ИНСТИТУТ ПО МАТЕМАТИКА И ИНФОРМАТИКА

INSTITUTE OF MATHEMATICS AND INFORMATICS

Секция Анализ, Геометрия и Топология

Section Analysis, Geometry, and Topology

БЪЛГАРСКА АКАДЕМИЯ НА НАУКИТЕ

BULGARIAN ACADEMY OF SCIENCES

Класически полиноми на Ермит и Лагер и разпределението на нулите на римановата ζ-функция

Петър Русев

Classical Hermite and Laguerre polynomials and the zero-distribution of Riemann's ζ-function

Peter Rusev

ПРЕПРИНТ № 6/2011
PREPRINT No 6/2011

София
Sofia

Септември 2011
September 2011

Classical Hermite and Laguerre polynomials and the zero-distribution of Riemann's ζ-function

PETER RUSEV

Abstract. Necessary and sufficient conditions for absence of zeros of the function $\zeta(s), s = \sigma + it$, in the half-plane $\sigma > \theta, 1/2 \le \theta < 1$ are proposed in terms of representations of holomorphic functions by series in Hermite and Laguerre polynomials as well as in terms of Fourier and Hankel integral transforms.

1. Representation of holomorphic functions by series of Hermite and Laguerre polynomials

The region of convergence of a series in Hermite polynomials

$$\sum_{n=0}^{\infty} a_n H_n(z), \tag{1.1}$$

as it is pointed out by G. Szegö [14, **9.2**, (5)], is a strip symmetricaly situated to the real axis. More precissely, let

$$\tau_0 = \max\{0, -\limsup_{n\to\infty}(2n+1)^{-1/2}\log(2n/e)^{n/2}|a_n|\}, \tag{1.2}$$

then:

If $\tau_0 = 0$, *then the series* (1.1) *diverges in the open set* $\mathbb{C} \setminus \mathbb{R}$. *If* $0 < \tau_0 \le \infty$, *then it is absolutely uniformly convergent on each compact subset of the strip* $S(\tau_0) := \{z \in \mathbb{C} : |\Im(z)| < \tau_0\}$ *and diverges in the open set* $\mathbb{C} \setminus \overline{S(\tau_0)}$.

The equality (1.2), which can be regarded as a formula of Cauchy-Hadamard type for series in Hermite polynomials, is a corollary of the asymptotic formula for these polynomials in the complex plane [14, Theorem 8.22.7].

The $\mathbb{C}$-vector space $\mathcal{H}(\tau_0), 0 < \tau_0 \le \infty$ of holomorphic functions, having an expansion of the kind (1.1) in the strip $S(\tau_0)$, is completely described by E. Hille [4]. In fact, he proved the following assertion:

Let $0 \le \tau < \infty, \overline{S}(\tau) := \{z \in \mathbb{C} : |\Im z| \le \tau$ *and define*

$$\eta(\tau; x, y) = x^2/2 - |x|(\tau^2 - y^2)^{1/2}, \quad z = x + iy \in \overline{S}(\tau). \tag{1.3}$$

Then, a complex function f *is in the space* $\mathcal{H}(\tau_0)$ *if and only if for each* $\tau \in [0, \tau_0)$ *there exist a positive constant* $H(f,\tau)$ *such that*

$$|f(z)| = |f(x+iy)| \le H(f,\tau)\exp(\eta(\tau; x, y)), \quad z = x + iy \in \overline{S}(\tau). \tag{1.4}$$

Moreover, if (1.1) *is the Hermite polynomial expansion of the function* f *in the strip* $S(\tau_0)$, *then*

$$a_n = (\sqrt{\pi}n!2^n)^{-1} \int_{-\infty}^{\infty} \exp(-x^2) H_n(x) f(x)\, dx, \quad n = 0, 1, 2, \dots \tag{1.5}$$

The region of convergence of a series in Laguerre polynomials is the interior of a parabola with focus at the zero-point and vortex at a point of the real and negative semi-axes [14, **9.2.** (5)]. More precissely, let

$$\lambda_0 = -\limsup_{n\to\infty} (2\sqrt{n})^{-1} \log |a_n| > 0 \tag{1.6}$$

and α be an arbitrart complex number, then:

The series

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z) \tag{1.7}$$

is absolutely uniformly convergent on each compact subset of the region

$$\Delta(\lambda_0) = \{z \in \mathbb{C} : \Re(-z)^{1/2} < \lambda_0\} \tag{1.8}$$

and diverges outside it [6, (**IV.2.1**)].

Let us note, that if $0 < \lambda_0 < \infty$, then $\Delta(\lambda_0)$ is just the interior of the papabola with focus at the origin and with vortex at the point $-\lambda_0^2$, and that $\Delta(\infty) = \mathbb{C}$.

Let $\mathcal{P}^{(\alpha)}(\lambda_0), 0 < \lambda_0 \leq \infty$ be the $\mathbb{C}$-vector space of even complex-valued functions which are holomorphic in the strip $S(\lambda_0)$ and have there a representation by a series of the kind

$$\sum_{n=0}^{\infty} a_n L_n^{(\alpha)}(z^2). \tag{1.9}$$

The growth of the functions in the space $\mathcal{P}^{(0)}(\lambda_0)$ is determined first by H. Pollard [5, THEOREM A] by means of Hille's function (1.3). In fact, Pollard proved that:

A complex-valued function f, *holomorphic in the strip* $S(\lambda_0), 0 < \lambda_0 \leq \infty$, *is in the space* $\mathcal{P}^{(0)}(\lambda_0)$ *if and only if to each* $\lambda \in [0, \lambda_0)$ *there corresponds a positive constant* $A(f, \lambda)$ *such that for each* $z = x + iy \in \overline{S}(\lambda)$,

$$|f(z)| = |f(x + iy)| \leq A(f, \lambda) \exp(\eta(\lambda; x, y)). \tag{1.10}$$

O. Szazs and N. Yeardly generalized Polard's criterion by showing that it holds for each $\alpha > -1$ [13, THEOREM A]. This means that the space $\mathcal{P}^{(\alpha)}(\lambda_0)$ does not depend on $\alpha > -1$. Hence, it can be denoted by $\mathcal{P}(\lambda_0)$ only.

Let $\mathcal{L}(\lambda_0), 0 < \lambda_0 \leq \infty$ be the $\mathbb{C}$-vector space of complex-valued functions f, holomorphic in the region $\Delta(\lambda_0)$, with the property that for each $\lambda \in [0, \lambda_0)$ there exists a positive constant $M(f, \lambda)$ such that for each $z \in \overline{\Delta}(\lambda) = \{z \in \mathbb{C} : \Re(-z)^{1/2} \leq \lambda\}$,

$$|f(z)| = |f(x+iy)| \leq M(f,\lambda)\varphi(\lambda; x, y), \tag{1.11}$$

where

$$\varphi(\lambda; x, y) \tag{1.12}$$

$$= \exp\left\{\frac{\sqrt{x^2+y^2}+x}{4} - \left[\frac{\sqrt{x^2+y^2}+x}{2}\left(\lambda^2 - \frac{\sqrt{x^2+y^2}-x}{2}\right)\right]^{1/2}\right\}$$

Since the image of the strip $S(\lambda_0)$ under the mapping $z \longmapsto z^2$ is the region $\Delta(\lambda_0)$, the following assertion holds true:

A complex function f, holomorphic in the region $\Delta(\lambda_0), 0 < \lambda \leq \infty$ is representable in this region by a series in Laguerre polynomials with parameter $\alpha > -1$ if and only if it is in the space $\mathcal{L}(\lambda_0)$. Moreover, if (1.7) is the Laguerre polynomial expansion of the function f in the region $\Delta(\lambda_0)$, then

$$a_n = \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}\int_0^\infty x^\alpha \exp(-x) L_n^{(\alpha)}(x) f(x)\, dx, \quad n = 0, 1, 2, \ldots. \tag{1.13}$$

Remark. Let $A_k = \{z \in \mathbb{C} : \Re z = k + 1/2, z \neq k + 1/2\}, k \in \mathbb{Z}$ and $\mathbb{A} = \bigcup_{k\in\mathbb{Z}} A_k$. Then, the above assertion remains true provided the parameter $\alpha \in \mathbb{C} \setminus \mathbb{A}$ [6, **(V.3.6)**].

There is another approach to the problem of expansion of holomorphic fuctions in series of Laguerre and Hermite polynomials. Namely, the integral representations

$$L_n^{(\alpha)}(z) = \frac{z^{-\alpha/2}\exp z}{n!}\int_0^\infty t^{n+\alpha}\exp(-t) J_\alpha(2\sqrt{zt})\, dt, n = 0, 1, 2, \ldots, \tag{1.14}$$

and

$$H_n(z) = \frac{2^n(-i)^n \exp z^2}{n!}\int_{-\infty}^\infty t^n \exp(-t^2 + 2izt)\, dt, \quad n = 0, 1, 2, \ldots \tag{1.15}$$

give rise to "translate" this problem in the language of Hankel and Fourier integral transforms. The role of "mediator" is playing by the class $\mathcal{G}(\gamma), -\infty < \gamma \leq \infty$ of entire functons G such that

$$\limsup_{|w|\to\infty} (2\sqrt{|w|})^{-1}(\log|G(w)| - |w|) \leq -\gamma. \tag{1.16}$$

Evidently, the class $\mathcal{G}(\gamma)$ consists of the entire functions G such that the estimate

$$|G(w)| = O(\exp|w| - 2(\gamma - \delta)\sqrt{|w|}), \quad w \in \mathbb{C} \tag{1.17}$$

holds whatever the positive δ be. Hence, $\mathcal{G}$ is a $\mathbb{C}$-vector space.

Closely related to $\mathcal{G}(\gamma)$ is the class $\mathcal{E}(\gamma), -\infty < \gamma \leq \infty$ of entire functions E such that

$$\limsup_{|w|\to\infty} (2|w|)^{-1}(\log|E(w)| - |w|^2) \leq -\gamma. \tag{1.18}$$

It is quite easy to prove that an entire function E is in $\mathcal{E}(\gamma)$ if and only if it has the form

$$E(w) = U(w^2) + wV(w^2), \quad w \in \mathbb{C}, \tag{1.19}$$

where the entire functions U and V are in the class $\mathcal{G}(\gamma)$. Indeed, if E has the form (1.19), then (1.18) is a corollary of (1.16). Conversly, if E satisfies (1.18), then we define

$$U(w) = (1/2)(E(w^{1/2}) + E(-w^{1/2}))$$

and

$$V(w) = (1/2)w^{-1/2}(E(w^{1/2}) - E(-w^{1/2})).$$

Furthermore, the entire functions U, V are in the class $\mathcal{G}(\gamma)$ and

$$E(w) = U(w^2) + wV(w^2), w \in \mathbb{C}.$$

The role of the spaces $\mathcal{G}(\gamma)$ and $\mathcal{E}(\gamma)$ is cleared up by the following assertions:

A complex function f, holomorphic in the region $\Delta(\lambda_0), 0 < \lambda_0 \leq \infty$, has an expansion in this region in a series of Laguerre polynomials with parameter $\alpha > -1$ if and only if it admits the representation

$$f(z) = z^{-\alpha/2} \exp z \int_0^\infty t^{\alpha/2} \exp(-t) G(t) J_\alpha(2\sqrt{zt})\, dt \tag{1.20}$$

in the region $z \in \Delta(\lambda_0) \setminus (\lambda_0^2, 0]$ *with a function* $G \in \mathcal{G}(\lambda_0)$ [7, Theorem 1; 8, THEOREM V; 6, **(VI.1.3)**].

A complex function f, holomorphic in the region $S(\tau_0), 0 < \tau_0 \leq \infty$, has an expansion there in a series of Hermite polynomials if and only if it admits the representation

$$(1.21) \qquad f(z) = \frac{1}{\sqrt{\pi}} \int_{-\infty}^{\infty} E(t) \exp\{-(t-iz)^2\}\, dt, z \in S(\tau_0)$$

with a function $E \in \mathcal{E}(\tau_0)$ [6, **(VI.4.1)**; 8, THEOREM VI].

Remark. The asertion [6, **(VI.1.2)**] is a criteron the power series

$$(1.22) \qquad \sum_{n=0}^{\infty} (n!)^{-1} a_n w^n$$

to define an entire function of the class $\mathcal{G}(\gamma)$. I states that this is the case if and only if

$$(1.23) \qquad \limsup_{n\to\infty} (2\sqrt{n})^{-1} \log |a_n| \leq -\gamma.$$

Futhermore, the above criterion and Stirling's formula yield that the power series (1.22) defines an entire function of the class $\mathcal{E}(\gamma)$ if and only if

$$(1.24) \qquad \limsup_{n\to\infty} (2n)^{-1} \log(2n/e)^{n/2} |a_n| \leq -\gamma.$$

2. Holomorphic extension by means of series in Laguerre and Hermite polynomials

For a complex-valued function f, defined on an interval $(a, b), -\infty \leq a < b \leq \infty$ of the real line is said that it is holomorphically extendable in the complex plane if there exist a domain $D \subset \mathbb{C}$ and a function F which is holomorphic in the region D and such that $F(x) = f(x)$ a.e. (almost everwhere) in (a, b). Evidently, the uniqueness of the holomorphic extension F is a direct consequence of the identity theorem for holomorphic functions of one complex variable.

Sufficient conditions for existence of holomorphic extensions of complex functions of a real variable in terms of Jacobi, Laguerre and Hermite polynomials are given in the paper [9] as well as in CHAPTER V of the monograph [6]. The assertions we need further are the following:

Suppose that for a mesurable complex-valued function b, defined on the interval $(0,\infty)$, there exist $r>0, \delta<1$ and $\alpha>-1$ such that the function $\exp(-\delta x)b(x)$ is essentially bounded on the interval (r,∞) and, moreover,

$$\int_0^r x^\alpha |b(x)|\,dx < \infty. \tag{2.1}$$

If

$$\lambda_0(b) = -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|b_n^{(\alpha)}(b)| > 0,$$

where

$$b_n^{(\alpha)}(b) = \int_0^\infty x^\alpha \exp(-x)L_n^{(\alpha)}(x)b(x)\,dx, \quad n=0,1,2,\ldots, \tag{2.2}$$

then b has a holomorphic extension. More precisely, there exists a function B holomorphic in the region $\Delta(\lambda_0(b))$ and such that $B(x)=b(x)$ a.e. in $(0,\infty)$. Moreover, for each $\lambda\in[0,\lambda_0(b))$ there exists a positive constant $M(b,\lambda)$ such that

$$|b(x)| \le M(b,\lambda)\exp(x/2-\lambda\sqrt{x}) \quad a.e. \quad in \quad (0,\infty). \tag{2.3}$$

Suppose that for a mesurable complex-valued function c, defined on the real line, there exist $r>0$ and $\delta<1$ such that the function $\exp(-\delta x^2)c(x)$ is essentially bounded for $|x|\ge r$ and, moreover

$$\int_{-r}^r |c(x)|\,dx < \infty.$$

If $\tau_0(g) = -\limsup_{n\to\infty}(2n+1)^{-1/2}\log(2n/e)^{-n/2}|c_n(c)| > 0$, where

$$c_n(c) = \int_{-\infty}^\infty \exp(-x^2)H_n(x)c(x)\,dx, \quad n=0,1,2,\ldots, \tag{2.4}$$

then c has holomorphic extension. More precisely, there exists a complex-valued function C which is holomorphic in the strip $S(\tau_0(g))$ and such that $C(x)=c(x)$ a.e. in $(-\infty,\infty)$. Moreover, to each $\tau\in[0,\tau_0(g))$ there correspond a positive number $N(c,\tau)$ such that

$$|c(x)| \le N(c,\tau)\exp(x^2/2-\tau|x|) \quad a.e. \quad in \quad (-\infty,\infty). \tag{2.5}$$

In order to prove the first of them we define the complex-valued function B by

$$B(z) = \sum_{n=0}^\infty \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)} b_n^{(\alpha)}(b)L_n^{(\alpha)}(z), \tag{2.6}$$

Then, Stirling's formula yields that

$$-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|(\Gamma(n+1)/\Gamma(n+\alpha+1))b_n^{(\alpha)}(b)| =$$

$$-\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|b_n^{(\alpha)}(b)| = \lambda_0(f).$$

Hence, the series in (2.6) converges absolutely uniformly on each compact subset of the region $\Delta(\lambda_0(b))$, i.e. the function B is holomorphic in this region. Futhermore,

$$\int_0^\infty B^{(\alpha)}(x)L_n^{(\alpha)}(x)\,dx = 0, \quad n = 0,1,2,\ldots, \tag{2.7}$$

where

$$B^{(\alpha)}(x) = x^\alpha \exp(-x)(B(x) - b(x)), \quad 0 < x < \infty. \tag{2.8}$$

Indeed,

$$\int_0^\infty B^{(\alpha)}(x)L_n^{(\alpha)}(x)\,dx$$

$$= \frac{\Gamma(n+1)}{\Gamma(n+\alpha+1)}b_n^{(\alpha)}\int_0^\infty x^\alpha\exp(-x)\{L_n^{(\alpha)}(x)\}^2\,dx - b_n^{(\alpha)} = 0, \quad n = 0,1,2,\ldots.$$

Since $\deg L_n^{(\alpha)} = n, \alpha \neq -1,-2,-3,\ldots n = 0,1,2,\ldots$, the system of Laguerre polynomials with parameter $\alpha > -1$ is linearly independent. Hence, it is a basis in the space of the algebraic polynomials with real coefficients. Then, from equalities (2.7) it follows that

$$\int_0^\infty B^{(\alpha)}(x)x^n\,dx = 0, \quad n = 0,1,2,\ldots. \tag{2.9}$$

Let $\hat{B}^{(\alpha)}(x)$ be the Fourier transform of the function $B^{(\alpha)}(x)$, i.e.

$$\hat{B}^{(\alpha)}(w) = \int_0^\infty B^{(\alpha)}(x)\exp(-iwx)\,dx. \tag{2.10}$$

There is a positive constant D such that the inequality $|b(x)| \leq D\exp(\delta x)$ holds a.e. in the interval (r,∞). Since the function B is in the space $\mathcal{L}(\lambda_0(b))$, from (1.12) it follows that $|B(x)| \leq M(b,0)\exp(x/2)$ for $x \in [0,\infty)$. Hence, $|B^{(\alpha)}(x)| \leq Q\exp(-(1-q)x)$ a.e. in (r,∞) where $Q = \max\{M(b,0), D\}$ and $q = \max(1/2,\delta)$. Therefore, the integral in (2.10) is uniformly convergent on each closed strip $\overline{S}(\tau) = \{w \in \mathbb{C} : |\Im w| \leq \tau\}$ with $\tau \in [0, 1-q)$, i.e. the function $\hat{B}^{(\alpha)}$ is holomorphic in the strip $S(1-q)$. Furtermore, because of the equalities (2.9), this function and each of its derivatives vanishes at the point $w = 0$. Hence, the function $B^{(\alpha)}$ is identically zero and the uniqueness property of Fourier transform yields that $B^{(\alpha)} \sim 0$, i.e. $B^{(\alpha)}(x) = 0$ a.e. in $(0, \infty)$ which immediately gives that $B(x) = b(x)$ a.e. in $(0, \infty)$. Since $\varphi(\lambda; x, 0) = x/2 - \lambda\sqrt{x}$ for $x \in [0, \infty)$, (2.3) is a consequence of (1.12).

The proof of the criterion for holomorphic extension by means of series in Hermite polynomials proceeds in a similar way. We define the function C by

$$C(z) = \sum_{n=0}^{\infty} (\sqrt{\pi} n! 2^n)^{-1} c_n(c) H_n(z). \tag{2.11}$$

Since

$$-\limsup_{n\to\infty} (2n+1)^{-1/2} \log(\sqrt{\pi} n! 2^n)^{-1} (2n/e)^{n/2} |c_n(c)| =$$
$$-\limsup_{n\to\infty} (2n+1)^{-1/2} \log(2n/e)^{-n/2} |c_n(c)| = \tau_0(c),$$

the series in (2.11) is absolutely uniformly convergent on each compact subset of the strip $S(\tau_0(c))$, i.e. its sum is in the space $\mathcal{H}(\tau_0(c))$. Moreover,

$$\int_{-\infty}^{\infty} T(x) H_n(x)\, dx = 0, \quad n = 0, 1, 2, \ldots,$$

where $T(x) = \exp(-x^2)(C(x) - c(x)), -\infty < x < \infty$. Futhermore, the Fourier transform $\hat{T}$ of the function T, which is holomorphic in the strip $S(1-q)$, turns out to be identically zero. Hence, $T \sim 0$ in $(-\infty, \infty)$ which yields that $C(x) = c(x)$ a.e. in $(-\infty, \infty)$.

3. Applications to the zero-distribution of Riemann's ζ-function

It is well-known that the function $\zeta(s)$, $s = \sigma + it$, defined in the half-plane $\sigma > 1$ by the Dirichlet series

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s}, \tag{3.1}$$

is analytically continuable in the whole complex plane as a meromorphic function with unique pole at the point $s = 1$. Indedeed, from (2.6) it follows that

$$(1 - 2^{1-s})\zeta(s) = \sum_{n=1}^{\infty} \frac{(-1)^n}{n^s}. \tag{3.2}$$

for $\Re s > 1$. But the series on the right-hand side is uniformly convergent on each closed half-plane $\sigma \geq \delta > 0$. Hence, its sum $Z(s)$ is a holomorphic function in the half-plane $\sigma > 0$. Assuming that $\zeta(s) = (1 - 2^{1-s})^{-1} Z(s)$ for $0 < \sigma \leq 1$ and $s \neq 1$, we obtain the continuation of the function $\zeta(s)$ as a meromorphic function in the half-plane $\sigma > 0$ with unique pole at the point 1. Furthermore, the functional equation

$$\pi^{-s/2}\Gamma(s/2)\zeta(s) = \pi^{-(1-s)}\Gamma((1-s)/2)\zeta(1-s)$$

holds in the strip $0 < \sigma < 1$. But, in fact, it realizes the continuation of the function $\zeta(s)$ as a holomorphic function on left of the imaginary axis. Its direct consequence is that this function has simple zeros at the non-zero poles of the function $\Gamma(s/2)$, i.e. at each of the points $-2k, k = 1, 2, 3, \ldots$. These are the so called trivial zeros of the function $\zeta(s)$.

It is well-known, that it has infinitely many zeros in the strip $0 < \Re s < 1$, called non-trivial. The conjecture that all the non-trivial zeros of the function $\zeta(s)$ are situated on the line $\sigma = 1/2$ is the famos hypothesis of Riemann. Till now it is neither proved, nor disproved. Moreover, it is not known whether these zeros are in a closed strip of the kind $1/2 - \delta \leq \sigma \leq 1/2 + \delta$ for some $\delta \in (0, 1/2)$. It is clear that this is true if and only if the function $\zeta(s)$ has no zeros in the half-plane $\sigma > \theta$ for some $\theta \in [1/2, 1)$.

It is also well-known that the function $\zeta(s)$ has no zeros on the closed half-plane $\sigma \geq 1$. Hence, there is a region Ω containing this half-plane and such that $\zeta(s) \neq 0$ for $s \in \Omega$. Therefore, the function

$$\Phi(s) = -\frac{\zeta'(s)}{\zeta(s)} - \frac{1}{s-1} \tag{3.3}$$

is holomorphic in the region Ω. Moreover, the integral representation

$$\Phi(s) = \int_1^\infty \frac{\psi(x) - x}{x^{s+1}}\, dx \tag{3.4}$$

holds on the closed half-plane $\sigma \geq 1$, where ψ is one of the Chebisheff functions [2, Section **3**]. More precissely, the integral in (3.4) is absolutely and uniformly convergent on this half-plane and the function Φ is bounded there. Indeed, since $\psi(x) - x = O(x \exp(-c(\log x)^{1/2})), c > 0$, as $x \to \infty$ [s.e.g. 3, Section 18. (1)], we have that for $\sigma \geq 1$ and $-\infty < t < \infty$,

$$|\Phi(s)| \leq \int_1^\infty \frac{|\psi(x) - x|}{x^{\sigma+1}}\, dx = O\left(\int_1^\infty x^{-1} \exp(-c(\log x)^{1/2})\, dx\right)$$

$$= O\left(\int_0^\infty \exp(-cx^{1/2})\, dx\right) = O(1).$$

It turns out that the function

$$\Phi(1+iz) = \int_1^\infty \frac{\psi(t)-t}{t^{2+iz}}\,dt, \quad z = x+iy, \tag{3.5}$$

is holomorphic on the closed half-plane $\Im z \le 0$. Moreover, it is bounded there and, in particular, on the real axis. Hence, there exist the Fourier-Hermite coefficients of the function $\Phi(1+ix), -\infty < x < \infty$, namely

$$a_n(\Phi) = \int_{-\infty}^{\infty} \exp(-x^2)H_n(x)\Phi(1+ix)\,dx, \quad n = 0,1,2,\ldots. \tag{3.6}$$

Let us define

$$A_n(\psi) = \int_0^\infty t^n \exp(-t^2/4 - t)(\psi(\exp t) - \exp t)\,dt, \quad 0,1,2\ldots, \tag{3.7}$$

then the eqialities

$$a_n(\Phi) = \sqrt{\pi}(-i)^n A_n(\psi), \quad n = 0,1,2,\ldots \tag{3.8}$$

hold [10, (3.6)]. If

$$\tau_0(\Phi) = -\limsup_{n\to\infty}(2n+1)^{-1/2}\log(2n/e)^{-n/2}|a_n(\Phi)|$$

and

$$T_0(\psi) = -\limsup_{n\to\infty}(2n+1)^{-1/2}\log(2n/e)^{-n/2}|A_n(\psi)|,$$

then, (3.8) yields that

$$\tau_0(\Phi) = T_0(\psi). \tag{3.9}$$

The first of our results concerning the distribution of the non-trivial zeros of Riemann's ζ-function is the following assertion:

The function $\zeta(s)$ has no zeros in the half-plane $\sigma > \theta, 1/2 \le \theta < 1$ if and only if $T_0(\psi) \ge 1-\theta$ [10, (I)].

If $T_0(\psi) \ge 1-\theta$, then (3.9) yields that $\tau_0(\Phi) \ge 1-\theta$. Hence, the function $\Phi(1+ix), -\infty < x < \infty$, has a holomorphic extension at least in the strip $S(1-\theta)$. This means that the function Φ has no poles in the half-plane $\sigma > \theta$, i.e. the fuction ζ has no zeros in this half-plane.

The assumption that $\zeta(s) \ne 0$ when $\sigma > \theta, 1/2 \le \theta < 1$ implies that $\psi(x) = x + O(x^\theta \log^2 x)$ as $x \to \infty$ [3, Section 18], i.e.

$$\psi(x) = x + O(x^{\theta+\varepsilon}), \quad x \to \infty, \tag{3.10}$$

whatever the positive ε be.

The proof that $T_0(\psi) \geq 1-\theta$ if $\zeta(s) \neq 0$ for $\sigma > \theta$, given in [10], is based on the asymptotic estimate (3.10), Hille's theorem and Cauchy-Hadamard's formula for series in Hermite polynomials. But, there is a more direct proof of this fact which avoids the whole "machinary" of Hermite's series representation of holomorphic functions including Hille's theorem. Indeed, from (3.7) and (3.10) it follows that

$$|A_n(\psi)| = O\left(\int_0^\infty t^n \exp(-t^2/4 - (1-\theta-\varepsilon)t)\, dt\right)$$

$$= O\left(2^{n/2}\int_0^\infty \exp(-t^2/2 - \sqrt{2}(1-\theta-\varepsilon)t)\, dt\right)$$

and the integral representation [12, **8.3**, (3)]

$$D_\nu(z) = \frac{\exp(-z^2/4)}{\Gamma(-\nu)}\int_0^\infty t^{-\nu-1}\exp(-t^2/2 - zt)\, dt, \ \Re\nu < 0,$$

of Weber-Hermite's function $D_\nu(z)$ gives that

$$|A_n(\psi)| = O\left(2^{n/2}\Gamma(n+1)D)_{-n-1}(\sqrt{2}(1-\theta-\varepsilon))\right).$$

Furthermore, Stirling's formula as well as T.M. Cherry's asymptotic formula [1, **8.4**, (5)]

$$D_\nu(z) = \frac{1}{\sqrt{2}}\exp((\nu/2)\log(-\nu) - \nu/2 - (-\nu)^{1/2}z)(1+O(|\nu|^{-1/2})),$$

$$|\arg(-\nu)| \leq \pi/2, \quad |\nu| \to \infty$$

yield that

$$(2n/e)^{-n/2}|A_n(\psi)| = O(\exp(-(2n+1)^{1/2}(1-\theta-\varepsilon))), \quad n \to \infty.$$

Hence, the inequality $T_0(\psi) \geq 1-\theta-\varepsilon$ holds for each positive $\varepsilon < 1-\theta$, i.e. $T_0(\psi) \geq 1-\theta$.

It is clear that $T_0(\psi) \leq 1/2$. Otherwise $\tau_0(\Phi) = T_0(\psi) > 1/2$ and the function $\Phi(1+ix), -\infty < x < \infty$ would have a holomorphic extension at least in the strip $S(\tau_0(\Phi))$ which is impossible. Hence, we may allow us to formulate the following assertion:

Riemann's hypothesis is true if and only if $T_0(\psi) = 1/2$ [10, (II)].

The next assertion is "insipred" by the integral represenation (1.21) of the functins from the space $\mathcal{H}(\tau_0), 0 < \tau_0 \leq \infty$. It sais that:

The function $\zeta(s)$ has no zeros in the half-plane $\sigma > \theta, 1/2 \leq \theta < 1$ if and only if the Fourier transform of the function

$$\exp(-x^2/4)\Phi(1+ix), \quad -\infty < x < \infty, \tag{3.11}$$

is of the form

$$\sqrt{2}\exp(-u^2)E(u) \tag{3.12}$$

with a function $E \in \mathcal{E}(1-\theta)$ [10, (III)].

If $\zeta(s) \neq 0$ when $\sigma > \theta$, then the function $\Phi(1+iz) \in \mathcal{H}(\tau_0(\Phi))$. Hence, the representation

$$\Phi(1+iz) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} E(u)\exp(-(u-iz)^2)\,du$$

holds in the strip $S(\tau_0(\Phi))$ with $E \in \mathcal{E}(\tau_0(\Phi))$. Futhermore, if $z = x \in (-\infty, \infty)$, then (1.21) and the inversion formula for the Fourier transform yield that

$$\sqrt{2}\exp(-u^2)E(u) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} \Phi(1+ix)\exp(iux))\,dx. \tag{3.13}$$

It is quite easy to verify that $\lambda \geq \mu$ implies $\mathcal{E}(\lambda) \subset \mathcal{E}(\mu)$. Then, since $T_0(\psi) \geq 1-\theta$ and $\mathcal{E}(\tau_0(\Phi)) = \mathcal{E}(T_0(\psi))$, the entire function E is in the class $\mathcal{E}(1-\theta)$.

Conversly, let the Fourier transform of the function (3.11) be of the form (3.12) with $E \in \mathcal{E}(1-\theta)$. Then, (3.13) holds and again the inversion formula yields that

$$\Phi(1+ix) = \frac{1}{\sqrt{\pi}}\int_{-\infty}^{\infty} E(u)\exp(-(u+ix)^2)\,du, \quad -\infty < x < \infty.$$

Furthermore, whatever the positive $\varepsilon < 1-\theta$ be, the integral

$$\int_{-\infty}^{\infty} E(u)\exp(-(u+iz)^2)\,du$$

is uniformly converegent on the closed strip $\overline{S}(1-\theta-\varepsilon)$. This means that the function $\Phi(1+ix)$ has a holomorphic extension in the strip $S(1-\theta)$. Hence, the function $\zeta(s)$ has no zeros in the half-plane $\sigma > \theta$.

As a corollary of the last assertion we can formulate the following one:

Riemann's hypothesis is true if and only if the Fourier transform of the function $\exp(-x^2/4)\Phi(1+ix/2), -\infty < x < \infty$ is of the form $\exp(-u^2)E(u)$ with a function $E \in \mathcal{E}(1/2)$ [10].

As a consequence of the integral representation (1.20) it can be obtained a criterion a complex function to have an expansion in a series of the polynomials $\{L_n^{(\alpha)}(z^2)\}_{n=0}^{\infty}$. More preciesely:

An even complex function f, holomorphic in the strip $S(\lambda_0), 0 < \lambda_0 \le \infty$, is in the space $\mathcal{P}^{(\alpha)}(\lambda_0), \alpha > -1$ if and only if the representation

$$z^{\alpha+1/2}\exp(-z^2)f(z/\sqrt{2}) = \int_0^\infty t^{\alpha+1/2}\exp(-t^2)F(t^2/2)(zt)^{1/2}J_\alpha(zt)\,dt \tag{3.14}$$

holds in the half-strip $S^+(\lambda_0) = \{z \in S(\lambda_0) : \Re z > 0\}$ with a function $F \in \mathcal{G}(\lambda_0)$ [13, p.].

Let us suppose that the function $\zeta(s)$ has no zeros in the half-plane $\Re s > \theta, 1/2 \le \theta < 1$. Then, the function $\tilde{\Phi}(s) = \Phi(s) + \Phi(2-s)$ is holomorphic in the strip $\theta < \Re s < 2-\theta$ and is bounded in each closed half-strip $\theta + \varepsilon \le \Re s \le 2-\theta-\varepsilon$ provided $0 < \varepsilon < 1-\theta$. Hence, the even function $\Phi^*(z) = \Phi(1+iz) + \Phi(1-iz)$ is holomorphic in the strip $S(1-\theta)$. Moreover, it is bounded on each closed strip $\overline{S}(1-\theta-\varepsilon)$ with $\varepsilon \in (0, 1-\theta)$. This means that it is in the space $\mathcal{P}^{(\alpha)}(1-\theta)$ for each $\alpha > -1$, i.e. there is a function $F \in \mathcal{G}(1-\theta)$ such that

$$z^{\alpha+1/2}\exp(-z^2)\Phi^*(z/\sqrt{2}) = \int_0^\infty t^{\alpha+1/2}\exp(-t^2/2)F(t^2/2)(zt)^{1/2}J_\alpha(zt)\,dt \tag{3.15}$$

for $z \in S^+(1-\theta)$. Then, the inversion rule for the Hankel transform yields that

$$t^{\alpha+1/2}\exp(-t^2/2)F(t^2/2) = \int_0^\infty x^{\alpha+1/2}\exp(-x^2/2)\Phi^*(x/\sqrt{2})(tx)^{1/2}J_\alpha(tx). \tag{3.16}$$

Well, if $\zeta(s) \ne 0$ for $\Re s > \theta, 1/2 \le \theta < 1$, then the Hankel transform with kernel $w^{1/2}J_\alpha(w), \alpha > -1$, of the function in the left-hand side of (3.15) is the function in the left-hand side of (3.16). The converse is also true. Indeed, if the function F is in the class $\mathcal{G}(1-\theta), 1/2 \le \theta < 1$, then the asymptotic formula [1, **7.13.**,(3)] for the function $J_\alpha(z)$ yields that whatever $\varepsilon \in (0, 1-\theta)$ be, the integral in the right-hand side of (3.15) is uniformly convergent in the strip $S(\sqrt{2}(1-\theta-\varepsilon))$ and defines a holomorphic function in the strip $S(\sqrt{2}(1-\theta))$. This means that the function $\Phi^*(x)$ has a holomorphic extension in the strip $S(1-\theta)$, i.e. the function $\Phi(s)$ is analytically continuable in the half-plane $\Re s > \theta$. Hence, the function $\zeta(s)$ has no zeros in this half-plane. Thus it is proved that:

A nessesary and sufficient condition that $\zeta(s) \neq 0$ in the half-plane $\Re s > \theta, 1/2 \leq \theta < 1$, is the Hankel transform with kernel $w^{1/2}J_\alpha(w)$ of the function (3.15) *to be of the form* (3.16) *with a function $F \in \mathcal{G}(1-\theta)$* [11].

A direct consequence of the last assertion is the following criterion:

Riemann's hypothesis is true if and only if the Hankel transform with kernel $w^{1/2}J_\alpha(w)$ of the function (3.15) *is of the form* (3.16) *with $F \in \mathcal{G}(1/2)$* [11].

The absence of zeros of $\zeta(s)$ in the half-plane $\Re s > \theta, 1/2 \leq \theta < 1$ can be ensured also by the growth of the Fourier-Laguerre coefficients of the function $\Phi(1+i\sqrt{x}), 0leqx < \infty$. Indeed, let define

$$\lambda_0^{(\alpha)}(\Phi) = -\limsup_{n\to\infty}(2\sqrt{n})^{-1}\log|a_n^{(\alpha)}(\Phi)|,$$

where

$$a_n^{(\alpha)}(\Phi) = \int_0^\infty x^\alpha \exp(-x)L_n^{(\alpha)}(x)\Phi(1+i\sqrt{x})\,dx, \alpha > -1, n = 0, 1, 2, \ldots.$$

Then:

Riemann's ζ-function has no zeros in the half-plane $\Re s > \theta, 1/2 \leq \theta < 1$, if and only if $\lambda_0^{(\alpha)}(\Phi) \geq 1-\theta$ [12].

Since $\lambda_0^{(\alpha)}(\Phi) \leq 1/2$ whatever $\alpha > -1$ be, one can formulate the following criterion:

Riemann's hypothesis holds true if and only if $\lambda_0^{(\alpha)}(\Phi) = 1/2$ for some $\alpha > -1$ [12].

References

[1] H. Bateman, A. Erdélyi. Higher transcendental functions II, *Mc Graw-Hill Company*, 1953.

[2] K. Chandrasecharan. Introduction to analytic number theory, *Springer*, 1968.

[3] H. Davenport. Multiplicative number theory, *Markham Publishing Company*, 1953.

[4] E. Hille. Contribution to the theory of Hermitian series: The represetation problem, *Trans. Amer. Math. Soc.* **47** (1940), 80-94.

[5] H. Polard. Representation of analytic functions by a Laguerre series, *Ann. of Math.* (2) **43** (1947), 358-365.

[6] P. Rusev. Classical orthogonal polynomials and their associated functions in complex domain, *Bulgaran Academic Monographs* (10), Marin Drinov Academic Publishing House, Sofia, 2005.

[7] —— On the representation of analytic functions by Laguerre series, *Dokl. Akad. Nauk SSSR*, Tom **240**, No 5 (1978), 1025-1027 (Russian); *Soviet Math. Dokl.* Vol. 19 (1978), No 3, 713-715.

[8] —— Expansion of analytic functions in series of classical orthogonal polynomials, *Complex Analysis, Banach Center Publications*, **11**, Warszawa (1983), 287-298.

[9] —— Holomorphic extenson by means of series in Jacobi, Laguerre and Hermite polynomials, *C. R. Acad. Bulgare Sci.* **51**, 5-6 (1998), 13-16.

[10] —— Hermie polynomials and the zero-distribution of Riemann's ζ-function, *Fract. Calc. Appl. Analysis*, **13**, No 1 (2010), 85-95.

[11] —— Hankel's Transforms and Riemann's Hypothesis, EJPAM, **3**, No 6 (2009), 1113-1117.

[12] —— Laguerre polynomials and the nonreal zeros of Riemann's ζ-function. *C. R. Acad. Bulgare Sci.* **63**, No 11 (2010), 1547-1550.

[13] O. Szasz and Y. Yeardly. The representation of an analytic function by general Laguerre series, *Pacific J. Math.* **8**, 3 (1958), 621-633.

[14] G. Szegö. Orthogonal polynomials, *AMS Colloquium publications* 23, 1939.

Acknowledgement. This paper is partially supported by Project D ID 02/25/2009 "Integral Transforms Methods, Special Functions and Applications", National Science Fund - Ministry of Education, Yuoth and Science, Bulgaria.

Institute of Mathematics and Informatics
Bulgarian Academy of Sciences
Acad.G. Bonchev Str., Block 8
1113 Sofia, Bulgaria
E-mail: pkrusev@math.bas.bg