of unity and ii): η is a root of unity of order m_0 and satisfies (1.3). Then i): for any m and ii): for any $m, m \leq m_0 - 1$ there holds

$$e_{n,m}(u) \rightarrow H_\eta(u) + \sum_{k=0}^{m-1} \frac{\prod_{j=1}^k (1 - \eta^j) (-1)^{k+1} u^k}{B_k(u) B_{k+1}(u)} \text{ as } n \rightarrow \infty$$

uniformly inside the domains described by Theorem 1), excluding \mathcal{B} , where \mathcal{B} is the set of the zeros of the polynomials $B_k = B_k(z, \eta), k = 1, \dots, m$

Set $\delta(m, \eta) := \min \{|z| : B_m(z, \eta) = 0, k = 1, \dots, m\}$. From Theorem 2, we get

Corollary 1: With the assumptions of Theorem 2, for any $\varepsilon, 0 < \varepsilon < 1$, the Padé error function $f - \pi_{n,m}$ has for n sufficiently large not more than a finite number of zeros in $0 < |z| < \delta(m, \eta)(1 - \varepsilon) \cdot |a_n/a_{n+1}|$.

The next results characterize the limiting behaviour of the error functions as $n \rightarrow \infty$, m - fixed, in the case when the numbers η_n tend to $\eta = 1$ in the way described by (1.3).

Denote by $E_{n,m}(u)$ the error function $f - \pi_{n,m}$ normalized as follows:

$$E_{n,m}(u) := \frac{(f - \pi_{n,m})(ua_n/a_{n+1})}{(ua_n/a_{n+1})^{n+m+1} \cdot (-1)^m \cdot D(n+1, m+1) / D(n, m)}$$

In the present paper, we prove

Theorem 3: Let $m \in \mathbb{N}$ be fixed and $f \notin \mathcal{R}_m$. Assume that $a_j \neq 0$ for j large; assume, further that η_n admits the expansion (1.3) with $\eta = 1, c_1 \neq 0$ and $|\eta_n| \leq 1$ for all $n \in \mathbb{N}$ sufficiently large.

Then

$$E_{n,m}(u) \rightarrow (1 - u)^{-2m-1} \text{ as } n \rightarrow \infty$$

uniformly inside $\{u : |u| < 1\}$.

From Theorem 3, we have

Corollary 2: With the assumptions of Theorem 3, for each fixed $m \in \mathbb{N}$ and any $\varepsilon, 0 < \varepsilon < 1$, the Padé error function $(f - \pi_{n,m})(z)$ has no zeros in $0 < |z| < |a_n/a_{n+1}| \cdot (1 - \varepsilon)$ for n sufficiently large.

Recall that under our assumptions each Padé error function of order (n, m) has a zero at $z = 0$ of order $m + n + 1$.

Further, we consider the special case when $\eta = 1$ and the first coefficient c_1 in (1.3) is a real negative number. Under this additional condition, the next result provides information about the existence of "extraneous" zeros of the normalized Padé error functions $E_{n,m}(u)$ and about the limiting behaviour of those zeros as $n \rightarrow \infty$, as well.

Theorem 4: If $\eta = 1$ and $c_1 < 0$, then $u = 1$ is a limit point of zeros of $\{E_{n,m}(u)\}_{n=1}^\infty$.

For $n \in \mathbb{N}$, we denote by P_n the set of the zeros of $E_{n,m}(u)$. Set $P_n := \{\xi_{n,k}\}$ with the normalization $|1 - \xi_{n,k}| \leq |1 - \xi_{n,k+1}|, k = 1, \dots$. From Theorem 4, we have

$$\text{dist}(P_n, 1) \rightarrow 0, \text{ as } n \rightarrow \infty.$$

For any positive ε , denote by $\iota_n(\varepsilon)$ the number of the zeros of $\xi_{n,k}$ which lie in the disk of radius ε , centered at $u = 1$. In the present paper we prove

Theorem 5: Under the conditions of Theorem 4, for any ε , we have

$$\liminf_{n \rightarrow \infty} \frac{k_n(\varepsilon)}{n} > 0. \quad (1.5)$$

For a number ε , $0 < \varepsilon < 1$, we denote by $\mathcal{A}_n(\varepsilon)$ the annulus $(1 - \varepsilon)|a_n/a_{n+1}| < |z| < (1 + \varepsilon)|a_n/a_{n+1}|$. From the last theorem, we have

Corollary 3: If $\eta = 1$ and $c_1 < 0$, then for any ε , $0 < \varepsilon < 1$ the Padé error function $f - \pi_{n,m}$ has, for n large enough, extraneous zeros which are situated in the annulus $\mathcal{A}_n(\varepsilon)$. Their number $k_n(\varepsilon)$ satisfies, as $n \rightarrow \infty$, condition (1.5).

Set

$$R := \liminf_{n \rightarrow \infty} |a_n/a_{n+1}|.$$

(Notice that $\rho(f) \geq R$.)

Obviously, if $R = \infty$, then f is an entire function. If in addition the conditions of Theorem 4 are fulfilled, then each Padé error function $f - \pi_{n,m}$ has, for n large enough, extraneous zeros and they go to infinity, as $n \rightarrow \infty$, with the speed of $|a_n/a_{n+1}|$.

Further, if $\eta = 1$ and $0 < R < \infty$, (in this case $\rho(f) > 0$), then the set $\{z : |z| < R\} - 0$ does not contain, in view of Theorem 3, accumulation points of the zeros of $(f - \pi_{n,m})$ as $n \rightarrow \infty$ (recall that each Padé error function has a zero at $z = 0$ of order $m + n + 1$). If in addition the parameter c_1 in (1.3) is a negative number, then, in accordance with Theorem 4, the circle $\{z : |z| = R\}$ contains accumulation points of zeros of $(f - \pi_{n,m})(z)$ as $n \rightarrow \infty$. If $R = \limsup_{n \rightarrow \infty} |a_n/a_{n+1}|$, then $R = \rho(f)$ and all the extraneous zeros of $f - \pi_{n,m}$ tend to the circle $\{z : |z| = R\}$.

Finally, if $\rho(f) = 0$, then $z = 0$ is an accumulation point of extraneous zeros of $f - \pi_{n,m}$, as $n \rightarrow \infty$.

Important functions to which Theorem 4 may be applied are the exponential function (see [10])

$$f(z) = \exp z = \sum_{j=0}^{\infty} z^j/j!$$

and the Mittag-Leffler function of order λ , $\lambda > 0$, (see [4])

$$f(z) = \sum_{j=0}^{\infty} z^j/\Gamma(1 + j/\lambda), \quad \lambda > 0.$$

The Padé error function for e^{-z} has been considered in [6].

2. Preliminaries

Lemma 1: For any n and m , there holds

$$\frac{a_{n+m}}{a_n} = \left(\frac{a_{n+1}}{a_n} \right)^m \prod_{j=1}^{m-1} \eta_{n+m-j}^j.$$

The proof will be omitted.

Set

$$D_{n,m} := \frac{D(n,m)}{a_n^m}$$

The following lemma is of essential importance for all the considerations in the present paper.

Lemma 2 (see [7]): Let f be a formal power series, with $a_j \neq 0$ for j large. Assume that η_j has the asymptotic expansion (1.3) with $c_1 \neq 0$. Then for $m = 1, 2, \dots$ we have

$$D_{n,m} = (-c_1/n)^{m(m-1)/2} \cdot \prod_{j=1}^{m-1} j^{m-j} \cdot \{1 + \alpha(1,m)/n + o(1/n)\} \text{ as } n \rightarrow \infty. \quad (3.2)$$

and

$$\lim_{n \rightarrow \infty} Q_{n,m}(ua_n/a_{n+1}) = B_m(u).$$

If (1.2) holds for a number η that is not a root of unity of order m then

$$\lim_{n \rightarrow \infty} D_{n,m} = \prod_{j=1}^{m-1} (1 - \eta^j)^{m-j}$$

We set

$$I_{n,j} := \prod_{l=0}^j \eta_{n+1+l}$$

The next lemma describes the asymptotic behaviour of $I_{n,j}$ as $n \rightarrow \infty$ for j "small". Before presenting it, we introduce for a given function g and a fixed number $p, p \in \mathbb{N}$, the operator

$$\nabla^p g(x) := \sum_{i=0}^p (-1)^i \binom{p}{i} g(x-i),$$

with

$$\nabla^0 f(x) := f(x) \quad (3.2)$$

and

$$\nabla f(x) := \nabla^1 f(x)$$

Lemma 3 (see [5]): Assume that η_j has the asymptotic expansion (1.3) with $\eta = 1$ and $c_1 \neq 0$. Let N be an arbitrary positive integer. Then

a): for each $j, j+1 \leq n/N$ we have as $n \rightarrow \infty$

$$I_{n,j} = 1 + \sum_{s=1}^N (j+1) \cdot Q_{s-1}(j) / (n+1)^s + M_{N+1}(j,n), \quad (3.4)$$

where Q_s is a polynomial of degree $\leq s$; $\quad \spadesuit \quad (3.5)$

b): for each fixed $p, p = 0, 1, \dots$ and for any $j, j + 1 < (n - p) / (N + 2p)$ the remainder $M_{N+1}(j, n)$ behaves as follows:

$$(n + 1)^{N+1+p} |\nabla^p M_{N+1}(j, n)| \leq C_{N+1,p} (j + 1)^{N+1}$$

with suitable positive constants $C_{N+1,p}$ which do not depend on n and j .

We notice that

$$Q_0(j) = c_1 \tag{2.1}$$

and

$$Q_1(j) = (jc_1^2 - jc_1 + 2c_2) / 2 \tag{2.2}$$

(In what follows, we shall denote by $C_{a,b,\dots}$ and $C(\dots)$ positive constants that do not depend on n .)

3. Proofs of the results.

Proof of Theorem 1: (See [11])

Applying Lemma 1 to the normalized error function $W_n(u)$, we easily get

$$W_n(u) = 1 + \sum_{j=1}^{\infty} d_{n,j} u^j,$$

where

$$d_{n,j} := \prod_{l=1}^j \eta_{n+j+1-l}^l \tag{3.1}$$

First, we notice that under condition (1.2), for any fixed positive integer k there holds

$$1 + \sum_{j=1}^k d_{n,j} u^j \rightarrow \sum_{j=0}^k \eta^{j(j+1)/2} z^j \text{ as } n \rightarrow \infty \tag{3.2}$$

We first consider case i).

Suppose that $|\eta| < 1$. Condition (1.2) ensures the existence of a positive number δ with

$$|\eta| < 1 - \delta. \tag{3.3}$$

Let ε and R be arbitrarily fixed positive numbers. Obviously, there is an integer k such that

$$R(1 - \delta)^{(k+1)/2} \leq 1/2 \tag{3.4}$$

and

$$(1/2)^k < \varepsilon/2 \tag{3.5}$$

We rewrite $W_n(u)$ as follows:

$$W_n(u) = 1 + \sum_{j=1}^{k-1} d_{n,j} u^j + \sum_{j=k}^{\infty} d_{n,j} u^j := W_{n,1,k}(u) + W_{n,2,k}(u).$$

From (3.3), we have for every n large enough ($n > n_\delta$) the inequality

$$|d_{n,j}| \leq (1 - \delta)^{j(j+1)/2}$$

From the last inequality, we get for $n > n_\delta$

$$\|W_{n,2,k}(u)\|_{|u| \leq R} \leq \left(R(1 - \delta)^{(k+1)/2} \right)^k \cdot \sum_{l=0}^{\infty} R^l (1 - \delta)^{(l^2 + l(1+2k))/2}$$

Making now use of (3.4) and (3.5), for every $n > n_\delta$ we obtain

$$\|W_{n,2,k}(u)\|_{|u| \leq R} < \varepsilon.$$

Together with (3.2), this inequality yields statement i) of Theorem 1.

Assume now that the conditions of case ii) hold; then for every n large enough ($n > n_o$) we have

$$|d_{n,j}| \leq 1$$

Repeating now the previous considerations, we come to statement ii).

Proof of Theorem 2:

From the equality (see [2])

$$(\pi_{n,k} - \pi_{n,k+1})(z) = (-1)^{k+1} \frac{z^{n+k+1} D(n+1, k+1)}{\{Q_{n,k} Q_{n,k+1}\}(z) D(n, k)}$$

we get

$$\begin{aligned} (f - \pi_{n,m})(z) &= (f - S_n)(z) + \sum_{k=0}^{m-1} (\pi_{n,k} - \pi_{n,k+1})(z) \\ &= (f - S_n)(z) + \sum_{k=0}^{m-1} (-1)^{k+1} \cdot \frac{D(n+1, k+1)}{D(n, k)} \cdot \frac{z^{n+k+1}}{Q_{n,k}(z) Q_{n,k+1}(z)}, \end{aligned}$$

so that

$$e_{n,m}(u) = W_n(u) + \sum_{k=0}^{m-1} \frac{(-1)^{k+1} u^k D_{n+1,k+1}}{D_{n,k} \cdot (Q_{n,k} Q_{n,k+1})(u a_n / a_{n+1})}.$$

The statement of Theorem 2 follows now from Lemma 2 and from Theorem 1.

Proof of Theorem 3.

Set

$$Q_{n,m} = \sum_{k=0}^m q_{k,n,m} z^{m-k}.$$

(recall that $q_{m,n,m} = 1$). Completing technical transformations we rewrite $E_{n,m}$ as follows

$$E_{n,m}(u) = \frac{1 + \sum_{j=1}^{\infty} d_{n,j} F_{j,n,m} u^j}{Q_{n,m}(ua_n/a_{n+1})}. \quad (4.1)$$

In the last formula

$$F_{j,n,m} = (-1)^m \frac{D_{n,m}}{D_{n+1,m+1}} \cdot \sum_{k=0}^m \frac{a_n^{m-k}}{a_{n+1}^{m-k}} \cdot q_{k,n,m} \cdot \prod_{l=1}^k \eta_{n+k+j+1-l}^l \cdot \prod_{l=1}^j \eta_{n+j+1-l}^k \quad (4.2)$$

Indeed, from the definition of Padé approximants we have

$$\frac{(f \cdot Q_{n,m} - P_{n,m})(ua_n/a_{n+1})}{(ua_n/a_{n+1})^{n+m+1}} = \sum_{j=0}^{\infty} (ua_n/a_{n+1})^j \sum_{k=0}^m a_{n+k+j+1} q_{k,n,m}$$

Applying now Lemma 1, we easily come to (4.1).

From Sylvester's identity (see [2]), there results the recurrence formulas

$$q_{k,n,m} = q_{k-1,n,m-1} - \tilde{D}(n,m) q_{k,n-1,m-1},$$

$$q_{0,n,m} = -\tilde{D}(n,m) q_{0,n-1,m-1}$$

where

$$\tilde{D}(n,m) := D(n-1, m-1) D(n+1, m) / D(n, m-1) D(n, m)$$

Making use of these formulas, we obtain

$$F_{j,n,m} := \frac{-D(n,m) D(n+1,m)}{D(n+1,m+1) D(n,m-1)} \cdot \frac{a_{n+1}}{a_n} \cdot \{F_{j+1,n,m-1} \cdot I_{n,j} - F_{j+1,n-1,m-1}\}. \quad (4.3)$$

The next step in the proof is to establish by induction on m that

a) for every $N \in \mathbb{N}$ and for $j + m < n / (N + 3m - 1)$ the following expansion is valid as $n \rightarrow \infty$

$$F_{j,n,m} = \prod_{l=1}^m \frac{(j+l)}{m!} + \sum_{s=1}^N \frac{\mathcal{P}_{s,m}(j)}{(n+1)^s} + \mathcal{N}_{N+1,m}(j,n), \quad (4.4)$$

where $\mathcal{P}_{s,m}(j)$ is a polynomial of degree not exceeding $m + s$;

b) for every j , satisfying $j + m < n/(N + 1 + m + 2(p + m - 1))$ there holds, as $n \rightarrow \infty$,

$$|(n + 1)^{N+p+1} \cdot \nabla^p \mathcal{N}_{N+1,m}(j, n)| < C_{m,N+1,p} \cdot j^{N+m+1} \quad (4.5)$$

with $C_{m,N+1,p}$ a positive constant not depending on n and on j ;

c) for numbers j with $j + m \geq n/3m$ we have for $n \rightarrow \infty$

$$|F_{j,n,m}| \leq c_m \cdot j^m \quad (4.6)$$

Check the hypothesis for $m = 1$. The direct calculation gives

$$F_{j,n,1} = \frac{I_{n,j} - 1}{\eta_{n+1} - 1} \quad (4.7)$$

In accordance with (1.3), we may write for n sufficiently large

$$1/(\eta_{n+1} - 1) = \left\{ 1 + \sum_{i=1}^N g_i / (n + 1)^i + o(1/n^N) \right\} \cdot (n + 1) / c_1$$

with

$$g_1 = -c_2 / c_1$$

Using Lemma 3, (2.1) and (4.7), we get for $F_{j,n,1}$

$$F_{j,n,1} := \left\{ (j + 1) \left\{ 1 + \sum_{s=1}^N \frac{Q_s(j)}{c_1 (n + 1)^s} \right\} + \frac{(n + 1) \cdot M_{N+2}(j, n)}{c_1} \right\} \cdot \left\{ 1 + \sum_{i=1}^{N+1} \frac{g_i}{(n + 1)^i} + o(1/n^N) \right\}.$$

The last formula can be rewritten as follows

$$F_{j,n,1} = j + 1 + \sum_{s=1}^N \mathcal{P}_{s,1}(j) / (n + 1)^s + \mathcal{N}_{N+1,1}(j, n)$$

We easily verify that $\deg \mathcal{P}_{s,1}(j) \leq s + 1$. Further, we see that $\mathcal{N}_{N+1,1}(j, n)$ depends on $(n + 1) \cdot M_{N+2}(j, n)$. Let now $p \in \mathbb{N}$ be fixed. Since for $j + 1 < (n - p) / (N + 1 + 2p)$ the remainder $M_{N+2}(j, n)$ behaves in the way described by Lemma 3, then $\mathcal{N}_{N+1,1}(j, n)$ satisfies as $n \rightarrow \infty$ the induction hypothesis (4.5) for numbers j with $j + 1 < n / (N + 2 + 2p)$ and $C_{1,N+1,p}$ being a suitable positive constant.

Also, for $j + 1 \geq n/3$, (4.7) implies (4.6) for $m = 1$ with a suitable positive constant.

Thus the assertion is proved for $m = 1$.

Set now

$$\mathcal{D}_{n,m} := \frac{-D(n, m) D(n + 1, m)}{D(n + 1, m + 1) D(n, m - 1)} \cdot \frac{a_{n+1}}{a_n}$$

Before proving the induction hypothesis for an arbitrary number m , we consider the asymptotic behaviour of $\mathcal{D}_{n,m}$ as $n \rightarrow \infty$ and m is fixed. From Lemma 2, there follows for every $N \geq 1$ the representation

$$\mathcal{D}_{n,m} = \frac{n+1}{mc_1} \cdot \left\{ 1 + \sum_{i=1}^N \frac{\alpha_{i,m}}{(n+1)^i} + o\left(1/(n+1)^N\right) \right\}, \text{ as } n \rightarrow \infty.$$

Suppose (4.4) - (4.6) are valid for some m . Let $N \in \mathbf{N}$ be fixed. Then in view of (4.3) we may write

$$\begin{aligned} F_{j,n,m+1} &= \frac{n+1}{(m+1)c_1} \left\{ 1 + \sum_{i=1}^N \frac{\alpha_{i,m+1}}{(n+1)^i} + o\left(1/(n+1)^N\right) \right\} \cdot \\ &\quad \left\{ \left\{ 1 + \sum_{s=1}^{N+1} (j+1) \cdot Q_{s-1}(j)/(n+1)^s + M_{N+2}(j,n) \right\} \cdot \right. \\ &\quad \left. \left\{ \prod_{l=2}^{m+1} \frac{j+l}{m!} + \sum_{s=1}^{N+1} \frac{\mathcal{P}_{s,m}(j+1)}{(n+1)^s} + \mathcal{N}_{N+2,m}(j+1,n) \right\} - \right. \\ &\quad \left. \left\{ \prod_{l=2}^{m+1} \frac{j+l}{m!} + \sum_{s=1}^{N+1} \frac{\mathcal{P}_{s,m}(j+1)}{n^s} + \mathcal{N}_{N+2,m}(j+1,n-1) \right\} \right\} \end{aligned}$$

Using (2.1) and (2.2), we rewrite this formula in the required form, namely

$$F_{j,n,m+1} = \prod_{l=1}^{m+1} \frac{(j+l)}{(m+1)!} + \sum_{s=1}^N \mathcal{P}_{s,m+1}(j)/(n+1)^s + \mathcal{N}_{N+1,m+1}(j,n)$$

where obviously each polynomial $\mathcal{P}_{s,m+1}$ is of degree not exceeding $m+s+1$. Further, we see that the remainder $\mathcal{N}_{N+1,m+1}(j,n)$ depends on the difference $(n+1) \cdot \{\mathcal{N}_{N+2,m}(j+1,n) - \mathcal{N}_{N+2,m}(j+1,n-1)\} = (n+1) \cdot \nabla \mathcal{N}_{N+2,m}(j+1,n)$. From the definition of ∇^p we easily get

$$\nabla^p \{(n+1) \cdot \nabla \mathcal{N}_{N+2,m}(j+1,n)\} = (n+1) \cdot \nabla^{p+1} \mathcal{N}_{N+2,m}(j+1,n) - p \cdot \nabla^p \mathcal{N}_{N+2,m}(j+1,n-1).$$

Therefore, for $j+1+m < \mu_{n,p} := \min(n/(N+2+m+2(p+m)), (n-1)/(N+2+m+2(p+m-1)))$ the term $\nabla^p \mathcal{N}_{N+1,m+1}(j,n)$ behaves in the way described by (4.6). The observation that $\mu_{n,p} = n/(N+2+m+2(p+m))$ for n large establishes (4.5) for $m+1$.

Further, for $j+1+m \geq n/(3m+3)$ we easily check that

$$|F_{j,n,m+1}| \leq c_{m+1} j^{m+1}.$$

This proves the induction hypothesis (4.4) - (4.6) for $m+1$.

We notice for $j+m < n/3m$ the validity of the following inequalities

$$F_{j,n,m} = \prod_{l=1}^m \frac{(j+l)}{m!} + \mathcal{N}_{1,m}(j,n) \tag{4.8}$$

with

$$|n \cdot \mathcal{N}_{1,m}(j, n)| < C_m \cdot j^{m+1} \text{ as } n \rightarrow \infty. \quad (4.9)$$

We now are in position to prove Theorem 2.

Set, for any $k \in \mathbb{N}$

$$e_{n,m,k}(u) := \sum_{j=0}^{k-1} d_{n,j} F_{j,n,m} u^j$$

and

$$\mathcal{E}_{n,m,k}(u) := \sum_{j=k}^{\infty} d_{n,j} F_{j,n,m} u^j \quad (5.1)$$

From (4.1), we have

$$E_{n,m}(u) = \frac{e_{n,m,k}(u) + \mathcal{E}_{n,m,k}(u)}{Q_{n,m}(ua_n/a_{n+1})}. \quad (4.1)'$$

Let δ be a fixed positive number. For $n \in \mathbb{N}$ sufficiently large we write

$$\mathcal{E}_{n,m,k}(e^{-\delta}) := \left\{ \sum_{j=k}^{j < n/3m-m} + \sum_{j \geq n/3m-m}^{\infty} \right\} d_{n,j} F_{j,n,m} e^{-\delta j}$$

Applying (4.4)-(4.6), we establish that

$$\|\mathcal{E}_{n,m,k}\|_{|u| \leq e^{-\delta}} \leq C (e^{-k\delta} + e^{-n\delta})$$

where C is a constant independent of k and n . On the other hand, for any fixed j we have

$$F_{j,n,m} \rightarrow \prod_{l=1}^m \frac{(j+l)}{m!} \text{ as } n \rightarrow \infty$$

so that, for any fixed k we may write

$$e_{n,m,k}(u) \rightarrow \sum_{j=0}^{k-1} \prod_{l=1}^m \frac{(j+l)}{m!} \text{ as } n \rightarrow \infty.$$

Hence the statement of Theorem 3 follows easily. Indeed,

$$\sum_{j=0}^{\infty} u^j \prod_{l=1}^m (j+l)/m! = \frac{1}{m!} \cdot (u^m(1-u)^{-1})^{(m)} = (1-u)^{-m-1}. \quad (5.8)$$

This result, (4.1)' and Lemma 2 establish the statement of Theorem 3. (5.9)

Proof of Theorem 4.

In our further considerations, we shall deal with the functions $\mathcal{E}_{n,m}(u) := \mathcal{E}_{n,m,0}(u)$. Recall that m is fixed and $n \rightarrow \infty$. In accordance with (3.1) we have

$$d_{n,j} := \prod_{l=1}^j \eta_{n+j+1-l} \text{ as } n \rightarrow \infty. \quad (5.1)$$

Arguing in the same way as in [5], we shall establish that for every δ small enough there exists a positive integer n_δ such that for any $n > n_\delta$ the inequality

$$\operatorname{Re} \mathcal{E}_{n,m}(e^{2\delta}) \geq C(0) \cdot e^{n \cdot \alpha(\delta)} \quad (5.1)$$

is valid, where

$$\alpha(\delta) := -\delta^2 / (-c_1 + \delta)$$

and $C(0)$ is a positive constant, independent of n and δ .

For convenience, we shall use the notation $c_1 := -2d_1$. In the conditions of Theorem 4, $d_1 > 0$. In [5], the validity of the following inequalities for each n large ($n > n_0$) was established.

$$|\eta_n| \leq 1 - d_1/n \quad (5.2)$$

and

$$|\eta_n| \leq |\eta_{n+1}|. \quad (5.3)$$

The two last inequalities lead to

$$|d_{n,j}| \leq (|\eta_{n+j}|)^{j(j+1)/2} \leq (1 - d_1/(n+j))^{j(j+1)/2}. \quad (5.4)$$

This inequality means that each function $\mathcal{E}_{n,m}(u)$ admits an analytic continuation in the disk $\{u, |u| < e^{d_1/2}\}$.

Let ε be a fixed positive number, $\varepsilon < 1$.

In our further considerations, we will assume that for $n > n_0$ the following inequalities are fulfilled:

$$(n/d_1) |\log(1 - d_1/n)| \geq 1 - \varepsilon, \quad (5.5)$$

and

$$|\operatorname{Im} \eta_n| \leq C(1) \cdot \operatorname{Re} \eta_n / n^2 \quad (5.6)$$

for a suitable positive constant $C(1)$. Without loss of generality, we may assume that $C(1) > 1$. In accordance with Theorem 2, (4.8), (4.9) and (4.6), we may also write

$$|\mathcal{N}_{1,m}(j, n)| \leq C(1) \cdot j^{m+1}/n \quad (5.7)$$

for $j + m < n/3m$ and

$$|F_{j,n,m}| \leq C(1) j^m \quad (5.8)$$

otherwise. Select a positive number δ_0 such that

$$0 < 6C(1)m!\delta_0 / (d_1(1 - \varepsilon) - 6\delta_0) < 1/3. \quad (5.9)$$

and set

$$d(\varepsilon, \delta_0) := d_1(1 - \varepsilon) - 6\delta_0.$$

In what follows, we shall assume that each $n > n_0$ satisfies the inequality

$$\operatorname{Re} \eta_n \geq 1 - (2d_1 + \delta_0)/n > 0. \quad (5.10)$$

Let δ be a positive number such that $\delta < \delta_0$. Obviously, there is an integer $n_\delta, n_\delta > n_0$, such that for any $n \geq n_\delta$ the inequalities

$$\operatorname{Re} \eta_n \geq 1 - (2d_1 + \delta)/n \quad (5.11)$$

and

$$\frac{(n+1)}{2d_1 + \delta} \cdot \left| \log \left(1 - \frac{2d_1 + \delta}{n+1} \right) \right| < 2 \quad (5.12)$$

are fulfilled. Set $j_1(\delta) := 6\delta/d(\varepsilon, \delta)$. For $j > j_1(\delta) \cdot n$ we obtain, in accordance with (5.4) and (5.5) that

$$|d_{n,j}| \leq e^{-3j\delta},$$

which implies with respect to (5.9), (5.7) and to the choice of δ the inequality

$$\left\| \sum_{j=j_1(\delta)n}^{\infty} d_{n,j} F_{j,n,m} u^j \right\|_{|u|=e^{2\delta}} \leq C(\delta_0) e^{-\delta j_1(\delta)n} \quad (5.13)$$

On the other hand, we have for $j+1 \leq j_1(\delta) \cdot n$ by (5.3), (5.6) and the choice of δ

$$\left| \frac{d_{j,n}}{\prod_{l=1}^j (\operatorname{Re} \eta_{n+j+1-l})} - 1 \right| \leq C(2) \cdot \delta^2$$

where $C(2) = \frac{18C(1)}{d_{\varepsilon,\delta}^2}$. Notice that the choice of δ ensures that $C(2) \cdot \delta_0^2 < 1/2$.

Further, the last inequality with respect to $d_{n,j}$ leads to

$$(1 - C(2)\delta^2) \prod_{l=1}^j (\operatorname{Re} \eta_{n+j+1-l})^l \leq \operatorname{Re} d_{n,j} \leq (1 + C(2)\delta^2) \prod_{l=1}^j (\operatorname{Re} \eta_{n+j+1-l})^l$$

and

$$|\operatorname{Im} d_{n,j}| \leq C(2)\delta^2 \prod_{l=1}^j (\operatorname{Re} \eta_{n+j+1-l})^l$$

Next, we assume without loss of generality that the integer n_δ satisfies the additional condition

$$j_1(\delta) \cdot n_\delta + m - 1 < n_\delta/3m \quad (5.14)$$

Using now (4.8), (5.7) and the last inequalities, we get

$$\operatorname{Re} F_{j,n,m} \cdot \operatorname{Re} d_{n,j} - \operatorname{Im} F_{j,n,m} \cdot \operatorname{Im} d_{n,j} \geq Q_{\delta_0}(j) \cdot \prod_{l=1}^j (\operatorname{Re} \eta_{n+j+1-l})^l.$$

with

$$Q_{\delta_0}(j) := (1 - C(2)\delta_0^2) \cdot \left(\prod_{l=1}^m (j+l)/m! - C(1)j^m \cdot \frac{\delta_0}{d(\varepsilon, \delta_0)} \right) - C(2)\delta_0^2 C(1)j^m \cdot \frac{6\delta_0}{d(\varepsilon, \delta_0)}$$

As we see, Q_{δ_0} is a polynomial of degree exactly m and all its coefficients are positive. Further, in view of (5.10) and to the choice of δ_0 we may write

$$\operatorname{Re} F_{j,n,m} \cdot \operatorname{Re} d_{n,j} - \operatorname{Im} F_{j,n,m} \cdot \operatorname{Im} d_{n,j} > 0. \quad (5.15).$$

Recall that the last inequality is valid for $n > n_\delta$ (compare with (5.14)) for any j with $j+1 < j_1(\delta) \cdot n$. Set now $j_2(\delta) := \delta/(2d_1 + \delta)$ and consider $\mathcal{E}_{n,m,\delta}(e^{2\delta}) := \sum_{j=0}^{j_2(\delta)(n+1)-1} d_{n,j} F_{j,n,m}(e^{2\delta j})$. From (5.12) we get

$$j+1 < \frac{2\delta}{|\log(1 - \frac{2d_1 + \delta}{n+1})|}$$

which immediately implies

$$\left(1 - \frac{2d_1 + \delta}{n+1}\right)^{j(j+1)/2} > e^{-\delta j}.$$

Taking into account (5.11), we conclude from this inequality that

$$\prod_{l=1}^j (\operatorname{Re} \eta_{n+j+1-l})^l > e^{-\delta j}.$$

Recall that the last inequality holds for $j < j_2(\delta)(n+1) - 1$ and for n "large". Now, combining (4.13), (5.15) and the last result, we get

$$\operatorname{Re} \mathcal{E}_{n,m}(e^{2\delta}) > \sum_{j=0}^{j_2(\delta)(n+1)-1} Q_{\delta_0}(j) e^{\delta j} - C(\delta_0) e^{-6(\delta)^2 n/d(\varepsilon, \delta)}$$

Inequality (5.1) results from here.

Now, it easily follows that the point $u = 1$ attracts, as $n \rightarrow \infty$, at least one zero of the sequence $E_{n,m}(u)$. Before presenting the proof, we introduce the notation $U_a(r)$; that is the open disk of radius r , centered at the point a ; further, we set $\Gamma_a(r) := \partial U_a(r)$.

Assume now the contrary that there is a disk $U_1(e^{-\rho}), 1 + e^{-\rho} < e^{d_1/2}$ such that $\mathcal{E}_{n,m}(u) \neq 0$ there. Set $\tau := \log(1 + e^{-\rho})$. Let θ be an arbitrary positive number with $1 + e^{-\rho} \cdot e^\theta < e^{d_1/2}$ and $0 < 1 - e^{-\rho} \cdot e^\theta$. Set $\tau(\theta) := \log(1 + e^{-\rho+\theta})$. Without loss of generality we may assume that the number $\tau(\theta)/2$ satisfies inequality (5.9). In the notations of the preceding considerations, we introduce for n "large" in the previous sense the series A_n and B_n as follows:

$$A_n(u) := \sum_{j+1=j_1(\tau(\theta)/2) \cdot n}^{\infty} d_{n,j} F_{j,n,m} u^j$$

and

$$B_n(u) := \sum_{j=0}^{j_1(\tau(\theta)/2) \cdot n} d_{n,j} F_{j,n,m} u^j$$

Repeating the same considerations as above (see(5.13)), we establish that

$$\|A_n\|_{U_0(e^{\tau(\theta)})} \leq C(\tau(\theta)) e^{-n\tau(\theta)j_1(\tau(\theta)/2)} \quad (5.16).$$

On the other hand, for B_n we easily get

$$\|B_n\|_{U_0(e^{\tau(\theta)})} \leq C(\tau(\theta)) e^{n\tau(\theta)j_1(\tau(\theta)/2)} \quad (5.17).$$

Set now $V := U_0(1) \cup U_1(e^{-\rho})$ and let X_n be the regular branch of $(\mathcal{E}_{n,m})(u)^{1/n}$ determined by the condition $X_n(0) = 1$. Inequalities (5.16) and (5.17) ensure the uniform boundedness of the sequence $\{X_n\}$ in V . By Theorem 2 and by the theorem of uniqueness for holomorphic functions,

$$X_n \rightarrow 1 \quad (6.2)$$

uniformly inside $U_0(1)$ and therefore, inside V , as well. On the other hand, by (5.1) for any $\delta < \tau$ the inequality $X_n(e^\delta) \geq \exp n \frac{\delta^2}{2(4d_1+\delta)}$ is valid for each n large enough. This contradicts with the last result about the convergence of the sequence $X_n(u)$. The contradiction we obtained establishes the statement of Theorem 5.

Proof of Theorem 5.

Preserving the notations of Theorem 4, denote now by $\xi_{n,k}, k = 1, \dots, \iota_n$ the zeros of $\mathcal{E}_{n,m}(u)$ in $U_1(e^{-\rho})$. By Theorem 3, $\iota_n \geq 1$.

We shall show that

$$\liminf_{n \rightarrow \infty} \iota_n/n > 0. \quad (6.3)$$

Suppose to the contrary that there is an infinite sequence $\Lambda, \Lambda \subset \mathbf{N}$ such that

$$\lim_{n \rightarrow \infty, n \in \Lambda} \iota_n/n = 0. \quad (6.1)$$

Set

$$q_n(u) := \prod_{k=1}^{\iota_n} \left(1 - \frac{u}{\xi_{n,k}}\right)$$

and

$$\chi_n(u) := \left\{ \frac{\mathcal{E}_{n,m}(u)}{q_n(u)} \right\}^{1/n}$$

with $\chi_n(0) = 1$.

Consider the sequence $\{\chi_n\}_{n \in \Lambda}$.

For $q_n(u)$ we have

$$\min_{u \in \Gamma_0(e^{\tau(\theta)})} |q_n(u)| \geq \left\{ \frac{e^{-\rho}(e^\theta - 1)}{(1 + e^{-\rho})} \right\}^{t_n}$$

Combine now the last equality, (5.16) and (5.17). By virtue of (6.1) and by the maximum principle for holomorphic functions, the sequence $\{\chi_n\}_{n \in \Lambda}$ is uniformly bounded on V . (recall that accordingly to the geometric construction and to the choice of θ , $V \subset U_0(\tau(\theta))$).

Select now a positive number r with $r < 1 - e^{-\rho}e^\theta$. For $u \in U_0(r)$, we obviously have

$$\left\{ \frac{(1 - e^{-\rho} + r)}{(1 - e^{-\rho})} \right\}^{t_n} \geq |q_n(u)| \geq \left\{ \frac{(1 - e^{-\rho} - r)}{(1 - e^{-\rho})} \right\}^{t_n}$$

Therefore, in view of Theorem 4 and of (6.2), we may write

$$\chi_n \rightarrow 1 \text{ as } n \rightarrow \infty, n \in \Lambda$$

on the disk $U_0(r)$. Therefore

$$\chi_n \rightarrow 1 \text{ as } n \rightarrow \infty, n \in \Lambda \quad (6.2)$$

uniformly inside the domain V .

Select a positive number ε_0 such that

$$\varepsilon_0 < \frac{e^{-\rho}}{4}$$

Set $\Omega(\varepsilon_0) := \bigcup_{n \in \Lambda} \bigcup_{k=1}^{t_n} \left\{ u, |u - \xi_{n,k}| < \frac{\varepsilon_0}{t_n \cdot n^2} \right\}$. Obviously,

$$mes_1(\Omega(\varepsilon_0)) < \varepsilon_0 < \frac{e^{-\rho}}{4}. \quad (6.3)$$

Further, for $u \in U_1(e^{-\rho}) - \Omega(\varepsilon_0)$ we have

$$\left\{ \frac{2e^{-\rho}}{(1 - e^{-\rho})} \right\}^{t_n} \geq |q_n(u)|$$

The choice of ε_0 and (6.3) ensure the existence of a positive number $\delta, \delta < \tau$ such that $e^\delta \in U_1(e^{-\rho}) - \Omega(\varepsilon_0)$. Applying (5.1) to that number δ , using the last estimate and (6.1),

we conclude that $\chi_n(e^\delta) > e^{n\delta^2/2(4d_1+\delta)}$ for n large enough. This inequality forms a contradiction with (6.2). Consequently, (1.5) is valid and Theorem 5 is true.

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