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Abstract. In this paper we investigate the application of a self-verified general-purpose *parametric* fixed-point iteration method to linear systems involving nonlinear dependencies. The inclusion method is combined with a simple interval arithmetic technique providing inner and outer bounds for the range of monotone rational functions. The arithmetic on proper and improper intervals is considered as an intermediate computational tool for eliminating the dependency problem in range computation and for obtaining inner estimations by outwardly rounded interval arithmetic. Therefore the target problems to be solved are restricted to linear systems whose input data are rational functions of uncertain parameters varying within given intervals. Supporting software tools with result verification, developed in the environment of *Mathematica*, are presented. The discussed methodology and software tools can be applied to a wide range of practical problems leading to linear systems with dependent uncertain data, in particular applications to FEM models of mechanical structures. Beside some illustrative examples, an advanced practical application from structural engineering mechanics is solved by the discussed parametric iteration, as well as by a combination of interval techniques based on the parametric solver; the results are compared to literature data produced by other methods. A comparison of different measures of overestimation is done.

Keywords: linear systems, interval parameters, nonlinear dependencies, automatic result verification, structural steel frames.

AMS subject classification: 15A06, 65G20

1 Introduction

Solving parametric linear systems involving uncertainties in the parameters is an important part of the solution to many scientific and engineering problems.

Consider linear algebraic system

$$A(p) \cdot x = b(p), \tag{1a}$$

where $A(p)$ is an $n \times n$ matrix, $b(p)$ is an n-dimensional vector and $p = (p_1, \ldots, p_k)^\top$ is a k-dimensional parameter vector. The elements of $A(p)$ and $b(p)$ are, in general, nonlinear functions of the parameters

$$a_{ij}(p) = a_{ij}(p_1, \ldots, p_k), \tag{1b}$$

$$b_i(p) = b_i(p_1, \ldots, p_k), \qquad i, j = 1, \ldots, n. \tag{1c}$$

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The parameters are considered to be unknown or uncertain and varying within prescribed intervals

$$p \in [p] = ([p_1], \ldots, [p_k])^\top. \tag{1d}$$

When the parameters vary within a box $[p] \in \mathbb{IR}^k$ the set of solutions, called parametric solution set is

$$\Sigma^p = \Sigma\,(A(p), b(p), [p]) \;:=\; \{x \in \mathbb{R}^n \mid A(p) \cdot x = b(p) \text{ for some } p \in [p]\}\,. \tag{2}$$

In general, a solution set has very complicated structure, and does not need even to be convex. The parametric solution set Σ^p is bounded if $A(p)$ is nonsingular for every $p \in [p]$. For a nonempty bounded set $\Sigma \subseteq \mathbb{R}^n$, define interval hull $\Box : P\mathbb{R}^n \to \mathbb{IR}^n$ by

$$\Box\,\Sigma \;:=\; [\inf \Sigma,\ \sup \Sigma] \;=\; \cap\{[x] \in \mathbb{IR}^n \mid \Sigma \subseteq [x]\}.$$

It is well-known that

$$\Sigma\,(A(p), b(p), [p]) \subseteq \Sigma\,([A], [b])\,, \tag{3}$$

where

$$[A] = A([p]) := \Box\{A(p) \in \mathbb{R}^{n\times n} \mid p \in [p]\} \qquad [b] = b([p]) := \Box\{b(p) \in \mathbb{R}^n \mid p \in [p]\}$$

are the non-parametric interval matrix, resp. vector, that correspond to the parametric ones (the elements of $A([p])$, $b([p])$ are assumed to be independent) and $\Sigma\,([A], [b])$ is defined in (14). Since it is quite expensive to obtain Σ^p or $\Box\,\Sigma^p$, the solution of interest is seeking an interval vector $[y] \in \mathbb{IR}^n$ such that $[y] \supseteq \Box\,\Sigma^p \supseteq \Sigma^p$, and the goal is $[y]$ to be as narrow as possible.

Parametric linear systems are common in practice, where the coefficients in the system are described by complicated dependencies between different factors (cf. [2, 3, 16]). The result-verification methods based on interval analysis combine the rigor in algorithm design with the controlled rounding of computer arithmetic to guarantee that the stated problem has (or does not have) a solution in an enclosing interval. In view of the relation (3) and the increasing demand of the marketplace, the development of efficient result-verification methods for sharp enclosure of $\Box\,\Sigma^p$ is highly appreciated. Probably the first general purpose method computing outer and inner bounds for $\Box\,\Sigma^p$ is based on the fixed-point interval iteration theory developed by S. Rump. In [30, Theorem 4.8] Rump applies the general verification theory for systems of nonlinear equations and explicitly states the method for solving parametric linear systems involving affine-linear dependencies. This method was generalized in [22] by proving that a sharp enclosure of the iteration matrix expands the scope of application of the method over problems involving the so-called column-dependent matrices. Meanwhile, there were many attempts (mainly in applied sciences) to construct suitable methods for solving parameter dependent interval linear systems [3, 9, 10, 14, 15, 16]. While Rump's parametric fixed-point iteration is reformulated in [3] in terms of factored parameters, Muhanna and Mullen use construction methods, based on the application of FEM in structural mechanics, to overcome the overestimation due to coupling and multiple occurrences of interval parameters [14, 16]. The methods developed by Kolev [9, 10] are based on an expansion of the interval multiplication operation but they are not designed as self-verification methods. Recently, a new efficient method with result verification was proposed by Neumaier and Pownuk for the special case of parametric linear systems involving a particular structure of the dependencies that arise in the analysis of truss structures [19]. We do not intend to give here a complete overview of methods used for solving linear systems with dependent data. For other approaches in solving mechanical problems involving uncertainties see e.g. [15], [16] and the literature

cited therein. Most of the methods developed so far address linear systems involving affine-linear dependencies between the parameters. In [3] and [10] nonlinear dependencies are handled by linearizing the elements of $A(p), b(p)$ and then solving parametric system with affine-linear dependencies.

As stated in [30], the inclusion theorems for systems of nonlinear equations or (with obvious modifications) the corresponding theorems for linear equations can be applied directly even for linear systems involving nonlinear dependencies between the parameters in $A(p)$, $b(p)$. In this paper we combine the inclusion theory, developed by S. Rump in [27, 30], with methods for sharp range estimation of continuous and monotone rational functions, based on the arithmetic of proper and improper intervals, in order to compute inner and outer bounds for $\square\,\Sigma^p(A(p), b(p), [p])$ where the elements of $A(p)$, $b(p)$ are rational functions of the parameters p. Section 2 contains some basic properties of generalised interval arithmetic that will be used for the purpose of this work, rounded interval operations, and theorems for range computation. Section 3 summarizes the evolution of the inclusion theorems related to different kinds of linear systems, and the background of our approach in solving parametric linear systems with nonlinear dependencies. One goal of this work is to provide a free software for verified enclosure of the parametric solution set in the environment of technical computing system *Mathematica* [31]. The software tools, we describe here, are intended and used for exploring the behavior of the fixed-point iteration method applied to linear systems involving nonlinear dependencies. Because it is free and open-source software with result verification, it could be used as a benchmark for other methodology and performance investigations related to parametric linear systems. Computer implementation of the algorithm for solving linear systems whose data are rational functions of interval parameters is discussed in Section 4 where an illustrative example is also given. Section 5 contains a detailed case study of a small but realistic practical problem coming from structural engineering and initially solved in [2] by methods based on the EBE approach [14]. Methodology and software tools presented in this paper are applied to the structural steel frame system and the results obtained by various methods are compared. Some speculations with different measures of overestimation assessing the quality of the solution enclosures are discussed. It is demonstrated that the parametric fixed-point iteration method is superior to the EBE approach for small parameter tolerances, providing in addition guaranteed inclusions in floating point at no additional cost of human preprocessing.

Below we use the following notations. $\mathbb{R}^n, \mathbb{R}^{n\times m}$ denote the set of real vectors with n components and the set of real $n \times m$ matrices, respectively. By normal (*proper*) interval we mean a real compact interval $[a] = [a^-, a^+] := \{a \in \mathbb{R} \mid a^- \le a \le a^+\}$. By $\mathbb{IR}^n, \mathbb{IR}^{n\times m}$ we denote interval n-vectors and interval $n \times m$ matrices. The end-point functionals $(\cdot)^-, (\cdot)^+$, the mid-point function $\mathrm{mid}(\cdot)$, where $\mathrm{mid}([a^-, a^+]) := (a^- + a^+)/2$, and the diameter (width) function $\omega(\cdot)$, where $\omega([a^-, a^+]) := a^+ - a^-$, are applied to interval vectors and matrices componentwise. The absolute value of a matrix $A = (a_{ij})$ is denoted by $|A| = (|a_{ij}|)$; for $[a] \in \mathbb{IR}$, $|[a]| := \max\{|a| \mid a \in [a]\}$. For two matrices of the same size matrix (vector) inequalities $A \le B$ and the interval subset relations $[A] \subseteq [B]$ are understood componentwise. $A < B$ if $A \le B$ and $A \ne B$, analogously $[A] \subset [B]$ if $[A] \subseteq [B]$ and $[A] \ne [B]$. The above matrix notations apply to vectors, considered as one-column matrices, as well. $\varrho(A)$ is the spectral radius of a matrix A, I denotes the identity matrix. For interval quantities $[A], [B]$, operations between them are always interval operations. The result is the smallest interval quantity containing the corresponding result when using power set operations. For example,

$$[A] \in \mathbb{IR}^{n\times n},\ [b] \in \mathbb{IR}^n \ :\ [A]\cdot[b] := \cap\{[c] \in \mathbb{IR}^n \mid \forall\, a \in [A], \forall\, b \in [b] \ :\ a \cdot b \in [c]\}.$$

We assume the reader is familiar with conventional interval arithmetic on proper intervals (cf. [1], [13], [18]).

2 The Arithmetic on Proper and Improper Intervals

In this section we present only those basic facts from generalised interval arithmetic which are necessary to use it as an intermediate computational tool for handling conventional interval problems. The set of proper intervals $\mathbb{IR}$ is extended in [8] by the set $\{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}, a^- \geq a^+\}$ of *improper* intervals obtaining thus the set $\mathbb{I}^*\mathbb{R} = \{[a^-, a^+] \mid a^-, a^+ \in \mathbb{R}\}$ of all ordered couples of real numbers called here generalised intervals. The conventional arithmetic and lattice operations, order relations and other functions are isomorphically extended onto the whole set of proper and improper intervals [8]. The theory of *modal interval analysis* imposes a logical-semantic background on generalized intervals (considered there as modal intervals), defines semantic extensions of continuous rational functions, establishes their properties, thus making possible to compute the semantic extensions and to give a logical meaning to the interval results [5]. By modal interval analysis the algebraic completion of conventional interval arithmetic turns into a valuable and practically useful computing theory. Normal (proper) intervals are a special case of generalised (and modal) intervals, and the conventional interval arithmetic can be obtained as a projection of generalised interval arithmetic on $\mathbb{IR}$.

"*Dual*" is an important monadic operator that reverses the end-points of the intervals and expresses an element-to-element symmetry between proper and improper intervals in $\mathbb{I}^*\mathbb{R}$. For $[a] = [a^-, a^+] \in \mathbb{I}^*\mathbb{R}$, its dual is defined by $\mathtt{Dual}([a]) = [a^+, a^-]$. `Dual` is applied componentwise to vectors and matrices. For $[a], [b] \in \mathbb{I}^*\mathbb{R}$ and $\circ \in \{+, -, \times, /\}$,

$$\mathtt{Dual}(\mathtt{Dual}([a])) = [a], \qquad \mathtt{Dual}([a] \circ [b]) = \mathtt{Dual}([a]) \circ \mathtt{Dual}([b]). \tag{4}$$

The generalised interval arithmetic structure possesses group properties with respect to $+$ and $\times$ operations: for $[a], [b] \in \mathbb{I}^*\mathbb{R}$, $0 \notin [b]$

$$[a] - \mathtt{Dual}([a]) = 0, \qquad [b]/\mathtt{Dual}([b]) = 1. \tag{5}$$

Lattice operations are closed with respect to the inclusion relation; handling of norm and metric are very similar to norm and metric in linear spaces [8]. For more details on the theory, implementation and applications of generalized interval arithmetic consult [5], [8], [26].

2.1 Inner and Outer Estimations in Floating Point

Let $\mathbb{F} \subset \mathbb{R}$ denote the set of floating-point numbers on a computer, respectively $\mathbb{IF}, \mathbb{I}^*\mathbb{F}$ denote the corresponding sets of floating-point intervals. Denote by $\nabla, \Delta : \mathbb{R} \longrightarrow \mathbb{F}$ the directed roundings toward $-\infty$, resp. $+\infty$ as specified by the IEEE floating-point standard (cf. [6]). For intervals $[a] = [a^-, a^+] \in \mathbb{IR}$, outward ($\Diamond$) and inward ($\bigcirc$) roundings $\Diamond, \bigcirc : \mathbb{IR} \longrightarrow \mathbb{IF}$ are defined as

$$\Diamond([a]) := [\nabla(a^-),\ \Delta(a^+)] \supseteq [a], \qquad \bigcirc[a] := [\Delta(a^-),\ \nabla(a^+)] \subseteq [a]. \tag{6}$$

If $\circ \in \{+, -, \times, /\}$ is an arithmetic operation in $\mathbb{IR}$ and $[a], [b] \in \mathbb{IF}$, the corresponding computer operations $\diamondsuit, \odot : \mathbb{IF} \times \mathbb{IF} \longrightarrow \mathbb{IF}$ are defined by

$$[a] \diamondsuit [b] := \Diamond([a] \circ [b]) = [\nabla(([a] \circ [b])^-),\ \Delta(([a] \circ [b])^+)] \supseteq [a] \circ [b], \tag{7}$$

$$[a] \odot [b] := \bigcirc([a] \circ [b]) = [\Delta(([a] \circ [b])^-),\ \nabla(([a] \circ [b])^+)] \subseteq [a] \circ [b]. \tag{8}$$

Obtaining inner approximations on a computer in conventional interval arithmetic is possible only if the four interval arithmetic operations are implemented with inward rounding $\odot$ in addition to the four $\diamondsuit$ operations. Since most of the wide-spread interval packages do not support inwardly rounded

interval arithmetic, we use an alternative computational technique based on the properties of the algebraic extension of conventional interval arithmetic.

For intervals $[a] = [a^-, a^+] \in \mathbb{I}^*\mathbb{R}$, outward ($\Diamond$) and inward ($\bigcirc$) roundings $\Diamond, \bigcirc : \mathbb{I}^*\mathbb{R} \longrightarrow \mathbb{I}^*\mathbb{F}$ are defined by the same formulae (6) preserving the inclusion properties (6) in $\mathbb{I}^*\mathbb{F}$. Analogously, if $\circ \in \{+, -, \times, /\}$ is an arithmetic operation in $\mathbb{I}^*\mathbb{R}$ and $[a], [b] \in \mathbb{I}^*\mathbb{F}$, the corresponding computer operations $\diamondsuit$, $\odot$: $\mathbb{I}^*\mathbb{F} \times \mathbb{I}^*\mathbb{F} \longrightarrow \mathbb{I}^*\mathbb{F}$ are defined by formulae (7), (8) also preserving the inclusion properties (7), (8) of the arithmetic in $\mathbb{I}^*\mathbb{F}$.

The following additional properties show that inner numerical approximations can be obtained at no additional cost only by outward directed rounding and the `Dual` operator in $\mathbb{I}^*\mathbb{F}$ [4, 5].

$$\text{For } [a] \in \mathbb{I}^*\mathbb{R}, \qquad \bigcirc([a]) = \texttt{Dual}(\Diamond(\texttt{Dual}[a])). \tag{9}$$

$$\text{For } [a], [b] \in \mathbb{I}^*\mathbb{F},\ \circ \in \{+, -, \times, /\}, \qquad [a] \odot [b] = \texttt{Dual}(\texttt{Dual}[a] \diamondsuit \texttt{Dual}[b]). \tag{10}$$

In order to obtain inner estimations of proper interval problems, the above properties can be also applied in computing environments not supporting the arithmetic on proper and improper intervals (cf. [24]).

2.2 Elimination of the Dependency Problem

Let $f : D_f \subseteq \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued rational function continuous in a domain $[x] \in \mathbb{IR}^n$, $[x] \subseteq D_f$. Denote the range of f over $[x] \in \mathbb{IR}^n$ by $r_f([x]) = \{f(x) \mid x \in [x]\}$. Since f is continuous $r_f([x])$ is an interval $r_f([x]) \in \mathbb{IR}^n$. If a rational function is totally monotonic for all its variables over a box $[x]$, the exact lower and upper bounds of the range can be obtained in exact arithmetic evaluating the function at suitable end-points of the intervals according to the monotonicity type. The inclusion properties of conventional (proper) interval arithmetic can be used for sharp range inclusion only if all the incidences of a variable in a function depending on a multi-incident variable have the same monotonicity type, otherwise the dependency causes overestimation. This is the deficiency of conventional interval arithmetic in bounding the range of a monotone function, and one of the great advantages of generalised interval arithmetic, presented in the next Theorems of this section.

Define a generalized rational extension of f as the interval function $R_f([x]) : \mathbb{I}^*\mathbb{R}^n \longrightarrow \mathbb{I}^*\mathbb{R}$ defined by the syntactic expression of f where real variables are replaced by generalized intervals and real operations are replaced by operations between generalized intervals. When $[x] \in \mathbb{IR}^n$, $R_f([x])$ is the classical natural interval extension of f.

In modal interval analysis inner and outer estimations of functional ranges over a box are connected to an enhanced interpretation of quantified propositions which has many promising applications [5] but this is out of the scope of this work. Here we specify how to eliminate the dependency problem by using generalised interval arithmetic in range computation over a domain of proper intervals. The corresponding formulation for the domain of modal intervals can be found in [5] and the literature cited therein. For a better understanding, first we give a simple one-dimensional case and then formulate the general theorems.

We need the following definitions. A real function $f(x, y) : \mathbb{R}^{1+m} \longrightarrow \mathbb{R}$ is x-uniformly monotonic for x on a domain $([x], [y])$ if it is monotonic for x on $[x]$, and it keeps the same monotonicity for all $y \in [y]$. A real function f is x-totally monotonic for a multi-incident variable $x \in \mathbb{R}$ if f is uniformly monotonic for this variable and for each one of its incidences (considering each incidence as an independent variable).

Theorem 2.1. *Let $f(x, a)$ be a rational function multi-incident on a. Let $[x], [a] \in \mathbb{IR}$, f be a-totally monotonic on $[x] \times [a]$, and there exist a splitting $a' = (a'_1, .., a'_p)$, $a'' = (a''_1, .., a''_q)$ of the incidences of a. Let $f^*(x, a', a'')$ correspond to the expression of f with explicit reference to the incidences of*

a and $f^*(x,a',a'')$ be continuous on $[x]\times[a']\times[a'']$. Suppose that $f^*(x,a',a'')$ is $\leq$-isotone for any component of a' and $\leq$-antitone for any component of a'' on $[x]\times[a']\times[a'']$, then

- *if $f(x,a)$ is $\leq$-isotone for a on $[x]\times[a]$,*

$$r_f([x],[a]) = R_{f^*}\left([x],[a'],\mathtt{Dual}([a''])\right) \subseteq R_f([x],[a]);$$

- *if $f(x,a)$ is $\leq$-antitone for a on $[x]\times[a]$,*

$$r_f([x],[a]) = R_{f^*}\left([x],\mathtt{Dual}([a']),[a'']\right) \subseteq R_f([x],[a]).$$

If $f : D\subseteq\mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued rational function it can be specified by several expressions which are isomorphic by equivalent algebraic transformations. Suppose that we have chosen f to be represented by an expression with hopefully minimal number of the variable incidences. In Theorem 2.1 and hereafter the expressions of $f^*(x,a',\mathtt{Dual}(a''))$, resp. $f^*(x,\mathtt{Dual}(a'),a'')$, where the Dual-operator is involved at the incidences of a having monotonicity type opposite to the global monotonicity of f with respect to a, will be called *dual-transformed expressions* of f. A dual-transformed expression f^* corresponds to a quantified proposition connecting the function f, the interval vector of variables, and the range enclosing interval. The evaluation of f^* in generalised interval arithmetic provides better range estimation for monotone functions with multi-incident arguments. Equivalently, we may also say $R_{f^*}([x],[a'],\mathtt{Dual}([a'']))$, resp. $R_{f^*}([x],\mathtt{Dual}([a']),[a''])$, is a dual-transformed generalised extension of f over the domain of proper and improper intervals.

The following theorem is a special case of Theorem 5.2 (*-Partially Optimal Coertion) from [5].

Theorem 2.2. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a real-valued rational function continuous in a given interval vector $[x]\in\mathbb{IR}^n$, and multi-incident on its variables. Let $R_f([x])$ be defined on $[x]$ and let there exist splitting $x=(xn,xt)$ such that f be totally monotonic for the components of xt. Let $[xt^*]$ be the enlarged vector of $[xt]$, such that each incidence of every component of xt is included in $[xt^*]$ as independent component, but transformed into its dual if the corresponding incidence-point has a monotonicity type opposite to the global one of the corresponding xt-component. Then*

$$r_f([x]) \subseteq R_f\left([xn],[xt^*]\right)) \subseteq R_f([x]).$$

In case a function f is totally monotonic for all its variables, we have a sharp range estimation, specified in more details by the following theorem. For a set of indices $\mathcal{I}=\{i_1,\ldots,i_n\}$, the vector $(x_{i_1},\ldots,x_{i_n})^\top$ will be denoted by $x_{\mathcal{I}}$.

Theorem 2.3. *Let $f : \mathbb{R}^n \longrightarrow \mathbb{R}$ be a rational function continuous in a given interval vector $[x]\in\mathbb{IR}^n$, multi-incident on its variables and totally monotonic on all variables. Define two sets of indices $\mathcal{P}=\{i_1,\ldots,i_q\}$, $\mathcal{N}=\{i_1,\ldots,i_r\}$ such that $\mathcal{P}\cap\mathcal{N}=\emptyset$, $\mathcal{P}\cup\mathcal{N}=\{1,\ldots,n\}$, and f be $\leq$-isotone for $x_i : i\in\mathcal{P}$, f be $\leq$-antitone for $x_i : i\in\mathcal{N}$. Let for each variable x_i, $1\leqq i\leqq n$, there exist splitting $x_i'=(x_{i1}',\ldots,x_{ik}')$, $x_i''=(x_{i1}'',\ldots,x_{im}'')$ of the incidences of x_i. Let $f^*(x_{\mathcal{P}}',x_{\mathcal{P}}'',x_{\mathcal{N}}',x_{\mathcal{N}}'')$ correspond to the expression of f with explicit reference to the incidences of every variable, f^* be continuous on $[x_{\mathcal{P}}']\times[x_{\mathcal{P}}'']\times[x_{\mathcal{N}}']\times[x_{\mathcal{N}}'']$, and $R_{f^*}([x_{\mathcal{P}}'],[x_{\mathcal{P}}''],[x_{\mathcal{N}}'],[x_{\mathcal{N}}''])$ be defined. If $f^*(x_{\mathcal{P}}',x_{\mathcal{P}}'',x_{\mathcal{N}}',x_{\mathcal{N}}'')$ is $\leq$-isotone for the components of $x_{\mathcal{P}}',x_{\mathcal{N}}'$ and $\leq$-antitone for the components of $x_{\mathcal{P}}'',x_{\mathcal{N}}''$, then*

$$r_f([x]) = R_{f^*}\left([x_{\mathcal{P}}'],\mathtt{Dual}([x_{\mathcal{P}}'']),\mathtt{Dual}([x_{\mathcal{N}}']),[x_{\mathcal{N}}'']\right) \subseteq R_f([x]). \tag{11}$$

The application of the above theorem is illustrated in Section 4.1 below.

In general, a dual-transformed expression of a function f on variables $x \in \mathbb{R}^n$ will be denoted by $f^*(x_{\mathcal{I}}, \mathtt{Dual}(x_{\mathcal{J}}))$, where $\mathcal{I}$ and $\mathcal{J}$ are index sets involving the indexes of the incidences for all the variables x such that $\mathcal{I}$ contains the indexes of those variable instances which are not dual-transformed, $\mathcal{J}$ contains the indexes of those variable instances which are dual-transformed according to the application of Theorem 2.1 or Theorem 2.2 or Theorem 2.3.

The inclusion properties of rounded generalised intervals, rounded interval operations in $\mathbb{I}^*\mathbb{F}$, and their inclusion properties allow a rigorous implementation of Theorems 2.1, 2.2, 2.3 on the computer providing inner and outer inclusion of the true range

$$\bigcirc R_{f^*}\,(\bigcirc[x_{\mathcal{I}}], \mathtt{Dual}(\bigcirc[x_{\mathcal{J}}])) \subseteq r_f([x]) \subseteq \Diamond R_{f^*}\,(\Diamond[x_{\mathcal{I}}], \mathtt{Dual}(\Diamond[x_{\mathcal{J}}]))\,. \tag{12}$$

Hereafter we use the notation that an arithmetic expression in parentheses preceded by a rounding symbol $(\bigcirc, \Diamond)$ implies that all operations are performed in floating-point in the specified rounding mode. The notation $\Diamond f([x])$ indicates that the argument of $\Diamond f([x])$ is a floating-point interval and floating-point interval operations with outward rounding are to be used for evaluation.

Furthermore, by applying the remarkable properties (9), (10) it is possible to obtain inner inclusion only by outwardly rounded operations. Thus, the left inclusion relation in (12) becomes

$$\bigcirc R_{f^*}\,(\bigcirc[x_{\mathcal{I}}], \mathtt{Dual}(\bigcirc[x_{\mathcal{J}}])) = \mathtt{Dual}\,(\Diamond R_{f^*}\,(\Diamond(\mathtt{Dual}[x_{\mathcal{I}}]), \mathtt{Dual}(\Diamond(\mathtt{Dual}[x_{\mathcal{J}}])))) \subseteq r_f([x]). \tag{13}$$

3 Inclusion Theorems

In this section we give a brief summary of the theory of the enclosure methods for our problem. The inclusion theorems for the solution set of a parametric linear system present a direct consequence from the inclusion theory for nonparametric problems developed by S. Rump and discussed in many works (cf. [27, 28, 29, 30] and the literature cited therein).

The basic idea of combining the Krawczyk-operator [11] and the existence test by Moore [12] is further elaborated by S. Rump in [27] who proposed several improvements leading to the following inclusion theorem for the solution to an interval linear system $[A] \cdot x = [b]$.

Theorem 3.1 ([27, 30]). *Let $[A] \in \mathbb{IR}^{n\times n}$, $R \in \mathbb{R}^{n\times n}$, $[b], [y] \in \mathbb{IR}^n$, $\tilde{x} \in \mathbb{R}^n$ be given. Define $[z] \in \mathbb{IR}^n$, $[C] \in \mathbb{IR}^{n\times n}$ by*

$$\begin{aligned} [z] &:= R \cdot ([b] - [A] \cdot \tilde{x})\,, \\ [C] &:= I - R \cdot [A]. \end{aligned}$$

Define $[v] \in \mathbb{IR}^n$ by means of the following Einzelschrittverfahren

$$1 \le i \le n \;:\; [v_i] := \{[z] + [C] \cdot [u]\}_i, \text{ where } u := (v_1, \ldots, v_{i-1}, y_i, \ldots, y_n)^{\top}.$$

If $[v] \subsetneqq [y]$, then R and every matrix $A \in [A]$ are regular, and for every $A \in [A]$, $b \in [b]$ the unique solution $\widehat{x} = A^{-1}b$ of $Ax = b$ satisfies $\widehat{x} \in \tilde{x} + [v]$. Define the solution set Σ by

$$\Sigma\,([A], [b]) := \{x \in \mathbb{R}^n \mid \exists\, A \in [A]\, \exists b \in [b] : Ax = b\}. \tag{14}$$

Then with $[d] := [C] \cdot [v] \in \mathbb{IR}^n$ the following inner estimation of $\square\,\Sigma$ holds true

$$[\tilde{x} + [z]^- + [d]^+,\ \tilde{x} + [z]^+ + [d]^-] \subseteq [\inf(\Sigma),\ \sup(\Sigma)]\,.$$

When aiming to compute an inclusion of the solution to a given problem, an iteration scheme is proven to be very useful [27, 29]. Several improvements in the inclusion theory have lead to a self-verification algorithm which computes an enclosure of the solution to a system of linear equations and proves the existence and uniqueness of the solution within the bounds computed in finite precision arithmetic. The most important features of the iterative algorithm providing an automatic result verification by the computer will be discussed in Section 4.

When applying Theorem 3.1 to the solution of an interval linear system with matrix $[A] \in \mathbb{IR}^{n\times n}$ and right hand side $[b] \in \mathbb{IR}^n$, we assume A and b to vary componentwise independently within $[A]$ and $[b]$. However, in most practical applications there are complicated dependencies between the components of A and possibly the components of b. The main reason for the dependency is that the errors in several different components may be caused by same parameters (see Section 5). The restriction of componentwise variation to parameter variation usually shrinks the size of the solution set significantly. Computing verified inclusions for the solution set of an interval linear system with data dependencies was first considered by C. Jansson [7]. He treated systems with symmetric and skew-symmetric matrices, as well as dependencies in the right hand side, by improving the inclusion Theorem 3.1 to account for the dependencies in the system. In [30, Theorem 4.8] S. Rump gives a straightforward generalization to affine-linear dependencies in the matrix and the right hand side.

In case of parameter dependent linear systems, Theorem 2.4 from [30], or with obvious modifications, Theorem 3.1 can be applied directly. In order to obtain sharp inclusions, the problem is to obtain sharp bounds for the ranges of functions $z(p) := -R \cdot f(p, \tilde{x}) = R \cdot (b(p) - A(p) \cdot \tilde{x})$ and $C(p) := I - R \cdot A(p)$ on the domain $[p] \in \mathbb{IR}^k$ because a straightforward evaluation (natural interval extension) causes overestimation. The affine-linear dependencies between the parameters in $A(p), b(p)$ allow an explicit representation of the ranges of $z(p), C(p)$ by interval expressions, as it is stated by the following theorem.

Theorem 3.2. *Consider parametric linear system (1a) where $A(p), b(p)$ are defined by*

$$a_{ij}(p) := a_{ij}^{(0)} + \sum_{\nu=1}^{k} p_\nu a_{ij}^{(\nu)}, \qquad b_i(p) := b_i^{(0)} + \sum_{\nu=1}^{k} p_\nu b_i^{(\nu)}, \quad i,j = 1,\ldots,n.$$

Let $R \in \mathbb{R}^{n\times n}$, $[y] \in \mathbb{IR}^n$, $\tilde{x} \in \mathbb{R}^n$ be given and define $[z] \in \mathbb{IR}^n$, $[C] \in \mathbb{IR}^{n\times n}$ by

$$\begin{aligned} [z] &:= R \cdot (b^{(0)} - A^{(0)}\tilde{x}) + \sum_{\nu=1}^{k} [p_\nu](R \cdot b^{(\nu)} - R \cdot A^{(\nu)} \cdot \tilde{x}), \\ [C] &:= I - R \cdot A^{(0)} - \sum_{\nu=1}^{k} [p_\nu](R \cdot A^{(\nu)}), \end{aligned}$$

where $A^{(0)} := \left(a_{ij}^{(0)}\right), \ \ldots, A^{(k)} := \left(a_{ij}^{(k)}\right) \in \mathbb{R}^{n\times n}$, $b^{(0)} := (b_i^{(0)}), \ldots, b^{(k)} := (b_i^{(k)}) \in \mathbb{R}^n$.
Define $[v] \in \mathbb{IR}^n$ by means of the following Einzelschrittverfahren

$$1 \le i \le n \ : \ [v_i] := \{[z] + [C] \cdot [u]\}_i, \qquad u := (v_1, ..., v_{i-1}, y_i, ..., y_n)^\top.$$

If $[v] \subsetneqq [y]$, then R and every matrix $A(p), p \in [p]$ are regular, and for every $p \in [p]$ the unique solution $\widehat{x} = A^{-1}(p)b(p)$ of (1a) satisfies $\widehat{x} \in \tilde{x} + [v]$.
With $[d] := [C] \cdot [v] \in \mathbb{IR}^n$ and the solution set Σ^p, defined by (2), the following inner estimation holds true

$$\left[\tilde{x} + [z]^- + [d]^+, \ \tilde{x} + [z]^+ + [d]^-\right] \subseteq [\inf(\Sigma^p), \ \sup(\Sigma^p)]\,.$$

The above theorem generalizes Theorem 4.8 from [30] by requiring computation of the range of $C(p)$ instead of using an interval extension $C([p])$ (cf. [22]). Although a sharp enclosure of the iteration matrix $[C]$ is required also by other authors [3], the necessity of this improvement is not well justified therein. The generalization of Theorem 4.8 from [30] is first proven theoretically and by several numerical examples in [21, 22]. Indeed, for a class of so-called column-dependent parametric matrices (cf. [21]), the following relation holds

$$[Cp] := \square\{C(p) \mid p \in [p]\} \subset C([p]) =: [C],$$

which implies $|[Cp]| < |[C]|$. If in addition, $|[Cp]| + |[C]|$ is irreducible, from the theory of nonnegative matrices it follows that $\varrho(|[Cp]|) < \varrho(|[C]|)$. Thus the range enclosure of $C(p)$ will provide convergence of the iteration method for $\varrho(|[Cp]|) < 1$, while a worse enclosure (e.g. $C([p])$) may not for some column-dependent parametric matrices and some interval domains for the parameters. Examples demonstrating the expanded scope of application of the generalized inclusion Theorem 3.2 can be found in [21, 22, 24, 25].

In case of arbitrary nonlinear dependencies between the parameters in a linear system we can give only a general formulation of the inclusion theorem, as given bellow.

Theorem 3.3. *Consider parametric linear system defined by (1a–1d). Let* $R \in \mathbb{R}^{n\times n}$, $[y] \in \mathbb{IR}^n$, $\tilde{x} \in \mathbb{R}^n$ *be given and define* $[z] \in \mathbb{IR}^n$, $[C] \in \mathbb{IR}^{n\times n}$ *by*

$$\begin{aligned} [z] &:= \square\{R\,(b(p) - A(p)\tilde{x}) \mid p \in [p]\}, \\ [C] &:= \square\{I - R \cdot A(p) \mid p \in [p]\}. \end{aligned}$$

Define $[v] \in \mathbb{IR}^n$ *by means of the following Einzelschrittverfahren*

$$1 \le i \le n \;:\; [v_i] := \{[z] + [C] \cdot [u]\}_i, \qquad u := (v_1, ..., v_{i-1}, y_i, ..., y_n)^\top.$$

If $[v] \subsetneqq [y]$, *then* R *and every matrix* $A(p)$ *with* $p \in [p]$ *are regular, and for every* $p \in [p]$ *the unique solution* $\widehat{x} = A^{-1}(p)b(p)$ *of (1a–1d) satisfies* $\widehat{x} \in \tilde{x} + [v]$.

By the same reasons as in the case of affine-linear dependencies, the computation of $\square\{C_{ij}(p) \mid p \in [p]\}$, i,j=1,...,n, is important when the matrix $A(p)$ involves column dependencies.

In case of arbitrary nonlinear dependence of the coefficients on the uncertain parameters, computing $[z]$ and $[C]$ in Theorem 3.3 requires sharp range enclosure for nonlinear functions. This is a key problem in interval analysis and there exists a huge amount of methods and techniques devoted to this problem, no one being universal. In what follows we will restrict ourselves to linear systems where the elements of $A(p)$ and $b(p)$ are rational functions of the uncertain parameters. In this case the elements of $z(p)$ and $C(p)$ are also rational functions of p. Computing enclosure of $\square z(p) = \square\{R\,(b(p) - A(p)\tilde{x}) \mid p \in [p]\}$, respectively of $\square C(p) = \square\{I - R \cdot A(p) \mid p \in [p]\}$ can be performed by computing enclosure of the range for every component function $z_i(p), C_{ij}(p)$, defined by

$$z_i(p) := \sum_{\nu=1}^{n} r_{i\nu} b_\nu(p) - \sum_{\mu,\nu=1}^{n} r_{i\mu}\tilde{x}_\nu a_{\mu\nu}(p), \tag{15}$$

$$C_{ij}(p) := \delta_{ij} - \sum_{\nu=1}^{n} r_{i\nu} a_{\nu j}(p), \tag{16}$$

where $\delta_{ij} = \{1 \text{ if } i = j,\ 0 \text{ otherwise}\}$, $R = (r_{ij})$, $i, j = 1, \ldots, n$. Our goal will be to use Theorem 2.3, or at least Theorem 2.2, for enclosing the ranges of $z_i(p)$, $C_{ij}(p)$. It may seem impossible to prove

the total monotonicity for the parameters since this requires enclosing the range of the derivative. However, since $[z]$ is an enclosure of the residual, its components should have small width for reasonable parameter tolerances. Therefore, we may hope that will be able to prove the total monotonicity of $z_i(p)$ with respect to the parameters, even for large parameter intervals. Furthermore, the expressions of $z_i(p)$ can be considered as an analogy of the expression $f(x) = x - x$, for which we have $\omega(R_f([x])) = 2\omega([x])$, while $[x] - \mathtt{Dual}([x]) = 0$ and $r_f([x]) = R_f([x], \mathtt{Dual}([x]))$. Although sharp enclosure of the elements of $[C]$ is necessary only for column-dependent parametric matrices, the diameter of $[C]$ is also not too big and we may prove the required total monotonicity. That is why, for the computation of $[z_i]$, $[C_{ij}]$ from Theorem 3.3 we apply Theorem 2.3 or Theorem 2.2. A combination of Theorem 2.2 with a preliminary algebraic simplification of the expressions (15), (16) will be helpful for reducing the number of the incidences. It may seem quite restrictive to require that the elements of $z(p)$, $C(p)$ be monotone functions of the parameters on some interval domains. However, we shall demonstrate in Section 5 that there are realistic practical problems which can be solved successfully by this approach. Of course, more sophisticated techniques for range enclosure will possibly expand the scope of the application of Theorem 3.3.

The above theorems define how to compute an outer enclosure of the solution set of an interval linear system, i.e. an interval vector which is verified to contain the exact solution set hull, respectively the true solution set of the system. However, it is important to know the quality of the computed enclosure, in other words: how much such an enclosure overestimates the exact hull of the solution set. The amount of overestimation can be approximated by an inner inclusion of the solution set hull which is a componentwise inner estimation of the solution set [17, 28].

Definition 3.1. *An interval vector $[x] \in \mathbb{IR}^n$ is called componentwise inner approximation for some set $\Sigma \in \mathbb{R}^n$ if*

$$\inf_{\sigma \in \Sigma} \sigma_i \leq x_i^- \quad \text{and} \quad x_i^+ \leq \sup_{\sigma \in \Sigma} \sigma_i, \qquad \text{for every } 1 \leq i \leq n.$$

The inner box (interval vector) $[x] \in \mathbb{IR}^n$ has the property that the projection to every coordinate is an inner inclusion of the corresponding projection of the true solution set Σ. We estimate every *component* of the solution set from inside. An inner inclusion of the solution set hull should be distinguished from an inner inclusion of the solution set, that is $[x] \subseteq [\inf(\Sigma), \sup(\Sigma)]$ but $[x] \not\subseteq \Sigma$. The outer and inner inclusions of the solution set hull together with the true solution set for the problem from Example 4.1 are displayed in Figure 1.

Basing on ideas developed in [17], a cheap method for computing rigorous inner inclusion of the solution set hull is proposed in [28]. The next theorem, resp. the second part of Theorems 3.1, 3.2, establish how to compute the componentwise inner estimation of the (parametric) solution set.

Theorem 3.4. *Let $A(p) \cdot x = b(p)$, where $A(p) \in \mathbb{R}^{n \times n}$, $b(p) \in \mathbb{R}^n$, $p \in [p] \in \mathbb{IR}^k$, and $R \in \mathbb{R}^{n \times n}$, $\tilde{x} \in \mathbb{R}^n$, $[y] \in \mathbb{IR}^n$ be given. Define*

$$\begin{aligned} [z] &:= \square\{R \cdot (b(p) - A(p) \cdot \tilde{x}) \mid p \in [p]\}, \\ [\Delta] &:= [C] \cdot [y], \quad \text{where } [C] := \square\{I - R \cdot A(p) \mid p \in [p]\}. \end{aligned}$$

Let the solution set $\Sigma^p = \Sigma(A(p), b(p), [p])$ be defined as in (2) and assume

$$[z] + [\Delta] \subsetneqq [y].$$

Then

$$[\tilde{x} + [z]^- + [\Delta]^+,\ \tilde{x} + [z]^+ + [\Delta]^-] \subseteqq \square\Sigma^p \subseteqq \tilde{x} + [z] + [\Delta]$$

or, in coordinate notations, for all $i = 1, \ldots, n$ *there exists* $x^-, x^+ \in \Sigma^p$ *with*

$$\begin{array}{rcl} \tilde{x}_i + [z_i]^- + [\Delta_i]^- & \leqq & x_i^- \leqq \tilde{x}_i + [z_i]^- + [\Delta_i]^+ \quad \text{and} \\ \tilde{x}_i + [z_i]^+ + [\Delta_i]^- & \leqq & x_i^+ \leqq \tilde{x}_i + [z_i]^+ + [\Delta_i]^+. \end{array}$$

In order to have a guaranteed inner inclusion all the computations should be done in computer arithmetic with directed roundings.

The method from Theorem 3.4 has its limits. When widening the intervals for the parameters, respectively the interval components of the linear system, the inner inclusion becomes smaller and smaller, and finally vanishes. The latter means that no quantitative measure for the quality of the outer enclosure can be given. For wide parameter intervals empty inner inclusion usually means bad outer enclosure and, when further widening the input intervals, the outer solution enclosure will fail at a certain point. Numerical examples demonstrating this effect can be found in [25]. The same result of empty inner inclusion intervals can be obtained also for very tight parameter intervals due to the rounding errors in computing inner approximations. A necessary and sufficient condition for a non-empty inner inclusion is provided by the following relation [20]

$$\omega([\Delta_i]) \leqq \omega([z_i]),$$

where the notations are as in Theorem 3.4, $[\Delta_i]$ is computed with outward rounding and $[z_i]$ is computed with inward rounding.

When somehow we have sharpen the outer solution enclosure $\square\, \Sigma^p \subseteq [\hat{v}] \subseteq [v] = \tilde{x} + [z] + [\Delta]$, then the improved outer estimation $[\hat{v}]$ can replace $[v]$ in Theorem 3.4 to get an improved inner estimation of $\square\, \Sigma^p$. Numerical example demonstrating this property can be found in [20].

4 Algorithm and Implementation

The basic goals of self-validating methods are to deliver rigorous results by computations in finite precision arithmetic, including the proof of existence (and possibly uniqueness) of a solution. In order to achieve this goal the inclusion theorems from the preceding section should be verifiable on computers. With the definitions of rounded floating-point interval arithmetic (cf. Section 2.1) and due to its inclusion properties, the following Theorem holds true.

Theorem 4.1. *Consider parametric linear system defined by (1a–1d) with* $p \in [p] \in \mathbb{IF}^k$. *Let* $R \in \mathbb{F}^{n\times n}$, $[y] \in \mathbb{IF}^n$, $\tilde{x} \in \mathbb{F}^n$ *be given. Define*

$$\begin{array}{rcl} z(p) & := & R\,(b(p) - A(p)\tilde{x})\,, \\ C(p) & := & I - R \cdot A(p) \end{array}$$

and suppose that the elements of $z(p)$, $C(p)$ *are real-valued rational functions. With the notations and the assumptions of Theorems 2.2, 2.3 define* $[z] \in \mathbb{IF}^n$, $[C] \in \mathbb{IF}^{n\times n}$ *by*

$$\begin{array}{rcl} [z_i] & = & \Diamond z_i^*([p_{\mathcal{I}_i}], \mathtt{Dual}[p_{\mathcal{J}_i}]) \\ [C_{ij}] & = & \Diamond C_{ij}^*([p_{\mathcal{I}_{ij}}], \mathtt{Dual}[p_{\mathcal{J}_{ij}}]), \qquad i,j = 1, \ldots, n. \end{array}$$

Define $[v] \in \mathbb{IF}^n$ *by means of the following Einzelschrittverfahren*

$$1 \leq i \leq n \;:\; [v_i] := \{\Diamond([z] + [C] \cdot [u])\}_i, \qquad u := (v_1, ..., v_{i-1}, y_i, ..., y_n)^\top.$$

If $[v] \subsetneqq [y]$, *then* R *and every matrix* $A(p)$ *with* $p \in [p]$ *are regular, and for every* $p \in [p]$ *the unique solution* $\widehat{x} = A^{-1}(p)b(p)$ *of (1a–1d) satisfies* $\widehat{x} \in \Diamond(\tilde{x} + [v])$.
With $[d] := \Diamond([C] \cdot [v]) \in \mathbb{IF}^n$ *the following inner estimation of* $\square\, \Sigma^p$ *holds true*

$$\left[\triangle\left(\tilde{x} + (\bigcirc[z])^- + [d]^+\right),\ \triangledown(\tilde{x} + (\bigcirc[z])^+ + [d]^-)\right] \subseteq [\inf(\Sigma^p),\ \sup(\Sigma^p)]\,.$$

In the implementation of the above Theorem we choose $R \approx A^{-1}(p_m)$ and $\tilde{x} \approx A^{-1}(p_m) \cdot b(p_m)$, where $p_m = \mathrm{mid}([p])$.

The inclusions of the residual vector $[z]$ and the iteration matrix $[C]$ should be sharp. Suppose that `rangeExpr(f(p), [p])` is a function which verifies the conditions of Theorem 2.2 and yields the corresponding dual-transformed expression $f^*(p_{\mathcal{I}}, \mathtt{Dual}p_{\mathcal{J}})$ for a rational function $f(p)$ continuous on $[p]$. The evaluation of this expression in rounded generalised interval arithmetic for $p \in [p]$ gives a corresponding inner/outer inclusion of the true range of $f(p)$, as presented in (12), (13). In order to reduce the number of occurrences of the parameters, a rearrangement of the expressions (15), (16) (e.g. by algebraic simplification) is usually helpful as a preliminary step before applying the function `rangeExpr`.

When aiming to compute a self-verified enclosure of the solution to a linear system by the above inclusion methods, an iteration scheme, usually called fixed-point iteration, is proven to be very useful [27, 29]. To force $[V] \subsetneqq [Y]$, the concept of *ε-inflation* is introduced. Epsilon inflation blows up the intervals somewhat in order to "catch" a nearby fixed-point. For a real interval $[a]$, *ε-inflation* is defined by

$$\mathrm{blow}([a], \varepsilon) = \begin{cases} [a] + \omega([a])[-\varepsilon,\ \varepsilon], & \text{if } \omega([a]) > 0 \\ [\mathrm{pred}(a),\ \mathrm{succ}(a)], & \text{if } \omega([a]) = 0, \end{cases}$$

where $\mathrm{pred}(a)$, $\mathrm{succ}(a)$ are the predecessor and successor of a floating-point number a in the floating-point screen. The *ε-inflation* is applied to interval vectors componentwise. Usually, $\varepsilon = 0.1$ or $\varepsilon = 0.2$ are reasonable values. A complete overview of the convergence behavior of the affine iteration with the ε-inflation is given in [27]. The classical verification step is extended by an Einzelschrittverfahren which accelerates the iteration.

The following algorithm computes guaranteed inclusions of the solution set hull to a linear system whose input data are rational functions of interval parameters.

Algorithm 4.1. *Inner and Outer Inclusions of the Parametric Solution Set Hull*

1. *Initialization.*
   $\check{p} := \mathrm{mid}\ ([p])$; $\check{A} := A(\check{p})$; $\check{b} := b(\check{p})$;
   Compute $R \approx \check{A}^{-1}$; $\tilde{x} = R \cdot \check{b}$.

2. *Enclosures.*

   2.1 Compute the analytic expressions
   $z(p) := R\,(b(p) - A(p) \cdot \tilde{x})$;
   $C(p) := I - R \cdot A(p)$;

   2.2 Apply algebraic simplification to $z(p)$ and $C(p)$ in order to reduce the number of incidences of the variables.

   2.3 Obtain the corresponding dual-transformed expressions. For $i, j = 1, \ldots, n$
   $z_i^*(p_{\mathcal{I}_i}, \mathtt{Dual}(p_{\mathcal{J}_i})) = \mathtt{rangeExpr}(z_i(p), [p])$
   $C_{ij}^*(p_{\mathcal{I}_{ij}}, \mathtt{Dual}(p_{\mathcal{J}_{ij}})) = \mathtt{rangeExpr}(C_{ij}(p), [p])$;

2.4 For $i, j = 1, \ldots, n$
$[z]_i = \Diamond z_i^*([p_{\mathcal{I}_i}], \texttt{Dual}[p_{\mathcal{J}_i}])$;
$[C]_{ij} = \Diamond C_{ij}^*([p_{\mathcal{I}_{ij}}], \texttt{Dual}[p_{\mathcal{J}_{ij}}])$;

3. *Verification.*

$[x] := [z]$;
repeat
$[y] := [x] := \text{blow}([x], \varepsilon)$
for $i = 1$ to n **do** $[x_i] := \Diamond([z_i] + [C_i] \cdot [x])$
until $[x] \subsetneqq [y]$ or max iteration exceeded

If $[x] \subsetneqq [y]$ **then** $\square\Sigma^p \subseteq \Diamond(\tilde{x} + [x])$;
else the algorithm fails.

4. *Inner Estimation of the Outer Enclosure.* (If $[x] \subsetneqq [y]$)
$[y] = \texttt{Dual}\,(\Diamond\,(\tilde{x} + \Diamond z^*(\texttt{Dual}[p_{\mathcal{I}}], [p_{\mathcal{J}}]) + [C] \cdot [x]))$;
for $i = 1$ to n, **If** $[y_i] \notin \mathbb{IR}$ **then** $[y_i] = \emptyset$;
$[y] \subseteq \square\Sigma^p \subseteq \Diamond(\tilde{x} + [x])$.

By the next example we illustrate the above algorithm.

Example 4.1. *Consider a linear system the elements of whose matrix are rational functions of five interval parameters*

$$Ap = \begin{pmatrix} -(p_1 + p_2)/p_4 & p_5 \\ p_2 p_4 & p_3/p_5 \end{pmatrix}; \quad bp = \begin{pmatrix} 1 \\ 1 \end{pmatrix};$$

where $p_1, p_3 \in [0.96,\ 1.04]$, $p_2 \in [1.92,\ 2.08]$, $p_4, p_5 \in [0.48,\ 0.52]$.

In the initialization step we have

$$\check{p} = (1, 2, 1, 0.5, 0.5)^\top; \qquad R = \begin{pmatrix} -0.16 & 0.04 \\ 0.08 & 0.48 \end{pmatrix}; \qquad \tilde{x} = (-0.12,\ 0.56)^\top$$

After an algebraic simplification, the residual vector is

$$z(p) = \begin{pmatrix} -0.12 + 0.0192(p_1 + p_2)/p_4 + 0.0048 p_2 p_4 - 0.0224 p_3/p_5 + 0.0896 p_5 \\ 0.56 - 0.0096(p_1 + p_2)/p_4 + 0.0576 p_2 p_4 - 0.2688 p_3/p_5 - 0.0448 p_5 \end{pmatrix}.$$

Function `rangeExpr` proves the total monotonicity of both components with respect to all the parameters. For the first component function we have one incidence of p_3 and the same monotonicity for all the incidences of p_2 and p_5, so that there will be no dual-transformation for these parameters. The first component function is globally $\leq$-antitone with respect to p_4 while $\leq$-isotone for the first p_4 incidence and $\leq$-antitone w.r.t. the second p_4 incidence. Analogously, the second component function is globally $\leq$-isotone with respect to p_2, p_5 and has different monotonicity w.r.t their incidences. Thus, the `rangeExpr` function yields the following dual-transformed expressions for the residual function

$$z^*(p, \texttt{Dual}) = \begin{pmatrix} -0.12 + 0.0192(p_1 + p_2)/p_4 + 0.0048 p_2 \texttt{Dual}(p_4) - 0.0224 p_3/p_5 + 0.0896 p_5 \\ 0.56 - 0.0096(p_1 + \texttt{Dual}(p_2))/p_4 + 0.0576 p_2 p_4 - 0.2688 p_3/p_5 - 0.0448 \texttt{Dual}(p_5) \end{pmatrix}.$$

The evaluation of the above $z^*(p, \texttt{Dual})$ in generalised interval arithmetic gives the following enclosure with outwardly rounded arithmetic (for simplicity the results are presented with 6 digits accuracy)

$$[z] = ([-0.0143946,\ 0.0148305],\ [-0.0500198,\ 0.0466349])^\top.$$

The interval evaluation of $z(p)$ overestimates the evaluation of $z^*(p, \texttt{Dual})$ by 2.5%, resp. 9.1% for the vector components. The evaluation of $C^*(p, \texttt{Dual})$ in generalised interval arithmetic gives the following enclosure

$$[C] = \begin{pmatrix} [-0.079936,\ 0.0739102] & [-0.00986667,\ 0.00935385] \\ [-0.0514757,\ 0.0509653] & [-0.0784,\ 0.0722462] \end{pmatrix}.$$

For this particular example, the natural interval extension of $C(p)$ is also convergent. The verification iteration converges in one iteration yielding the following outer solution enclosure

$$\square\,\Sigma^p \subseteq ([-0.136242,\ -0.103329],\ [0.505062,\ 0.611791])^\top.$$

The corresponding inner inclusion of the solution set hull is

$$([-0.132555,\ -0.107016],\ [0.515136,\ 0.601717])^\top \subseteq \square\,\Sigma^p.$$

The obtained inner and outer inclusions of the solution set hull are presented on Figure 1 together

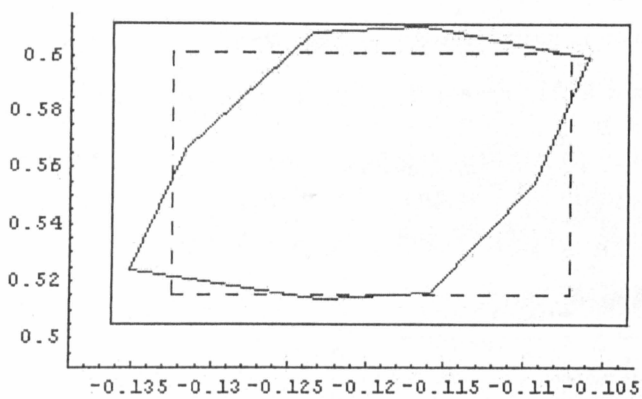

Figure 1: Solution set of the system from Example 4.1 and its inner (dashed line) and outer inclusions by the parametric fixed-point iteration.

with the true solution set. The true overestimation of the exact solution set hull is 11.19 %, 9.51 % for the components.

Interval methods and the algorithm discussed in this paper are implemented in the environment of *Mathematica* [31]. The *Mathematica* package `IntervalComputations 'LinearSystems'` contains a collection of functions which compute guaranteed inclusions for the solution set of an interval linear system [23]. The particular solvers differ upon the type of the linear system to be solved and the implemented solution method. Now, the existing functions are updated to handle sparse arrays as input data, and several new functions are implemented.

`ParametricSSolve[Ap, bp, tr]` computes a guaranteed enclosure of the solution set to a parametric linear system `Ap.x = bp` involving nonlinear dependencies by the algorithm presented above in this section. The parameters and their interval values are specified by a list `tr` of transformation

rules[1]. The function can take two optional arguments affecting the computational process, respectively the output of the function. Options in *Mathematica* are set by giving rules of the form `name -> value`. Each rule must appear after all the other arguments of a function. Rules for different options can be given in any order. If no explicit rule is given for a particular option, a default setting for that option is used. `InnerEstimation` is an option associated to everyone of the iterative solvers of interval linear systems. `InnerEstimation`, when set to `True`, specifies the computing of component-wise inner approximation of the solution set in addition to the outer enclosure. The option is set to `False` by default. Even set to `True`, the option is active only if the *Mathematica* package `IntervalComputations` `'GeneralisedIntervals'` is available [26]. `Refinement` is an option also associated to everyone of the iterative solvers of interval linear systems. `Refinement` set to `True` implies an iterative refinement procedure applied to the computed outer solution enclosure. The default setting is `False`.

Since traditional numerical systems do not have integrated symbolic capabilities with which to perform symbolic preprocessing, the role of computer algebra systems increase. Such environments allow not only convenient symbolic-numeric representation of the input parameter-dependent data but provide also a variety of functions for converting expressions from one form to another. Simplification is the "killer application" of computer algebra (application that everyone wants but only one can). The *Mathematica* function `Simplify` tries to find the simplest form of an expression by applying various standard algebraic transformations. The newly developed function `ParametricSSolve` which solves parametric linear systems involving nonlinear dependencies applies algebraic simplification to the expressions of the residual function $z(p)$ (15) and the iteration matrix $C(p)$ (16) as a symbolic preprocessing step before the application of the function `rangeExpr`.

`rangeExpr[expr, pars, pvals]` is a newly developed function which verifies the conditions of Theorem 2.2 and generates the corresponding dual-transformed expression for an input expression `expr` which should be a rational function of a number of variables specified by a list `pars`. The interval values for the parameters are specified by a list `pvals` of rules having the form `name -> value`. `rangeExpr` uses symbolic differentiation and algebraic simplification when trying to find the monotonicity type of `expr` with respect to parameters.

Approaching to parametric linear systems with nonlinear dependencies, the integration of symbolic-algebraic and self-validating numerical computations based on interval arithmetic is found to be a fruitful synergism. The power of *Mathematica* to support rigorous exact and/or variable precision interval computations, the functionality of a generalized interval arithmetic package and the tools provided by the presented interval problem solving package, make a suitable environment for exploration and solving real-life parametric problems with uncertainty.

In order to provide a broad access to the above solvers for interval linear systems a web interface is developed and can be found at

http://cose.math.bas.bg/webComputing/

Accessing the webComputing pages users enter or upload data, choose between different options, and submit data to build up a sequence of results in a numeric, symbolic, graphics or combined form. The end-users do not need to buy, install, and maintain software; they do not need to develop user software or to learn different software applications training time being considerably reduced. They can be certain that use the most recent version. The technical professionals and interval researchers can easily explore newly developed methods; compare the efficiency of different methods and software tools; teach interval methods involving students in an active exploration by doing. Since algebraic computations are time consuming and web*Mathematica* applications have a fixed time limit for using the *Mathematica* kernel, the nonlinear parametric web solver is suitable only for small size problems, while large problems involving affine-linear dependencies can be solved remotely. The parametric web

[1] *Mathematica* transformation rules have the form `name -> value`.

solvers allow uploading data files from the client machine onto the server. For a parametric system, 3 data files (containing the matrix, the right-hand side vector and the rules for the parameters) are required. Present restriction to the maximum size of a data file is 4MB. Matrix/vector data in a file presently should be specified by *Mathematica* lists, or as sparse arrays [31]. Future enhancement of the solvers include different data formats, downloading the generated results on the client machine and combining/reusing the results from different pages.

5 Practical Case Study

The case study, considered here, illustrates the method presented in this paper as well as its combination with other interval techniques for obtaining sharp bounds to the solution of linear systems involving nonlinear dependencies between uncertain parameters. This is a small but realistic problem coming from structural engineering where the analysis of linear mechanical systems, in particular frames, by Finite Element methods leads to parametric linear systems.

Example 5.1. *A simple one-bay structural steel frame, initially considered in [2], is presented in Fig. 2.*

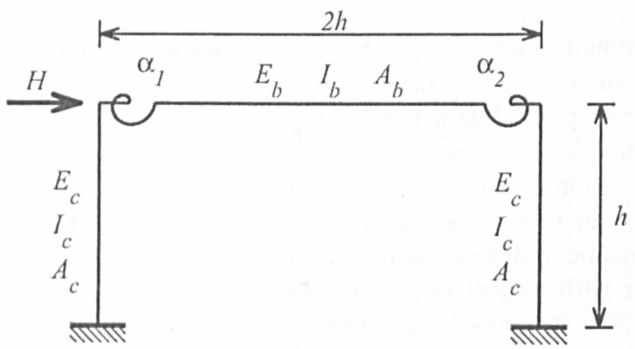

Figure 2: One-bay structural steel frame (after [2]).

Applying conventional methods for analysis of frame structures the authors of [2] have assembled a system of linear equations

$$K \cdot x = F,$$

where K is the following global stiffness matrix

$$\begin{pmatrix}
\frac{AbEb}{Lb} + \frac{12EcIc}{Lc^3} & 0 & \frac{6EcIc}{Lc^2} & 0 & 0 & -\frac{AbEb}{Lb} & 0 & 0 \\
0 & \frac{AcEc}{Lc} + \frac{12EbIb}{Lb^3} & 0 & \frac{6EbIb}{Lb^2} & \frac{6EbIb}{Lb^2} & 0 & -\frac{12EbIb}{Lb^3} & 0 \\
\frac{6EcIc}{Lc^2} & 0 & \alpha + \frac{4EcIc}{Lc} & -\alpha & 0 & 0 & 0 & 0 \\
0 & \frac{6EbIb}{Lb^2} & -\alpha & \alpha + \frac{4EbIb}{Lb} & \frac{2EbIb}{Lb} & 0 & -\frac{6EbIb}{Lb^2} & 0 \\
0 & \frac{6EbIb}{Lb^2} & 0 & \frac{2EbIb}{Lb} & \alpha + \frac{4EcIc}{Lc} & 0 & -\frac{6EbIb}{Lb^2} & -\alpha \\
-\frac{AbEb}{Lb} & 0 & 0 & 0 & 0 & \frac{AbEb}{Lb} + \frac{12EcIc}{Lc^3} & 0 & \frac{6EcIc}{Lc^2} \\
0 & -\frac{12EbIb}{Lb^3} & 0 & -\frac{6EbIb}{Lb^2} & -\frac{6EbIb}{Lb^2} & 0 & \frac{AcEc}{Lc} + \frac{12EbIb}{Lb^3} & -\frac{6EbIb}{Lb^2} \\
0 & 0 & 0 & 0 & -\alpha & \frac{6EcIc}{Lc^2} & -\frac{6EbIb}{Lb^2} & \alpha + \frac{4EcIc}{Lc}
\end{pmatrix}$$

whose elements are, in general, nonlinear functions of the following parameters: material properties E_b, E_c, cross sectional properties I_b, I_c, A_b, A_c, lengths L_b, L_c, and the joint stiffness α. The right hand

Table 1: Parameters involved in the steel frame example, their nominal values, and worst case uncertainties.

parameter		nominal value	uncertainty
Young modulus	E_b	$29 * 10^6$ lbs/in^2	$\pm 348 * 10^4$
	E_c	$29 * 10^6$ lbs/in^2	$\pm 348 * 10^4$
Second moment	I_b	510 in^4	± 51
	I_c	272 in^4	± 27.2
Area	A_b	10.3 in^2	± 1.3
	A_c	14.4 in^2	± 1.44
External force	H	5305.5 lbs	± 2203.5
Joint stiffness	α	$2.77461 * 10^9$ lb-in/rad	$\pm 1.26504 * 10^9$
Length	L_c	144 in	
	L_b	288 in	

side vector $F = (H, 0, 0, 0, 0, 0, 0, 0)^\top$ in this example is considered to depend only on the applied loading H. Typical nominal parameter values and the corresponding worst case uncertainties, as proposed in [2], are shown in Table 1.

In [2] all the parameters, except the lengths, are considered to be uncertain and varying within given intervals. Replacing L_b and L_c with their nominal values we come to the problem of solving a parametric linear system

$$K(p) \cdot x = F(p), \tag{17}$$

where the vector of the uncertain parameters is $p = (E_b, E_c, I_b, I_c, A_b, A_c, \alpha, H)^\top$, the right hand side vector is $F(p) = (H, 0, 0, 0, 0, 0, 0, 0)^\top$, and the parametric matrix $K(p)$ is

$$\begin{pmatrix}
\frac{A_b E_b}{288} + \frac{E_c I_c}{248832} & 0 & \frac{E_c I_c}{3456} & 0 & 0 & -\frac{A_b E_b}{288} & 0 & 0 \\
0 & \frac{A_c E_c}{144} + \frac{E_b I_b}{1990656} & 0 & \frac{E_b I_b}{13824} & \frac{\mathrm{EbIb}}{13824} & 0 & -\frac{\mathrm{EbIb}}{1990656} & 0 \\
\frac{\mathrm{EcIc}}{3456} & 0 & \alpha + \frac{\mathrm{EcIc}}{36} & -\alpha & 0 & 0 & 0 & 0 \\
0 & \frac{\mathrm{EbIb}}{13824} & -\alpha & \alpha + \frac{\mathrm{EbIb}}{72} & \frac{\mathrm{EbIb}}{144} & 0 & -\frac{\mathrm{EbIb}}{13824} & 0 \\
0 & \frac{\mathrm{EbIb}}{13824} & 0 & \frac{\mathrm{EbIb}}{144} & \alpha + \frac{\mathrm{EcIc}}{36} & 0 & -\frac{\mathrm{EbIb}}{13824} & -\alpha \\
-\frac{\mathrm{AbEb}}{288} & 0 & 0 & 0 & 0 & \frac{\mathrm{AbEb}}{288} + \frac{\mathrm{EcIc}}{248832} & 0 & \frac{\mathrm{EcIc}}{3456} \\
0 & -\frac{\mathrm{EbIb}}{1990656} & 0 & -\frac{\mathrm{EbIb}}{13824} & -\frac{\mathrm{EbIb}}{13824} & 0 & \frac{\mathrm{AcEc}}{144} + \frac{\mathrm{EbIb}}{1990656} & -\frac{\mathrm{EbIb}}{13824} \\
0 & 0 & 0 & 0 & -\alpha & \frac{\mathrm{EcIc}}{3456} & -\frac{\mathrm{EbIb}}{13824} & \alpha + \frac{\mathrm{EcIc}}{36}
\end{pmatrix}.$$

In what follows we solve the system (17) by the parametric iteration algorithm given in Section 4 and by other interval techniques based on the parametric iteration. The results will be compared to what is obtained by alternative methods based on the Element-By-Element approach [2], [16].

To judge about the quality of an outer enclosure we need a measure of overestimation. As it was discussed at the end of Section 3, the parametric iteration is internally equipped with an inner projection inclusion allowing estimation of the quality of the outer inclusion. For two intervals $[a], [b] \in \mathbb{IR}$ such that $[a] \subseteq [b]$, the standard measure of overestimation that is usually applied is the percentage by which $[b]$ overestimates the interval $[a]$, defined as $\mathcal{O}_\omega : \mathbb{IR} \times \mathbb{IR} \longrightarrow \mathbb{R}_+$

$$\mathcal{O}_\omega([a], [b]) := 100(1 - \omega([a])/\omega([b])).$$

Distance-based measures of overestimation are sometimes used in the engineering literature (e.g., [16]). $\mathcal{O}_d : \mathbb{IR} \times \mathbb{IR} \longrightarrow \mathbb{R} \times \mathbb{R}$ is defined by

$$\mathcal{O}_d([a], [b]) := 100\left(1 - a^-/b^-,\ 1 - a^+/b^+\right).$$

Since we will compare our results to those obtained in [2], we will need the measure of overestimation used therein. For $[a], [b] \in \mathbb{IR}$, $[a] \subseteq [b]$ and $c \in \mathbb{R}$, $c \in [a]$, define $\mathcal{O}_c : \mathbb{IR} \times \mathbb{IR} \times \mathbb{R} \longrightarrow \mathbb{R}_+$ by

$$\mathcal{O}_c([a], [b], c) := 100 \left(b^- - a^- + a^+ - b^+\right) / c.$$

Overestimation measures are applied to interval vectors componentwise.

In order to compare the results generated by different methods, we strictly follow the structure system and the uncertainties for the parameters considered in [2]. Initially, the system (17) is solved with parameter uncertainties which are 1% of the values presented in the last column of Table 1,

$$E_b, E_c \in [28965200,\ 29034800],\ I_b \in [509.49,\ 510.51],\ I_c \in [271.728,\ 272.272],\ A_b \in [10.287,\ 10.313]$$
$$A_c \in [14.3856,\ 14.4144],\quad \alpha \in [276195960,\ 278726040],\quad H \in [5283.465,\ 5327.535]. \tag{18}$$

A straightforward replacement of the parameters in the system matrix and in the right hand side vector with their interval values result in a nonparametric interval linear system where all the elements in the matrix and the right hand side vector are assumed to vary independently within their intervals. This approach, called naïve interval approach, obscures and does not account for the dependencies between the parameters in solving practical problems which by their physical nature depend on many parameters. It is well-known that the solution of a naïve interval system greatly overestimates the solution of the original parametric linear system (cf. [2]). Nevertheless, many recent papers still compare the results of parametric methods to the corresponding naïve interval solution (e.g. [2], [14], [16]). In this paper, instead comparing to the naïve solution, we will use a numerical estimation for the quality of the obtained parametric solution enclosure, the latter will be compared to the exact hull of the solution range for small size problems, and will apply a combination of interval techniques to achieve the sharpest possible bounds on the system response for large worst case parameter tolerances.

Table 2: Crisp mid-point solution and exact hull of the solution set for one-bay steel frame example with uncertain parameters (18). The results are rounded outwardly to 10 digits accuracy.

solution component	Mid-point solution – μ	Hull $[h]$	Result from [2]
1. $d2_x$	0.1532674393	[0.1490008147, 0.1576707570]	[0.15237484, 0.15476814]
2. $d2_y * 10^3$	0.3267821043	[0.3105885018, 0.3438472600]	[0.32940418, 0.33533906]
3. $r2_z * 10^3$	-0.9646668639	[-0.9903338964, -0.9397354520]	[-0.97085151, -0.95490139]
4. $r5_z * 10^3$	-0.4656795813	[-0.4818370784, -0.4500670762]	[-0.4638112, -0.45611532]
5. $r6_z * 10^3$	-0.4270205236	[-0.4421272879, -0.4124286781]	[-0.44930811, -0.4418354]
6. $d3_x$	0.1507136505	[0.1464976946, 0.1550647578]	[0.14985048, 0.15221127]
7. $d3_y * 10^3$	-0.6709042527	[-0.7091762809, -0.6348472886]	[-0.33533906, -0.32940418]
8. $r3_z * 10^3$	-0.9327734470	[-0.9576203294, -0.9086349791]	[-0.95100335, -0.93531196]

It is rigorously proven, by using a hybrid numerical approach combining verified parametric linear solver and proven global and local monotonicity properties of the parametric solution [23], that the exact hull of the parametric solution set can be obtained by the combinatorial approach. In combinatorial approach, the exact bounds for the parametric system response are obtained as minimum/maximum of the solutions to all point linear systems corresponding to an exhaustive combination of the extreme values of the interval parameters. This way, by a rigorous application of the monotonicity approach, we can find the exact hull of the parametric solution set to our system involving the uncertain interval

parameters (18). The results presented in Table 2 are obtained first in rational arithmetic and then rounded to 10 digits of accuracy.

The parametric linear system (17) is solved by the present parametric fixed-point iteration. The system involves eight uncertain parameters which are considered to vary independently within tolerance intervals (18). One single execution of the parametric solver function guarantees the outer enclosure of the system response which could be observed from any combination of the values of cross-sectional properties, loading, material properties and connections. Table 3 shows the results obtained in just one single execution of the parametric solver function: the guaranteed outer enclosure of the system response and an inner estimation of the outer enclosure allowing to estimate the quality of the outer enclosure.

Table 3: One-bay steel frame example with uncertain parameters (18): outer enclosure $[u]$ of the solution set and its inner estimation $[v]$ computed by the present parametric fixed-point iteration.

solution component	solution inclusions	$\mathcal{O}_\omega([v],[u])$
1. $d2_x$		1.66 %
outer	[0.1522222356, 0.1543123381]	
inner	[0.1522396156, 0.1542949581]	
2. $d2_y * 10^3$		1.15 %
outer	[0.3237760067, 0.3297873075]	
inner	[0.3238105628, 0.3297527514]	
3. $r2_z * 10^3$		9.17 %
outer	[-0.9719730914, -0.9573591990]	
inner	[-0.9713028984, -0.9580293920]	
4. $r5_z * 10^3$		17.31 %
outer	[-0.4693539781, -0.4620039136]	
inner	[-0.4687177899, -0.4626401019]	
5. $r6_z * 10^3$		27.08 %
outer	[-0.4306060526, -0.4234337856]	
inner	[-0.4296350039, -0.4244048344]	
6. $d3_x$		1.68 %
outer	[0.1496821686, 0.1517448275]	
inner	[0.1496994940, 0.1517275021]	
7. $d3_y * 10^3$		1.59 %
outer	[-0.6773978325, -0.6644092806]	
inner	[-0.6772948472, -0.6645122658]	
8. $r3_z * 10^3$		6.80 %
outer	[-0.9398187649, -0.9257267319]	
inner	[-0.9393393872, -0.9262061096]	

The obtained solution inclusions of (17)–(18) are used for comparison of different measures of overestimation and the results are displayed in Table 4.

Table 4: One-bay steel frame example with uncertain parameters (18): comparison of overestimation measures in %. $\mathcal{O}_c([h],[u_i])$ are after [2], $i = 3$ – Table V, $i = 2$ – Table IV, $i = 1$ – Table III, respectively, dash means no available data.

solution comp.	$\frac{1}{2}\mathcal{O}_\omega$ $([v],[u])$	$\mathcal{O}_\omega$ $([h],[u])$	$10^2\mathcal{O}_d$ $([h],[u])$	$\mathcal{O}_c$ $([h],[u],\mu)$	$\mathcal{O}_c$ $([\tilde{h}],[u_3],\mu)$	$\mathcal{O}_c$ $([\tilde{h}],[u_2],\mu)$	$\mathcal{O}_c$ $([\tilde{h}],[u_1],\mu)$
1. $d2_x$	0.83	0.83	-0.75, 0.38	0.011	0.29	0.40	78.02
2. $d2_y$	0.57	0.57	-0.86, 0.20	0.011	0.004	0.13	85.38
3. $r2_z$	4.58	4.31	3.01, -3.53	0.065	0.75	0.84	81.18
4. $r5_z$	8.65	7.73	5.89, -6.31	0.122	1.62	1.63	85.32
5. $r6_z$	13.54	11.99	9.81, -10.32	0.201	–	–	–
6. $d3_x$	0.84	0.84	-0.76, 0.39	0.011	–	–	–
7. $d3_y$	0.79	0.79	0.33, -1.21	0.015	–	–	–
8. $r3_z$	3.40	3.23	2.19, -2.70	0.049	–	–	–

The most precise estimation of the quality of an outer enclosure $[u]$ is given by $\mathcal{O}_w([h],[u])$ which gives the percentage by which $[u]$ overestimates the exact solution set hull $[h] = \square\Sigma^p$. However, obtaining $[h]$ is an NP-hard problem. That is why, $[u]$ should be compared to an easy computable inner estimation of $[h]$. The presented parametric fixed-point method provides a guaranteed inner estimation $[v]$ of the hull at no additional cost. Since the computation of $[v]$ uses the computed outer enclosure $[u]$ in a "symmetric " way, it can be expected that $[v]$ is almost symmetric to $[u]$ with respect to the exact solution set hull. That is why, $\frac{1}{2}\mathcal{O}_w([v],[u]) \approx \mathcal{O}_w([h],[u])$ as demonstrated in Table 4. Note that $\mathcal{O}_w([v],[u])$ is an upper bound on the true overestimation $\mathcal{O}_w([h],[u])$. So, if $[v] \neq \emptyset$, the present method provides also a good measure for the quality of the solution enclosure by $\frac{1}{2}\mathcal{O}_\omega([v],[u])$.

Often, for an inner estimation of the solution set hull, the authors use a solution $[\tilde{v}]$ obtained by the combinatorial approach or the monotonicity approach. In general, both approaches give a solution $[\tilde{v}] \subseteq \square\Sigma^p$. Since the size of underestimation is unpredictable and could be considerable, a measure Overestimation$([\tilde{v}],[u])$ could be arbitrary weak. When rigorously applied (by proven monotonicity properties), both approaches give the exact solution set hull in exact arithmetic. Usually, the exact solution set hull for structural engineering problems can be described by the combinatorial approach. In this case, the combinatorial approach applied in floating-point arithmetic without directed rounding may give a solution $[\tilde{v}] \not\subseteq \square(\Sigma^p)$ (compare the results in Table 2) and $\mathcal{O}_\omega([\tilde{v}],[u])$ would not be correct.

Distance-based measure $\mathcal{O}_d$ gives two numbers with different signs corresponding to the end-points of the intervals. As demonstrated in Table 4, the distance-based measure $\mathcal{O}_d([h],[u])$ yields values which are two orders of magnitude less than the overestimation measure $\mathcal{O}_\omega([h],[u])$. The other overestimation measure $\mathcal{O}_c([h],[u],\mu)$ is also not comparable to $\mathcal{O}_\omega([h],[u])$ giving values with one order of magnitude less than the latter.

The last three columns in Table 4 present the quality of the solution enclosures obtained in [2] by the application of EBE approach [16] to the system (17)–(18). The application of the EBE approach was successively improved in [2] by applying subdistributivity property and scaling which has resulted in improved solution enclosures measured by $\mathcal{O}_c([\tilde{h}],[u_i],\tilde{\mu})$, where $[\tilde{h}]$ is the solution set hull from the last column of Table 2, $[u_i]$ is the corresponding solution enclosure. Comparing the best solution

enclosure, obtained by the EBE approach — $\mathcal{O}_c([\tilde{h}], [u_3], \tilde{\mu})$, to the quality $\mathcal{O}_c([h], [u], \mu)$ of the solution enclosure obtained by the present parametric method, we see the superiority of the present method by one order of magnitude.

It is well-known that the parametric fixed-point iteration gives sharper solution enclosures for smaller interval tolerances. To illustrate this effect we subdivide the ranges (18) of some wide interval-valued parameters and obtain the enclosure of system response as a hull of the solution enclosures in all sub-domains. Table 5 compares the results obtained after the application of the subdivision approach. The first column in the table demonstrates the improvement (between 0.37% and 3.05%) in the solution enclosure obtained by subdivision of the intervals. The overestimation for the different components of the system response is different ranging from 0.2% to 9.22%.

Table 5: One-bay steel frame example with uncertain parameters (18) solved by subdivision of the parameter intervals $(E_b, E_c, I_b, I_c, A_b, A_c, \alpha, H)^\top$ correspondingly into $(5, 5, 1, 1, 1, 1, 5, 1)^\top$ equal subintervals. Inner $[v_s]$ and outer $[u_s]$ inclusions of the solution set hull are compared to the exact hull $[h]$ (Table 2) and to the inner and outer inclusions $[v], [u]$ (Table 3) computed without subdivision.

solution component	$\mathcal{O}_\omega$ $([u_s], [u])$	$\frac{1}{2}\mathcal{O}_\omega$ $([v_s], [u_s])$	$\mathcal{O}_\omega$ $([h], [u_s])$	$\mathcal{O}_c$ $([h], [u_s], \mu)$
1. $d2_x$	0.53 %	0.30 %	0.30 %	0.004 %
2. $d2_y$	0.37 %	0.20 %	0.20 %	0.004 %
3. $r2_z$	1.36 %	2.96 %	3.00 %	0.045 %
4. $r5_z$	2.34 %	5.82 %	5.52 %	0.085 %
5. $r6_z$	3.05 %	9.71 %	9.22 %	0.150 %
6. $d3_x$	0.54 %	0.31 %	0.31 %	0.004 %
7. $d3_y$	0.49 %	0.30 %	0.30 %	0.006 %
8. $r3_z$	1.18 %	2.05 %	2.08 %	0.031 %

The presented parametric fixed-point iteration fails in solving the parametric linear system (17) for the worst case (over 40%) parameter uncertainties given in Table 1. For very large parameter uncertainties the iteration matrix is not strongly regular as required by the method. But we can solve the problem by subdividing the parameter intervals. As small are the sub-domains (as many are the subdivisions) as better will be the solution enclosure. For completeness, in Table 6 we present the combinatorial solution $[\tilde{h}]$ to the system (17) with worst-case uncertain parameters. Since we have not proven that the range of the system response coincides with the combinatorial solution, the latter should be considered as an inner estimation of the hull.

Inclusions (inner and outer) of the solution set hull are obtained by subdivision of the worst-case parameter intervals $(E_b, E_c, I_b, I_c, A_b, A_c, \alpha, H)^\top$ correspondingly into $(2, 2, 1, 1, 1, 1, 6, 6)^\top$ equal subintervals. The quality of the obtained outer enclosure is presented in Table 7. Although the inner estimations for some components are empty set intervals, the corresponding outer enclosure is computed with 20% to 56% overestimation of the combinatorial solution. These results show that even for comparatively large parameter intervals, having 6% to 10% uncertainty, the presented parametric fixed-point iteration is able to enclose the solution.

Table 6: Combinatorial solution to one-bay steel frame example with worst-case uncertainty in the parameters (Table 1). The results are rounded outwardly to 10 digits accuracy.

solution component	combinatorial solution $[\tilde{h}]$
1. $d2_x * 10$	[0.7075232547, 2.941366859]
2. $d2_y * 10^3$	[0.1104840303, 0.7566847593]
3. $r2_z * 10^3$	[-2.0031328402, -0.4304905453]
4. $r5_z * 10^3$	[-0.8879147528, -0.1862032536]
5. $r6_z * 10^3$	[-0.8212198352, -0.1701250948]
6. $d3_x * 10$	[0.6939942045, 2.8996174023]
7. $d3_y * 10^3$	[-1.7622106815, -0.2355509106]
8. $r3_z * 10^3$	[-1.9522427316, -0.4139722766]

Table 7: One-bay steel frame example with worst-case parameter uncertainties (Table 1) solved by subdivision of the parameter intervals $(E_b, E_c, I_b, I_c, A_b, A_c, \alpha, H)^\top$ correspondingly into $(2, 2, 1, 1, 1, 1, 6, 6)^\top$ equal subintervals. Inner $[v_s]$ and outer $[u_s]$ inclusions of the solution set hull are compared to the combinatorial solution $[\tilde{h}]$ (Table 6).

solution component	$\frac{1}{2}\mathcal{O}_\omega([v_s],[u_s])$	$\mathcal{O}_\omega([\tilde{h}],[u_s])$
1. $d2_x$	–	27.72 %
2. $d2_y$	24.13 %	20.21 %
3. $r2_z$	–	38.21 %
4. $r5_z$	–	55.45 %
5. $r6_z$	–	56.20 %
6. $d3_x$	–	27.95 %
7. $d3_y$	35.20 %	32.81 %
8. $r3_z$	–	39.61 %

6 Conclusion

In this work we reported newly developed software tools for solving parametric linear systems whose input data are rational functions of interval parameters.

Our goal was to study the behavior of the parametric fixed-point iteration in cases when we can compute the range hull of the residual vector and the iteration matrix. To this end the iteration method was combined with a technique for sharp range enclosure based on the arithmetic of proper and improper intervals. The latter is also used for computing a numerical estimation for the quality of the solution set enclosure. Although the *Mathematica* package 'GeneralizedIntervals' contains indispensable tools for solving many problems, the usage of methods based on generalised interval arithmetic is by no means obligatory for the application of the parametric fixed-point iteration. Furthermore, combining the iteration method with more sophisticated advanced tools for range enclosure will certainly expand its scope of application to systems involving more complicated dependencies.

We have demonstrated the feasibility of the general-purpose parametric fixed-point iteration to solve linear systems involving nonlinearly dependent data. As it is known in the case of affine-linear dependencies, the method provides very sharp and guaranteed inclusions for the solution set hull when the parameter intervals have relatively small widths. It was demonstrated by the examples that for small intervals the method is superior to other, although not self-verified, methods like the EBE approach [14, 16] generating sharper inclusions. Even for quite large parameter uncertainties, the interval subdivision will guarantee the feasibility of the method and the accuracy of the inclusions at a price of bigger computing time. It is hoped that an application of more clever branch-and-bound strategies will reduce the computing time. So, the biggest drawback of the parametric fixed-point iteration is the requirement of strong regularity of the parametric matrix, while the most attractive feature of the discussed methodology and software tools consists in the fact that they yield validated inclusions computed by a finite precision arithmetic.

Since the discussed parametric method is the only, by now, general-purpose self-verified method, it presents an indispensable tool for computer-assisted proof of global and local monotonicity properties of a linear system solution with respect to the parameters. Basing on these properties, a guaranteed and highly accurate enclosure of the solution set hull can be computed. This was shortly demonstrated by the examples and a detailed presentation of this approach is forthcoming.

Contrary to other approaches for modelling uncertain mechanical systems (e.g. the EBE approach [14, 16] which applies special techniques at the level of constructing the linear system to be solved in order to reduce the dependencies), the present method requires no preliminary specialized construction methods. For example, there is no need to overcome the coupling as in the EBE approach. Present method is highly automated since engineers need to apply only conventional methods for obtaining the linear system in a parametric form by software tools widely available in modern computing environments (Matlab, *Mathematica*, etc.). Uncertainties in all the system parameters (e.g., material, load and geometry properties) can be considered and handled simultaneously. A combination of interval methods can ensure very sharp bounds for the system response range. Furthermore, the present method and all the methods combined to obtain sharp bounds for the system response, are implemented rigorously in software tools which are freely available and ready for application.

The present approach is also applicable to other uncertainty theories which rely on interval arithmetic for computations, such as fuzzy set theory, random set theory, or probability bounds theory.

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