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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР

INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

# On a maximal sequence associated with simple branching processes

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# On a maximal sequence associated with simple branching processes

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**Abstract**

The number $Y_n$ of offspring of the most productive particle in the $n$th generation of a Bienaymé-Galton-Watson process is considered. The asymptotic behaviour of $Y_n$ as $n \to \infty$ may be viewed as an extreme value problem for i.i.d. random variables with random sample size. Limit theorems for both $Y_n$ and $EY_n$ provided the offspring mean is finite are proved using some convergence results for branching processes as well as transfer theorems for maxima.

*Key words:* Bienaymé-Galton-Watson branching process; max-stability; max-semistability; maximum with random sample size; transfer theorems.
*AMS 1991 subject classification:* primary 60J80, secondary 60G70.

## 1 Introduction.

Let $\{Z_n\}$ be a Bienaymé-Galton-Watson process which can be defined by the reccurence

$$Z_n = \sum_{i=1}^{Z_{n-1}} X_i(n), \qquad n = 1, 2, \ldots; \quad Z_0 \equiv 1,$$

where $\{X_i(n)\}$, $i, n = 1, 2, \ldots$ are nonnegative, independent and identicaly distributed, integer-valued random variables.

Denote by $f(s) = Es^{X_i(n)}$ the offspring generating function and by $f_n(s)$ the $n$th functional iterate of $f(s)$ i.e. $f_n(s) = f(f_{n-1}(s))$, $n = 1, 2, \ldots$, $f_0(s) = s$, $0 \le s \le 1$. Additionally let $F(x) = P(X_i(n) \le x)$ be the distribution function of the 'offspring variable' which has mean $0 < m < \infty$ and variance $0 < \sigma^2 \le \infty$.

Define

$$Y_n = \max_{1 \le i \le Z_{n-1}} X_i(n), \quad n = 1, 2, \ldots; \quad Y_0 \equiv 1 .$$

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It is clear that this definition is equivalent to

$$P(Y_n \le x) = \sum_{k=0}^{\infty} P(Z_{n-1} = k) F^k(x) = f_{n-1}(F(x)). \tag{1.1}$$

The study of the sequence $\{Y_n\}$ might be motivated in different ways. There have been several recent works developing results for certain kinds of extremes in branching processes, and investigating $Y_n$ is perhaps plausible as a contribution to this program. Alternatively, a natural interpretation within the demografical framework, for example, may be given. Indeed, the random variable under question is the number of offspring in families having the largest numbers of children. Thus the asymptotic behaviour of $Y_n$ provides some information about the influence of number of offspring of these families on the size of whole generation. Most close to our consideration is a recent paper by Arnold and Villaseñor (1996), written quite independent of the work reported here.

## 2 Transfer limit theorems.

Recall (see e.g. Resnick (1987), prop.0.3) that a nondegenerate distribution function $H(s)$ is max-stable iff for a distribution function $F(x)$ there exist sequences of real numbers $\{a_n\}_1^{\infty}$, $(a_n > 0)$ and $\{b_n\}_1^{\infty}$ such that

$$\lim_{n\to\infty} F^n(a_n x + b_n) = H(x) \ , \tag{2.1}$$

weakly. If (2.1) holds then $F(x)$ is said to belong to the domain of attraction of $H(x)$; in our notation, $F \in MSD(H)$. According to the classical Gnedenko's result, $H(x) = \exp\{-h(x)\}$, say, is of the type of one of the following three classes:

$$\begin{cases} (i) \;\; h(x) = (-x)^a & \text{for } x \in (-\infty, 0) \ , \quad = 1 \quad \text{for } x \in [0, \infty) \ , \\ (ii) \;\; h(x) = x^{-a} & \text{for } x \in (0, \infty) \ , \quad = 0 \quad \text{for } x \in (-\infty, 0] \ , \\ (iii) \;\; h(x) = \exp\{-x\} & \text{for } x \in (-\infty, \infty) \ , \end{cases} \tag{2.2}$$

where $a > 0$. It is well-known (see e.g. Resnick (1987), p.54) that $F \in MSD(\exp\{-x^{-a}\})$, $a > 0$ if and only if for $x > 0$,

$$1 - F(x) = x^{-a} L(x) \ , \tag{2.3}$$

where $L(x)$ is a slowly varying at infinity function (s.f.v.).

Further on let us have the following three sequences:

(a) $\{\xi_i(n)\}$ - independent and identically distributed for any fixed $n$ random variables;

(b) $\{\nu(n)\}$ - nonnegative integer-valued random variables;

(c) $\{i(n)\}$ - positive integers such that $i(n) \to \infty$ as $n \to \infty$.

Assume that $\nu(n)$ is independent of $\xi_i(n)$ , $k = 1, 2, \ldots$ for any fixed $n$.

For convenience we shall formulate here a transfer limit theorem for a maximum with random sample size (see Galambos (1987), thm 6.2.2 and Gnedenko and Gnedenko (1982)).

**Theorem 2.1** *Assume that for $x \in R$,*

$$\lim_{n\to\infty} P(\max_{1 \le i \le i(n)} \xi_i(n) \le x) = \Phi(x)$$

*and for* $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{\nu(n)}{i(n)} \le x\right) = A(x)\ ,$$

*where* $\Phi(x)$ *and* $A(x)$ *are distribution functions. Then for* $x \in R$,

$$\lim_{n\to\infty} P(\max_{1\le i\le \nu(n)} \xi_i(n) \le x) = \int_0^\infty (\Phi(x))^y dA(y)\ .$$

Further on we will denote by $L(x)$ and $L_1(x)$ certain s.v.f. and by $[a]$ the greatest integer less than or equal to $a$. Denote by $r(n) : R \to R$ a function, tending to $+\infty$ with $n$. Finally, with the convention that the infimum of an empty set is equal to $+\infty$, we define the (left continuous generalized) inverse $F^{\leftarrow} : R \to R$ of $F$ by $F^{\leftarrow}(y) := \inf\{x \in R : F(x) \ge y\}$.

Let $\{\eta_i(n)\}$ be the sequence of random variables as in (a) which have a common distribution function $F(x)$. This additonal assumption allows us to prove below a theorem which seems to be of independent interest.

**Theorem 2.2** *Assume that (2.1) (with (2.2)) holds. Suppose that there exists a positive random variable* $\nu$ *with* $\varphi(u) = E\exp\{-u\nu\}$, $u > 0$, *such that*

$$\lim_{n\to\infty} \frac{\nu(n)}{r(n)} = \nu\ , \tag{2.4}$$

*weakly.*

*(i)If* $F \in MSD(\exp\{-x^{-a}\})$, $a > 0$ *then for* $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le \nu(n)} \eta_i(n)}{d_n} \le x\right) = \varphi(x^{-a}) \tag{2.5}$$

*where* $d_n = (1/(1-F))^{\leftarrow}(r(n))$ *satisfies as* $n \to \infty$,

$$d_n \sim (r(n))^{1/a} L_1(r(n))\ , \tag{2.6}$$

*for* $L_1(x)$ *determined (asymptotically uniquely) by* $\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1$ *where* $L(x)$ *is defined in (2.3).*

*(ii) Suppose that convergence in (2.4) holds in probability. Then for* $x \in R$,

$$\lim_{n\to\infty} P\left(\frac{\max\limits_{1\le i\le \nu(n)} \eta_i(n) - b_{r(n)}}{a_{r(n)}} \le x\right) = \varphi(h(x))\ . \tag{2.7}$$

**Proof.** (i) Denote $U(x) = 1/(1-F(x))$ and $d_n = U^{\leftarrow}(r(n))$. We shall prove, using some arguments by Resnick (1987), p.15, that

$$U(d_n) = 1/(1-F(d_n)) \sim r(n)\ , \tag{2.8}$$

as $n \to \infty$. Indeed, by the definition of $U^{\leftarrow}$ it follows that $z < U^{\leftarrow}(r(n))$ iff $U(z) < r(n)$. For $\varepsilon > 0$ setting $z = U^{\leftarrow}(r(n))(1-\varepsilon)$ and then $z = U^{\leftarrow}(r(n))(1+\varepsilon)$ we obtain

$$\frac{U(U^{\leftarrow}(r(n)))}{U(U^{\leftarrow}(r(n))(1+\varepsilon))} \le \frac{U(U^{\leftarrow}(r(n)))}{r(n)} \le \frac{U(U^{\leftarrow}(r(n)))}{U(U^{\leftarrow}(r(n))(1-\varepsilon))}\ .$$

Since $F \in MSD(\exp\{-x^{-a}\}), a > 0$ is equivalent to (2.3), we get $U(x) \sim x^a/L(x)$ as $x \to \infty$. Thus

$$(1+\varepsilon)^{-a} \leq \liminf_{n\to\infty} \frac{U(U^{\leftarrow}(r(n)))}{r(n)} \leq \limsup_{n\to\infty} \frac{U(U^{\leftarrow}(r(n)))}{r(n)} \leq (1-\varepsilon)^{-a}$$

and (2.8) follows since $\varepsilon > 0$ is arbitrary.

Set $x_0 = \sup\{x : F(x) < 1\}$. Under conditions (i) it is not difficult to get $x_0 = \infty$ (see also Resnick (1987), p.15). Hence from (2.8), $d_n \to \infty$ as $n \to \infty$ and now from (2.3) we obtain for any $x > 0$, that

$$\lim_{n\to\infty} r(n)(1 - F(d_n x)) = \lim_{n\to\infty} \frac{1 - F(d_n x)}{1 - F(d_n)} = x^{-a} . \tag{2.9}$$

On the other hand, for $x > 0$,

$$\begin{aligned} P(\max_{1\leq i\leq \nu(n)} \eta_i(n) \leq x) &= \sum_{k=0}^{\infty} P(\nu(n) = k) P(\cap_{i=1}^{k}\{\eta_i(n) \leq x\}) \\ &= \sum_{k=0}^{\infty} P(\nu(n) = k) \prod_{i=1}^{k} P(\eta_i(n) \leq x) \\ &= \sum_{k=0}^{\infty} P(\nu(n) = k) F^k(x) \\ &= v_n(F(x)) , \end{aligned} \tag{2.10}$$

where $v_n(s) = Es^{\nu(n)}$ and by (2.4)

$$\lim_{n\to\infty} v_n(\exp\{-u/r(n)\}) = \varphi(u) , \qquad u > 0 . \tag{2.11}$$

Therefore, by (2.9) - (2.11), for $x > 0$ as $n \to \infty$,

$$\begin{aligned} P(\max_{1\leq i\leq \nu(n)} \eta_i(n) \leq d_n x) &= v_n(\exp\{\ln F(d_n x)\}) \\ &= v_n(\exp\{-(1 - F(d_n x))(1 + o(1))\}) \\ &= v_n\left(\exp\{-x^{-a}(r(n))^{-1}(1 + o(1))\}\right) \\ &\to \varphi(x^{-a}) . \end{aligned} \tag{2.12}$$

Furthermore, since $U(x) \sim x^a/L(x)$ as $x \to \infty$ we get

$$d_n = U^{\leftarrow}(r(n)) \sim (r(n))^{1/a} L_1(r(n)) , \tag{2.13}$$

as $n \to \infty$, where (cf. Seneta (1976), lemma 1.10, p.27) $\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1$ as $x \to \infty$. The asymptotically uniquiness of $L_1(x)$ follows by a result due to de Bruijn (see e.g. Bojanic and Seneta (1971), p.307).

Now, from (2.12) and (2.13), using the continuouty of $\varphi(x)$, we see that

$$\lim_{n\to\infty} P\left(\frac{\max_{1\leq i\leq \nu(n)} \eta_i(n)}{(r(n))^{1/a} L_1(r(n))} \leq x\right)$$

$$= \lim_{n\to\infty} P\left(\frac{\max_{1\leq i\leq \nu(n)} \eta_i(n)}{(r(n))^{1/a} L_1(r(n))} \leq \frac{d_n x}{(r(n))^{1/a} L_1(r(n))}\right) = \varphi(x^{-a}) .$$

(ii) The assertion is a consequence of Exercise 2 on p. 360 in Galambos(1987).

## 3 Limit theorems for $Y_n$.

**(A) Subcritical process** $(0 < m < 1)$. It is known (see e.g. Athreya and Ney, thm 1, p.16) that for subcritical processes

(3.1) $$\lim_{n\to\infty} P(Z_n = j \mid Z_n > 0) = p_j, \qquad j = 0, 1, \ldots,$$

where $\{p_j\}$ is a probability distribution with generating function $\gamma(s) = \sum_{j=0}^{\infty} p_j s^j$, $|s| \le 1$, which is the unique solution of the equation

(3.2) $$\gamma(f(s)) = m\gamma(s) + 1 - m, \qquad \gamma(0) = 0\ ,$$

where $f(s)$ is the offspring generating function. Using (3.1) we prove the following theorem.

**Theorem 3.1** *If* $m < 1$ *then for* $x \ge 0$,

$$\lim_{n\to\infty} P(Y_n \le x \mid Z_{n-1} > 0) = \gamma(F(x)),$$

*where* $\gamma$ *is the unique solution of (3.2) among the probability generating functions.*

**Proof.** Using (1.1) and (3.1) we obtain for $x \ge 0$ as $n \to \infty$,

$$\begin{aligned} P(Y_n > x \mid Z_{n-1} > 0) &= \frac{1 - f_{n-1}(F(x))}{1 - f_{n-1}(0)} = 1 - \frac{f_{n-1}(F(x)) - f_{n-1}(0)}{1 - f_{n-1}(0)} \\ &= 1 - E\left(F^{Z_{n-1}}(x) \mid Z_{n-1} > 0\right) \to 1 - \gamma(F(x)). \end{aligned}$$

**(B) Critical process** $(m = 1)$. Let the offspring generating function satisfy

(3.3) $$f(s) = s + (1-s)^{1+\alpha}\mathsf{L}(1/(1-s)),$$

for $0 < \alpha \le 1$, where $0 \le s \le 1$ and $\mathsf{L}(x)$ is a s.v.f. Slack has proved (see e.g. Bingham et al. (1987), p.395) that (3.3) is a necessary and sufficient condition for

(3.4) $$\lim_{n\to\infty} P(Q_n Z_n > y \mid Z_n > 0) = P(Z > y), \qquad y \ge 0,$$

where $Q_n = P(Z_n > 0)$, and $Z$ has Laplace transform

(3.5) $$\varphi(u) = Ee^{-uZ} = 1 - (1 + u^{-\alpha})^{-1/\alpha}, \quad u > 0.$$

In other words, (3.3) holds iff

(3.6) $$\lim_{n\to\infty} (1 - f_n(\exp\{-uQ_n\}))/Q_n = 1 - \varphi(u), \qquad u > 0.$$

In addition, if (3.3) is true then

(3.7) $$Q_n = n^{-1/\alpha} M(n),$$

where $M(n)$ is a s.v.f. and

(3.8) $$\lim_{x\to\infty} M^{\alpha}(x)\mathsf{L}(x^{1/\alpha}/M(x)) = 1/\alpha\ .$$

The special case $\alpha = 1$ is particularly important: here $\varphi(u) = 1/(1+u)$, so the limit law is exponential. If $\sigma^2 < \infty$ then (3.3) holds with $\alpha = 1$ and $\mathsf{L}(x)$ asymptotically constant. If $0 < \alpha < 1$ then (3.3) is equivalent to (2.3) with $a = 1 + \alpha$ and $L(x) \sim \alpha\mathsf{L}(x)/\Gamma(1-\alpha)$ as $x \to \infty$, where $\Gamma(x)$ is the Euler Gamma function (see Bingham and Doney (1974), thm A, p.716). Note that (3.3) does not necessarily imply (2.3) with some $a$ in the boundary case $\alpha = 1$.

We proceed to study the asymptotic behaviour of

$$P(Y_n > a_n x + b_n \mid Z_{n-1} > 0) = \frac{1 - f_{n-1}(F(a_n x + b_n))}{Q_{n-1}} \tag{3.9}$$

as $n \to \infty$, where $\{a_n\}_1^\infty$, $(a_n > 0)$ and $\{b_n\}_1^\infty$ are certain sequences of real numbers.

Let us first consider the case $\sigma^2 < \infty$. Applying Theorem 2.1 we get the following result.

**Theorem 3.2** *Assume that $m = 1$ and $\sigma^2 < \infty$. If (2.1) (with (2.2)) holds then for any $x \in R$,*

$$\lim_{n\to\infty} P\left(\frac{Y_n - b_n}{a_n} \le x \mid Z_{n-1} > 0\right) = \left(1 + \frac{\sigma^2}{2}h(x)\right)^{-1}. \tag{3.10}$$

**Proof.** We have (see e.g. Athreya and Ney (1972), p.20) for $x \ge 0$,

$$\lim_{n\to\infty} P\left(\frac{Z_{n-1}}{n} \le x \mid Z_{n-1} > 0\right) = P(Z \le x)$$

weakly, where $\varphi(u) = E\exp\{-uZ\} = 1/(1+\sigma^2 u/2)$.

Now, appealing to Theorem 2.1 with $i(n) = n$, $\nu(n) \equiv Z_{n-1}$ and $\xi_i(n) \equiv (X_k(n) - b_n)/a_n$, we get, for any real $x$,

$$\lim_{n\to\infty} P\left(\frac{Y_n - b_n}{a_n} \le x \mid Z_{n-1} > 0\right) = \int_0^\infty \exp\{-yh(x)\}dP(Z \le y) = \varphi(h(x)).$$

**Example.** Under conditions of Theorem 3.2, if additionally (2.3) holds then

$$\lim_{n\to\infty} P\left(\frac{Y_n}{n^{1/a}L_1(n)} \le x \mid Z_{n-1} > 0\right) = \left(1 + \frac{\sigma^2}{2}x^{-a}\right)^{-1}, \tag{3.11}$$

where $\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1$ for $L(x)$ defined by (2.3). Indeed, since (2.3) holds we have $F(x) \in MSD(\exp\{-x^{-a}\})$. Now, (see also (2.13)) $b_n = 0$ and $a_n = (1/(1-F))^{-}(n) \sim n^{1/a}L_1(n)$ as $n \to \infty$, which, appealing to (3.10), implies (3.11).

Let us now consider the case $\sigma^2 = \infty$.

**Theorem 3.3** *Assume that $\sigma^2 = \infty$ and (3.3) holds.*

*(i) If $0 < \alpha < 1$ then for $x > 0$,*

$$\lim_{n\to\infty} P\left(\frac{Y_n}{d_n} \le x \mid Z_{n-1} > 0\right) = 1 - (1 + x^{\alpha(1+\alpha)})^{-1/\alpha}, \tag{3.12}$$

*where* $d_n = (1/(1-F))^{\leftarrow}(Q_n^{-1})$ *and as* $n \to \infty$,

$$d_n \sim Q_n^{-1/(1+\alpha)} L_1(Q_n^{-1}) \tag{3.13}$$

*and* $Q_n$ *satisfies (3.7) and* $\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1$, *where* $L(x)$ *is from (2.3).*

(ii) *If* $\alpha = 1$ *and (2.3) holds, then (3.12) is still valued.*

(iii) *If* $\alpha = 1$ *and*

$$\lim_{n\to\infty} \frac{P(X_1(1) > n)}{P(X_1(1) > n+1)} = 1 \tag{3.14}$$

*then for* $x > 0$,

$$\lim_{n\to\infty} P\left(Q_n U(Y_n) \le x \mid Z_{n-1} > 0\right) = 1 - (1+x)^{-1}, \tag{3.15}$$

*where* $Q_n$ *satisfies (3.7) and* $U(x) = 1/(1-F(x))$.

**Proof.** (i) Since $0 < \alpha < 1$ then (3.3) is equivalent to (2.3) with $a = 1+\alpha$ (see the notes after (3.8)). Hence $F \in MSD(\exp\{-x^{-(1+\alpha)}\}$, $x > 0$. To make use Theorem 2.2 we let $\eta_i(n) \equiv X_i(n)$ and $\nu_n = Z_{n-1} I\{Z_{n-1} > 0\}$, where $I\{A\}$ denote the indicator variable of the event $A$. Appealing to (3.4) and (3.5) it is easily verified that Theorem 2.2 implies (3.12).

(ii) Since $\sigma^2 = \infty$ and (2.3) holds we have that $a \le 2$ (see e.g. Stoyanov et al. (1988), ex. 17.10, p.122). If $a < 2$ then (see the notes after (3.8)) (3.3) holds with $0 < \alpha < 1$. Hence $a = 2$ and, similarly to (i), we obtain (3.12).

(iii) Since $\lim_{n\to\infty} Q_n = 0$ we have (cf. Leadbetter et al. (1983), p.24) that (3.14) is necessary and sufficient condition for the existence of a sequence $\{u_n\}$ such that for $x > 0$,

$$\lim_{n\to\infty} \frac{1-F(u_n)}{Q_n} = x\,. \tag{3.16}$$

Since $\lim_{n\to\infty}(1-F(u_n)) = 0$ we get by (1.1) and (3.6) that

$$\begin{aligned} P(Y_n > u_n \mid Z_{n-1} > 0) &= \frac{1 - f_{n-1}(F(u_n))}{Q_{n-1}} = \frac{1 - f_{n-1}(\exp\{\ln F(u_n)\})}{Q_{n-1}} \\ &= \frac{1 - f_{n-1}(\exp\{-(1-F(u_n))(1+o(1))\})}{Q_{n-1}} \\ &= \frac{1 - f_{n-1}(\exp\{-xQ_{n-1}(1+o(1))\})}{Q_{n-1}} \\ &\to 1 - \varphi(x), \end{aligned}$$

as $n \to \infty$, where $\varphi(x)$ is defined by (3.5). Now, from (3.16) ($u_n$ are chosen to be not integers) using Lemma 2.2.1 by Galambos (1987) one can obtain for $x > 0$,

$$\begin{aligned} &\lim_{n\to\infty} P\left(\frac{1-F(Y_n)}{Q_{n-1}} \le x \mid Z_{n-1} > 0\right) \\ &= \lim_{n\to\infty} P\left(\frac{1-F(Y_n)}{Q_{n-1}} \le \frac{1-F(u_n)}{Q_{n-1}} + x - \frac{1-F(u_n)}{Q_{n-1}} \mid Z_{n-1} > 0\right) \\ &= \lim_{n\to\infty} P(F(Y_n) \ge F(u_n) \mid Z_{n-1} > 0) \\ &= \lim_{n\to\infty} P(Y_n > u_n \mid Z_{n-1} > 0) = 1 - \varphi(x)\,. \end{aligned}$$

From here, taking into account (3.5), we get (3.15).

**Examples.** Now, we shall consider two examples when the normalizing constants $d_n$ can be obtained explicitly. In the first example for the offspring generating function $f(s)$ we put in (3.3) $\mathsf{L}(s) = \log s$, while in the second one we assume that the "tail" of the distribution function of the 'offspring variable' satisfies (2.3) with $L(x) = \log x$.

(i) Assume that $0 < \alpha < 1$ and (3.3) holds with $\mathsf{L}(x) = \log x$. Then by Bojanic and Seneta (1971), p.309, $Q_n \sim \alpha^{-1/\alpha} n^{-1/\alpha} (\log n)^{-1/\alpha}$ as $x \to \infty$. One can see that $L(x) = c \log x$ (see the notes after(3.8)) and hence $L_1(x) \sim \log x$ as $x \to \infty$. Now, it is not difficult to obtain that (3.12) holds with $d_n = \alpha^{(1-\alpha(1+\alpha))/(\alpha(1+\alpha))} n^{1/(\alpha(1+\alpha))} (\log n)^{(1+\alpha(1+\alpha))/(\alpha(1+\alpha))}$.

(ii) Let $\alpha = 1$ and (2.3) holds with $L(x) = \log x$. By Bingham and Doney (1974), prop. A(ii) we have that $\mathsf{L}(x) = \int_1^x \log u / u du = (\log x)^2 / 2$. Now, using (3.7) and (3.8) one can obtain that $2Q_n \sim n(\log n)^2$ as $n \to \infty$. Finally, $L_1(x) = \log x$. Therefore,

$$\lim_{n\to\infty} P\left(\frac{Y_n}{\sqrt{n/2}\,(\log n)^2} \le x \mid Z_{n-1} > 0\right) = 1 - (1 + x^2)^{-1} .$$

**(C) Supercritical process** $(1 < m < \infty)$. It is well-known (see e.g. Athreya and Ney (1972), p.30) that if $1 < m < \infty$ then there exists a sequence of constants $\{C_n\}$ with $\lim_{n\to\infty} C_n = \infty$ such that $\{Z_n / C_n\}$ converges almost surely to a non-degenerate limit $W$. The Laplace transform $\psi(u) = E \exp\{-uW\}$, $u > 0$, of the limiting random variable, is the unique (up to a scale factor) solution of the equation

$$\psi(u) = f\left(\psi\left(\frac{u}{m}\right)\right) . \tag{3.17}$$

The constants $C_n$ take the form (see Cohn (1982), thm 4)

$$C_n = m^n / L_2\left(m^n\right) , \tag{3.18}$$

where $L_2(x) = \int_0^x P(W > y) dy$ is a s.v.f.

Further, we will also require the following extension of the class of max-stable distributions. A nondegenerate distribution function $G(s)$ is max-semistable (under linear transformation) iff for a distribution function $F(x)$ there exist two sequences of real numbers $\{\bar{a}_k\}_1^\infty$, $(\bar{a}_k > 0)$ and $\{\bar{b}_k\}_1^\infty$ such that

$$\lim_{k\to\infty} F^k(\bar{a}_k x + \bar{b}_k) = G(x) , \tag{3.19}$$

weakly, where $k$ runs over the sequence of positive integers $k(1) < k(2) < \ldots$ subject to the condition

$$\lim_{n\to\infty} \frac{k(n+1)}{k(n)} = r \ge 1 . \tag{3.20}$$

The case $r = 1$ corresponds to max-stable laws. If (3.19) (with (3.20)) holds, then $F(x)$ is said to belong to the domain of attraction of $G(x)$; in our notation, $F \in MSSD(G)$. By Theorem 2 in Grinevich (1992), $G(x) = \exp\{-g(x)\}$, say, is of the type of one of the following three classes:

$$\begin{cases} (i) \ \ g(x) = (u - x)^{\beta} \pi(\ln(u - x)) & \text{for } x \in (-\infty, u) , \\ (ii) \ \ g(x) = (x - u)^{-\beta} \pi(\ln(x - u)) , & \text{for } x \in (u, \infty) , \\ (iii) \ \ g(x) = \exp\{-\beta x\} \pi(x) , & \text{for } x \in (-\infty, \infty) , \end{cases} \tag{3.21}$$

for $u \in R$, $\beta = | c \ln r |$, where $c$ is certain constant, and $\pi(x)$ is periodic positive and bounded function satisfying certain conditions (see Grinevich (1992), thm 2). Necessary and sufficient conditions for $F \in MSSD(G)$ are established by Grinevich (1993).

We shall prove the following result.

**Theorem 3.4** *Assume that* $1 < m < \infty$.

*(i) If (2.1) (with (2.2)) holds then for* $x \in R$,

$$\lim_{n\to\infty} P\left(\frac{Y_n - b_{C_n}}{a_{C_n}} \le x\right) = \psi(h(x)) . \tag{3.22}$$

*(ii) If (3.19) (with (3.21)) holds and* $r = m$ *then for* $x \in R$,

$$\lim_{n\to\infty} P\left(\frac{Y_n - \bar{b}_{[C_n]}}{\bar{a}_{[C_n]}} \le x\right) = \psi(g(x)) .$$

**Proof.** (i) Since $\{Z_n/C_n\}$ tends almost surely, and hence in probability, to $W$ and $\lim_{n\to\infty} C_{n+1}/C_n = m > 1$ we get (3.22) from Theorem 2.2(ii) with $\eta_i(n) \equiv X_i(n)$, $\nu(n) = Z_{n-1}$ and $r(n) = C_n$.

(ii) Since $(C_{n+1} - 1)/C_n \le [C_{n+1}]/[C_n] \le C_{n+1}/(C_n - 1)$ we get $\lim_{n\to\infty}[C_{n+1}]/[C_n] = \lim_{n\to\infty} C_{n+1}/C_n = m > 1$. Hence $F \in MSSD(\exp\{-g(x)\})$, where $k(n) = [C_n]$. Further, we have for $x > 0$,

$$P\left(\frac{Z_n}{C_n} \le x\right) \ge P\left(\frac{Z_n}{[C_n]} \le x\right) \ge P\left(\frac{Z_n}{C_n} \le x(1 - \frac{1}{C_n})\right) .$$

Since $\{Z_n/C_n\}$ tends almost surely, and hence weakly, to $W$ which has an absolutely continuous distribution on $(0, \infty)$ (see e.g. Athreya and Ney, cor. 12.1, p.52) we get for $x > 0$,

$$\lim_{n\to\infty} P\left(\frac{Z_n}{[C_n]} \le x\right) = P(W \le x) .$$

Now, it is easily verified that by Theorem 2.1, setting $i(n) = [C_n]$, $\nu(n) \equiv Z_{n-1}$ and $\xi_i(n) \equiv (X_i(n) - \bar{b}_{[C_n]})/\bar{a}_{[C_n]}$, it follows that for any real $x$,

$$\lim_{n\to\infty} P\left(\frac{Y_n - \bar{b}_{[C_n]}}{\bar{a}_{[C_n]}} \le x\right) = \int_0^\infty \exp\{-yg(x)\} dP(W \le y) = \psi(g(x)) .$$

**Remark.** It is known (cf. Galambos (1987), cor. 2.4.1) that if (3.14) does not hold then there are no constants $a_n > 0$ and , $b_n$ so that $(Y_n - b_n)/a_n$ may tend weakly to a nondegenerate limit. So, if $F \in MSD(H)$ then (3.14) is fulfilled. Hence the assumptions of Theorems 3.2 - 3.4 also imply (3.14). It is not difficult to verify that (3.14) is not true for geometric and Poisson distributions (see e.g. Galambos (1987)). On the other hand, by Theorems 3.48 and 3.50 in Wilms (1994) we have that if the distribution function $F_X$ of a random variable $X$ belongs to the domain of attraction of $H(x) = \exp\{-x^{-a}\}$, $x > 0$, $a > 0$ then so does $F_{[X]}$. Furthermore, $F_X \in MSD(\exp\{-\exp\{-x\}\})$, $x \in R$ iff $F_{[X]} \in MSD(\exp\{-\exp\{-x\}\})$, $x \in R$ provided $x_0 = \sup\{x : F_X(x) < 1\} = \infty$.

# 4 Asymptotic behaviour of $EY_n$.

Using the results of Section 3 and an approach to study the moment convergence of sample extremes (see e.g. Resnick (1987), p.77) we shall investigate the asymptotic behaviour of $EY_n$. More precisely we shall obtain conditions on $F(x)$ only, under which,

$$\text{(4.1)} \qquad \lim_{n\to\infty} E\left(\frac{Y_n - b_n}{a_n} \mid Z_{n-1} > 0\right) = EY < \infty ,$$

provided

$$\text{(4.2)} \qquad \lim_{n\to\infty} P\left(\frac{Y_n - b_n}{a_n} \le x \mid Z_{n-1} > 0\right) = P(Y \le x) = R(x) \text{ , say,}$$

for some sequences $\{b_n\}_1^\infty$, $\{a_n\}_1^\infty$, $(a_n > 0)$ and distribution function $R(x)$.

Set $V_n = \mid (Y_n - b_n)/a_n \mid$. If (4.2) holds then it is clear that for any $N > 0$,

$$\lim_{n\to\infty} E\left(V_n I\{V_n \le N\} \mid Z_{n-1} > 0\right) = \int_0^N x dR(x) .$$

Write

$$\text{(4.3)} \qquad \begin{aligned} &\left| E(V_n \mid Z_{n-1} > 0) - \int_0^\infty x dR(x) \right| \\ &\le \mid E(V_n \mid Z_{n-1} > 0) - E(V_n I\{V_n \le N\} \mid Z_{n-1} > 0) \mid \\ &+ \left| E(V_n I\{V_n \le N\} \mid Z_{n-1} > 0) - \int_0^N x dR(x) \right| \\ &+ \left| \int_0^N x dR(x) - \int_0^\infty x dR(x) \right| . \end{aligned}$$

To prove (4.1) it is sufficient to show

$$\text{(4.4)} \qquad \lim_{N\to\infty} \limsup_{n\to\infty} E(V_n I\{V_n > N\} \mid Z_{n-1} > 0) = 0$$

(then the right hand side of (4.3) has limit equal to 0). We have

$$\begin{aligned} E(V_n I\{V_n > N\} \mid Z_{n-1} > 0) &= E\left( \sum_{j=1}^{V_n} I\{V_n > N\} \mid Z_{n-1} > 0 \right) \\ &= E\left( \sum_{j=1}^{N} I\{V_n > N\} \mid Z_{n-1} > 0 \right) + E\left( \sum_{j=N+1}^{\infty} I\{V_n > N, V_n > j\} \mid Z_{n-1} > 0 \right) \\ &= NP(V_n > N \mid Z_{n-1} > 0) + \sum_{j=N+1}^{\infty} P(V_n > j \mid Z_{n-1} > 0) \\ &= A_n(N) + B_n(N) \text{ , say.} \end{aligned}$$

Let us now consider separately three cases: $0 < m < 1$, $m = 1$ and $1 < m < \infty$.

**(A) Subcritical process.** Set $b_n = 0$ and $a_n = 1$ in (4.2) and hence $V_n = Y_n$. Applying Theorem 3.1 we prove the following result.

**Theorem 4.1** *If $0 < m < 1$ and $EX_1(1)\log(1+X_1(1)) < \infty$ then*

$$\lim_{n\to\infty} E(Y_n \mid Z_{n-1} > 0) = \sum_{k=1}^{\infty}(1-\gamma(F(k)) < \infty\ ,$$

*where $\gamma$ is the unique solution of (3.2) among the probability generating functions.*

**Proof.** From Theorem 3.1 we have

$$EY = \sum_{k=1}^{\infty}(1-\gamma(F(k)) \leq m\gamma'(1) < \infty\ ,$$

since $\gamma'(1) < \infty$ iff $EX_1(1)\log(1+X_1(1)) < \infty$ (cf. Athreya and Ney (1972), cor.2. p.45). Hence, appealing to Theorem 3.1,

$$\lim_{N\to\infty}\limsup_{n\to\infty} A_n(N) = \lim_{N\to\infty} NP(Y > N) = 0\ . \tag{4.5}$$

Consider $B_n(N)$. We get for $j = 1, 2, \ldots$

$$\begin{aligned}
P(Y_n > j \mid Z_{n-1} > 0) &= \sum_{k=1}^{\infty} P(Z_{n-1} = k \mid Z_{n-1} > 0) P(\max_{1\leq i\leq k} X_i(n) > j \mid Z_{n-1} > 0) \\
&= \sum_{k=1}^{\infty} P(Z_{n-1} = k \mid Z_{n-1} > 0)(1 - F^k(j)) \\
&\leq \sum_{k=1}^{\infty} P(Z_{n-1} = k \mid Z_{n-1} > 0)k(1 - F(j)) \\
&= (1 - F(j))E(Z_{n-1} \mid Z_{n-1} > 0)\ .
\end{aligned} \tag{4.6}$$

Therefore

$$\begin{aligned}
\lim_{N\to\infty}\limsup_{n\to\infty} B_n(N) &= \lim_{N\to\infty}\limsup_{n\to\infty} \sum_{j=N+1}^{\infty} P(Y_n > j \mid Z_{n-1} > 0) \\
&= \lim_{N\to\infty} const. \sum_{j=N+1}^{\infty} (1 - F(j)) = 0\ .
\end{aligned} \tag{4.7}$$

By (4.5) and (4.7) it follows that (4.4) holds and hence also (4.1).

**(B) Critical process.** We shall prove the following theorem. Let $B(x,y)$ be the Beta function.

**Theorem 4.2** *Assume that $m = 1$ and (2.3) holds with $a > 1$.*

*(i) If $\sigma^2 < \infty$ then*

$$\lim_{n\to\infty} \frac{1}{(\sigma^2/2)^{1/a} n^{1/a} L_1(n)} E\left(Y_n \mid Z_{n-1} > 0\right) = (2/\sigma^2)^{1/a} \int_0^\infty \left(1 + \frac{\sigma^2}{2}x^{-a}\right)^{-1} dx = B\left(\frac{1}{a}, 1 + \frac{1}{a}\right),$$

*where $\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1$.*

*(ii) If $\sigma^2 = \infty$ and $1 < a \leq 2$ then*

$$\lim_{n\to\infty} \frac{a-1}{n^{1/(a(a-1))} L_2(n)} E\left(Y_n \mid Z_{n-1} > 0\right) = (a-1)\int_0^\infty (1 + x^{a(a-1)})^{-1/(a-1)} dx = B\left(\frac{1}{a}, 1 + \frac{1}{a(a-1)}\right)$$

*where $L_2(n) \sim n^{1/(a(a-1))} L_1(Q_n^{-1})/Q_n^{1/a}$ as $n \to \infty$ is s.v.f. since (3.7) holds.*

**Proof.** Under the condition (2.3), from Theorems 3.2 and 3.3 we get (4.2) with $b_n = 0$ and $a_n = d_n$, where $d_n = (1/(1-F))^{\leftarrow}(Q_n^{-1})$. Hence $V_n = Y_n/d_n$. Now, from Theorems 3.2 and 3.3 it follows that

$$(4.8)\quad \lim_{N\to\infty} \limsup_{n\to\infty} A_n(N) = \lim_{N\to\infty} \limsup_{n\to\infty} NP(\frac{Y_n}{d_n} > N \mid Z_{n-1} > 0) = \lim_{N\to\infty} NP(Y > N) = 0,$$

since the integrals in the second parts of (i) and (ii) are finite.

On the other hand, similarly to (4.6), one can obtain

$$(4.9)\quad P(\frac{Y_n}{d_n} > j \mid Z_{n-1} > 0) \le (1 - F(jd_n))E(Z_{n-1} \mid Z_{n-1} > 0) = \frac{1 - F(jd_n)}{Q_{n-1}} = \frac{(1 - F(jd_n))(1 - F(d_n))}{(1 - F(d_n))Q_{n-1}}.$$

By the properties of regularly varying functions we have for a given $\varepsilon > 0$ and large $n$ the following inequality:

$$(4.10)\quad \frac{1 - F(jd_n)}{1 - F(d_n)} \le (1 + \varepsilon)j^{-a+\varepsilon}.$$

Additionally, from (2.8) and (3.13) we get

$$(4.11)\quad \limsup_{n\to\infty} \frac{1 - F(d_n)}{Q_n} < \infty.$$

Now, from (4.9) – (4.11) we get

$$(4.12)\quad \lim_{N\to\infty} \limsup_{n\to\infty} B_n(N) = \lim_{N\to\infty} \limsup_{n\to\infty} \sum_{j=N+1}^{\infty} P(\frac{Y_n}{d_n} > j \mid Z_{n-1} > 0) = \lim_{N\to\infty} \sum_{j=N+1}^{\infty} \frac{1+\varepsilon}{j^{a-\varepsilon}} = 0.$$

Finally, from (4.8) and (4.12) we obtain (4.4) and hence (4.1) holds.

**Example.** Let $m = 1$ and $1 - F(x) \sim x^{-2} \log x$ as $x \to \infty$. Then similarly to Example (ii) after Theorem 3.3, one can obtain by Theorem 4.2(ii) that

$$\lim_{n\to\infty} \frac{1}{\sqrt{n/2}\,(\log n)^2} E(Y_n \mid Z_{n-1} > 0) = \frac{\pi}{2}.$$

**(C) Supercritical process ($1 < m < \infty$).**

**Theorem 4.3** *Assume $1 < m < \infty$ and (2.3) holds with $a > 1$. Then*

$$(4.13)\quad \lim_{n\to\infty} \frac{1}{m^{n/a} L_1(m^n)} E(Y_n \mid Z_{n-1} > 0) = \int_0^{\infty} (1 - \psi(x^{-a}) dx,$$

*where $\lim_{x\to\infty} L_1(x/L(x))/L(x) = 1$ and the Laplace transform $\psi(x)$ is the unique (up to a scale factor) solution of the equation (3.17).*

**Proof.** Applying Theorem 3.4(i) (note that under (2.3) with $a > 1$ we have $C_n = m^n$) we get (4.2) with $b_n = 0$ and $a_n = m^{n/a} L_1(m^n)$. Hence $V_n = Y_n / a_n$ (see also (2.6)). Now, from Theorem 3.4(i) it follows that

$$(4.14)\quad \lim_{N\to\infty} \limsup_{n\to\infty} A_n(N) = \lim_{N\to\infty} \limsup_{n\to\infty} NP(\frac{Y_n}{a_n} > N \mid Z_{n-1} > 0) = \lim_{N\to\infty} NP(Y > N) ,$$

where $P(Y > N) = (1 - \psi(x^{-a}))/(1-q)$, setting $\lim_{n\to\infty} P(Z_n > 0) = 1 - q$. Since, under (2.3), $1 - \psi(u) \sim u$ as $u \to 0$ (cf. Athreya and Ney (1972), p.27) we have that the integral in the second part of (4.13) is finite. Now, by (4.14) it follows that

$$(4.15)\quad \lim_{N\to\infty} \limsup_{n\to\infty} A_n(N) = 0 .$$

On the other hand, similarly to (4.6), one can obtain

$$(4.16)\quad P(\frac{Y_n}{a_n} > j \mid Z_{n-1} > 0) \le (1 - F(ja_n)) E(Z_{n-1} \mid Z_{n-1} > 0) = \frac{(1 - F(ja_n)) m^n}{Q_{n-1}}$$
$$= \frac{(1 - F(ja_n))(1 - F(a_n)) m^n}{(1 - F(a_n)) Q_{n-1}}$$

Since (2.3) holds with $a > 1$, we have that in (3.18) $C_n = m^n$ and hence from (2.8) we get

$$(4.17)\quad \limsup_{n\to\infty} \frac{1 - F(a_n)}{m^n} < \infty.$$

Now, appealing to (4.10), from (4.16) and (4.17) we obtain for $\varepsilon > 0$,

$$(4.18)\quad \lim_{N\to\infty} \limsup_{n\to\infty} B_n(N) = \lim_{N\to\infty} \limsup_{n\to\infty} \sum_{j=N+1}^{\infty} P(\frac{Y_n}{a_n} > j \mid Z_{n-1} > 0)$$
$$= \lim_{N\to\infty} C_2 \sum_{j=N+1}^{\infty} \frac{1+\varepsilon}{j^{a-\varepsilon}} = 0 .$$

Finally, from (4.15) and (4.18) we get (4.4) and hence (4.1) holds.

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