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НЕРАВЕНСТВА ОТ ТИП НА ДАФИН И ШЕФЕР II

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INEQUALITIES OF DUFFIN-SCHAEFFER TYPE II

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Abstract

Recently we proved that if a polynomial f of degree n has smaller absolute value than T_n, the n-th Tchebycheff polynomial of the first kind, at $n+1$ points located in $[-1,1]$ and separated by the zeros of T_n, then $\|f'\| \le \|T_n'\|$, where $\|\cdot\|$ is the uniform norm in $[-1,1]$. This result extends both the famous A. Markov inequality and its refinement given by Duffin and Schaeffer. Here we prove analogous extension for the higher order derivatives.

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1. Introduction and Statement of the Result

Denote by π_n the set of all algebraic polynomials of degree not exceeding n, and by $\|\cdot\|$ the supremum norm in $[-1,1]$, i.e., $\|f\| := \sup_{x\in[-1,1]} |f(x)|$.

The classical inequality of the brothers Markov ([17, 18]) reads as follows.

Theorem A. *If $f \in \pi_n$ satisfies $\|f\| \le 1$, then*

$$\|f^{(k)}\| \le T_n^{(k)}(1) \quad \text{for} \quad k = 1, \dots, n. \tag{1.1}$$

The equality occurs only if $f(x) = \gamma T_n(x) = \gamma \cos(n \arccos x)$, $|\gamma| = 1$.

In 1941 Duffin and Schaeffer [10] found a beautiful extension of the Markov inequality, proving that (1.1) remains true under the weaker assumption

$$|f(\eta_j)| \le 1, \; j = 0, \dots, n, \tag{1.2}$$

where $\eta_j = \cos\frac{j\pi}{n}$ are the extreme points of the Tchebycheff polynimial $T_n(x)$. Since $|T_n(\eta_j)| = 1$ for $j = 0, \dots, n$, the result of Duffin and Schaeffer may be viewed as a comparison type theorem: the assumption $|f| \le |T_n|$ at the points $\{\eta_j\}_0^n$ induces the inequalities $\|f^{(k)}\| \le \left\|T_n^{(k)}\right\|$ for $k = 1, \dots, n$. This observation motivated the author to formulate in [21] the following definition:

Definition. A polynomial $Q \in \pi_n$ and a mesh $\Delta = \{t_\nu\}_{\nu=0}^n$, $(t_0 > t_1 > \dots > t_n)$ are said to admit an inequality of the Duffin-and Schaeffer-type if the assumption $f \in \pi_n$ and $|f| \le |Q|$ at the points from Δ implies $\left\|f^{(k)}\right\| \le \left\|Q^{(k)}\right\|$ for $k = 1, \dots, n$.

For some inequalities of the Duffin- and Schaeffer-type we refer the reader to [4], [5], [14], [19], [21, 22, 23, 25, 26], [28], [30], [31]. A detailed account on the Markov-type inequalities is given in the survey paper [3]. The interested reader may find exhaustive exposition on the topic in [6], [7], [16], [20], and the new book of Rahman and Schmeisser [29].

In a recent paper [24], the author proved the following theorem.

THEOREM B. *Let* $\{t_\nu\}_{\nu=0}^n$ *satisfy* $1 \ge t_0 > \xi_1 > t_1 > \cdots > \xi_n > t_n \ge -1$, *where* $\{\xi_\nu\}_{\nu=1}^n$ *are the zeros of* T_n, *i.e.,* $\xi_\nu = \cos((2\nu-1)\pi/(2n))$. *If* $f \in \pi_n$ *and*

$$|f(t_\nu)| \le |T_n(t_\nu)| \quad \textit{for} \quad \nu = 0, \ldots, n,$$

then

$$\|f'\| \le \|T_n'\| \,. \tag{1.3}$$

Moreover, equality in (1.3) is possible if and only if $f = cT_n$ *with* $|c| = 1$.

The aim of this paper is to extend Theorem B to higher order derivatives. Namely, we prove the following theorem.

THEOREM 1.1. *Under the assumptions of Theorem B, we have*

$$\left\|f^{(k)}\right\| \le \left\|T_n^{(k)}\right\| \quad \textit{for} \quad k = 1, \ldots, n. \tag{1.4}$$

Moreover, equality in (1.4) is possible if and only if $f = cT_n$ *with* $|c| = 1$.

Theorem 1.1 reveals that for $Q = T_n$ each mesh Δ of points in $[-1, 1]$ which are separated by the zeros of T_n admits an inequality of the Duffin- and Schaeffer-type. On the other hand, it has been shown in [24, Section 2] that the set of all meshes Δ enjoying this property cannot be substantially larger.

The paper is organized as follows: In Section 2 we prove a general inequality (Theorem 2.1), which is the main ingredient of the proof of Theorem 1.1. The proof of Theorem 1.1 is given in Section 4. In Section 3 we establish some properties of $T_n^{(k)}$, the derivatives of the Tchebycheff polynomial of the first kind. Section 5 contains some comments and remarks.

2. A General Inequality

We shall exploit some observation from V. Markov's paper [18] about the zero interlacing inheritance of algebraic polynomials. Let us start with a definition:

DEFINITION. Let p and q be two algebraic polynomials having only real and simple zeros. The zeros of p and q are said to *interlace* if one can trace all of them, switching alternatively from a zero of p to a zero of q and vice versa, and moving only in one direction. If, in addition, no zero of p coincides with a zero of q, then the zeros of p and q are said to *interlace strictly.*

Obviously, the interlacing is possible only if p and q are of the same degree or of degrees which differ by one. In the latter case, if, e.g., p is of degree $n+1$, q is of degree n, and the zeros of p and q interlace strictly, we shall say shortly that the zeros of q *separate* the zeros of p.

The following simple lemma, due to V. Markov [18], asserts that the zero interlacing property of two polynomials is inherited by their derivatives (for a proof, the reader may consult [27, Lemma 2.7.1] or [31]).

LEMMA 2.1. *Let p and q ($p \ne q$) be algebraic polynomials having only real and simple zeros. If the zeros of p and q interlace, then the zeros of p' and q' interlace strictly.*

In particular, Lemma 2.1 shows that each zero of p' is a strictly monotone function of any zero of p. We apply Lemma 2.1 to derive the main result in this section.

THEOREM 2.1. *Let Q be an algebraic polynomial of degree n, having n distinct real zeros located in $(-1,1)$. Assume that $f \in \pi_n$ satisfies the inequality $|f| \le |Q|$ at a set of $n+1$ distinct points in $[-1,1]$, which are separated by the zeros of Q. If $k \in \{1,2,\dots,n-1\}$, then*

$$\left\|f^{(k)}\right\| \le \max_{0\le r\le n-k+1} \max_{x\in[\tau_{r+1},\tau_{r-1}]} \left|\varphi_r(x)\right|. \tag{2.1}$$

where

$$\varphi_r(x) := \frac{1-\tau_r x}{x-\tau_r} Q^{(k)}(x),$$

$\tau_1 > \tau_2 > \cdots > \tau_{n-k}$ *are the zeros of* $Q^{(k)}(x)$, *and* $\tau_{-1} = \tau_0 := 1$, $\tau_{n-k+1} = \tau_{n-k+2} := -1$.

Proof. Note first that $-\varphi_0(x) = \varphi_{n-k+1}(x) = Q^{(k)}(x)$. Without loss of generality we may assume that the coefficients of Q are real. Let $t_0 > t_1 > \cdots > t_n$ be the $n+1$ points in $[-1,1]$ which are separated by the zeros of Q, and at which $|f| \le |Q|$. Set $\omega(x) := (x-t_0)(x-t_1)\dots(x-t_n)$ and $\omega_\nu(x) := \omega(x)/(x-t_\nu)$, $\nu = 0,\dots,n$. From the Lagrange interpolation formula and the triangle inequality we infer

$$\left|f^{(k)}(x)\right| = \left|\sum_{\nu=0}^{n} \frac{\omega_\nu^{(k)}(x)}{\omega_\nu(t_\nu)} f(t_\nu)\right| \le \sum_{\nu=0}^{n} \left|\frac{\omega_\nu^{(k)}(x)}{\omega_\nu(t_\nu)}\right| . \left|f(t_\nu)\right| \le \sum_{\nu=0}^{n} \left|\frac{\omega_\nu^{(k)}(x)}{\omega_\nu(t_\nu)}\right| . \left|Q(t_\nu)\right|. \tag{2.2}$$

For each pair of indices (i,j), $0 \le i < j \le n$, the zeros of $\omega_i(x)$ interlace with the zeros of $\omega_j(x)$, the zeros of $\omega_i(x)$ being smaller than or equal to the corresponding zeros of $\omega_j(x)$. This observation and Lemma 2.1 imply the following arrangement for the zeros $\{\gamma_\mu^{(\nu)}\}_{\mu=1}^{n-k}$ of $\omega_\nu^{(k)}(x)$, $\nu = 0,\dots,n$:

$$1 > \gamma_1^{(n)} > \gamma_1^{(n-1)} > \cdots > \gamma_1^{(0)} > \gamma_2^{(n)} > \cdots > \gamma_2^{(0)} > \cdots > \gamma_{n-k}^{(n)} > \cdots > \gamma_{n-k}^{(0)} > -1. \tag{2.3}$$

Setting

$$I_0 := (\gamma_1^{(n)}, 1),\ I_\nu := (\gamma_{\nu+1}^{(n)}, \gamma_\nu^{(0)})\ \ (\nu = 1,\dots,n-k-1)\ \text{ and }\ I_{n-k} := (-1, \gamma_{n-k}^{(0)}),$$

it is easy to see from (2.3) that if x belongs to some of the intervals $\{I_\nu\}_{\nu=0}^{n-k}$, then

$$\operatorname{sign}\{\omega_0^{(k)}(x)\} = \operatorname{sign}\{\omega_1^{(k)}(x)\} = \cdots = \operatorname{sign}\{\omega_n^{(k)}(x)\} \ne 0.$$

By our assumption, the values $\{Q(t_\nu)\}_{\nu=0}^{n}$ alternate in sign, and so do $\{\omega_\nu(t_\nu)\}_{\nu=0}^{n}$. Therefore, for $x \in I_\nu$ $(\nu = 0,\dots,n-k)$, the last upper bound for $\left|f^{(k)}(x)\right|$ in (2.2) is equal to $\left|Q^{(k)}(x)\right|$ and is attained only if $f = cQ$ with some constant c, $|c| = 1$.

Our next observation is that the intervals $\{I_\nu\}_{\nu=0}^{n-k}$ are separated by the zeros $\{\tau_\nu\}_{\nu=1}^{n}$ of $Q^{(k)}$. Indeed, by assumption, the zeros of Q interlace with the zeros of both ω_0 and ω_n, and are located between the corresponding zeros of ω_0 and ω_n. Then Lemma 2.1 implies interlacing (and the same arrangement) for the zeros of $\omega_0^{(k)}$, $Q^{(k)}$ and $\omega_n^{(k)}$.

Now we iterate the above reasoning: For $\nu = 0,1,\dots,n-k$, we select a point from the interval I_ν (denote it again by t_ν). The new points $\{t_\nu\}_{\nu=0}^{n-k}$ satisfy

$$1 > t_0 > \tau_1 > t_1 > \cdots > \tau_{n-k} > t_{n-k} > -1, \tag{2.4}$$

and

$$\left|f^{(k)}(t_\nu)\right| \le \left|Q^{(k)}(t_\nu)\right| \ \text{ for }\ \nu = 0,1,\dots,n-k. \tag{2.5}$$

Moreover, in (2.5) equality occurs for either ν if and only if $f = cQ$ with some constant c, $|c| = 1$. Denote by $\{\ell_\nu\}_{\nu=0}^{n-k}$ the Lagrange fundamental polynomials for interpolation at the points $\{t_\nu\}_{\nu=0}^{n-k}$.

Assume first that $x_0 \in (t_r, t_{r-1})$ for some $r \in \{1, \dots, n-k\}$, then

$$\text{sign}\,\{\ell_\nu(x_0)\} = (-1)^{r-1-\nu} \quad \text{for} \quad \nu = 0, 1, \dots, r-1$$

and

$$\text{sign}\,\{\ell_\nu(x_0)\} = (-1)^{\nu-r} \quad \text{for} \quad \nu = r, \dots, n-k.$$

Apart from a multiplier ± 1, the sequences $\{\varphi_r(t_\nu)\}_{\nu=0}^{n-k}$ and $\{\ell_\nu(x_0)\}_{\nu=0}^{n-k}$ have the same sign pattern, therefore

$$\begin{aligned}|\varphi_r(x_0)| &= \sum_{\nu=0}^{n-k} |\ell_\nu(x_0)|.|\varphi_r(t_\nu)| > \sum_{\nu=0}^{n-k} |\ell_\nu(x_0)|.|Q^{(k)}(t_\nu)| \\ &\geq \sum_{\nu=0}^{n-k} |\ell_\nu(x_0)|.|f^{(k)}(t_\nu)| \geq |f^{(k)}(x_0)|.\end{aligned}$$

Moreover, the same inequality holds if $x_0 = t_r$ or $x_0 = t_{r-1}$, therefore

$$|f^{(k)}(x)| < |\varphi_r(x)| \quad \text{if} \quad x \in [t_r, t_{r-1}], \quad r = 1, \dots, n-k. \tag{2.6}$$

On the other hand, if x_0 belongs to $[-1, t_{n-k})$ or $(t_0, 1]$, then both the sequences $\{Q^{(k)}(t_\nu)\}_{\nu=0}^{n}$ and $\{\ell_\nu(x_0)\}_{\nu=0}^{n-k}$ alternate in sign, thus

$$|Q^{(k)}(x_0)| = \sum_{\nu=0}^{n-k} |\ell_\nu(x_0)|.|Q^{(k)}(t_\nu)| \geq \sum_{\nu=0}^{n-k} |\ell_\nu(x_0)|.|f^{(k)}(t_\nu)| \geq |f^{(k)}(x_0)|.$$

Hence

$$|f^{(k)}(x)| \leq |Q^{(k)}(x)| = |\varphi_0(x)| = |\varphi_{n-k+1}(x)| \quad \text{if } x \in [-1, t_{n-k}) \quad \text{or} \quad x \in (t_0, 1]. \tag{2.7}$$

Now (2.1) follows from (2.6), (2.7) and (2.4). The proof of Theorem 2.1 is complete. □

An obvious consequence of Theorem 2.1 is the following corollary.

COROLLARY 2.1. *If, under the assumptions of Theorem 2.1, for some* $k \in \{1, \dots, n-1\}$,

$$\max_{1 \leq r \leq n-k} \max_{x \in [\tau_{r+1}, \tau_{r-1}]} |\varphi_r(x)| \leq \left\| Q^{(k)} \right\|,$$

then

$$\left\| f^{(k)} \right\| \leq \left\| Q^{(k)} \right\|.$$

3. Some Properties of $T_n^{(k)}$

3.1 The ultraspherical polynomials and $T_n^{(k)}$

The ultraspherical polynomial $P_n^{(\lambda)}$ is the n-th orthogonal polynomial in $[-1, 1]$ with respect to the weight functions $w(x) = (1 - x^2)^{\lambda - 1/2}$, $(\lambda > -1/2)$. Some well-known properties of the ultraspherical polynomials are (see, e.g. [33, Chapt. 5]) :

(i) The function $y = P_n^{(\lambda)}(x)$ satisfies the second order differential equation

$$(1-x^2)y'' - (2\lambda+1)xy' + n(n+2\lambda)y = 0.$$

(ii) If $\lambda > 0$, then the relative maxima of $|P_n^{(\lambda)}|$ increase as the distance between their abscissae and the origin increases, and $\left\|P_n^{(\lambda)}\right\| = P_n^{(\lambda)}(1)$ (the opposite behaviour of the local maxima prevails when $-1/2 < \lambda < 0$).

Apart from a constant factor, $T_n^{(k)}$ is equal to the ultraspherical polynomial $P_{n-k}^{(k)}$ (this observation applies also to the case $k = 0$). In particular, $\left\|T_n^{(k)}\right\| = T_n^{(k)}(1)$, and the differential equation for $T_n^{(k)}$ reads as

$$(1-x^2)T_n^{(k+2)}(x) - (2k+1)xT_n^{(k+1)}(x) + (n^2-k^2)T_n^{(k)}(x) = 0. \qquad (3.1)$$

From (3.1) we have the following relation between the norms of $T_n^{(k)}$ and $T_n^{(k+1)}$:

$$T_n^{(k+1)}(1) = \frac{n^2-k^2}{2k+1}T_n^{(k)}(1). \qquad (3.2)$$

3.2 An upper estimate for the largest local extremum of $T_n^{(k)}$

According to what was said above, the local maxima of $|T_n^{(k)}|$ increase as their abscissae increase in absolute value. It seems an interesting task to compare the largest local maximum of $|T_n^{(k)}|$ and $\left\|T_n^{(k)}\right\|$. Such a comparison is given in the next lemma.

LEMMA 3.1. *Let $1 \le k \le n-2$, and let τ be the largest zero of $T_n^{(k+1)}$. Then*

$$|T_n^{(k)}(\tau)| \le \frac{1}{2k+1}T_n^{(k)}(1). \qquad (3.3)$$

Moreover,

$$|T_n^{(k)}(\tau)| \le \frac{8}{(2k+1)(2k+7)}T_n^{(k)}(1) \ \ \text{if} \ \tau \ge \tilde{\tau}, \ \ \text{where} \ \tilde{\tau} := \frac{2k-1}{2k+7}.$$

Proof. From the differential equations for $T_n^{(k)}$ and $T_n^{(k+1)}$, and $T_n^{(k+1)}(\tau) = 0$ we infer

$$(1-\tau^2)T_n^{(k+3)}(\tau) = (2k+3)\tau T_n^{(k+2)}(\tau), \ (1-\tau^2)T_n^{(k+2)}(\tau) = -(n^2-k^2)T_n^{(k)}(\tau). \qquad (3.4)$$

On using (3.4) and the fact that $T_n^{(k+2)}$ is convex in $[\tau, 1]$, we obtain

$$\begin{aligned} T_n^{(k+1)}(1) &= \int_\tau^1 T_n^{(k+2)}(t)\,dt \ge (1-\tau)T_n^{(k+2)}((1+\tau)/2) \\ &\ge (1-\tau)\left[T_n^{(k+2)}(\tau) + \frac{1-\tau}{2}T_n^{(k+3)}(\tau)\right] \\ &= (1-\tau)\left[1 + \frac{(2k+3)\tau}{2(1+\tau)}\right]T_n^{(k+2)}(\tau) \\ &= (n^2-k^2)\frac{(2k+5)\tau+2}{2(1+\tau)^2}|T_n^{(k)}(\tau)|. \end{aligned}$$

This inequality combined with (3.2) implies

$$\left|T_n^{(k)}(\tau)\right| \le \frac{1}{2k+1}T_n^{(k)}(1)g(\tau), \quad \text{where} \quad g(x) := \frac{2(1+x)^2}{(2k+5)x+2}. \tag{3.5}$$

The function $g(x)$ is continuous in $[0,1]$, and has a unique point of local extremum therein, which is a minimum. Therefore

$$\max_{x\in[0,1]} g(x) = \max\{g(0),\ g(1)\} = 1,$$

and this together with (3.5) proves (3.3). The remaining claim of the lemma follows from the fact that $g(\tilde{\tau}) = g(1) = 8/(2k+7)$. The proof is complete. □

Notice that (3.3) holds true for $k = 0$, too. In the cases $k = 0$ and $k = n-2$ it becomes an equality.

3.3 An inequality relating $T_n^{(k)}$ and $T_n^{(k+1)}$

As is seen from (3.2), we have $\left\|T_n^{(k)}\right\| < \left\|T_n^{(k+1)}\right\|$ for $k = 1, \ldots, n-1$. It turns out that, for $2 \le k \le n-2$, the factor $1-x^2$ suffices for the reverse inequality to hold true, i.e., the function

$$\psi_k(x) := (1-x^2)T_n^{(k+1)}(x)$$

already has a smaller uniform norm in $[-1,1]$ than $T_n^{(k)}$. Specifically, we prove the following lemma.

LEMMA 3.2. *For every $k \in \{2, 3, \ldots, n-2\}$ we have*

$$\|\psi_k\| < \left(1 - \frac{1}{(2k+1)^2}\right)\left\|T_n^{(k)}\right\|. \tag{3.6}$$

For the proof of Lemma 3.2, we shall need some information about the local extrema of $\psi_k(x)$. We show below that, despite of the factor $1-x^2$, the local extrema of $\psi_k(x)$ still have the same behaviour as those of $T_n^{(k)}$.

LEMMA 3.3. *If $2 \le k \le n-2$, then the local maxima of $|\psi_k(x)|$ increase as $|x|$ increases.*

Proof. The proof of Lemma 3.3 makes use of the following beautiful result of Sonin - Pòlya (see, e.g., [33, Paragraph 7.31]):

LEMMA 3.4. *Let $u(x)$ satisfy the differential equation*

$$(p(x)u')' + P(x)u = 0, \tag{3.7}$$

where $p(x) > 0$, $P(x) > 0$, and both functions $p(x)$ and $P(x)$ have continuous derivative in an interval (a,b). Then the relative maxima of $|u|$ in (a,b) form an increasing or decreasing sequence according as $p(x)P(x)$ is decreasing or increasing in (a,b).

Using (3.1), one readily verifies that $u = \psi_k(x)$ satisfies the differential equation (3.7) with

$$p(x) := (1-x^2)^{k-1/2}, \quad P(x) := (1-x^2)^{k-5/2}\Big[[n^2-(k-1)^2](1-x^2) - 2(2k-1)\Big]. \tag{3.8}$$

To apply Lemma 3.3 to $u = \psi_k(x)$, we check first that the function $P(x)$ defined in (3.8) is positive in the interval $(-\xi, \xi)$, where ξ is the largest critical point of $\psi_k(x)$. Indeed, assume the contrary, then $P(x) < 0$ for every $x \in (\xi, 1)$. The function

$$q(x) := p(x)\psi_k(x)\psi_k'(x)$$

satisfies $q(\xi) = q(1) = 0$, and by Rolle's theorem there would exist a point $\eta \in (\xi, 1)$ such that $h'(\eta) = 0$. However, we have

$$q'(x) = \big(p(x)\psi_k'(x)\big)'\psi_k(x) + p(x)\big(\psi_k'(x)\big)^2 = p(x)\big(\psi_k'(x)\big)^2 - P(x)\psi_k^2(x) > 0 \text{ in } (\xi, 1),$$

a contradiction. This means that the largest critical point ξ of $\psi_k(x)$ satisfies

$$1 - \xi^2 > \frac{2(2k-1)}{n^2 - (k-1)^2}, \tag{3.9}$$

and $P(x) > 0$ in $(-\xi, \xi)$. An easy calculation reveals that

$$(p(x)P(x))' = -4x(1-x^2)^{2k-4}\Big[(k-1)[n^2-(k-1)^2](1-x^2) - (2k-1)(2k-3)\Big].$$

Then (3.9) shows that, for $k \ge 2$, $(p(x)P(x))' > 0$ in $(-\xi, 0)$ and $(p(x)P(x))' < 0$ in $(0, \xi)$. Now the claim of Lemma 3.3 follows from Lemma 3.4. □

Notice that the local maxima of $|\psi_1|$ have just the opposite behaviour. This can be seen from the above expression for $(p(x)P(x))'$ and application of Lemma 3.4.

Proof of Lemma 3.2. The case $k = n-2$ is easily verified, making use of the representation $T_n^{(n-2)}(x) = 2^{n-3}n![2x^2 - 1/(n-1)]$. We therefore assume that $2 \le k \le n-3$. According to Lemma 3.3, we have $\|\psi_k\| = \psi_k(\xi)$, where ξ is the last zero of ψ_k'. From $\psi_k'(\xi) = 0$ and the differential equation (3.1) we find

$$T_n^{(k+2)}(\xi) = \frac{2\xi}{1-\xi^2}T_n^{(k+1)}(\xi), \quad T_n^{(k+1)}(\xi) = \frac{n^2-k^2}{(2k-1)\xi}T_n^{(k)}(\xi). \tag{3.10}$$

From (3.10), we have

$$\psi_k(\xi) = \eta\, T_n^{(k)}(\xi), \tag{3.11}$$

where

$$\eta := \frac{(n^2-k^2)(1-\xi^2)}{(2k-1)\xi}.$$

We consider separately two cases.

Case I: $3 \le k \le n-3$. Making use of (3.10) and the fact that $T_n^{(m)}(x)$ $(m \ge k+1)$ are positive to the right ξ, we obtain

$$\begin{aligned} T_n^{(k)}(1) - T_n^{(k)}(\xi) &\ge (1-\xi)T_n^{(k+1)}((1+\xi)/2) \ge (1-\xi)\left[T_n^{(k+1)}(\xi) + \frac{1-\xi}{2}T_n^{(k+2)}(\xi)\right] \\ &= \frac{(1-\xi)(1+2\xi)}{1+\xi}T_n^{(k+1)}(\xi) = \frac{(1+2\xi)\eta}{(1+\xi)^2}T_n^{(k)}(\xi), \end{aligned}$$

and hence

$$T_n^{(k)}(\xi) \le \frac{1}{1 + \frac{(1+2\xi)\eta}{(1+\xi)^2}}T_n^{(k)}(1).$$

Together with (3.11) this yields

$$\psi_k(\xi) \le \frac{\eta}{1+\frac{(1+2\xi)\eta}{(1+\xi)^2}} T_n^{(k)}(1) < \frac{\eta}{1+\frac{3}{4}\eta} T_n^{(k)}(1),$$

and obviously Lemma 3.2 will be proved if we show that

$$\frac{\eta}{1+\frac{3}{4}\eta} \le 1 - \frac{1}{(2k+1)^2},$$

or, equivalently, if

$$\frac{(n^2-k^2)(1-\xi^2)}{\xi} \le \frac{4k(k+1)(2k-1)}{k^2+k+1}. \tag{3.12}$$

To prove (3.12) we need an estimate for ξ from below, and we derive it from (3.10) and the differential equation for $T_n^{(k+1)}$ as follows:

$$\begin{aligned} 0 < (1-\xi^2)T_n^{(k+3)}(\xi) &= (2k+3)\xi T_n^{(k+2)}(\xi) - [n^2-(k+1)^2]T_n^{(k+1)}(\xi) \\ &= \Big[2(2k+3)\xi^2/(1-\xi^2) - n^2 + (k+1)^2\Big] T_n^{(k+1)}(\xi). \end{aligned}$$

Since $T_n^{(k+1)}(\xi) > 0$, the term in the square brackets must be positive, too, and this yields the (equivalent) inequalities

$$\xi^2 > \frac{n^2-(k+1)^2}{n^2-k^2+2k+5}, \quad 1-\xi^2 < \frac{2(2k+3)}{n^2-k^2+2k+5}$$

(compare with (3.9)). We see from these inequalities that (3.12) will hold true if

$$\frac{\Big[(n^2-k^2+2k+5)(n^2-k^2-2k-1)\Big]^{1/2}}{n^2-k^2} \ge \frac{(k^2+k+1)(2k+3)}{2k(k+1)(2k-1)}. \tag{3.13}$$

By our assumption, the quantity $y = n^2-k^2$ belongs to $[3(2k+3), \infty)$. A straightforward calculation shows that the left-hand side of (3.13), considered as a function of y for a fixed k, has exactly one local extremum in this interval, which is a maximum. Consequently, the minimum on the left is attained for $y = 3(2k+3)$ or $y = \infty$. It is easy to see that this minimum is greater than 1 for $k = 3$, and greater than $2\sqrt{2}/3$ for $k \ge 4$. The right-hand side of (3.13) decreases with respect to k; it is less than 1 for $k = 3$, and is less than or equal to $33/40$ for $k \ge 4$. Hence (3.13) is established, as $33/40 < 2\sqrt{2}/3$. Lemma 3.2 is proved in the case $3 \le k \le n-3$.

Case II: $k = 2$. The cases $5 \le n \le 10$ are verified numerically, so we assume in what follows that $n \ge 11$. Since $T_n^{(m)}(x)$ $(m \ge 3)$ are convex to the right of ξ, it follows from a familiar property of the trapezium rule that

$$T_n''(1) - T_n''(\xi) \ge \frac{1-\xi}{2}\big[T_n'''(\xi) + T_n'''(1)\big] - \frac{(1-\xi)^2}{12}\big[T_n^{(4)}(1) - T_n^{(4)}(\xi)\big].$$

This inequality together with (3.2) and (3.10) yields

$$T_n''(\xi) \le \frac{1 - \frac{1}{10}z + \frac{n^2-9}{420(n^2-4)}z^2}{1+\frac{4\xi+3}{18\xi(1+\xi)}z} T_n''(1),$$

where

$$z := (n^2-4)(1-\xi).$$

Hence

$$\begin{aligned}\psi_2(x) &< \frac{z - \frac{1}{10}z^2 + \frac{n^2-9}{420(n^2-4)}z^3}{\frac{3\xi}{1+\xi} + \frac{4\xi+3}{6(1+\xi)^2}z} T_n''(1) \\ &< \frac{z - \frac{1}{10}z^2 + \frac{1}{420}z^3}{\frac{3\xi}{1+\xi} + \frac{7}{24}z} T_n''(1).\end{aligned} \tag{3.14}$$

In view of (3.14), the case $k = 2$ of Lemma 3.2 will be settled if we manage to prove that

$$\frac{z - \frac{1}{10}z^2 + \frac{1}{420}z^3}{\frac{3\xi}{1+\xi} + \frac{7}{24}z} \le \frac{24}{25},$$

or, equivalently, if

$$h(z) := \frac{1}{420}z^3 - \frac{1}{10}z^2 + \frac{18}{25}z \le \frac{72\xi}{25(1+\xi)}. \tag{3.15}$$

To this end, we prove that ξ, the largest zero of $\psi_2'(x)$, satisfies

$$\cos\frac{1.1\pi}{n} < \xi < \cos\frac{1.04\pi}{n} \quad \text{for } n \ge 11. \tag{3.16}$$

On using $T_n(\cos\alpha) = \cos(n\alpha)$, $T_n'(\cos\alpha) = n\sin(n\alpha)/\sin\alpha$ and the differential equation (3.1), we find the following representation of $\psi_2'(x)$:

$$\psi_2'(\cos\alpha) = \frac{n^2\cos(n\alpha)}{\sin^2\alpha}\Big[\frac{\tan(n\alpha)}{n\tan\alpha}(9\cot^2\alpha - 4n^2 + 7) - 9\cot^2\alpha + n^2 - 4\Big]. \tag{3.17}$$

To prove (3.16), it suffices to show that for $n \ge 11$

$$\psi_2'(\cos(1.04\pi/n)) > 0 \quad\text{and}\quad \psi_2'(\cos(1.1\pi/n)) < 0.$$

Indeed, this guarantees the existence of a zero of ψ_2' in $(\cos(1.1\pi/n), \cos(1.04\pi/n))$, and we claim that ξ is there. If this was not the case, then ψ_2' would have at least three zeros located to the right of $\cos(1.1\pi/n)$. Since the zeros of ψ_2 and T_n'' interlace, so do the zeros of ψ_2' and T_n'''. Therefore, if ψ_2' had three zeros to the right of $\cos(1.1\pi/n)$, then T_n''' would have at least two zeros there, which is false.

Our next arguments involve the well-known fact that

$$\{n\sin(\alpha/n)\}_{n=4}^{\infty} \nearrow \alpha, \quad \{n\tan(\alpha/n)\}_{n=4}^{\infty} \searrow \alpha, \quad 0 < \alpha < 2\pi.$$

To prove that $\psi_2'(\cos(1.1\pi/n)) > 0$, we need to show that the term in the square brackets in (3.17) is negative for $\alpha = 1.1\pi/n$. We observe first that

$$9\cot^2(1.1\pi/n) - 4n^2 + 7 < \cot^2(1.1\pi/n)\Big[9 - 1.21\pi^2(4n^2-7)/n^2\Big] < 0.$$

Therefore, for $n \ge 11$ we have

$$\begin{aligned}&\frac{\tan(1.1\pi)}{n\tan(1.1\pi/n)}\big[9\cot^2(1.1\pi/n) - 4n^2 + 7\big] - 9\cot^2(1.1\pi/n) + n^2 - 4 \\ &\le \big[9\cot^2(1.1\pi/n) - 4n^2 + 7\big]/11 - 9\cot^2(1.1\pi/n) + n^2 - 4 \\ &< \frac{7}{11}\cot^2(1.1\pi/n)\Big[\big(n\tan(1.1\pi/n)\big)^2 - \frac{90}{7}\Big] \\ &\le \frac{7}{11}\cot^2(1.1\pi/n)\Big[\big(11\tan(0.1\pi)\big)^2 - \frac{90}{7}\Big] < 0.\end{aligned}$$

Analogously, $\psi_2'(\cos(1.04\pi/n)) < 0$ exactly when the term in the square brackets in (3.17) is positive. The estimation is done as follows:

$$
\begin{aligned}
&\frac{\tan(1.04\pi)}{n\tan(1.04\pi/n)}\left[9\cot^2(1.04\pi/n)-4n^2+7\right]-9\cot^2(1.04\pi/n)+n^2-4\\
&\geq \frac{\tan(0.04\pi)}{1.04\pi}\left[9\cot^2(1.04\pi/n)-4n^2+7\right]-9\cot^2(1.04\pi/n)+n^2-4\\
&> 0.038666\left[9\cot^2(1.04\pi/n)-4n^2+7\right]-9\cot^2(1.04\pi/n)+n^2-4\\
&> 0.845336\cot^2(1.04\pi/n)\left[(n^2-4.412)\tan^2(1.04\pi/n)-10.235\right]\\
&> 0.845336\cot^2(1.04\pi/n)\left[(1.04\pi)^2(n^2-4.412)/n^2-10.235\right]>0.
\end{aligned}
$$

With this (3.16) is established. Now we proceed with the proof of (3.15). At first, we estimate $z=(n^2-4)(1-\xi)$ with the help of (3.16).

$$
\begin{aligned}
(n^2-4)(1-\xi) &> (n^2-4)(1-\cos(1.04\pi/n)) = 2\frac{n^2-4}{n^2}\left(n\sin(1.04\pi/(2n))\right)^2\\
&\geq 234\sin^2(1.04\pi/22) > 5.123,
\end{aligned}
$$

$$
\begin{aligned}
(n^2-4)(1-\xi) &< (n^2-4)(1-\cos(1.1\pi/n)) = 2\frac{n^2-4}{n^2}\left(n\sin(1.1\pi/(2n))\right)^2\\
&< 1.21\pi^2/2 < 5.972.
\end{aligned}
$$

A straightforward calculation shows that the function $h(z)$ in (3.15) is monotone decreasing in the interval $(5.123, 5.972)$, and therefore

$$h(z) < h(5.123) < 1.385.$$

For the right-hand side of (3.15) we have, in view of (3.16),

$$\frac{72\xi}{25(1+\xi)} > \frac{72\cos(0.1\pi)}{25(1+\cos(0.1\pi))} > 1.403.$$

With this (3.15) is established, and the proof of *Case II* of Lemma 3.2 is completed . ☐

3.4 An upper bound for the largest zero of ultraspherical polynomials

There have been many publication devoted to the extreme zeros of the classical orthogonal polynomials. For earlier results the reader may consult Szegö's book [33, Chapt.6], while for some newer contributions in this direction we refer (without any claim for completeness) to [1], [2], [8, 9], [11], [12], [13], [15]. We give below an upper bound for the largest zero $x_{n1}(\lambda)$ of the ultraspherical polynomial $P_n^{(\lambda)}$, due to Elbert and Laforgia [12] (see also [11]):

$$x_{n1}^2(\lambda) \leq \frac{(n-1)(n+2\lambda+1)}{(n+\lambda)^2}, \quad \lambda > 0. \tag{3.18}$$

The best so far upper bound for $x_{n1}(\lambda)$ when the parameter λ is large was obtained recently by Area, Dimitrov, Godoy and Ronveaux ([1, Theorem 4.3]). However, this upper bound looks too complicated to be quoted here.

Although the above mentioned estimates may turn out good enough for our purposes, we decided to incorporate another upper bound for $x_{n1}(\lambda)$. This decision is motivated by two reasons. The first one is that our proof is very elementary. The more important, our estimate is sharper than (3.18), and the numerical experiments indicate that it is superior even to the estimate given in [1, Theorem 4.3].

LEMMA 3.5. *For any $\lambda > -1/2$, the following estimate holds true:*

$$x_{n1}^2(\lambda) \le \frac{(n-1)(n+2\lambda+1)}{(n+\lambda)^2+3\lambda+5/4+3(\lambda+1/2)^2/(n-1)}. \tag{3.19}$$

Proof. The proof exploits an idea from [33, Paragraph 6.2], based on the following observation of Laguerre: If f is a polynomial of degree n having only real and distinct zeros, and $f(x_0)=0$, then

$$3(n-2)[f''(x_0)]^2 - 4(n-1)f'(x_0)f'''(x_0) \ge 0.$$

We substitute in this inequality $f = P_n^{(\lambda)}$ and $x_0 = x_{n1}(\lambda)$, and replace $f'(x_0)$ and $f'''(x_0)$ from the differential equations for f and f' as follows:

$$f'(x_0) = \frac{1-x_0^2}{(2\lambda+1)x_0}f''(x_0),\ \ f'''(x_0) = \Big[\frac{(2\lambda+3)x_0}{1-x_0^2} - \frac{(n-1)(n+2\lambda+1)}{(2\lambda+1)x_0}\Big]f''(x_0).$$

Then we cancel out the factor $[f''(x_0)]^2$ and solve the resulting inequality with respect to x_0^2. This yields exactly (3.19). The proof of Lemma 3.5 is complete. □

As we already mentioned, $T_n^{(k)}$ is equal, apart from a constant multiplier, to the ultraspherical polynomial $P_{n-k}^{(k)}$. We shall need the following consequence from Lemma 3.5.

COROLLARY 3.1. *The largest zero $\tau_1 = \tau_{n1}(k)$ of $T_n^{(k)}$ satisfies*

$$\tau_1^2 \le \frac{n^2-(k+1)^2}{n^2+3k+5/4+3(k+1/2)^2/(n-k-1)}.$$

For $k=2$ and $k=3$ we obtain sharper upper bounds, taking advantage of the explicit form of T_n and T_n' and arguing as in the proof of (3.16).

LEMMA 3.6. *(i) The largest zero $\tau_1 = \tau_{n1}(2)$ of T_n'' satisfies*

$$\tau_1 \le \cos\frac{1.43\pi}{n}.$$

(ii) The largest zero $\tau_1 = \tau_{n1}(3)$ of T_n''' satisfies

$$\tau_1 \le \cos\frac{7\pi}{4n}.$$

We leave the proof to the reader.

4. PROOF OF THEOREM 1.1

As was already mentioned in Section 1, the case $k=1$ of Theorem 1.1 has been established in [24]. The cases $k=n-1$ and $k=n$ follow trivially, as in these cases $\|f^{(k)}\|$ is attained for $x=1$ or $x=-1$ (see [24, Section 4]). Therefore, we assume that $2 \le k \le n-2$. In what follows, we adopt the following notation: $\tau_1 > \tau_2 > \cdots > \tau_{n-k}$ are the zeros of $T_n^{(k)}$, $\tau_{-1} = \tau_0 := 1$, $\tau_{n-k+1} = \tau_{n-k+2} := -1$, and

$$\varphi_\nu(x) := \frac{1-\tau_\nu x}{x-\tau_\nu}T_n^{(k)}(x), \quad \nu = 0,1,\ldots,n-k+1. \tag{4.1}$$

We need one more lemma.

LEMMA 4.1. *(i) If $2 \le r \le n-k-1$, then*

$$\max_{x\in[\tau_{r+1},\tau_{r-1}]} \big|\varphi_r(x)\big| < T_n^{(k)}(1).$$

(ii) If $r = 1$ or $r = n-k$, then

$$\max_{x\in[\tau_{r+1},\tau_{r-1}]} \big|\varphi_r(x)\big| = T_n^{(k)}(1),$$

and the maximum is attained only for $x = \tau_0$ and $x = \tau_{n-k+1}$, respectively.

Proof. We begin with the proof of part (i). Due to symmetry, we shall restrict our study only to half of the polynomials $\{\varphi_r\}$, say, those with indices $1 \le r \le \lfloor (n-k+1)/2 \rfloor$. It is clear that $\max_{x\in[\tau_{r+1},\tau_{r-1}]} |\varphi_r(x)| = \big|\varphi_r(\theta)\big|$, where $\theta := \theta_r$ is the unique zero of φ_r' in the interval (τ_{r+1}, τ_{r-1}). We verify first the case $r = (n-k+1)/2$ (i.e., $n-k$ is odd). In this case $\theta = 0$, and

$$\big|\varphi_r(0)\big| = \big|T_n^{(k+1)}(0)\big| = \big[n^2-(k-1)^2\big]\big|T_n^{(k-1)}(0)\big| < \frac{n^2-(k-1)^2}{2k-1} T_n^{(k-1)}(1) = T_n^{(k)}(1),$$

where we have used the differential equation for $T_n^{(k-1)}$, the fact that $x = 0$ is a point of a local maximum for $|T_n^{(k-1)}|$ (not the largest one), and Lemma 3.1.

Next, we assume that $2 \le r \le (n-k)/2$ (and, as a consequence, $0 < \tau_r < 1$). With the help of (3.1) we find

$$2\varphi_r'(\tau_r) = (1-\tau_r^2)T_n^{(k+2)}(\tau_r) - 2\tau_r T_n^{(k+1)}(\tau_r) = (2k-1)\tau_r T_n^{(k+1)}(\tau_r).$$

This yields $\operatorname{sign}\{\varphi_r'(\tau_r)\} = \operatorname{sign}\{\varphi_r(\tau_r)\} = \operatorname{sign}\{T_n^{(k+1)}(\tau_r)\} = (-1)^{r-1}$, and we deduce that $\theta \in (\tau_r, \tau_{r-1})$. From $\varphi_r'(\theta) = 0$ we get

$$(1-\tau_r\theta)(\theta-\tau_r)T_n^{(k+1)}(\theta) = (1-\tau_r^2)T_n^{(k)}(\theta), \tag{4.2}$$

whence $\operatorname{sign}\{T_n^{(k)}(\theta)\} = \operatorname{sign}\{T_n^{(k+1)}(\theta)\} = (-1)^{r-1}$. Since $\operatorname{sign}\{T_n^{(k+1)}(\tau_{r-1})\} = (-1)^{r-2}$, we conclude that $T_n^{(k+1)}$ has a zero (i.e., $T_n^{(k)}$ has a local extremum) in (θ, τ_{r-1}). On using (4.2) we find a second representation of $|\varphi_r(\theta)|$:

$$\begin{aligned} \big|\varphi_r(\theta)\big| &= \frac{1-\tau_r\theta}{\theta-\tau_r}\big|T_n^{(k)}(\theta)\big| \\ &= \frac{(1-\tau_r\theta)^2}{1-\tau_r^2}\big|T_n^{(k+1)}(\theta)\big|. \end{aligned} \tag{4.3}$$

Now we are prepared to prove part (i) of the lemma. We consider two possibilities.

If $(1-\tau_r\theta)/(\theta-\tau_r) < 2k+1$, then we take into account that $T_n^{(k)}$ has a local extremum to the right of θ, and apply Lemma 3.1 to obtain

$$\max_{x\in[\tau_{r+1},\tau_{r-1}]} \big|\varphi_r(x)\big| = \big|\varphi_r(\theta)\big| = \frac{1-\tau_r\theta}{\theta-\tau_r}\big|T_n^{(k)}(\theta)\big| < (2k+1)\frac{T_n^{(k)}(1)}{2k+1} = T_n^{(k)}(1).$$

On the other hand, if $(1-\tau_r\theta)/(\theta-\tau_r) \ge 2k+1$, then we make use of the second representation of $|\varphi_r(\theta)|$ in (4.3) and Lemma 3.2 to arrive at the same conclusion:

$$\begin{aligned} \max_{x\in[\tau_{r+1},\tau_{r-1}]} \big|\varphi_r(x)\big| = \big|\varphi_r(\theta)\big| &= \frac{(1-\tau_r\theta)^2}{1-\tau_r^2}\big|T_n^{(k+1)}(\theta)\big| \\ &= \frac{(1-\tau_r\theta)^2}{(1-\tau_r^2)(1-\theta^2)}\big|\psi_k(\theta)\big| = \frac{(1-\tau_r\theta)^2}{(1-\tau_r\theta)^2-(\theta-\tau_r)^2}\big|\psi_k(\theta)\big| \\ &\le \frac{1}{1-(2k+1)^{-2}}\,\|\psi_k\| < T_n^{(k)}(1). \end{aligned}$$

With this part (i) of the lemma is established.

For part (ii), it suffices to consider only φ_1, in view of the symmetry. We shall show that $\varphi_1(x)$ is monotone increasing in $[\tau_2, \tau_0] = \tau_2, 1]$. This is all we need, as $\varphi_1(\tau_2) = 0$ and $\varphi_1(1) = T_n^{(k)}(1)$. So, our goal is to prove that the last point of local extremum of φ_1 is situated to the right of $x = 1$, i.e., that $\varphi_1'(1) > 0$. It is easily seen with the help of (3.2) that this is equivalent to the inequality

$$\frac{1+\tau_1}{1-\tau_1} < \frac{n^2-k^2}{2k+1}. \tag{4.4}$$

In the proof of (4.4) we shall distinguish between the cases $k = 2$, $k = 3$ and $k \geq 4$.

Case 1: $k = 2$. The verification for $4 \leq n \leq 8$ can be performed by a computer. According to Lemma 3.6 (i), $\tau_1 < \cos(1.43\pi/n)$, and (4.4) will hold true if

$$(n^2-4)\tan^2\frac{1.43\pi}{2n} > 5.$$

This inequality is verified numerically for $9 \leq n \leq 21$. For $n \geq 22$ we have

$$(n^2-4)\tan^2\frac{1.43\pi}{2n} > \frac{n^2-4}{4n^2}(1.43\pi)^2 > \frac{480}{1936}(1.43\pi)^2 > 5.$$

Case 2: $k = 3$. Again, the cases of small n ($5 \leq n \leq 7$) can be verified by a computer. In view of Lemma 3.6 (ii), (4.4) will hold true in this case if

$$(n^2-9)\tan^2\frac{7\pi}{8n} > 7.$$

This inequality is verified numerically for $8 \leq n \leq 11$, while for $n \geq 12$ we have

$$(n^2-9)\tan^2\frac{7\pi}{8n} > \frac{49(n^2-9)}{64n^2}\pi^2 \geq \frac{6615}{9216}\pi^2 > 7.$$

Case 2: $k \geq 4$. The inequality (4.4) is equivalent to

$$\tau_1 < \frac{n^2-(k+1)^2}{n^2-k^2+2k+1},$$

and, in view of Corollary 3.1, this inequality will be certainly true if

$$\frac{n^2-(k+1)^2}{n^2+3k+5/4+3(k+1/2)^2/(n-k-1)} \leq \Big(\frac{n^2-(k+1)^2}{n^2-k^2+2k+1}\Big)^2. \tag{4.5}$$

Clearly, (4.5) is a consequence of

$$\frac{n^2-(k+1)^2}{n^2+3k+5/4} \leq \Big(\frac{n^2-(k+1)^2}{n^2-k^2+2k+1}\Big)^2.$$

It is an easy exercise to verify this last inequality, and the situations in which it is true are described in the next table (note that, by assumption, $n \geq k+2$).

k	$n \geq$
4	13
5 – 6	10
7 – 10	$k+3$
≥ 11	$k+2$

Finally, in the cases not covered by this table, (4.5) is verified to be true directly. □

Proof of Theorem 1.1. Theorem 2.1 applied to $Q = T_n$ with $k \in \{2, \dots, n-2\}$, implies that, for any $f \in \pi_n$ satisfying the assumptions of Theorem 1.1,

$$\left\| f^{(k)} \right\| \le \max_{0 \le r \le n-k+1} \max_{x \in [\tau_{r+1}, \tau_{r-1}]} |\varphi_r(x)|,$$

with $\{\varphi_\nu\}_0^{n-k+1}$ as defined in (4.1) (and $-\varphi_0 = \varphi_{n-k+1} = T_n^{(k)}$). Lemma 4.1 asserts that the maximum on the right is equal to $T_n^{(k)}(1)$, and is attained only for $x = \pm 1$. With this the inequality of Theorem 1.1 is established. It remains to clarify in which cases it becomes an equality. Tracing back the proof of Theorem 2.1, we see that, for $x \in [-1, 1]$, the equality $|f^{(k)}(x)| = T_n^{(k)}(1)$ is possible only if $x = \pm 1$, and only if $f = cT_n$, $|c| = 1$. □

5. Remarks and Comments

1. In Theorem 1.1, the assumption that the "check" points $\{t_\nu\}_0^n$ interlace strictly with the zeros of T_n was imposed only to avoid unimportant complications in the proof (the same applies to Theorem B). Moreover, a closer look to the proof of Theorem 1.1 reveals that we have established a more general statement. Namely, the following theorem holds true.

THEOREM 5.1. *Let $Q = P_n^{(\lambda)}$, where $\lambda \in \mathbb{N}_+$, and let $f \in \pi_n$ satisfy $|f| \le |Q|$ at $n+1$ distinct points in $[-1, 1]$, which are separated by the zeros of Q. Then*

$$\left\| f^{(k)} \right\| \le \left\| Q^{(k)} \right\| \quad \textit{for} \quad k = 1, \dots, n,$$

and equality occurs if and only if $f = cQ$, $|c| = 1$.

To see this, one only has to realize that the polynomials $\{\varphi_\nu(x)\}_{\nu=0}^{n-k+1}$ that appear in Theorem 2.1, and depend on Q and k, remain unchanged if Q is replaced by $Q^{(m)}$ and k by $k - m$, $0 \le m \le k$. Since the ultraspherical polynomial $Q = P_n^{(\lambda)}$, $\lambda \in \mathbb{N}$, is equal, apart from a constant multiplier, to $T_{n+\lambda}^{(\lambda)}$, we can derive the analogue of Lemma 4.1 for the polynomials $\{\varphi_\nu(x)\}_{\nu=0}^{n-k+1}$ generated by $Q = P_n^{(\lambda)}$, from Lemma 4.1 itself (notice that the restriction $k \ge 2$ drops in this case).

2. Let us mention that the method of proof of the case $k = 1$ in Theorem 1.1 (i.e., Theorem B) given in [24] is hardly applicable to the higher order derivatives. On the other hand, our approach in this paper does not work in the case $k = 1$, as in this case Lemma 4.1 is not true.

3. As was pointed out in [24], inequalities of the Duffin- and Schaeffer-type may find application to the estimation of the round-off error in the Lagrange differentiation formulae. Let

$$f^{(k)}(x) \approx \sum_{\nu=0}^{n} \ell_\nu^{(k)}(x) f(t_\nu) =: S(f; x)$$

be an interpolatory differentiation formula, where the nodes $\{t_\nu\}_0^n$ lie in $[-1, 1]$ and interlace with the zeros of $T_n(x)$. Assume that, instead of the true values $\{f(t_\nu)\}_0^n$ we have in our disposal inaccurate data $\{\tilde{f}(t_\nu)\}_0^n$, and know that $|f(t_\nu) - \tilde{f}(t_\nu)| \le \epsilon_\nu$, $\nu = 0, \dots, n$. Then, for the round-off error $R(f; x) := |S(f - \tilde{f}; x)|$ we have the following sharp estimate:

$$R(f; x) \le M.T_n^{(k)}(1) \quad \text{for every} \quad x \in [-1, 1], \quad \text{where } M = \max_{0 \le \nu \le n} \epsilon_\nu / |T_n(t_\nu)|.$$

In view of Theorem 5.1, a similar estimate for the round-off error holds if T_n is replaced by ultraspherical polynomial $Q = P_n^{(\lambda)}$, $\lambda \in \mathbb{N}$. Regarding the truncation error in the interpolatory differentiation formulas (i.e., the error caused by the fact that f is not necessarily polynomial), we refer the reader to the nice work of Shadrin [32].

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