ИНСТИТУТ ПО МАТЕМАТИКА
И ИНФОРМАТИКА

INSTITUTE OF MATHEMATICS
AND INFORMATICS

Секция Биоматематика

Section Biomathematics

БЪЛГАРСКА
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BULGARIAN
ACADEMY
OF SCIENCES

Явна характеризация на един клас параметрични множества от решения

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PREPRINT № 1/2007

Sofia
July 2007

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Abstract

Consider a parameter-dependent linear system $A(p) \cdot x = b(p)$, where the elements of the matrix and the right-hand side vector depend affine-linearly on a m-tuple of parameters $p = (p_1, \ldots, p_m)$ which vary within given intervals $p \in ([p_1], \ldots, [p_m])$.

It is a fundamental problem of considerable practical importance how to describe the parametric solution set $\Sigma(A(p), b(p), [p]) := \{x \in \mathcal{R}^n \mid \exists p \in [p], A(p)x = b(p)\}$ by a logical combination of inequalities depending of the coordinates. So far, in the general case of arbitrary affine-linear parameter dependencies, the solution set description can be obtained by a lengthy (and not unique) parameter elimination process. Recently, explicit descriptions of the symmetric and skew-symmetric solution sets were given.

We introduce a new classification of the parameters with respect to the way they participate in the equations of the system and give numerical characterization for each class of parameters. This paper considers a class of parametric linear systems, where each uncertain parameter occurs in only one equation of the system and does not matter how many times within that equation. For such systems, a simple explicit characterization of the parametric solution set is derived. The obtained explicit parametric solution set characterization generalizes the famous Oettly-Prager theorem for non-parametric linear systems. The new characterization is illustrated by some numerical examples and compared to other approaches as Fourier-Motzkin like parameter elimination and quantifier elimination used also for characterizing the parametric solution set.

MSC: 65F05, 65G99

AMS Subject classification: 15A06, 65F05, 65G10

Keywords: Linear systems, solution set, interval parameters

1 Introduction

Consider the linear algebraic system

$$A(p) \cdot x = b(p), \tag{1}$$

where the elements of the $n \times n$ matrix $A(p)$ and the vector $b(p)$ are affine-linear functions

$$a_{ij}(p) := a_{ij,0} + \sum_{\mu=1}^{m} a_{ij,\mu} p_\mu, \qquad b_i(p) := b_{i,0} + \sum_{\nu=1}^{m} b_{i,\mu} p_\mu, \tag{2}$$

$$a_{ij,\mu}, b_{i,\mu} \in \mathbb{R}, \qquad \mu = 0, \ldots, m, \; i, j = 1, \ldots, n$$

*This work is presented at the International Conference "60 Years Institute of Mathematics and Informatics", Bulgarian Academy of Sciences, July 6–8, 2007, Sofia, Bulgaria.

of m parameters. The parameters are considered to be uncertain and varying within given intervals

$$p \in [p] = ([p_1], \ldots, [p_m])^\top. \tag{3}$$

Such systems are common in many engineering analysis or design problems, models in operational research, linear prediction problems, etc., where there are complicated dependencies between the coefficients of the system. The uncertainties in the model parameters could originate from an inexact knowledge of these parameters, measurement imprecision, or round-off errors. Linear systems with interval input data are applicable also to uncertainty theories which rely on interval arithmetic for computations, such as fuzzy set theory, random set theory, or probability bounds theory.

The set of solutions to (1)–(3), called parametric solution set, is

$$\Sigma^p = \Sigma\left(A(p), b(p), [p]\right) \quad := \quad \{x \in \mathbb{R}^n \mid \exists p \in [p], A(p)x = b(p)\}\,. \tag{4}$$

Denote by $\mathbb{R}^n, \mathbb{R}^{n\times m}$ the set of real vectors with n components and the set of real $n\times m$ matrices, respectively. A real compact interval is $[a] = [a^-, a^+] := \{a \in \mathbb{R} \mid a^- \leq a \leq a^+\}$. By $\mathbb{IR}^n, \mathbb{IR}^{n\times m}$ we denote the sets of interval n-vectors and interval $n \times m$ matrices, respectively. For $[a] = [a^-, a^+]$, define mid-point $\check{a} := (a^- + a^+)/2$ and radius $a^\Delta := (a^+ - a^-)/2$. The end-point functionals $(\cdot)^-, (\cdot)^+$, as well as the mid-point and radius functionals are applied to interval vectors and matrices componentwise.

The well-known non-parametric interval linear system $[A]x = [b]$, which is the most studied in the interval literature, can be considered as a special case of the parametric linear system with n^2+n independent parameters $a_{ij} \in [a_{ij}]$, $b_i \in [b_i]$, $i, j = 1, \ldots, n$. For a parametric system (1–3), the corresponding non-parametric one with $[A] = A([p]) = ([a_{ij}]) \in \mathbb{IR}^{n\times n}$, $[b] \in \mathbb{IR}^n$ can be obtained as

$$[a_{ij}] = a_{ij}([p]) = a_{ij,0} + \sum_{\mu=1}^{m} a_{ij,\mu}[p_\mu], \qquad [b_i] = b_i([p]) = b_{i,0} + \sum_{\mu=1}^{m} b_{i,\mu}[p_\mu],\ i,j = 1, \ldots, n.$$

The non-parametric solution set, called also united solution set, is defined as

$$\Sigma\left([A], [b]\right) \quad := \quad \{x \in \mathbb{R}^n \mid \exists A \in [A], \exists b \in [b], A \cdot x = b\}\,. \tag{5}$$

In general, $\Sigma\left(A(p), b(p), [p]\right) \subseteq \Sigma\left([A], [b]\right)$ since the elements of $[A]$, $[b]$ are perturbed independently in contrast to $A(p)$, $b(p)$ with $p \in [p]$. The non-parametric solution set is well studied with a lot of results concerning its characterization and properties, for a summary see e.g. [1]. In particular, the famous Oettly-Prager theorem [5] characterizes the non-parametric solution set by the inequalities

$$|A(\check{p})x - b(\check{p})| \quad \leq \quad A^\Delta([p])|x| + b^\Delta([p]). \tag{6}$$

It is a fundamental problem of considerable practical importance how to describe the parametric solution set (4) by a logical combination of inequalities depending of the coordinates. Such a description can be used for visualization of the parametric solution set, exploring some of its properties, and even for computing componentwise boundaries. So far, in the general case of arbitrary affine-linear parameter dependencies, the solution set description can be obtained by a lengthy (and not unique) parameter elimination process [2] shortly recalled in Section 2. Recently, explicit descriptions of the symmetric and skew-symmetric solution sets were given in [3].

The goal of this paper is to give an explicit characterization of the parametric solution set for another class of parametric systems. To this end we introduce a classification of the parameters with respect to the way they participate in the equations of the system. In Section 3 a definition and numerical characterization is given for each class of parameters. In Section 4 the parameter elimination process is studied in details for a class of parametric linear systems, where each uncertain parameter occurs in only one equation of the system and does not matter how many times within that equation. For such systems we give a simple explicit characterization of the parametric solution set which generalizes the famous Oettly-Prager theorem. Some important properties regarding the elimination of zero and 1st class parameters are explored. The explicit parametric solution set characterization is illustrated in Section 5 on some numerical examples and its advantages are compared to other approaches as Fourier-Motzkin like parameter elimination and quantifier elimination which can be also used for characterizing the parametric solution set.

2 Fourier-Motzkin like Elimination of Parameters

The parametric solution set (4) is characterized by the following trivial set of inequalities

$$\Sigma^p = \{x \in \mathbb{R}^n \mid \exists p_\mu \in \mathbb{R}, \mu = 1, \ldots, m : (7)\text{–}(8) \text{ hold}\},$$

where

$$\sum_{j=1}^{n}\left(a_{ij0} + \sum_{\mu=1}^{m} a_{ij\mu}p_\mu\right) x_j \;\le\; b_{i0} + \textstyle\sum_{\mu=1}^{m} b_{i\mu}p_\mu \;\le\; \sum_{j=1}^{n}\left(a_{ij0} + \sum_{\mu=1}^{m} a_{ij\mu}p_\mu\right) x_j, \quad i = 1, \ldots, n, \tag{7}$$

$$p_\mu^- \le p_\mu \le p_\mu^+, \qquad \mu = 1, \ldots, m. \tag{8}$$

Starting from such a description of the parametric solution set, Theorem 2.1 below shows how the parameters in this set can be eliminated successively in order to obtain a description of the parametric solution set not involving p_μ, $\mu = 1, \ldots, m$.

Theorem 2.1 (Alefeld et al. [2]). *Let $f_{\lambda\mu}$, g_λ, $\lambda = 1, \ldots, k$ (≥ 2), $\mu = 1, \ldots, m$, be real-valued functions of $x = (x_1, \ldots, x_n)^\top$ on some subset $D \subseteq \mathbb{R}^n$. Assume that there is a positive integer $k_1 < k$ such that: $f_{\lambda 1}(x) \not\equiv 0$ for all $\lambda \in \{1, \ldots, k\}$; $f_{\lambda 1}(x) \ge 0$ for all $x \in D$ and all $\lambda \in \{1, \ldots, k\}$; for each $x \in D$ there is an index $\beta^* = \beta^*(x) \in \{1, \ldots, k_1\}$ with $f_{\beta^* 1}(x) > 0$ and an index $\gamma^* = \gamma^*(x) \in \{k_1 + 1, \ldots, k\}$ with $f_{\gamma^* 1}(x) > 0$. For m parameters $p_1, \ldots, p_m$ varying in $\mathbb{R}$ and for x varying in D define the sets S_1, S_2 by*

$$\begin{aligned} S_1 &:= \{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = 1, \ldots, m : (9), (10) \text{ hold}\}, \\ S_2 &:= \{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = 2, \ldots, m : (11) \text{ holds}\}, \end{aligned}$$

where inequalities (9), (10) and (11), respectively, are given by

$$g_\beta(x) + \sum_{\mu=2}^{m} f_{\beta\mu}(x)p_\mu \le f_{\beta 1}(x)p_1, \quad \beta = 1, \ldots, k_1, \tag{9}$$

$$f_{\gamma 1}(x)p_1 \le g_\gamma(x) + \sum_{\mu=2}^{m} f_{\gamma\mu}(x)p_\mu, \quad \gamma = k_1 + 1, \ldots, k \tag{10}$$

and

$$g_\beta(x)f_{\gamma 1}(x) + \sum_{\mu=2}^{m} f_{\beta\mu}(x)f_{\gamma 1}(x)p_\mu \;\le\; g_\gamma(x)f_{\beta 1}(x) + \sum_{\mu=2}^{m} f_{\gamma\mu}(x)f_{\beta 1}(x)p_\mu,$$
$$\beta = 1, \ldots, k_1, \; \gamma = k_1 + 1, \ldots, k. \tag{11}$$

(Trivial inequalities such as $0 \le 0$ can be omitted.) Then $S_1 = S_2$.

Theorem 2.1 defines the transition from inequalities (9), (10) to inequalities (11), where the parameter p_1 does not occur. The assertion of Theorem 2.1 remains true if the inequalities in (9), (10) and the inequalities in (11) are supplemented by inequalities which do not contain the parameter p_1, as long as these inequalities are the same in both cases. The parameter elimination process based on Theorem 2.1 resembles the so-called Fourier-Motzkin elimination of variables, see e.g. [8]. As demonstrated below and in [7], it is a lengthy and not unique process. Therefore, explicit parametric solution set characterizations are of particular interest.

3 Classification of the Parameters

In the initial description of the parametric solution set (7)–(8) we have $2n$ inequalities (7) and $2m$ so-called parameter inequalities (8). In addition, a restricted domain for the parametric solution set could be specified by a set D of so-called "domain inequalities" which do not involve the parameters p. In this section we classify the parameters involved in the system into three classes with respect to the way they participate in the parametric system. A definition and numerical characterization will be given for each class of parameters. Our goal is to reveal the specific way by which the elimination of each class of parameters updates the set of characterizing inequalities.

With the notations

$$A^\mu := (a_{ij,\mu}) \in \mathbb{R}^{n\times n}, \quad b^\mu := (b_{i,\mu}) \in \mathbb{R}^n, \qquad \mu = 0,\ldots,m$$

the system (1) can be rewritten equivalently as

$$\left(A^0 + \sum_{\mu=1}^{m} p_\mu A^\mu\right) x \;=\; b^0 + \sum_{\mu=1}^{m} p_\mu b^\mu .$$

For a matrix $A \in \mathbb{R}^{n\times n}$, $A_{m\bullet}$ denotes the m-th row of A.

Definition 3.1. *A parameter p_μ, $1 \le \mu \le m$, is of class zero if it is involved only in the right-hand side and only in one equation of the system (1).*

A parameter p_μ is of class zero iff $A^\mu = 0 \in \mathbb{R}^{n\times n}$ and only one component of the numerical vector b^μ is nonzero ($b_i^\mu \neq 0$ for exactly one i, $1 \le i \le n$). For example, the parameter p_3 involved in the system from Example 5.1 is of class zero. It is obvious that the elimination of every one parameter of class zero, by applying Theorem 2.1, removes one couple of parameter inequalities (8) and updates the inequalities (7) without changing their number.

Definition 3.2. *A parameter p_μ, $1 \le \mu \le m$, is of 1st class if it is not of class zero and occurs in only one equation of the system (1) does not matter how many times within that equation.*

A parameter p_μ is of 1st class iff $A^\mu \neq 0$ and $b^\mu - A^\mu x$ has only one nonzero component (that is $b_i^\mu - A_{i\bullet}^\mu x \neq 0$ for exactly one i, $1 \le i \le n$). For example, the parameters p_1 and p_2 involved in the system from Example 5.1 are parameters of 1st class. All parameters, except p_n, involved in the system from Example 5.2 are parameters of 1st class. The elimination of every one parameter of 1st class removes one couple of parameter inequalities (8). We demonstrate in the next section that eliminating the parameters of 1st class by applying Theorem 2.1 to the end-point parameter inequalities (8) updates the inequalities (7) expanding exponentially their number. A more efficient elimination procedure will be defined for parameters of zero and 1st class and an explicit characterization of the solution set to a special class of parametric linear systems will be derived in Section 4.

Definition 3.3. *A parameter p_μ, $1 \le \mu \le m$, is of 2nd class if it is involved in more than one equation of the system (1).*

A parameter p_μ is of 2nd class iff the vector $b^\mu - A^\mu x$ has more than one nonzero components. During the elimination procedure by Theorem 2.1 the elimination of each parameter of 2nd class removes one couple of parameter inequalities (8), updates those inequalities (7) corresponding to the nonzero components of $b^\mu - A^\mu x$ and expands the total number of characterizing inequalities.

In a subsequent article [7] we expand the analysis of the elimination process in the general case of a system involving all three classes of parameters and give conditions under which the parameter elimination is unique with respect to the resulting number of inequalities.

4 A Special Class of Parametric Systems

In this section we analyze the elimination of the parameters of zero and 1st class, specify an elimination procedure which is efficient with respect to the number of solution set characterizing inequalities, and by generalizing the famous Oettly-Prager characterization of solution set for systems involving only parameters of zero and 1st class, we characterize the considered special class of parametric solution sets by $2n$ explicit inequalities. We start by introducing some notations.

For $\lambda \in \mathbb{R}$, define $\operatorname{sign}(\lambda) := \{+ \text{ if } \lambda \geq 0, - \text{ if } \lambda < 0\}$ and apply the sign functional to vectors and matrices componentwise. Denote by $U(s) := \{\operatorname{sign}(u) \mid u \in \mathbb{R}^s, |u| = (1, \dots, 1)^\top\}$ the set of all s-dimensional sign vectors, where the absolute value $|u|$ is understood componentwise. The set $U(s)$ consists of $\operatorname{Card}(U(s)) = 2^s$ elements. For $\sigma \in \{+,-\}$ and $\prec \in \{\leq, \geq\}$, denote $a \prec^\sigma 0$ to be equivalent to $a \prec 0$ if $\sigma = +$ and to be equivalent to $-a \prec 0$ if $\sigma = -$. These relations are applied to vectors componentwise.

Let all the uncertain parameters participating in the system (1–3) are of zero and 1st class and $l \leq n$ equations involve all m uncertain parameters. Denoting by i_j, $1 \leq j \leq l$, the indexes of the equations involving uncertain parameters, let us suppose that

$$(m_{i_1} + r_{i_1}) + (m_{i_2} + r_{i_2}) + \cdots + (m_{i_l} + r_{i_l}) = m,$$

where $n_{i_j} := m_{i_j} + r_{i_j}$, $1 \leq j \leq l$, is the number of the parameters participating in the i_j-th equation and $0 \leq r_{i_j} \leq n_{i_j}$ of these parameters occur only in the right-hand side of this equation.

Denote by $\mathcal{K}(r_{i_j}) := \{k_1, \dots, k_{r_{i_j}}\}$ the set of indexes of the parameters involved only in the r.h.s. of the equation i_j and $\mathcal{K}(m_{i_j}) := \{k_1, \dots, k_{m_{i_j}}\}$ be the set of indexes of the other parameters involved in equation i_j, such that $\mathcal{K}(m_{i_j}) \bigcap \mathcal{K}(r_{i_j}) = \emptyset$, $\mathcal{K}_{i_j} := \mathcal{K}(m_{i_j}) \bigcup \mathcal{K}(r_{i_j})$, and $\operatorname{Card}(\mathcal{K}_{i_j}) = n_{i_j} \geq 1$.

Let fix a i_j, for which $m_{i_j} \geq 1$, $r_{i_j} \geq 1$. For a fixed $\mu \in \mathcal{K}_{i_j}$ denote $c(\mu, x) := b^\mu - A^\mu x$ and its i_j-th component is $c_{i_j}(\mu, x) := b^\mu_{i_j} - A^\mu_{i_j \bullet} x$. Then, for a fixed μ, from $p^-_\mu \leq p_\mu \leq p^+_\mu$ we obtain

$$c_{i_j}(\mu, x) p_\mu^{-\sigma_{i_j}} \leq c_{i_j}(\mu, x) p_\mu \leq c_{i_j}(\mu, x) p_\mu^{\sigma_{i_j}}, \qquad \text{where } \sigma_{i_j} = \operatorname{sign}(c_{i_j}(\mu, x)). \tag{12}$$

Denote $\mathcal{K} := \{1, \dots, m\} \setminus \mathcal{K}_{i_j}$,

$$A := b^0 - A^0 x + \sum_{\substack{\mu=1 \\ \mu \notin \mathcal{K}_{i_j}}}^{m} (b^\mu - A^\mu x) \qquad \text{and}$$

$$\Xi := \left(\sigma_{i_j}(\mu)\right)_{\mu \in \mathcal{K}(m_{i_j})}, \qquad \text{where } \sigma_{i_j}(\mu) = \operatorname{sign}(c_{i_j}(\mu, x)) \in \{+, -\}.$$

Inequalities (7) can be rewritten equivalently as

$$\begin{aligned} 0 \leq & b_{i0} - \sum_{j=1}^{n} a_{ij0} x_j + \sum_{\mu=1}^{m} p_\mu \left(b_{i\mu} - \sum_{j=1}^{n} a_{ij0} x_j \right) \wedge \\ & b_{i0} - \sum_{j=1}^{n} a_{ij0} x_j + \sum_{\mu=1}^{m} p_\mu \left(b_{i\mu} - \sum_{j=1}^{n} a_{ij0} x_j \right) \leq 0, \qquad i = 1, \dots, n. \end{aligned} \tag{13}$$

Then, the application of Theorem 2.1 to the i_j-th couple of inequalities (13) and the inequalities (12) for $\mu \in \mathcal{K}_{i_j}$ removes the parameters p_μ, $\mu \in \mathcal{K}_{i_j}$. Thus we obtain the following equivalent representation for the parametric solution set

$$\begin{aligned}\Sigma^p &= \{x \in \mathbb{R}^n \mid \exists p_\mu \in \mathbb{R}, \mu = 1, \ldots, m, \mu \notin \mathcal{K}_{i_j} : \\ &\bigvee_{\Xi \in U(m_{i_j})} \left(0 \le A_{i_j \bullet} + \sum_{\nu \in \mathcal{K}(m_{i_j})} c_{i_j}(\nu, x) p_\nu^{\sigma_{i_j}(\nu)} + \sum_{\nu \in \mathcal{K}(r_{i_j})} b_{i_j}^\nu p_\nu^{\sigma(b_{i_j}^\nu)} \wedge \right. \\ &\quad A_{i_j \bullet} + \sum_{\nu \in \mathcal{K}(m_{i_j})} c_{i_j}(\nu, x) p_\nu^{-\sigma_{i_j}(\nu)} + \sum_{\nu \in \mathcal{K}(r_{i_j})} b_{i_j}^\nu p_\nu^{-\sigma(b_{i_j}^\nu)} \le 0 \ \wedge \\ &\quad \left.\left(\bigwedge_{\nu \in \mathcal{K}(m_{i_j})} c_{i_j}(\nu, x) \ge^{\sigma_{i_j}(\nu)} 0 \right)\right) \wedge \left(\bigwedge_{s=1, s \ne i_j}^{n} A_{s\bullet} = 0 \ \bigwedge_{\mu \notin \mathcal{K}_{i_j}} p_\mu^- \le p_\mu \le p_\mu^+ \right) \Big\}.\end{aligned}$$

Expanding this result over all the parameters in the system we obtain the following explicit characterization of the parametric solution set.

$$\begin{aligned}\Sigma^p &= \{x \in \mathbb{R}^n \mid \\ &\bigvee_{\Xi \in U(\sum_{j=1}^l m_{i_j})} \left(\bigwedge_{j=1}^{l} \left(0 \le A_{i_j \bullet} + \sum_{\nu \in \mathcal{K}(m_{i_j})} c_{i_j}(\nu, x) p_\nu^{\sigma_{i_j}(\nu)} + \sum_{\nu \in \mathcal{K}(r_{i_j})} b_{i_j}^\nu p_\nu^{\sigma(b_{i_j}^\nu)} \wedge \right.\right. && (14) \\ &\quad A_{i_j \bullet} + \sum_{\nu \in \mathcal{K}(m_{i_j})} c_{i_j}(\nu, x) p_\nu^{-\sigma_{i_j}(\nu)} + \sum_{\nu \in \mathcal{K}(r_{i_j})} b_{i_j}^\nu p_\nu^{-\sigma(b_{i_j}^\nu)} \le 0 \ \wedge \\ &\quad \left.\left.\left(\bigwedge_{\nu \in \mathcal{K}(m_{i_j})} c_{i_j}(\nu, x) \ge^{\sigma_{i_j}(\nu)} 0 \right)\right)\right) \wedge \left(\bigwedge_{\substack{s=1 \\ s \notin \cup_{j=1}^l \mathcal{K}_{i_j}}}^{n} A_{s\bullet} = 0 \right) \}. && (15)\end{aligned}$$

Thus, applying the elimination process, as described in [2], we obtain an explicit characterization of the solution set to a parametric system involving only zero and 1st class parameters that consists of $2^{\sum_{j=1}^l m_{i_j}} \left(2l + \sum_{j=1}^l m_{i_j}\right)$ inequalities and $n - l$ equalities. The number of characterizing inequalities can be reduced if we replace the set of $2^{\sum_{j=1}^l m_{i_j}} \left(\sum_{j=1}^l m_{i_j}\right)$ inequalities in (15) by the set consisting of $\sum_{j=1}^l m_{i_j}$ inequalities $\bigwedge_{j=1}^l \bigwedge_{\mu \in \mathcal{K}(m_{i_j})} |c(\mu, i_j)| \ge 0$.

$$\begin{aligned}\Sigma^p &= \{x \in \mathbb{R}^n \mid \\ &\bigvee_{\Xi \in U(\sum_{j=1}^l m_{i_j})} \left(\bigwedge_{j=1}^{l} \left(0 \le A_{i_j \bullet} + \sum_{\nu \in \mathcal{K}(m_{i_j})} c_{i_j}(\nu, x) p_\nu^{\sigma_{i_j}(\nu)} + \sum_{\nu \in \mathcal{K}(r_{i_j})} b_{i_j}^\nu p_\nu^{\sigma(b_{i_j}^\nu)} \wedge \right.\right. && (16) \\ &\quad \left.\left. A_{i_j \bullet} + \sum_{\nu \in \mathcal{K}(m_{i_j})} c_{i_j}(\nu, x) p_\nu^{-\sigma_{i_j}(\nu)} + \sum_{\nu \in \mathcal{K}(r_{i_j})} b_{i_j}^\nu p_\nu^{-\sigma(b_{i_j}^\nu)} \le 0 \right)\right) \\ &\bigwedge_{\substack{s=1 \\ s \notin \cup_{j=1}^l \mathcal{K}_{i_j}}}^{n} A_{s\bullet} = 0 \ \bigwedge_{j=1}^{l} \bigwedge_{\nu \in \mathcal{K}(m_{i_j})} |b_{i_j}^\nu - A_{i_j \bullet}^\nu x| \ge 0 \}. && (17)\end{aligned}$$

Thus, the last characterization of the considered parametric solution set involves $2^{\sum_{j=1}^l m_{i_j}} 2l + \sum_{j=1}^l m_{i_j}$ inequalities and $n - l$ equalities. As seen, the number of characterizing inequalities grows exponentially with the number of 1st class parameters.

In order to reduce further the number of inequalities characterizing the solution set of a linear system involving only parameters of zero and 1st class, instead of parameter inequalities $p_\mu^- \le p_\mu \le p_\mu^+$, we shall use the equivalent inequalities $\check{p}_\mu - p_\mu^\Delta \le p_\mu \le \check{p}_\mu + p_\mu^\Delta$ for the parameters of 1st class. Thus, for $\lambda \in \mathbb{R}$ we have

$$\lambda\check{p}_\mu - |\lambda| p_\mu^\Delta \;\le\; \lambda p_\mu \;\le\; \lambda\check{p}_\mu + |\lambda| p_\mu^\Delta. \tag{18}$$

Then the elimination process, based on the relations (18) instead of relations (12), leads to the following inequalities

$$\bigwedge_{j=1}^{l} \left(A_{i_j\bullet} + \sum_{\nu\in\mathcal{K}(r_{i_j})} b_{i_j}^\nu p_\nu^{-\sigma(b_{i_j}^\nu)} + \sum_{\nu\in\mathcal{K}(m_{i_j})} c_{i_j}(\nu,x)\check{p}_\nu - \sum_{\nu\in\mathcal{K}(m_{i_j})} |c_{i_j}(\nu,x)| p_\nu^\Delta \le 0 \right.$$

$$\left. 0 \le A_{i_j\bullet} + \sum_{\nu\in\mathcal{K}(r_{i_j})} b_{i_j}^\nu p_\nu^{\sigma(b_{i_j}^\nu)} + \sum_{\nu\in\mathcal{K}(m_{i_j})} c_{i_j}(\nu,x)\check{p}_\nu + \sum_{\nu\in\mathcal{K}(m_{i_j})} |c_{i_j}(\nu,x)| p_\nu^\Delta \right) \tag{19}$$

$$\bigwedge_{\substack{s=1\\ s\notin\cup_{j=1}^{l}\mathcal{K}_{i_j}}}^{n} A_{s\bullet} = 0$$

Relations (19) can be rewritten in the following more general but equivalent forms

$$\bigwedge_{j=1}^{l} \left(b_{i_j}(\check{p}) - A_{i_j\bullet}(\check{p})x - \sum_{\mu\in\mathcal{K}_{i_j}} |b_{i_j}^\mu - A_{i_j\bullet}^\mu\, x| p_\mu^\Delta \le 0 \right.$$

$$\left. 0 \le b_{i_j}(\check{p}) - A_{i_j\bullet}(\check{p})x + \sum_{\mu\in\mathcal{K}_{i_j}} |b_{i_j}^\mu - A_{i_j\bullet}^\mu\, x| p_\mu^\Delta \right) \bigwedge_{s\notin\cup_{j=1}^{l}\mathcal{K}_{i_j}} b_s^0 - A_{s\bullet}^0 x = 0,$$

or

$$b(\check{p}) - A(\check{p})x - \sum_{\mu=1}^{m} |b^\mu - A^\mu\, x| p_\mu^\Delta \;\le\; 0 \;\le\; b(\check{p}) - A(\check{p})x + \sum_{\mu=1}^{m} |b^\mu - A^\mu\, x| p_\mu^\Delta. \tag{20}$$

This way, we have proven the following theorem which generalizes the famous Oettly-Prager solution set characterization (6) for parametric systems involving only parameters of zero and 1st class.

Theorem 4.1. *The solution set of the system (1)–(3) involving only uncertain parameters of zero and 1st class has the following explicit characterization*

$$\Sigma^p \;=\; \{x \in \mathbb{R}^n \mid |A(\check{p})x - b(\check{p})| \;\le\; \sum_{\mu=1}^{m} |b^\mu - A^\mu\, x| p_\mu^\Delta\}.$$

The solution set characterization (20), respectively Theorem 4.1, can be obtained by a parameter elimination process specified by the following proposition.

Proposition 4.1. *Let f_μ, g_1, g_2, $\mu = 1,\ldots,m$, be real-valued functions of $x = (x_1,\ldots,x_n)^\top$ on some subset $D \subseteq \mathbb{R}^n$. For m parameters $p_1,\ldots,p_m$ varying in R, where the ν-th parameter is of zero or 1st class, and for*

x varying in D, it holds $S_1 = S_2$ where the sets S_1, S_2 are defined by

$$\begin{aligned} S_1 \; := \; & \{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = 1, \ldots, m \; : \qquad p_\nu^- \le p_\nu \le p_\nu^+ \quad \wedge \\ & \qquad g_1(x) + \sum_{\substack{\mu=1\\ \mu\neq\nu}}^{m} f_\mu(x)p_\mu \;\le\; f_\nu(x)p_\nu \;\le\; g_2(x) + \sum_{\substack{\mu=1\\ \mu\neq\nu}}^{m} f_\mu(x)p_\mu\}, \\ S_2 \; := \; & \{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = 1, \ldots, m, \; \mu \neq \nu \; : \\ & \qquad g_1(x) + \sum_{\substack{\mu=1\\ \mu\neq\nu}}^{m} f_\mu(x)p_\mu \;\le\; f_\nu(x)\check{p_\nu} + |f_\nu(x)|\, p_\nu^{\triangle} \quad \wedge \\ & \qquad\qquad f_\nu(x)\check{p_\nu} - |f_\nu(x)|\, p_\nu^{\triangle} \;\le\; g_2(x) + \sum_{\substack{\mu=1\\ \mu\neq\nu}}^{m} f_\mu(x)p_\mu\}. \end{aligned}$$

Proof. The proposition follows from Theorem 2.1 and the equivalence of the relations $p_\nu^- \le p_\nu \le p_\nu^+$ and $\check{p}_\nu - p_\nu^{\triangle} \le p_\nu \le \check{p}_\nu + p_\nu^{\triangle}$. □

Being specialized for parameters of zero and 1st class, Proposition 4.1 removes the restriction of Theorem 2.1 for positiveness of the coefficient functions $f_{\lambda 1}$, $\lambda = 1, \ldots, k$ in (9), (10). Furthermore, utilizing the midpoint-radius representation of intervals, Proposition 4.1 allows updating the inequalities (7) without changing their number. Thus, Proposition 4.1 defines a more efficient elimination process for parameters of zero and 1st class than Theorem 2.1.

Based on Proposition 4.1, a simple explicit characterization of the parametric solution set of systems involving only parameters of zero and 1st class is derived. Theorem 4.1 characterizes the considered special class of parametric solution sets by $2n$ inequalities and this number does not depend on the number of the involved uncertain parameters. As being evident from the characterizations (14)–(15) or (19), the inequalities characterizing this special class of parametric solution sets are linear. Furthermore, the order of elimination of the parameters is not significant for the characterization of this special class parametric solution sets.

5 Numerical Examples

Section 4 demonstrates the advantage of an explicit characterization, provided by Theorem 4.1, of the solution set to parametric systems involving parameters of zero and 1st class in comparison to the Fourier-Motzkin like parameter elimination process defined in [2]. In this section we illustrate the explicit characterization of the solution set on some numerical examples. For comparison, we present the characterizations obtained by quantifier elimination in the environment of *Mathematica* [9].

Example 5.1. *Consider a simple parametric system*

$$\begin{pmatrix} p_1 & 1+p_1 \\ 1+p_2 & -2p_2 \end{pmatrix} \cdot x = \begin{pmatrix} p_3 \\ 1-3p_2 \end{pmatrix},$$

where $p_1, p_2 \in [0,1]$, $p_3 \in [-1,1]$.

Applying Theorem 4.1, the parametric solution set is defined by the following four inequalities

$$\begin{aligned} -x_2 - \frac{x_1+x_2}{2} - \frac{|-x_1-x_2|}{2} - 1 \le 0 \le -x_2 - \frac{x_1+x_2}{2} + \frac{|-x_1-x_2|}{2} + 1 \\ 1 - x_1 + \frac{-3-x_1+2x_2}{2} - \frac{|-3-x_1+2x_2|}{2} \le 0 \le 1 - x_1 + \frac{-3-x_1+2x_2}{2} + \frac{|-3-x_1+2x_2|}{2}. \end{aligned}$$

The parametric solution set is presented in Fig. 1 together with the corresponding non-parametric solution set.

For comparison we have run the Mathematica *function* `Resolve` *doing quantifier elimination and the resulting characterization by 181 inequalities is presented in Fig. 2.*

In the next example we consider an arbitrary large parametric system.

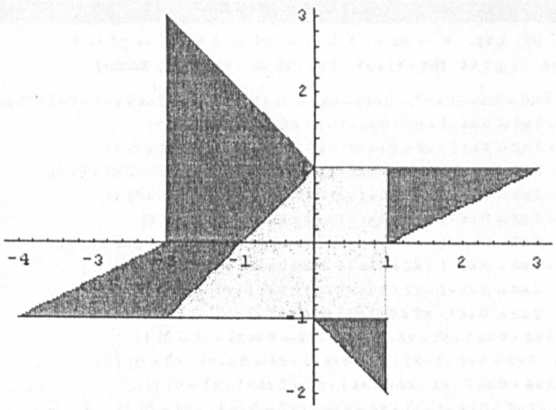

Figure 1: The parametric solution set (in blue or light gray) together with the corresponding non-parametric solution set (in red or dark gray) for the system from Example 5.1.

Example 5.2. *Consider the parametric system $A^\top(p)x = b(p)$, where $A(p) \in \mathbb{R}^{n\times n}$ is the Milnes matrix [4]*

$$A(p) = (a_{ij}(p)), \qquad a_{ij}(p) = \begin{cases} 1 & j \geq i, \\ p_j & j < i, \end{cases}$$

$b(p) = (p_1, \dots, p_n)^\top$, $\check{p}_i = 1/(i+1)$, $p_i^\triangle = 2\%\,\check{p}_i = 1/(50i+50)$.

According to Theorem 4.1 the solution set of this problem is characterized by the following system of $2n$ inequalities

$$\bigwedge_{i=1}^{n} \left((i+1)^{-1} - \sum_{j=1}^{i} x_j - \sum_{j=i+1}^{n} x_j/(i+1) - |1 - \sum_{j=i+1}^{n} x_j|/(50i+50) \leq 0 \wedge \right.$$

$$\left. 0 \leq (i+1)^{-1} - \sum_{j=1}^{i} x_j - \sum_{j=i+1}^{n} x_j/(i+1) + |1 - \sum_{j=i+1}^{n} x_j|/(50i+50) \right).$$

For $n = 20$, e.g., the solution set characterization, based on Theorem 2.1 and the end-point parameter inequalities (8), would require $2^{19}40 + 19 \approx 21 * 10^6$ inequalities.

The execution of the *Mathematica* function `Resolve` on the same problem for $n = 5$ took 7.859 seconds which was about 490 times slower than an interpreted implementation of Theorem 4.1.

6 Conclusion

Introducing a new classification of the parameters with respect to the way they participate in the system, we have proven explicit characterization of the solution set to a class of parametric linear systems, where each uncertain parameter occurs in only one equation of the system. Thus, besides for parametric systems involving symmetric or skew-symmetric matrix, the famous Oettly-Prager theorem for non-parametric linear systems is generalized for parametric systems involving zero and 1st class parameters. The latter parametric systems are characterized by $2n$ inequalities and the boundary of the corresponding parametric solution set consists of linear functions, contrary to the solution set characterization for systems involving symmetric or skew-symmetric matrix.

Although there exist other special methods for visualization of parametric solution sets [6], visualizing the regions determined by the explicit solution set characterizing inequalities is more straightforward.

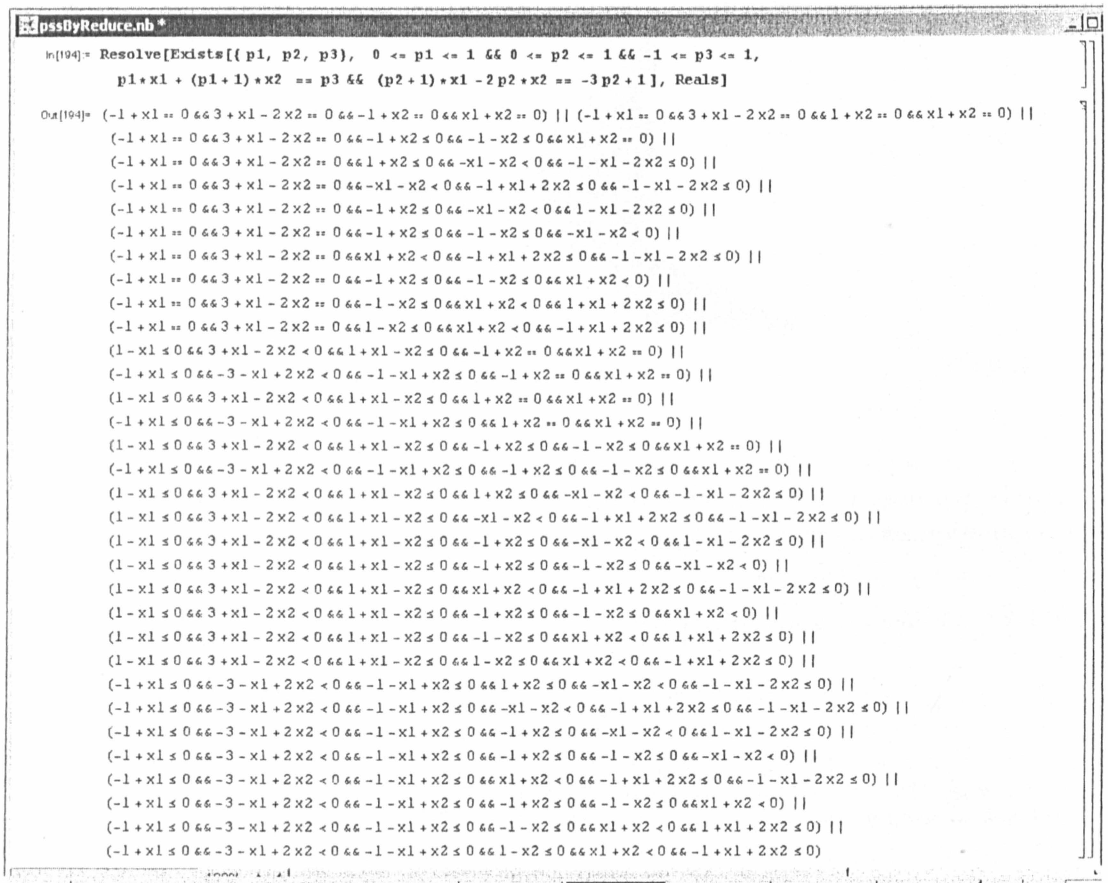

Figure 2: Characterization of the parametric solution set from Example 5.1 produced by the *Mathematica* function `Resolve` doing quantifier elimination.

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