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ИНСТИТУТ ПО МАТЕМАТИКА С ИЗЧИСЛИТЕЛЕН ЦЕНТЪР

INSTITUTE OF MATHEMATICS WITH COMPUTER CENTER

NOTES ON JONQUIÈRE POLYNOMIALS

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NOTES ON JONQUIÈRE POLYNOMIAlS

by

Peter Rusev

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0. Preliminaries

As in the paper [1] we assume Jonquière polynomials $\{P_n(z)\}_{n=0}^{\infty}$ to be defined by means of the equalities

$$\left(z\frac{d}{dz}\right)^n \frac{1}{(1-z)} = \frac{P_n(z)}{(1-z)^{n+1}}, \quad n = 0,1,2,\ldots, \quad z \in \boldsymbol{C} \setminus 1 \tag{0.1}$$

As it is mentioned in [1: 0. Introduction and Summary], these polynomials "*are of significance in various branches of mathematics, such as summability, analytic number theory and the theory of structure of polymers*". In [2: 0. Introduction and summary] is pointed out that Jonquière polynomials play an important role also in approximation theory.

We have e.g. that $P_0(z) = 1, P_1(z) = z, P_2(z) = z(z+1), P_3(z) = z(z^2+4z+1)$ etc. It is easy to prove that $\deg P_n = n$ for each $n = 0,1,2,\ldots$ and moreover that the "leading" coefficient of P_n i.e the coefficient of z^n in P_n is alwais equal to one. These assertions can be verified by means of the "recurrence" relation

$$P_{n+1}(z) = (n+1)zP_n(z) + z(1-z)P_n'(z), \quad n = 0,1,2,\ldots \tag{0.2}$$

which is a direct corollary of the defining equalities (0.1).

Another corollary of (0.2) is the equality $P_n(1) = n!, \quad n = 0,1,2,\ldots$. It gives rise to introduce the polynomials $\{E_n(z)\}_{n=0}^{\infty}$ by means of the equalities

$$E_n(z) = \frac{P_n(z)}{n!}, \quad n = 0,1,2,\ldots. \tag{0.3}$$

Then the relation (0.2) yields that

$$(n+1)E_{n+1}(z) = (n+1)zE_n(z) + z(1-z)E_n'(z), \quad n = 0,1,2,\ldots.$$

In this paper we are dealing with the system of modified Jonquière polynomials $\{E_n(z)\}_{n=0}^{\infty}$. In Section 1 we get an asymptotic formula for this system. By using it in Section 2 we consider series in Jonquière polynomials.

More precisely we point out that the well-known classical statements about power series as Abel's lemma, Cauchy-Hadamard's formula as well as Abel's theorem can be carried out on series in Jonqueère polynomials. It is shown also that the sungularities of series in these polynomials, on the boundaries of their regions of convergions, are simply connected with singularities of coresponding power series.

A class of summability methods of Poisson-Abel type is introduced in Section 3. It is proved that these methods are regular as well as that a statement like classical theorem of Tauber is valid for each of them.

In Section 4 we obtain two generating functions for the modified Jonquère polynomials and in Section 5 we discuss some "open problems".

Throughout this paper we use the notation $\tilde{\boldsymbol{C}}$ for the region $\boldsymbol{C} \setminus (-\infty, 0]$ (i.e. the complex plane cut along the nonpositive real semiaxis) and log for the the main branch of the logarithmic function in $\tilde{\boldsymbol{C}}$.

1. Asymptotic formula

We define the function ω in the region $\tilde{\boldsymbol{C}}$ as

$$\omega(z) = \frac{z-1}{\log z} \tag{1.1}$$

if $z \neq 1$ and $\omega(1) = 1$.

Since $\lim_{z\to 1} \omega(z) = 1$, it follows immediately that this function is holomorphic in the region $\tilde{\boldsymbol{C}}$. It is also evident that ω is nowhere zero in this region.

We define further

$$\eta_n(z) = (\omega(z))^{-n-1} E_n(z) - 1, \quad n = 0, 1, 2, \ldots \tag{1.2}$$

Then the following statement is true:

[1.2] *For each (nonempty) compact subset K of $\tilde{\boldsymbol{C}}$ there exist $0 < Q(K) < \infty$ and $0 < q(K) < 1$ such that the inequality*

$$|\eta_n(z)| \leq Qq^n \tag{1.3}$$

holds for each $n = 0, 1, 2, \ldots$ and each $z \in K$.

Proof. We are going to use series representations of Jonquière polynomials which appear as particular cases of the Lindelöf - Wirtinger expansions of Jonquere functions with a complex parameter [3, (7)].

In view of [1, (3.1)] and (0.3) we could write that

$$E_n(z) = \sum_{k=-\infty}^{\infty} \frac{(1-z)^{n+1}}{(2k\pi i - \log z)^{n+1}}, \tag{1.4}$$

provided that $z \in \check{C}$ and $n = 1, 2, 3, \ldots$.

Let us denote by $\tilde{E}_n(z)$ the right-hand side of (1.4). Then in order to verify the validity of the series representation (1.4) we have to prove that $\tilde{E}(z) = E_n(z)$ when $n = 1, 2, 3, \ldots$ and $z \in \check{C}$.

An easy computation gives that $(n+1)\tilde{E}_{n+1}(z) = (n+1)z\tilde{E}_n(z) + z(1-z)\tilde{E}_n'(z)$ and it remains to prove that $\tilde{E}_1(z) = z$ i.e. that the equality

$$\sum_{k=-\infty}^{\infty} \frac{1}{(2k\pi i - \log z)^2} = \frac{z}{(1-z)^2}$$

holds when $z \in \check{C}$. The last equality is equivalent to the following one

$$\sum_{k=-\infty}^{\infty} \frac{1}{(\zeta - 2k\pi i)^2} = \frac{\exp \zeta}{(1 - \exp \zeta)^2}, \tag{1.5}$$

provided that $\zeta \in S \setminus \{0\}$, where $S = \{\zeta \in C : |\operatorname{Im} \zeta| < \pi\}$.

It is known that the equality

$$\sum_{k=-\infty}^{\infty} \frac{1}{(w - k\pi i)^2} = \frac{1}{\operatorname{sh}^2 w} \tag{1.6}$$

holds when $w \neq k\pi i, k = 0, \pm 1, \pm 2, \ldots$. Then putting $w = \zeta/2 (\zeta \in S \setminus \{0\})$ in (1.6) we obtain (1.5).

As in [1, 3. Jonquière polynomilas] we rewrite (1.4) as

$$E_n(z) = (\omega(z))^{n+1} \Big\{ 1 + \sum_{k=-\infty, k \neq 0}^{\infty} \frac{(\log z)^{n+1}}{(\log z - 2k\pi i)^{n+1}} \Big\}. \tag{1.7}$$

Let us note that the above representation holds in the whole region $\check{C}$. Then (1.2) and (1.7) give that

$$\eta_n(z) = \sum_{k=-\infty, k \neq 0}^{\infty} \frac{(\log z)^{n+1}}{(\log z - 2k\pi i)^{n+1}}, \quad z \in \check{C}, \quad n \geq 1.$$

Let $1/2 < \mu < 1, \sigma(\mu) = \sum_{|k| \geq 1} |k|^{-2\mu}$ and $0 < \lambda < \min(1, (\sigma(\mu))^{-1})$. Since $K \subset \check{C}$ is compact, there exists a positive integer $k_0 = k_0(K)$ such that $|\log z (\log z - 2k\pi i)^{-1}| \leq \lambda |k|^{-\mu}$ whenever $|k| > k_0$ and $z \in K$. Let further $q_k = \max_{z \in K} |\log z (\log z - 2k\pi i)^{-1}|$, $k = \pm 1, \pm 2, \ldots$ and $q = \max\{q_{\pm 1}, q_{\pm 2}, \ldots, q_{\pm k_0}, \lambda\}$.

It is easy to see that $q_k < 1$ for each $k = \pm 1, \pm 2, \pm 3, \ldots$. Indeed, $K^* = \log K$ is a compact subset of the strip S and the image of K^*, under each of the linear transformation $T_k(\zeta) = \zeta(\zeta - 2k\pi i)^{-1}, |k| = 1, 2, 3, \ldots$, is a compact subset of the unit disk.

Then $(z \in K, n \geq 1)$

$$|\eta_n(z)| \leq \sum_{1 \leq |k| \leq k_0} |\log z(\log z - 2k\pi i)^{-1}|^{n+1} + \sum_{|k|>k_0} |\log z(\log z - 2k\pi i)^{-1}|^{n+1}$$

$$\leq \sum_{1 \leq |k| \leq k_0} q_k^{n+1} + \lambda^{n+1} \sum_{|k|>k_0} |k|^{-(n+1)\mu} \leq \sum_{1 \leq |k| \leq k_0} q_k^n + \lambda^n \leq (2k_0 + 1)q^n.$$

We define $Q = \max\{2k_0 + 1, \max_{z \in K} |(\omega(z))^{-1} - 1|\}$ and thus the validity of the inequality (1.3) is proved. As its corollary we have the following representation

$$E_n(z) = (\omega(z))^{n+1}\{1 + \eta_n(z)\}, \quad n = 0, 1, 2, \ldots \tag{1.8}$$

in the region $\tilde{C}$, which can be regarded as an asymptotic formula for the (modified) Jonquière polynomials in this region.

2. Series in Jonquière polynomials

2.1 We are going to describe the region as well as the mode of convergence of a series of the kind

$$\sum_{n=0}^{\infty} a_n E_n(z) \tag{2.1}$$

with arbitrary complex coefficients.

To that end we need to know more about the geometry of the mapping realized by the function ω defined by (1.1).

Let $0 \leq r < \infty, 0 < \delta < 1$ and define $u(r, \delta) = \operatorname{Re} \omega(r \exp i(\pi - \delta))$ and $v(r, \delta) = \operatorname{Im} \omega(r \exp i(\pi - \delta))$. Then

$$u(r) = \lim_{\delta \to 0} u(r, \delta) = -\frac{(1 + r)\log r}{(\log r^2) + \pi^2} \tag{2.2}$$

and

$$v(r) = \lim_{\delta \to 0} v(r, \delta) = \frac{\pi(1 + r)}{(\log r)^2 + \pi^2}. \tag{2.3}$$

Let L^+ be the curve with complex parametric equation $w = u(r) + iv(r), 0 \leq r < \infty$. Since $v(0) = 0, v(r) > 0$ and $\frac{dv}{dr} > 0$ when $r > 0$, L^+ is a Jordan arc starting at the zero point and, with exeption of this point, lying in the upper half-plane. Moreover, since $\lim_{r \to \infty} u(r) = -\infty$, L^+ goes to infinity. More precisely, $u(r) > 0$ when $0 < r < 1, u(1) = 0$ and $u(r) < 0$ when $1 < r < \infty$. That means the curve L^+ intersects the imaginary axis at the point $i\pi^{-1}$. Since $\lim_{r \to 0} \frac{dv}{dr}\left(\frac{du}{dr}\right)^{-1} = 0$, L^+ tuches the real axis at the origin.

Further as a corollary of the equalities (2.2) and (2.3) we can assert that whatever $0 < R < \infty$ be, the image of the segment $Z(R, \delta) : z = r \exp i(\pi - \delta), 0 \leq r \leq R$ is a Jordan arc similar to L^+, provided that δ is small enough.

A simple calculation yields that $\lim_{R \to \infty} \frac{d}{d\theta} \arg(\omega(R \exp i\theta)) = 1$ uniformly with respect to $\theta \in [0, \pi)$. That means the image of the circular arc $\Gamma(R, \delta) : z = R \exp i\theta, 0 \leq \theta \leq \pi - \delta$ is also a Jordan arc provided that R is large enough.

The function $\omega(x)$ is positive and strictly increasing when $x \geq 0$ i.e. it maps injectively the segment $[0, R]$ onto the segment $[0, \omega(R)]$.

Let $B^+(R, \delta)$ be the subregion of $\tilde{\boldsymbol{C}}$ whoose boundary is the union of the segments $[0, R], Z(R, \delta)$ and the circular arc $\Gamma(R, \delta)$.

As a corollary of the above considerations we can assert that the function ω is univalent in each region $B^+(R, \delta)$ provided that R is large and δ is small enough.

Let $B^-(R, \delta)$ be the image of the region $B^+(R, \delta)$ under the mapping $z \to \bar{z}$. Then by the principle of reflection the function ω is univalent in the region $B(R, \delta) = B^+(R, \delta) \bigcup B^-(R, \delta) \bigcup (0, R]$. Since $\bigcup_{R,\delta} B(R, \delta) = \tilde{\boldsymbol{C}}$, it follows that ω is univalent in the region $\tilde{\boldsymbol{C}}$. Moreover, the image G of $\tilde{\boldsymbol{C}}$ under ω is a simply connected region with boundary $L^+ \bigcup L^-$, where L^- is the image of L^+ under the reflection with respect to the real axis.

Remark. The curve $L^+(L^-)$ can be considered as the image of the upper (lower) bank of the cut $(-\infty,]$ under the mapping ω.

2.2 Let $0 < R < \infty$ and define $G(R) = \{w \in G : |w| < R$ i.e. $G(R) = G \bigcap U(0; R)$, where $U(0; R) = \{w \in \boldsymbol{C} : |w| < R\}$. Let further $D(R) = \omega^{-1}(G(R))$.

It is easy to prove that the equation $(u(r)^2) + (v(r))^2 = R^2$ i.e. $(1 + r)^2 = R^2((\log r)^2 + \pi^2)$ has unique (positive) solution $\rho(R)$. Let $\gamma(R) = \omega^{-1}(\partial U(0; R) \bigcap G)$, then $\partial D(R) = \gamma(R) \bigcup [-\rho(R), 0]$.

We assume that $G(0) = \emptyset$, $G(\infty) = G, D(0) = \emptyset$ and $D(\infty) = \tilde{\boldsymbol{C}}$. Let further $D^*(R) = \tilde{\boldsymbol{C}} \setminus \overline{D(R)}$ when $0 < R < \infty, D^*(0) = \tilde{\boldsymbol{C}}$ and $D^*(\infty) = \emptyset$.

Now we are able to answer the question about the region and the mode of convergence of a series of the kind (2.1).

[2.1] (Abel's lemma) *Let the series (2.1) be convergent at a point $\zeta \in \tilde{\boldsymbol{C}}$. Then it converges absolutely unuformly on each (nonempty) compact subset of the region $D(|\omega(\zeta)|)$.*

[2.2] (Cauchy - Hadamard Formula) *Let*

$$R = (\limsup_{n \to \infty} |a_n|^{1/n})^{-1}. \tag{2.4}$$

Then the series (2.1) converges absolutely uniformly on each (nonempty) compact subset of the region $D(R)$ and diverges at each point of the region $D^*(R)$.

Let $0 < \delta < 1, 0 < M < \infty, \zeta \in \tilde{\boldsymbol{C}}$ and define $A(\zeta, \delta, M) = \{z \in D(|\omega(\zeta)|) : 0 < |z - \zeta| \leq (1 - \delta)|\zeta|, |\omega(\zeta) - \omega(z)| \leq M(|\omega(\zeta)| - |\omega(z)|)\}$. Then we have also the following statement:

[2.3] (Abel's theorem) *If the series (2.1) converges at a point* $\zeta \in \tilde{C}$ *then it converges uniformly on each set of the kind* $A(\zeta, \delta, M)$.

Proof. We define $E_n^*(\zeta, z) = E_n(z)(E_n(\zeta))^{-1}, \quad n = 0, 1, 2, \ldots$.

Remark. Since all the roots of Jonquère polynomials are on the nonpositive real semiaxis, $E_n(\zeta) \neq 0$ for each $n = 0, 1, 2, \ldots$ and $\zeta \in \tilde{C}$.

As a corollary of the asymptotic formula (1.8) we have the representation

$$E_n^*(\zeta, z) - E_{n+1}^*(\zeta, z) = \left(\frac{\omega(z)}{\omega(\zeta)}\right)^{n+1} \left\{1 - \frac{\omega(z)}{\omega(\zeta)} + \mu_n(z, \zeta)\right\}$$

where $\lim_{n\to\infty} \mu(z, \zeta) = 0$ uniformly with respect to z on each compact subset of the region $\tilde{C}$.

Let $s_n(\zeta) = \sum_{k=0}^{n} a_k E_k(\zeta)(n = 0, 1, 2, \ldots)$. Without loss of generality we can assume that $\lim_{n\to\infty} s_n(\zeta) = 0$ i.e. that the sum of the series $\sum_{n=0}^{\infty} a_n E_n(\zeta)$ is equal to zero.

Since $a_n E_n(\zeta) = s_n(\zeta) - s_{n-1}(\zeta), n = 1, 2, 3, \ldots$, we have that

$$\sum_{k=n+1}^{n+p+1} a_k E_k(z) = \sum_{k=n+1}^{n+p+1} a_k E_k(\zeta) E_k^*(\zeta, z) = \sum_{k=n+1}^{n+p+1} (s_k(\zeta) - s_{k-1}(\zeta)) E_k^*(\zeta, z)$$

$$= s_{n+p+1}(\zeta) E_{n+p+1}(\zeta, z) - s_n(\zeta) E_{n+1}(\zeta, z) + \sum_{k=n}^{n+p} s_k(\zeta)(E_k(\zeta, z) - E_{k+1}(\zeta, z))$$

and further the proof proceeds as that of the classical Abel's theorem.

Corollary. *If the series (2.1) converges at a point* $\zeta \in \tilde{C}$, *then*

$$\lim_{z\to\zeta} \sum_{n=0}^{\infty} a_n E_n(z) = \sum_{n=0}^{\infty} a_n E_n(\zeta) \tag{2.5}$$

provided that $z \in A(\zeta, \delta, M)$.

Proof. The set $\tilde{A}(\zeta, \delta, M) = A(\zeta, \delta, M) \bigcup \{\zeta\}$ is compactt and the series (2.1) converges unoformly on it. Therefore the signs "lim" and "$\sum$" on the left-hand side of the equality (2.5) can be changed and thus it follows since $\lim_{z\to\zeta} E_n(z) = E_n(\zeta)$ for each $n = 0, 1, 2, \ldots$.

2.3 Let R be defined by the equality (2.4) and suppose that $0 < R < \infty$.Then the series (2.1) defines a complex function F, which is holomorphic in the region $D(R)$.

It is easy to prove that the series

$$\sum_{n=0}^{\infty} a_n(\omega(z))^n \tag{2.6}$$

converges absolutely uniformly on each compact subset of the region $D(R)$ and deverges at each point of $D^*(R)$. Let f be the complex function defined by the series (2.6).

[2.4] *A point $z_0 \in \gamma(R) = \partial D(R) \bigcap \tilde{C}$ is regular (singular) for the function F iff it is a regular (singular) point for f.*

Proof. Let $0 < r_0 < dist(z_0, [-\rho(R), 0])$. Then $U(z_0; r_0)$ is a neighbourhood of z_0 such that $\overline{U(z_0; r_0)} \subset \tilde{C}$. Then [1.1] gives that there exist $0 < Q = Q(z_0; r_0) < \infty$ and $0 < q = q(z_0; r_0) < 1$ such that $|\eta_n(z)| \leq Qq^n$ whenever $z \in \overline{U(z_0; r_0)}$ and $n = 0, 1, 2, \ldots$.

Whatever $\delta > 0$ be, there exists $0 < r(\delta) \leq r_0$ such that the inequality $|\omega(z)| \leq |\omega(z_0)|(1 + \delta)$ i.e. $|\omega(z)| \leq R(1 + \delta)$ holds when $z \in U(z_0; r(\delta))$. There exists a positive integer $N = N(\delta)$ such that $|a_n| \leq R^{-n}(1 + \delta^n)$ when $n \geq N$. Let us choose $\delta > 0$ so that $(1 + \delta)^2 q < 1$. Then the inequality $|a_n(\omega(z))^{n+1}\eta_n(z)| \leq (1 + \delta)Q((1 + \delta)^2 q)^n$ holds when $z \in U(z_0; r(\delta))$ and $n \geq N$. Therefore the series

$$\sum_{n=0}^{\infty} a_n(\omega(z))^{n+1}\eta_n(z)$$

is uniformly convergent in the neighbourhood $U(z_0; r(\delta))$. Let $\eta(z_0; z)$ be the holomorphic function defined by this series.

The asymptotic formula (1.10) gives that if $z \in D(R) \bigcap U(z_0; r(\delta))$ then

$$F(z) = \sum_{n=0}^{\infty} a_n E_n(z) = \sum_{n=0}^{\infty} a_n(\omega(z))^{n+1} + \sum_{n=0}^{\infty} a_n(\omega(z))^{n+1}\eta_n(z)$$

$$= \omega(z)f(z) + \eta(z_0; z)$$

and the statement follows simce ω is holomorphic and nowhere equal to zero in $\tilde{C}$.

Corollary. *A point $z_0 \in \gamma(R)$ is regular (singular) for the function F iff the point $w_0 = \omega(z_0)$ is regular (singular) for the power series*

$$\sum_{n=0}^{\infty} a_n w^n$$

Examples: (1) Let $a_n = (-1)^n$, $n = 0, 1, 2, \ldots$. Then each point of the curve $\gamma(1)$ is regular for the function defined in the region $D(1)$ by the series $\sum_{n=0}^{\infty}(-1)^n E_n(z)$.

(2) Let $\{k_n\}_{n=0}^{\infty}$ be an increasing sequence of positive integers such that $k_{n+1} \geq (1+\tau)k_n$, $n = 0, 1, 2, \ldots$ for some $\tau > 0$. Let $\{a_{k_n}\}_{n=0}^{\infty}$ be a sequence of complex number such that $\limsup_{n\to\infty} |a_{k_n}|^{1/k_n} = 1$. Then each point of $\gamma(1)$ is singular for the function defined in $D(1)$ by the series $\sum_{n=0}^{\infty} a_{k_n} E_{k_n}(z)$.

3. A class of summability methods

3.1 Let $\zeta \in \tilde{\boldsymbol{C}}$ be fixed, then a series

$$\sum_{n=0}^{\infty} a_n, \quad a_n \in \boldsymbol{C}, \quad n = 0, 1, 2, \ldots \tag{3.1}$$

is called (J, ζ)-summable with (J, ζ)-sum s if there exists

$$\lim_{t \to 1-0} \sum_{n=0}^{\infty} a_n E_n^*(\zeta, t\zeta) = s$$

Remark. It is supposed of course that the series $\sum_{n=0}^{\infty} a_n E_n^*(\zeta, z)$ converges in the region $D(|\omega(\zeta)|)$.

Before proving that each (J, ζ)-summation is regular we need to verify the validity of the following statement:

[3.1] *The inequality*

$$\mathrm{Re}\Big\{\frac{\zeta\omega'(\zeta)}{\omega(\zeta)}\Big\} > 0 \tag{3.2}$$

holds for each $\zeta \in \tilde{\boldsymbol{C}}$.

Proof. We have $\zeta\omega'(\zeta)/\omega(\zeta) = \zeta(\zeta-1)^{-1} - (\log\zeta)^{-1}$ when $\zeta \neq 1$. It is clear that $\zeta = 1$ is a regular point for the function $\zeta\omega'(\zeta)/\omega(\zeta)$ and since $\lim_{\zeta\to1}(\zeta(\zeta-1)^{-1} - (\log\zeta)^{-1}) = 1/2$, the inequality (3.2) holds at the point $\zeta = 1$.

Let $\zeta = r\exp i\theta (0 < r < \infty, -\pi < \theta < \pi)$ and define $U(r,\theta) = \mathrm{Re}\{\zeta\omega'(\zeta)/\omega(\zeta)\}$. Then

$$U(r,\theta) = \frac{r^2}{1 - 2r\cos\theta + r^2} - \frac{\log r}{(\log r)^2 + \theta^2}$$

when $(r,\theta) \neq (1,0)$ and $U(1,0) = 1/2$.

It is clear that the inequality $U(r,\theta) > 0$ holds when $0 < r < 1$ and $-\pi < \theta < \pi$. It holds also when $r = 1$ and $0 < |\theta| < \pi$ as well as when $r = 1$ and $\theta = 0$. Since $\lim_{r\to\infty} U(r,\theta) = 1$ uniformly with respect to $\theta \in (-\pi,\pi)$, this inequality holds when $r \geq r_0$ and $\theta \in (-\pi,\pi)$ provided that $r_0 > 1$ is large enough.

Let P be the (closed) rectangle in the (r,θ)-plane defined by the inequalities $1 \leq r \leq r_0, -\pi \leq \theta \leq \pi$. It is clear that the validity of the inequality $U(r,\theta) > 0$, when $1 < t < r_0$ and $-\pi < \theta < \pi$, will be proved if we verify its validity on the boundary of P.

We have already seen that this is really the fact on the vertical sides of P. Since $U(r,\pi) = U(r,-\pi)$, it remains to prove that the inequality

$$\frac{r^2}{(1+r)^2} - \frac{\log r}{(\log r)^2 + \pi^2} > 0 \tag{3.3}$$

holds when $1 < r < r_0$.

Let $r = \exp x$, then (3.3) is equivalent to

$$\frac{\exp 2x}{(1+\exp x)^2} - \frac{x}{x^2+\pi^2} > 0 \qquad (3.4)$$

provided that $0 < x < \log r_0$.

Now we will see that in fact (3.4) holds for each $x > 0$. Indeed, the function $\varphi_1(x) = (1+\exp x)^{-2}\exp 2x$ is increasing in the interval $[0,\infty)$ and $\varphi_1(0) = 1/4$. The function $\varphi_2(x) = x(x^2+\pi^2)^{-1}$ increases in the interval $[0,\pi]$ and decreases in $[\pi,\infty)$. That means the inequality $\varphi_2(x) \le \varphi_2(\pi)$ i.e. $\varphi_2(x) \le (2\pi)^{-1}$ holds when $0 < x < \infty$. Since $1/4 > (2\pi)^{-1}$, we have that $\varphi_1(x) > \varphi_2(x)$ when $0 < x < \infty$.

[3.2] *Each (J,ζ)-summation is regular.*

Proof. Since $E_0^*(\zeta, t\zeta) \equiv 1$, it is sufficient to prove that if the series (3.1) is convergent with usual sum equal to zero, then

$$\lim_{t\to 1-0} \sum_{n=0}^{\infty} a_n E_n^*(\zeta, t\zeta) = 0.$$

The proof of the validity of the above equality is quite similar to that of the statement [2.3]. But now we have to prove that the function $\varphi(t) = |\omega(\zeta) - \omega(t\zeta)|(|\omega(\zeta)| - |\omega(t\zeta)|)^{-1}$ is bounded when $t \to 1-0$.

Taylor's expansion gives that $\omega(t\zeta) = \omega(\zeta) - \zeta\omega'(\zeta)(1-t) + O((1-t)^2)$ and therefore $|1 - \omega(t\zeta)/\omega(\zeta)| = O(1-t)$ when $t \to 1-0$.

Further,

$$\left|\frac{\omega(t\zeta)}{\omega(\zeta)}\right|^2 = \frac{\omega(t\zeta)}{\omega(\zeta)}\overline{\left(\frac{\omega(t\zeta)}{\omega(\zeta)}\right)} = 1 - 2\,\mathrm{Re}\left\{\frac{\zeta\omega'(\zeta)}{\omega(\zeta)}\right\}(1-t) + O((1-t)^2)\ .$$

i.e

$$1 - \left|\frac{\omega(t\zeta)}{\omega(\zeta)}\right|^2 = 2\,\mathrm{Re}\left\{\frac{\zeta\omega'(\zeta)}{\omega(\zeta)}\right\}(1-t) + O((1-t)^2)$$

when $t \to 1-0$. Therefore, in view of (3.2), we have that

$$\varphi(t) = \frac{|1-\omega(t\zeta)/\omega(\zeta)|(1+|\omega(t\zeta)/\omega(\zeta)|)}{1-|\omega(t\zeta)/\omega(\zeta)|^2}$$

$$= O\Big(\frac{1-t}{2\,\mathrm{Re}\{\zeta\omega'(\zeta)/\omega(\zeta)\}(1-t) + O((1-t)^2)}\Big) = O(1)$$

when $t \to 1-0$.

3.2 The following statement is analogous to the theorem of Tauber for the (classical) Poisson - Abel summation.

[3.3] *If the series (3.1) is (J,ζ)-summable and moreover $\lim_{n\to\infty} na_n = 0$, then it is convergent.*

Proof. It is very much like to that in the "classical case". Let $S_n = \sum_{k=0}^{n} a_k$, $n = 0,1,2,\dots$ and define $F(\zeta,z) = \sum_{n=0}^{\infty} a_n E_n^*(\zeta,z)$ provided that $z \in D(|\omega(\zeta)|)$.

Let $0 < t < 1$, then

$$S_n - F(\zeta,t\zeta) = \sum_{k=1}^{n} a_k\{1 - E_k^*(\zeta,t\zeta)\} - \sum_{k=n+1}^{\infty} a_k E_k^*(\zeta,t\zeta).$$

As a corollary of the asymptotic formula (1.8) we obtain that

$$1 - E_k^*(\zeta,t\zeta) = 1 - \left(\frac{\omega(t\zeta)}{\omega(\zeta)}\right)^{k+1} + \lambda_k(\zeta,t),$$

where $\lambda_k(\zeta,t) = O(1-t)$ uniformly with respect to $k = 1,2,3,\dots$, when $t \to 1-0$.

If $\varepsilon_k = (k+1)|a_k|$, $k = 0,1,2,\dots$ then

$$\left|\sum_{k=0}^{n} a_k\{1 - E_k^*(\zeta,t\zeta)\}\right| \le \left|1 - \frac{\omega(t\zeta)}{\omega(\zeta)}\right| \sum_{k=1}^{n} \varepsilon_k + O(1-t)\sum_{k=1}^{n} \varepsilon_k$$

i.e.

$$\left|\sum_{k=0}^{n} a_k\{1 - E_k^*(\zeta,t\zeta)\}\right| = O(1-t)\sum_{k=1}^{n} \varepsilon_k$$

when $t \to 1-0$.

Let ε be an arbitrary positive number. Since $\lim_{k\to\infty} \varepsilon_k = 0$, we can choose the positive integer $N = N(\varepsilon)$ so that $\varepsilon_n < \varepsilon$ and $n^{-1}(\varepsilon_1 + \varepsilon_2 + \cdots + \varepsilon_n) < \varepsilon$ when $n > N$. Then

$$|S_n - F(\zeta,t\zeta)| \le O(1-t)\sum_{k=1}^{n} \varepsilon_k + O\left(\varepsilon n^{-1} \sum_{k=n+1}^{\infty} \left|\frac{\omega(t\zeta)}{\omega(\zeta)}\right|^{k+1}\right)$$

$$= O(1-t)\sum_{k=1}^{n} \varepsilon_k + \varepsilon n^{-1} O((1-t)^{-1})$$

and choosing $t = 1 - 1/n$ we obtain that $|S_n - F(\zeta,(1-1/n)\zeta)| = O(\varepsilon)$ when $n \ge N$.

4. Generating functions

4.1 We are going to use the series representations (1.4) for getting generating functions for the (modified) Jonquière polynomials.

A (formal) change of summations gives that

$$\sum_{n=1}^{\infty} E_n(z)w^n = \sum_{k=-\infty}^{\infty}\sum_{n=1}^{\infty}\frac{(1-z)^{n+1}w^n}{(2k\pi i-\log z)^{n+1}}$$

$$=\sum_{k=-\infty}^{\infty}\frac{(1-z)w^2}{(\log z-2k\pi i)(\log z+(1-z)w-2k\pi i)},$$

provided that $z\in \boldsymbol{C}\setminus\{1\}$ and $|w|$ is small enough.

Further we have that

$$\frac{1}{(\log z-2k\pi i)(\log z+(1-z)w-2k\pi i)}$$

$$=\frac{1}{(1-z)w}\left\{\frac{1}{\log z-2k\pi i}-\frac{1}{\log z+(1-z)w-2k\pi i}\right\}$$

and after some computation we obtain that

$$\sum_{n=1}^{\infty} E_n(z)w^n=(1-z)\left\{\frac{1}{\log z}+\sum_{k=1}^{\infty}\frac{2\log z}{(\log z)^2+4k^2\pi^2}\right.$$

$$\left.-\frac{1}{\log z+(1-z)w}-\sum_{k=1}^{\infty}\frac{2(\log z)^2}{(\log z+(1-z)w)^2+4k^2\pi^2}\right\}.$$

Then as a corollary of the series representation

$$\coth\zeta=\frac{1}{\zeta}+\sum_{k=1}^{\infty}\frac{2\zeta}{\zeta+k^2\pi^2}$$

we have that

$$\sum_{n=1}^{\infty}E_n(z)w^n=-\frac{z(1-\exp((1-z)w))}{1-z\exp((1-z)w)}. \tag{4.1}$$

Since $E_0(z)\equiv 1$, (4.1) gives that

$$\sum_{n=0}^{\infty}E_n(z)w^n=E(z,w),$$

where

$$E(z,w)=\frac{1-z}{1-z\exp((1-z)w)}. \tag{4.2}$$

If $z \in \check{C}$ is fixed then the function $E(z, w)$ is holomorphic in the disk $\{w \in C : |w| < |\omega(z)|^{-1}\}$. Indeed, the roots of the equation $1 - z\exp((1-z)w) = 0$ are $w_k(z) = (\omega(z))^{-1} + 2k\pi i(z-1)^{-1}$, $k \in Z$ and it is easy to see that $|w_0(z)| = |\omega(z)|^{-1} < |w_k(z)|$ for each $z \in \check{C} \setminus \{1\}$ and each $k \in Z^* = Z \setminus \{0\}$.

Let $w \neq 1$ then $\lim_{z\to 1} E(z, w) = (1-w)^{-1}$. If we assume that $E(1, w) = (1-w)^{-1}$ then $E(1, w)$ will be holomorphic in the unit disk i.e. in the disk $\{w : |w| < |\omega(1)|^{-1}\}$.

Let Ω be the region in the space C^2 defined by the conditions $z \in \check{C}$ and $|w| < |\omega(z)|^{-1}$. The function $E(z, w)$ is holomorphic in the region Ω and has there an expansion of the kind

$$E(z, w) = \sum_{n=0}^{\infty} A_n(z) w^n. \tag{4.3}$$

Remark. It is clear that all the coefficients of the "power" series in (4.3) are holomorphic in the region $\check{C}$.

The validity of the equality (4.2) will be verified if we show that $A_n(z) = E_n(z)$ for each $z \in \check{C}$ and each $n = 0, 1, 2, \ldots$. To that end it is sufficient to prove that the equality

$$(n+1)A_{n+1}(z) = (n+1)zA_n(z) + z(1-z)A_n'(z) \tag{4.4}$$

holds when $z \in C$, $\quad n = 0, 1, 2, \ldots$ and moreover that $A_0(z) \equiv 1$.

An easy computation shows that the function $E(z, w)$ satisfies the partial differential equation

$$(1 - zw)\frac{\partial E}{\partial w} - z(1-z)\frac{\partial E}{\partial z} - zE = 0 \tag{4.5}$$

in the region Ω.

But $\frac{\partial E}{\partial w} = \sum_{n=0}^{\infty}(n+1)A_{n+1}(z)w^n$, $zw\frac{\partial E}{\partial w} = \sum_{n=0}^{\infty}(n+1)zA_n(z)w^n$ and $\frac{\partial E}{\partial z} = \sum_{n=0}^{\infty} A_n'(z)w^n$ when $(z, w) \in \Omega$ and the equality (4.4) is a corollary of (4.5). Since $A_0(z) \equiv 1$, from (0.4) and (4.4) it follows that $A_n(z) = E_n(z)$ when $z \in \check{C}$ and $n = 0, 1, 2, \ldots$.

4.2 Since $E_n(0) = 0$ for each $n = 1, 2, 3, \ldots$, the polynomial $E_n(z)$ is not reciprical when $n \geq 1$, but the polynomials

$$F_n(z) = z^{-1}E_{n+1}(z), \quad n = 0, 1, 2, \ldots \tag{4.6}$$

have this property. Indeed, (4.1) and (4.6) give that

$$\frac{1 - \exp((1-z)w)}{1 - z\exp((1-z)w)} = \sum_{n=0}^{\infty} F_n(z)w^{n+1}, \quad z \in \check{C}.$$

We substitute z^{-1} for z and zw for w and thus obtain that

$$-\frac{1-\exp((1-z)w)}{1-z\exp((1-z)w)} = \sum_{n=0}^{\infty} zF_n(1/z)w^{n+1}.$$

Then the uniqueness of the Maclorin expansion of a holomorphic function gives that $z^n F_n(1/z) = F_n(z), \quad n = 0, 1, 2 \ldots$.

4.3 Let define the complex function $W(z,w)$ by means of the equality

$$W(z,w) = \sum_{n=0}^{\infty} \frac{E_{n+1}(z)}{n!} w^n \tag{4.7}$$

provided that $z \in \tilde{\boldsymbol{C}}$.

As a corollary of the asymptotic formula (1.8) we easily obtain that whatever $z \in \tilde{\boldsymbol{C}}$ be, $W(z,w)$ is an entire function of the complex variable w and moreover that W is of order $\rho = 1$ and type $\sigma = |\omega(z)|$.

A formal change of summations gives that if $z \in \tilde{\boldsymbol{C}} \setminus \{1\}$ then

$$W(z,w) = \sum_{k=-\infty}^{\infty} \frac{(1-z)^2}{(2k\pi i - \log z)^2} \exp\left\{\frac{(1-z)w}{2k\pi i - \log z}\right\}. \tag{4.8}$$

Remark. In fact (4.8) holds when $z = 1$.

If $z \in \tilde{\boldsymbol{C}}$ is fixed, then the series on the right-hand side of (4.8) converges (absolutely) uniformly on each compact subset of the complex plane. Therefore

$$\frac{1}{n!}\frac{\partial^n W(z,w)}{\partial W^n}\bigg|_{w=0} = \sum_{k=-\infty}^{\infty} \frac{(1-z)^{n+2}}{(2k\pi i - \log z)^{n+2}} = E_{n+1}(z)$$

and thus the validity of the series representation (4.1) of the function (4.7) is proved.

5. Comments

- It seems that the region of convergence of the series (2.1) is the "whole" interior of the Jordan curve $\gamma(R)\bigcup\{-\rho(R)\}$. This could be established if the asymptotics of Jonquière polynomials on compact subset of the ray $(-\infty, 0]$ is known.

- The statement [2.4] and especially its **Corollary** gives a full answer of the question about the singularities of the series (2.1) lying on the curve $\gamma(R)$. Statements like Jentsch's and Ostrowski's theorems also can be proved but other problems, as analytical continuation of Jonquière seires e.g. by Borel's or Mittag-Lefler's type methods, are still open.

- It seems that each two (J, ζ)-summability methods are equivalent. It is quite sure that each (J, ζ)-summation is stronger than any Cesaro's summation buth the "correlation" with the classical Poisson - Abel summation is not known.

- The generating functions (4.2) and (4.8) seems to be new ones. The author hopes that they will be usefull by studying the completeness of systems of Jonquière polynomials $\{E_{k_n}(z)\}_{n=0}^{\infty}$ in spaces of complex functions holomorphic in suitable subregions of the complex plane cut along the ray $(-\infty, 0]$.

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