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Abstract Consider linear systems whose input data are linear functions of uncertain parameters varying within given intervals. We are interested in an explicit description of the so-called AE parametric solution sets (where all universally quantified parameters precede all existentially quantified ones) by a set of inequalities not involving the parameters. This work presents how to obtain explicit description of AE parametric solution sets by combining a modified Fourier-Motzkin-type elimination of existentially quantified parameters with the elimination of the universally quantified parameters. Some necessary (and sufficient) conditions for existence of non-empty AE parametric solution sets are discussed, as well as some properties of the parametric AE solution sets, e.g. shape of the solution set and some inclusion relations. Explicit description of particular classes of AE parametric solution sets (tolerable, controllable, any 2D) are given. Numerical examples illustrate the solution sets and their properties.

Keywords linear systems · dependent data · AE solution set · tolerable solution set · controllable solution set

Mathematics Subject Classification (2000) 65F05 · 65G99

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1 Introduction

Consider the linear algebraic system

$$A(p) \cdot x = b(p), \qquad \text{where } p = (p_1, \ldots, p_m), \tag{1.1}$$

$$a_{ij}(p) := a_{ij,0} + \sum_{\mu=1}^{m} a_{ij,\mu} p_\mu, \qquad b_i(p) := b_{i,0} + \sum_{\nu=1}^{m} b_{i,\mu} p_\mu, \tag{1.2}$$

$$a_{ij,\mu}, b_{i,\mu} \in \mathbb{R}, \qquad \mu = 0, \ldots, m, \; i,j = 1, \ldots, n$$

and the parameters p_μ are considered to be uncertain, varying within given intervals $[p_\mu]$:

$$p \in [p] = ([p_1], \ldots, [p_m])^\top . \tag{1.3}$$

The dependencies between the parameters in (1.2) can be also nonlinear. Such systems are common in many engineering analysis or design problems, control engineering, robust Monte Carlo simulations, etc., where there are complicated dependencies between the model parameters which are uncertain. The set of solutions to (1.1)–(1.3), called united parametric solution set, is

$$\Sigma^p = \Sigma\left(A(p), b(p), [p]\right) := \{x \in \mathbb{R}^n \mid \exists p \in [p], A(p)x = b(p)\} . \tag{1.4}$$

The *(united) parametric* solution sets generalize the (united) non-parametric solution sets to interval linear systems; the elements of the matrix and the r.h.side in the latter are independent intervals. However, the solutions of many practical problems involving uncertain (interval) data have quantified formulation involving the universal logical quantifier ($\forall$) besides the existential quantifier ($\exists$). Examples of several mathematical problems formulated in terms of quantified solution sets can be found in [11] and in the vast literature on quantified constraints satisfaction problems, see e.g. [3] for references to applications in control engineering, electrical engineering, mechanical engineering, biology and various others.

In this work we focus on linear systems involving affine-linear dependencies between interval parameters and the quantified parametric solution sets where all universally quantified parameters precede all existentially quantified ones. Such solution sets are called *AE* parametric solution sets, after Shary [11]. *AE* parametric solution sets generalize both the united parametric solution set and the corresponding non-parametric *AE* solution sets. Our goal is to describe the parametric *AE* solution sets by inequalities not involving the interval parameters. This is a fundamental problem with considerable practical importance. The explicit description of a parametric solution set is useful for visualizing the solution set, for exploring the solution set properties which helps designing better (sharp and fast) numerical methods and for finding exact bounds for the solution which helps in testing new numerical methods.

The description of the parametric solution sets is related to quantifier elimination which stimulated a tremendous amount of research. Since Tarski's general theory [14] is EXPSPACE-hard [2], a lot of research is devoted to special cases with polynomial-time decidability. Apart from quantifier elimination, the only known general way of describing the united parametric solution set is a Fourier-Motzkin-type parameter elimination process proposed in [1] and modified in [8]. The non-parametric AE solution sets are studied by many authors, see [11] and the references given therein. With the exception of [12, 13], which consider some special cases of tolerable solution sets, and [9] considering also a special case, to our knowledge there are no other studies of the parametric AE solution sets.

In this paper (Section 4) we discuss how to obtain explicit description of parametric AE solution sets by a Fourier-Motzkin-type elimination of the existentially quantified parameters (called shortly E-parameters). The methodology for elimination of E-parameters is presented in Section 3. Explicit description of particular classes of parametric AE solution sets (tolerable, controllable, any 2D) are given in Section 5. Based on the explicit description or the properties of the parameter elimination process, in this section we prove several properties of the parametric AE solution sets. Some necessary and necessary and sufficient conditions for a parametric AE solution set to be non-empty are presented. Discussed are also the shape of the parametric AE solution sets and some inclusion relations. For simplicity of the notations we consider square systems. However, all the assertions in the paper are valid for rectangular systems. Numerical examples illustrate the parametric AE solution sets and their properties.

2 Notations

Denote by $\mathbb{R}^n, \mathbb{R}^{n\times m}$ the set of real vectors with n components and the set of real $n\times m$ matrices, respectively. A real compact interval is $[a] = [a^-, a^+] := \{a \in \mathbb{R} \mid a^- \leq a \leq a^+\}$. By $\mathbb{IR}^n, \mathbb{IR}^{n\times m}$ we denote the sets of interval n-vectors and interval $n\times m$ matrices, respectively. For $[a] = [a^-, a^+]$, define mid-point $\check{a} := (a^- + a^+)/2$ and radius $\hat{a} := (a^+ - a^-)/2$. These functionals are applied to interval vectors and matrices componentwise. For a given index set $\Pi = \{\pi_1, \ldots, \pi_k\}$, $p_\Pi = (p_{\pi_1}, \ldots, p_{\pi_k})$. $\bigwedge$ and $\wedge$ denote the logical "And". For a parametric matrix $A(p)$, resp. vector $b(p)$, depending on a number of parameters (1.3), $A([p]), b([p])$ denote the corresponding non-parametric matrix, resp. vector

$$a_{ij}([p]) := a_{ij,0} + \sum_{\mu=1}^{m} a_{ij,\mu}[p_\mu], \qquad b_i([p]) := b_{i,0} + \sum_{\nu=1}^{m} b_{i,\mu}[p_\mu].$$

Exactly one non-parametric system $A([p])x = b([p])$ corresponds to a parametric system $A(p)x = b(p)$. However, there are infinitely many parametric systems that correspond to a non-parametric system $[A]x = [b]$.

With the notations $A_{\bullet\bullet\mu} := (a_{ij,\mu}) \in \mathbb{R}^{n\times n}$, $b_{\bullet\mu} := (b_{i,\mu}) \in \mathbb{R}^n$, $\mu = 0, \ldots, m$, the system (1.1) can be rewritten equivalently as

$$\left(A_{\bullet\bullet 0} + \sum_{\mu=1}^{m} p_\mu A_{\bullet\bullet\mu} \right) x = b_{\bullet 0} + \sum_{\mu=1}^{m} p_\mu b_{\bullet\mu}.$$

For a matrix $A \in \mathbb{R}^{n\times n}$, $A_{m\bullet}$ denotes the m-th row of A.

Definition 2.1 A parameter p_μ, $1 \leq \mu \leq m$, is of 1st class if it occurs in only one equation of the system (1.1).

It does not matter how many times a 1st class parameter appears within an equation. A parameter p_μ is of 1st class iff the vector $b_{\bullet\mu} - A_{\bullet\bullet\mu}x$ has only one nonzero component (that is $b_{i\mu} - A_{i\bullet\mu}x \neq 0$ for exactly one i, $1 \leq i \leq n$).

Definition 2.2 A parameter p_μ, $1 \leq \mu \leq m$, is of 2nd class if it is involved in more than one equation of the system (1.1).

A parameter p_μ is of 2nd class iff the vector $b_{\bullet\mu} - A_{\bullet\bullet\mu}x$ has more than one nonzero components.

Definition 2.3 A parametric matrix is called *row-dependent*[1] if for some $\mu \in \{1,\ldots,m\}$ and some $i \in \{1,\ldots,n\}$, $\mathrm{Card}(\mathscr{J}) \geq 2$, where $\mathscr{J} := \{j \mid 1 \leq j \leq n, a_{ij,\mu} \neq 0\}$. A parametric matrix is called *row-independent* if for all $\mu \in \{1,\ldots,m\}$ and all $i \in \{1,\ldots,n\}$, $\mathrm{Card}(\mathscr{J}) < 2$.

A row-dependent parametric matrix is denoted by $A_{rd}(p)$ and a row-independent one by $A_{ri}(p)$. Examples of row-independent parametric matrices are the symmetric, skew-symmetric, Hankel, Toeplitz, Hurwitz matrices, as well as the non-parametric matrices.

Definition 2.4 For two parameter vectors $u \in [u] \in \mathbb{IR}^{m_1}$, $v \in [v] \in \mathbb{IR}^{m_2}$, such that $A([u]) = A([v]) = [A]$ and $b([u]) = b([v]) = [b]$, the system $A(u)x = b(u)$ involves more parameter dependencies than the system $A(v)x = b(v)$ if:

a) the system depending on u involves more row-dependencies than the system depending on v, or
b) it involves some 2nd class parameter u_μ having more non-zero components in the coefficient vector $A_{\bullet\bullet\mu}x - b_{\bullet\mu}$ than the corresponding parameter in the system depending on v.

3 Fourier-Motzkin Type Elimination of E-Parameters

The united parametric solution set (1.4) is characterized as follows by a trivial set of inequalities

$$\Sigma^p_{uni} = \{x \in \mathbb{R}^n \mid \exists p_\mu \in \mathbb{R}, \mu = 1,\ldots,m : \text{(3.1)–(3.2) hold}\}, \quad \text{where}$$

$$A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu=1}^{m} \left(A_{\bullet\bullet\mu}x - b_{\bullet\mu}\right) p_\mu \leq 0 \leq A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu=1}^{m} \left(A_{\bullet\bullet\mu}x - b_{\bullet\mu}\right) p_\mu \qquad (3.1)$$

$$p^-_\mu \leq p_\mu \leq p^+_\mu, \qquad \mu = 1,\ldots,m. \qquad (3.2)$$

Starting from a trivial description of Σ^p_{uni}, the following theorem shows how the existentially quantified parameters in this set of inequalities can be eliminated successively in order to obtain a new description not involving p_μ, $\mu = 1,\ldots,m$.

[1] by analogy with the column-dependent parametric matrices defined in [6].

Theorem 3.1 ([8]) *Let $g_\lambda(x), f_{\lambda\nu,1}(x), f_{\lambda\nu,2}(x), f_{\lambda\mu}(x)$, $\lambda = 1,\ldots,k(\geq n)$ be real-valued functions of $x = (x_1,\ldots,x_n)^\top$ on some subset $D \subseteq \mathbb{R}^n$. Assume that there exists a non-empty set $\mathcal{T} \subseteq \{1,\ldots,k\}$ such that $f_{\lambda m_1}(x) \not\equiv 0$ for all $\lambda \in \mathcal{T}$. For $m_1 \geq 1$ and the parameters p_μ, $\mu = m_1,\ldots,m$ varying in $\mathbb{R}$ and for x varying in D define the sets S_1, S_2 by*

$$S_1 := \{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = m_1,\ldots,m : (3.3), (3.4)\ hold\},$$
$$S_2 := \{x \in D \mid \exists p_\mu \in \mathbb{R}, \mu = m_1+1,\ldots,m : (3.5), (3.6), (3.7)\ hold\},$$

where inequalities (3.3), (3.4) and (3.5), (3.6), (3.7), respectively, are given by

$$g_\lambda(x) + \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,1}(x)\dot{p}_\nu \mp \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu + \sum_{\mu=m_1+1}^{m} f_{\lambda\mu}(x)p_\mu \leq -f_{\lambda m_1}(x)p_{m_1} \leq \cdots \qquad \lambda = 1,\ldots,k \tag{3.3}$$

$$\dot{p}_\mu - \hat{p}_\mu \leq p_\mu \leq \dot{p}_\mu + \hat{p}_\mu, \quad \mu = m_1,\ldots,m, \tag{3.4}$$

$$g_\lambda(x) + \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,1}(x)\dot{p}_\nu \mp \sum_{\nu=1}^{m_1-1} f_{\lambda\nu,2}(x)\hat{p}_\nu + f_{\lambda m_1}(x)\dot{p}_{m_1} \mp |f_{\lambda m_1}(x)|\hat{p}_{m_1} + \sum_{\mu=m_1+1}^{m} f_{\lambda\mu}(x)p_\mu \leq 0 \leq \cdots, \qquad \lambda = 1,\ldots,k \tag{3.5}$$

and for $\alpha, \beta \in \mathcal{T}$, $\alpha < \beta$

$$g_\alpha(x)f_{\beta m_1}(x) - g_\beta(x)f_{\alpha m_1}(x) + \sum_{\nu=1}^{m_1-1} \left(f_{\beta m_1}(x)f_{\alpha\nu,1}(x) - f_{\alpha m_1}(x)f_{\beta\nu,1}(x)\right)\dot{p}_\nu \mp \sum_{\nu=1}^{m_1-1} \left(|f_{\beta m_1}(x)|f_{\alpha\nu,2}(x) + |f_{\alpha m_1}(x)|f_{\beta\nu,2}(x)\right)\hat{p}_\nu + \sum_{\mu=m_1+1}^{m} \left(f_{\alpha\mu}(x)f_{\beta m_1}(x) - f_{\beta\mu}(x)f_{\alpha m_1}(x)\right)p_\mu \leq 0 \leq \cdots, \tag{3.6}$$

$$\dot{p}_\mu - \hat{p}_\mu \leq p_\mu \leq \dot{p}_\mu + \hat{p}_\mu, \quad \mu = m_1+1,\ldots,m. \tag{3.7}$$

The "$\cdots$" in the right side inequalities denotes the left side expression in the left inequality with the bottom sign in front of the terms involving a parameter radius. (Trivial inequalities which are true for any $x \in \mathbb{R}^n$ can be omitted.) Then $S_1 = S_2$.

The inequalities (3.5) are called *end-point inequalities* because they are obtained by combining (3.3) with (3.4). The inequalities (3.6) are called *cross inequality pairs* because they are obtained by combining two inequality pairs (3.3). Note that the resulting inequalities (3.5) and (3.6) have the form (3.3) which allows the elimination process to continue with the next parameters.

The parameter elimination process resembles the so-called Fourier-Motzkin elimination of variables, see e.g. [10]. It was first proposed in [1] in a form based on the parameter inequalities (3.2) which leads to a tremendous number of solution set characterizing inequalities. In order to reduce the number of characterizing inequalities, the modified parameter elimination in Theorem 3.1 is based on the equivalent parameter inequalities (3.4) in mid-point/radius representation. Thus, in the parameter elimination process we apply the following relation

$$\lambda \dot{p}_\mu - |\lambda| \hat{p}_\mu \le \lambda p_\mu \le \lambda \dot{p}_\mu + |\lambda| \hat{p}_\mu, \qquad \text{for } \lambda \in \mathbb{R},$$

without the necessity to consider the particular sign of λ. Therefore, the modified parameter elimination does not depend on a particular orthant. Furthermore, Theorem 3.1 gives a compact representation of the characterizing inequalities which will be illustrated bellow.

Consider the parametric system (1.1)–(1.3), the united parametric solution set of which is described by the trivial set of characterizing inequalities (3.1) and for $\mu = 1,\ldots,m$, (3.4). Let for $M_1 \cup M_2 = \{1,\ldots,m\}$, $M_1 \cap M_2 = \emptyset$, p_μ, $\mu \in M_1$ be 1st class E-parameters and p_μ, $\mu \in M_2$ be 2nd class E-parameters. By Theorem 3.1, the elimination of all p_μ, $\mu \in M_1$ updates the inequality pairs (3.1) so that they become

$$A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu \in M_1} (A_{\bullet\bullet\mu}x - b_{\bullet\mu})\, \dot{p}_\mu \mp \sum_{\mu \in M_1} |A_{\bullet\bullet\mu}x - b_{\bullet\mu}|\, \hat{p}_\mu + \sum_{\mu \in M_2} (A_{\bullet\bullet\mu}x - b_{\bullet\mu})\, p_\mu \le 0 \le \cdots. \quad (3.8)$$

The end-point inequality pairs (3.8) are equivalent to single absolute-value inequalities (3.9) and vice-versa

$$\left| A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu \in M_1} (A_{\bullet\bullet\mu}x - b_{\bullet\mu})\, \dot{p}_\mu + \sum_{\mu \in M_2} (A_{\bullet\bullet\mu}x - b_{\bullet\mu})\, p_\mu \right| \le \sum_{\mu \in M_1} |A_{\bullet\bullet\mu}x - b_{\bullet\mu}|\, \hat{p}_\mu. \quad (3.9)$$

Let for p_{ν_1}, $\nu_1 \in M_2$, $\mathcal{T}_{\nu_1} \subseteq \{1,\ldots,n\}$, $\mathrm{Card}(\mathcal{T}_{\nu_1}) = k$, be the index set of the inequalities (3.8), resp. (3.9), involving p_{ν_1}. By Theorem 3.1, the elimination of p_{ν_1} updates the end-point inequalities (3.8), resp. (3.9), which become

$$\left| A_{\bullet\bullet 0}x - b_{\bullet 0} + \sum_{\mu \in M_1 \cup \{\nu_1\}} (A_{\bullet\bullet\mu}x - b_{\bullet\mu})\, \dot{p}_\mu + \sum_{\mu \in M_2 \setminus \{\nu_1\}} (A_{\bullet\bullet\mu}x - b_{\bullet\mu})\, p_\mu \right| \le \sum_{\mu \in M_1 \cup \{\nu_1\}} |A_{\bullet\bullet\mu}x - b_{\bullet\mu}|\, \hat{p}_\mu \quad (3.10)$$

and for $\alpha,\beta \in \mathscr{T}_{\nu_1}$ generate $k(k-1)/2$ cross inequality pairs

$$\Delta_{0,\nu_1}(\alpha,\beta,x) + \sum_{\mu\in M_1} \Delta_{\mu,\nu_1}(\alpha,\beta,x)\dot{p}_\mu \mp \sum_{\mu\in M_1} \left(|f_{\nu_1}(\beta,x)|\,|f_\mu(\alpha,x)| + |f_{\nu_1}(\alpha,x)|\,|f_\mu(\beta,x)|\right)\hat{p}_\mu + \sum_{\mu\in M_2\setminus\{\nu_1\}} \Delta_{\mu,\nu_1}(\alpha,\beta,x)p_\mu \leq 0 \leq \cdots, \quad (3.11)$$

wherein $f_\mu(\alpha,x) := (A_{\alpha\bullet\mu}x - b_{\alpha\mu})$, similarly for $f_{\nu_1}(\alpha,x)$, $f_\mu(\beta,x)$, $f_{\nu_1}(\beta,x)$, and $\Delta_{\mu,\nu_1}(\alpha,\beta,x) := f_{\nu_1}(\beta,x)f_\mu(\alpha,x) - f_{\nu_1}(\alpha,x)f_\mu(\beta,x)$ for $\mu = \{0\}\cup M_1$ or $\mu = M_2 \setminus \{\nu_1\}$. The cross inequality pairs (3.11) also can be written as equivalent single absolute-value inequalities.

The elimination of the next 2nd class E-parameters updates similarly the the end-point inequalities (3.10) and introduces more cross inequalities. The cross inequalities can be more complicated than the inequalities (3.11). However the solution set characterizing inequalities (both end-point and cross inequalities), obtained by the Fourier-Motzkin type elimination of E-parameters, have the same general form which can be presented as follows.

For $\lambda \in \mathscr{T} := \{1,\ldots,n\}\cup\mathscr{T}_c$, where $\{1,\ldots,n\}$ is the index set of the end-point characterizing inequalities and $\mathscr{T}_c$ is the index set of the characterizing cross inequalities, the set of all solution set characterizing inequalities obtained by the Fourier-Motzkin type elimination of E-parameters is

$$\bigwedge_{\lambda\in\mathscr{T}} u_{\lambda,0}(x) + \sum_{\mu\in M_1} u_{\lambda,\mu}(x)\dot{p}_\mu - \sum_{\mu\in M_1} v_{\lambda,\mu}(x)\hat{p}_\mu \leq \sum_{\mu\in M_2} w_{\lambda,\mu}(x)p_\mu \leq u_{\lambda,0}(x) + \sum_{\mu\in M_1} u_{\lambda,\mu}(x)\dot{p}_\mu + \sum_{\mu\in M_1} v_{\lambda,\mu}(x)\hat{p}_\mu, \quad (3.12)$$

wherein M_1 is the index set of eliminated E-parameters, $u_{\lambda,0}(x)$, $u_{\lambda,\mu}(x)$, $v_{\lambda,\mu}(x)$, $w_{\lambda,\mu}(x)$ are corresponding real-valued functions of $x = (x_1,\ldots,x_n)^\top$, and M_2 is the index set of non-eliminated parameters. A more general representation of the inequality pairs (3.12) is

$$\bigwedge_{\lambda\in\mathscr{T}} u_\lambda(x,M_1) \leq \textstyle\sum_{\mu\in M_2} w_{\lambda,\mu}(x)p_\mu \leq v_\lambda(x,M_1). \quad (3.13)$$

4 Description of Parametric *AE* Solution Sets

Definition 4.1 Quantified solution sets to a parametric linear system $A(p)x = b(p)$, involving either affine-linear or nonlinear dependencies between the parameters $p = (p_1,\ldots,p_m)$, are sets of the form

$$\{x\in\mathbb{R}^n \mid (Q_1p_1\in[p_1])\ldots(Q_mp_m\in[p_m])(A(p)x = b(p))\},$$

where $Q_i \in \{\forall,\exists\}$, $i = 1,\ldots,m$.

The total number of quantified parametric solution sets exceeds 2^m since the existential and the universal quantifiers do not commute. In this work we consider only linear systems involving affine-linear dependencies between the uncertain parameters and quantified solutions sets of such systems where all occurrences of the universal quantifier precede all occurrences of the existential quantifier. After the terminology used in [11], we call these solution sets AE parametric solution sets. Thus, a parametric AE solution set of the system (1.1)–(1.3) is defined as

$$\Sigma_{AE}^{p} := \{x \in \mathbb{R}^n \mid (\forall p_{\mathscr{A}} \in [p_{\mathscr{A}}])(\exists p_{\mathscr{E}} \in [p_{\mathscr{E}}])(A(p)x = b(p))\}, \tag{4.1}$$

where $\mathscr{A}$ and $\mathscr{E}$ are the index sets $\mathscr{A} := \{t \mid \forall p_t \in [p_t]\}$, $\mathscr{E} := \{t \mid \exists p_t \in [p_t]\}$, such that $\mathscr{A} \cup \mathscr{E} = \{1, \ldots, m\}$, $\mathscr{A} \cap \mathscr{E} = \emptyset$. There are exactly 2^m parametric AE solution sets.

Theorem 4.1 *For given index sets $\mathscr{A} := \{t \mid \forall p_t \in [p_t]\}$ and $\mathscr{E} := \{t \mid \exists p_t \in [p_t]\}$, the parametric AE solution set (4.1) of the system (1.1)–(1.3) is described by the set of inequality pairs*

$$\bigwedge_{\lambda \in \mathscr{T}} \Bigg(u_\lambda(x, \mathscr{E}) - \sum_{\mu \in \mathscr{A}} \left(w_{\lambda,\mu}(x)\check{p}_\mu - |w_{\lambda,\mu}(x)|\hat{p}_\mu\right) \leq 0 \leq$$
$$v_\lambda(x, \mathscr{E}) - \sum_{\mu \in \mathscr{A}} \left(w_{\lambda,\mu}(x)\check{p}_\mu + |w_{\lambda,\mu}(x)|\hat{p}_\mu\right) \Bigg), \tag{4.2}$$

where

$$S_{\mathscr{E}} := \bigwedge_{\lambda \in \mathscr{T}} u_\lambda(x, \mathscr{E}) \leq \sum_{\mu \in \mathscr{A}} w_{\lambda,\mu}(x) p_\mu \leq v_\lambda(x, \mathscr{E})$$

is the set of inequality pairs obtained by Fourier-Motzkin type elimination of all E-parameters, $\mathscr{T} = \{1, \ldots, t\}$, $t \geq n$.

Proof

$$\begin{aligned}
\Sigma_{AE}^{p} &:= \{x \in \mathbb{R}^n \mid (\forall p_{\mathscr{A}} \in [p_{\mathscr{A}}])(\exists p_{\mathscr{E}} \in [p_{\mathscr{E}}])(A(p)x = b(p))\} \\
&= \left\{ x \in \mathbb{R}^n \mid (\forall p_{\mathscr{A}} \in [p_{\mathscr{A}}])\Big(\bigwedge_{\lambda \in \mathscr{T}} u_\lambda(x, \mathscr{E}) \leq \sum_{\mu \in \mathscr{A}} w_{\lambda,\mu}(x) p_\mu \leq v_\lambda(x, \mathscr{E})\Big) \right\} \\
&= \{x \in \mathbb{R}^n \mid (4.2)\}.
\end{aligned}$$

The first equality above follows from the Fourier-Motzkin type elimination of all E-parameters. The second equality follows from the distributivity of the universal quantifiers over conjunction, the parameter inequality pairs for the A-parameters and the relation

$$\forall p \in [p] : b_1 \leq f(p) \leq b_2 \Leftrightarrow b_1 \leq \min_{p \in [p]} f(p) \wedge \max_{p \in [p]} f(p) \leq b_2. \tag{4.3}$$

Corollary 4.1 *The elimination of the universally quantified parameters does not introduce new characterizing inequalities to the description of Σ_{AE}^{p} obtained by elimination of all existentially quantified parameters.*

Theorem 4.1 allows us to estimate also the shape of the parametric AE solution sets, i.e., the maximal degree of the polynomial equations describing the solution set boundary.

Corollary 4.2 *The elimination of the universally quantified parameters does not influence the shape of Σ^p_{AE} obtained by elimination of all existentially quantified parameters.*

The application of Theorem 4.1 will be illustrated in the next section where we consider some classes of parametric AE solution sets, give their explicit description and derive some of their properties.

5 Properties of the Parametric AE Solution Sets

We start this section by some general assertions which were proven in [9] not basing on the description of the parametric AE solution sets. The following theorem gives a set-theoretical description of AE parametric solution sets (4.1) and generalizes a corresponding theorem, c.f. [11, Theorem 3.1], for nonparametric AE solution sets.

Theorem 5.1 ([9])

$$\Sigma^p_{AE} = \bigcap_{p_{\mathscr{A}} \in [p_{\mathscr{A}}]} \bigcup_{p_{\mathscr{E}} \in [p_{\mathscr{E}}]} \{x \in \mathbb{R}^n \mid A(p_{\mathscr{A}}, p_{\mathscr{E}}) \cdot x = b(p_{\mathscr{A}}, p_{\mathscr{E}})\}.$$

Next theorem gives some analytic necessary conditions for a general AE parametric solution set to be nonempty.

Theorem 5.2 ([9]) *If a parametric AE solution set (4.1) is nonempty, then*

$$\sum_{\nu \in \mathscr{A}} (A_{\bullet\bullet\nu}x - b_{\bullet\nu})[p_\nu] \subseteq b_{\bullet 0} - A_{\bullet\bullet 0}x + \sum_{\mu \in \mathscr{E}} (b_{\bullet\mu} - A_{\bullet\bullet\mu}x)[p_\mu]. \tag{5.1}$$

The interval inclusion (5.1) is equivalent to the inequality

$$|A(\dot{p})x - b(\dot{p})| \le \sum_{\mu=1}^{m} \delta_\mu |A_{\bullet\bullet\mu}x - b_{\bullet\mu}|\hat{p}_\mu, \tag{5.2}$$

where $\delta_\mu := \{1 \text{ if } \mu \in \mathscr{E}, -1 \text{ if } \mu \in \mathscr{A}\}$.

The inequality (5.2) presents the end-point inequalities in the explicit characterization of a parametric AE solution set. The following theorem and corollary follow from Theorem 4.1 and a property proven in [8] that the elimination of 1st class E-parameters does not generate any cross inequalities.

Theorem 5.3 *A parametric AE solution set of the linear system (1.1)–(1.3) is nonempty iff the solution set describing inequalities (4.2), defined in Theorem 4.1, hold true.*

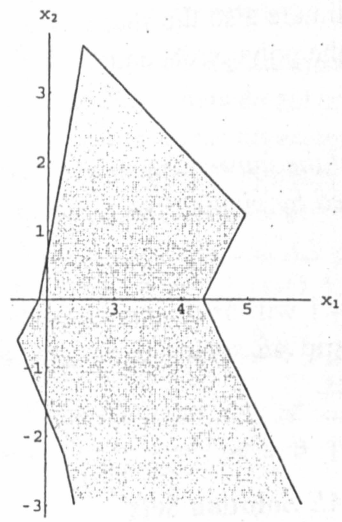

Fig. 5.1 The parametric AE solution set for the system from Example 5.1

Corollary 5.1 *Let the definition of a parametric AE solution set to the linear system (1.1)–(1.3) involve only 1st class existentially quantified parameters. Such parametric AE solution set is non-empty if and only if the inequality (5.2) holds true.*

In lots of situations it is not necessary to check the solution set describing inequalities and we can immediately say that the AE solution set is empty. For example, if there is an equation which does not involve any E-parameters, all parametric AE solution sets will be empty because there will be only negative terms in the right side of the corresponding end-point absolute-value inequality. If a parametric system involves only one 2nd class E-parameter which occurs in all equations of the system, all parametric AE solution sets will be also empty because the cross inequalities[2] describing the solution set will not contain this parameter. Therefore there will be only negative terms in the right sides of the corresponding absolute-value cross inequalities.

The non-empty parametric AE solution sets from Corollary 5.1 have linear shape but they are not convex in the general case.

Example 5.1 Consider the parametric linear system $A(p)x = b(q)$, where

$$A(p) = \begin{pmatrix} 2p_1 & p_{12} - p_1 \\ 2.5p_{21} + p_2 & p_2 \end{pmatrix}, \quad b(q) = \begin{pmatrix} 2q \\ 2q \end{pmatrix},$$

$$p_1 \in [\frac{1}{2}, \frac{3}{2}],\ p_2 \in [\frac{7}{10}, \frac{17}{10}],\ p_{12}, p_{21} \in [0,1],\ q \in [\frac{13}{6}, \frac{17}{6}].$$

The solution set $\Sigma_{\forall q \exists p_1, p_2, p_{12}, p_{21}}$ is presented on Fig. 5.1. Its boundary is linear but neither the whole solution set nor its intersection with the fourth orthant is convex. Furthermore, the solution set is unbounded in the fourth orthant.

It is well known that a parametric united solution set is a subset of its corresponding non-parametric solution set, but we have never seen a formal proof of this fact.

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[2] with respect to this parameter

Bellow, for the sake of completeness, we give the proof of a more general inclusion relation.

Theorem 5.4 *For two parameter vectors $u \in [u] \in \mathbb{IR}^{m_1}$, $v \in [v] \in \mathbb{IR}^{m_2}$, such that $A([u]) = A([v]) = [A]$ and $b([u]) = b([v]) = [b]$, if the system $A(u)x = b(u)$ involves more parameter dependencies than the system $A(v)x = b(v)$ then*

$$\Sigma_{uni}(A(u), b(u), [u]) \subseteq \Sigma_{uni}(A(v), b(v), [v]) \subseteq \cdots \subseteq \Sigma_{uni}([A], [b]).$$

Proof In case of Definition 2.4 a) the inclusion relation follows from the inequality

$$|A_{i\bullet\mu}x - b_{i\mu}| \leq \left| \sum_{j=1, j\neq k} a_{ij\mu}x_j - b_{i\mu} \right| + |a_{ik\mu}x_k|$$

applied to the right-sides of the corresponding absolute-value end-point inequalities involving the parameter u_μ having more row-dependencies.

In case of Definition 2.4 b), the elimination of the parameter having more non-zero components in the coefficient vector will generate additional characterizing cross inequalities which may additionally restrict the solution set.

5.1 Parametric Tolerable Solution Sets

Denote by $\Sigma_{tol}(A_{ri}(p), [p], [b])$ the tolerable solution set of a system involving a row-independent parametric matrix and a right-hand side vector with independent interval components. Since $A([p])$ is the interval hull of $A_{ri}(p)$ and in view of the Definition 2.3, by Theorem 4.1 the two tolerable solution sets $\Sigma_{tol}(A([p]), [b])$ and $\Sigma_{tol}(A_{ri}(p), [p], [b])$ have the same explicit representation

$$\Sigma_{tol}(A([p]), [b]) = \Sigma_{tol}(A_{ri}(p), [p], [b]) = \{x \in \mathbb{R}^n \mid |\check{A}x - \check{b}| \leq \hat{b} - \hat{A}|x|\}.$$

For $p = (p_1, \ldots, p_{m_1})$ and $q = (q_1, \ldots, q_{m_2})$, the general parametric tolerable solution set is defined by

$$\Sigma_{tol}(A(p), b(q), [p], [q]) := \{x \in \mathbb{R}^n \mid \forall p \in [p], \exists q \in [q], A(p)x = b(q)\}.$$

Proposition 5.1 *If $q_1, \ldots, q_{m_2}$ are 1st class parameters, then*

$$\Sigma_{tol}(A(p), b(q), [p], [q]) = \left\{ x \in \mathbb{R}^n \mid |\check{A}x - \check{b}| \leq \sum_{\mu=1}^{m_2} \hat{q}_\mu |b_{\bullet\mu}| - \sum_{\mu=1}^{m_1} \hat{p}_\mu |A_{\bullet\bullet\mu}x| \right\},$$

where $\sum_{\mu=1}^{m_2} \hat{q}_\mu |b_{\bullet\mu}| = rad(b([q]))$.

If the parametric tolerable solution set involves 2nd class E-parameters, then its description contains cross inequalities with respect to these parameters. However, since all 2nd class E-parameters (if any) are involved in the right-hand side of the system, the cross inequalities with respect to these parameters will be linear, which proves the following theorem.

Theorem 5.5 *The parametric tolerable solution sets have linear shape.*

Next we prove some inclusion relations between different tolerable solution sets.

Theorem 5.6 *Let $A_{ri}(u), A_{rd}(v) \in \mathbb{R}^{n\times n}$ and $[A] \in \mathbb{IR}^{n\times n}$ be such that for given parameter vectors $u \in [u] \in \mathbb{IR}^{m_1}$, $v \in [v] \in \mathbb{IR}^{m_2}$,*

$$A_{ri}([u]) = A_{rd}([v]) \subseteq [A].$$

If the parameters $q \in [q] \in \mathbb{IR}^{m_3}$ be of 1st class, then

$$\Sigma_{tol}([A], b([q])) \subseteq \Sigma_{tol}(A([u]), b([q])) = \Sigma_{tol}(A_{ri}(u), [u], b([q])) \subseteq \Sigma_{tol}(A_{rd}(v), [v], b([q])). \quad (5.3)$$

If $A(u)$, $A(v)$ be such that $A(v)$ involves more dependencies than $A(u)$ and $A([u]) = A([v])$, then for an arbitrary $q \in [q] \in \mathbb{IR}^{m_3}$ which may involve 2nd class E-parameters

$$\Sigma_{tol}(A(u), b(q), [u], [q]) \subseteq \Sigma_{tol}(A(v), b(q), [v], [q]). \quad (5.4)$$

Proof The equality relation in (5.3) follows from the equivalent explicit description of the two solution sets.
We prove $\Sigma_{tol}(A_{ri}(u), [u], b([q])) \subseteq \Sigma_{tol}(A_{rd}(v), [v], b([q]))$. Let for fixed $1 \leq i \leq n$ there exists a parameter p_λ, $\lambda \in \{1, \ldots, m\}$, such that $A(p)$ is row-dependent. Then the i-th characterizing inequality of $\Sigma_{tol}(A_{rd}(v), [v], b([q]))$ is

$$\left|A(\check{p})x - \check{b}\right|_i \leq \mathrm{rad}(b_i([q])) - \sum_{\mu=1}^{m} \left|\{A_{\bullet\bullet\mu}\}_i x\right| \hat{p}_\mu.$$

Since the i-th characterizing inequality of $\Sigma_{tol}(A_{ri}(u), [u], b([q]))$ is

$$\left|A(\check{p})x - \check{b}\right|_i \leq \mathrm{rad}(b_i([q])) - \sum_{\mu=1, \mu\neq\lambda}^{m} \left|\{A_{\bullet\bullet\mu}\}_i x\right| \hat{p}_\mu - (|\{A_{\bullet\bullet\lambda}\}_i|\,|x|)\hat{p}_\lambda,$$

the inclusion follows from $|\{A_{\bullet\bullet\lambda}\}_i x| \leq |\{A_{\bullet\bullet\lambda}\}_i|\,|x|$.
The inclusion relation in (5.4) follows by similar considerations for the characterizing cross inequalities for the 2nd class E-parameters.
We prove $\Sigma_{tol}([A], b([q])) \subseteq \Sigma_{tol}(A([u]), b([q]))$. If $[a_{ij}] \supseteq a_{ij}([u])$, there exist at least one interval $[t] \neq [0,0]$ such that $[a_{ij}] = a_{ij}([u]) + [t]$ and $\hat{a}_{ij} = \hat{a}_{ij}([u]) + \hat{t}$. Then the inclusion follows from $-\hat{a}_{ij} \leq -\hat{a}_{ij}([u])$.

Example 5.2 Consider the non-parametric interval linear system $[A]x = [b]$, where

$$[A] = \begin{pmatrix} [0,1] & [\frac{1}{2}, \frac{3}{2}] \\ [-2,0] & [1,2] \end{pmatrix}, \quad [b] = \begin{pmatrix} [-1,2] \\ [-3,3] \end{pmatrix}.$$

The non-parametric interval matrix $[A]$ presents an interval hull of the following parametric matrices (and of infinitely many other parametric matrices)

$$A_1 = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & 1+a_{11} \end{pmatrix}, \quad A_2 = \begin{pmatrix} a_{11} & a+\frac{1}{2} \\ -2a & 1+a_{11} \end{pmatrix}, \quad A_3 = \begin{pmatrix} a & a+\frac{1}{2} \\ -2a & 1+a \end{pmatrix},$$

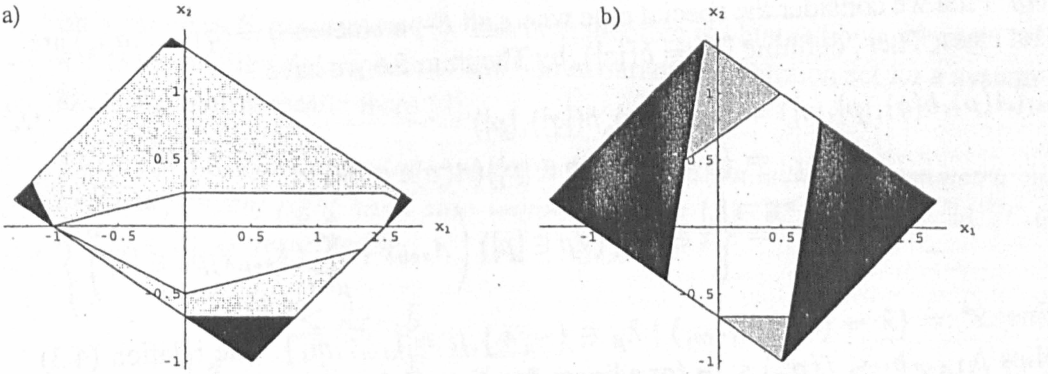

Fig. 5.2 Inclusion relations between the parametric tolerable solution sets from Example 5.2: a) the inclusions (5.5), b) the inclusions (5.6)

where $a_{11}, a \in [0,1]$, $a_{1,2} \in [\frac{1}{2}, \frac{3}{2}]$, $a_{21} \in [-2,0]$. Since both A_1 and A_2 are row-independent matrices with the same interval hull, the parametric tolerable solution sets $\Sigma_{tol}(A_1,[b])$ and $\Sigma_{tol}(A_2,[b])$ have the same explicit description which is equivalent to the description of the corresponding non-parametric tolerable solution set $\Sigma_{tol}([A],[b])$. The parametric matrix A_3 is row-dependent and has the same interval hull as the matrices A_1, A_2. Therefore, by Theorem 5.6 relation (5.3),

$$\Sigma_{tol}([A],[b]) = \Sigma_{tol}(A_1,[b]) = \Sigma_{tol}(A_2,[b]) \subseteq \Sigma_{tol}(A_3,[b]).$$

If we consider a system with matrix $[B] = \begin{pmatrix} [0,1] & [-4,1] \\ [-2,0] & [1,2] \end{pmatrix}$ which encloses the matrix $[A]$, we obtain the inclusions

$$\Sigma_{tol}([B],[b]) \subseteq \Sigma_{tol}([A],[b]) = \Sigma_{tol}(A_1,[b]) = \Sigma_{tol}(A_2,[b]) \subseteq \Sigma_{tol}(A_3,[b]). \quad (5.5)$$

The last inclusion chain is presented on Fig. 5.2 a), where $\Sigma_{tol}([B],[b])$ is the most inner white polyhedron, $\Sigma_{tol}([A],[b])$ is the polyhedron in light gray and $\Sigma_{tol}(A_3,[b])$ is the parallelogram with black corners.

Now, consider parametric systems involving the same matrices $[A], A_1, A_2, A_3$ and a right-hand side vector depending on a 2nd class parameter, that is $b(q) = (q_1, q_1 - q_2)^\top$, where $q_1, q_2 \in [-1,2]$ and $b([q]) = [b]$. For the tolerable solution sets of these systems we have the following inclusion relations

$$\Sigma_{tol}(V,b(q)) \overset{(5.4)}{\subseteq} \Sigma_{tol}(A_3,b(q)) \overset{Thm5.4}{\subseteq} \Sigma_{tol}(A_3,[b]) \quad (5.6)$$

$$\Sigma_{tol}(V,b(q)) \overset{Thm5.4}{\subseteq} \Sigma_{tol}(V,[b]) \overset{(5.3)}{\subseteq} \Sigma_{tol}(A_3,[b]),$$

wherein $V \in \{[A], A_1, A_2\}$. On Fig. 5.2 b) $\Sigma_{tol}(A_3,[b])$ is the black parallelogram, $\Sigma_{tol}(A_3,b(q))$ is the parallelogram in gray and $\Sigma_{tol}(V,b(q))$ is the most inner white polyhedron.

Next theorem gives a better description of the shape of the parametric tolerable solution set than Theorem 5.5.

Theorem 5.7 *The parametric tolerable solution set is a convex polyhedron.*

Proof First we consider the special case where all E-parameters $q = (q_1, \ldots, q_{m_2})$ are of 1st class. Then, defining $[b] := b([q])$, by Theorem 5.6 we have

$$\begin{aligned}\Sigma_{tol}(A(p), b(q), [p], [q]) &= \Sigma_{tol}(A(p), b([q]), [p]) \\ &= \{x \in \mathbb{R}^n \mid (\forall p \in [p])(A(p)x \in [b])\} \\ &= \left\{x \in \mathbb{R}^n \mid (\forall p \in [p]) \left(A_{\bullet\bullet 0}x + \sum_{\mu=1}^{m_1} (A_{\bullet\bullet\mu}x)p_\mu \in [b]\right)\right\}.\end{aligned}$$

Define $\mathscr{L} := \{\lambda = (\lambda_1, \ldots, \lambda_{m_1}) \mid \lambda_\mu \in \{-,+\}, \mu = 1, \ldots, m_1\}$. The relation (4.3) implies $\bigwedge_{\lambda \in \mathscr{L}} b_1 \leq f(p^\lambda) \leq b_2$ for a linear function $f(p)$ and $p \in [p] \in \mathbb{IR}^k$. Thus,

$$\Sigma_{tol}(A(p), b([q]), [p]) = \left\{x \in \mathbb{R}^n \mid \bigwedge_{\lambda \in \mathscr{L}} b^- \leq A_{\bullet\bullet 0}x + \sum_{\mu=1}^{m_1} (A_{\bullet\bullet\mu}x)p_\mu^{\lambda_\mu} \leq b^+\right\}, \quad (5.7)$$

which proves the theorem since a convex polyhedron is expressed as the solution set for a system of linear inequalities.

If the parametric tolerable solution set involves 2nd class E-parameters, their elimination will generate cross inequalities with respect to these parameters. However, since all 2nd class E-parameters are involved in the right-hand side of the system, all cross inequalities with respect to these parameters will be linear involving additional (new) affine-linear dependencies between the parameters p. Then, the proof will continue the same way as for 1st class E-parameters above but with an enlarged matrix A' having $n+k$ rows and a vector $[b'] \in \mathbb{IR}^{n+k}$, where k is the number of the cross inequalities.

The assertion of Theorem 5.7 and the left two relations in (5.3) are considered in [12, 13] for the special case of row-independent parametric matrix and right-hand side with independent components. Theorem 5.7 and relation (5.4) of Theorem 5.6 address the most general case of parametric tolerable solution sets. Note, that (5.7) gives another description of the parametric tolerable solution set by $n2^{m_1+1}$ inequalities. This description is equivalent to the description given in Proposition 5.1 that contains only n absolute-value inequalities.

5.2 Parametric Controllable Solution Sets

The general parametric controllable solution set is defined by

$$\Sigma_{cont}(A(p), b(q), [p], [q]) := \{x \in \mathbb{R}^n \mid (\forall q \in [q])(\exists p \in [p])(A(p)x = b(q))\},$$

where $p \in [p] \in \mathbb{IR}^{m_1}$, $q \in [q] \in \mathbb{IR}^{m_2}$ are two independent parameter vectors.

It follows from Theorem 4.1 that the explicit description of a parametric controllable solution set can be easily derived from the explicit description of the united parametric solution set for a system with the same parametric matrix and a right-hand side vector $[b] = b([q])$. So far we know the explicit description of the united parametric solution set for systems with symmetric or skew-symmetric matrix [4], as well as for arbitrary 2-dimensional parametric matrices [8] or systems involving only 1st class E-parameters [9]. The next theorem is obtained by applying Theorem 4.1 to the explicit description of the united parametric solution set for a system with skew-symmetric matrix from [4].

Theorem 5.8 *The controllable solution set to a system with skew-symmetric matrix and independent right-hand side vector* $\Sigma_{cont}^{skew} := \{x \in \mathbb{R}^n \mid \forall b \in [b], \exists A^{skew} \in [A], A^{skew}x = b\}$*, is described by*

$$|\dot{A}x - \dot{b}| \le \hat{A}x - \hat{b},$$

$$\left|\sum_{i=1}^{n} M_{i\bullet}x_i(u_i + v_i)\right| \le \sum_{i,j=1}^{n} |x_i x_j(u_1 - v_j)|\,\hat{a}_{ij} - \sum_{i=1}^{n} |x_i(u_i + v_i)|\,\hat{b}_i,$$

$$\forall u, v \in \{0,1\}^n \setminus \{0\}, u \preceq_{lex} v,$$

where $M = \dot{A}x - \dot{b}$.

For an arbitrary controllable solution set we have the following equality relation

$$\Sigma_{cont}(A(p), b(q), [p], [q]) = \Sigma_{cont}(A(p), b([q]), [p]),$$

which can be combined with the inclusion relations from Theorem 5.4.

It follows from Corollary 4.2 that the parametric controllable solution set has the same shape as the parametric united solution set for a system with the same parametric matrix and a right-hand side vector $[b] = b([q])$. In the special case when $A(p)$ involves only 1st class parameters the parametric controllable solution set has linear shape. An example of parametric controllable solution set is given in the next section.

5.3 2D Parametric *AE* Solution Sets

In [8] we studied the elimination of 2nd class existentially quantified parameters from two characterizing inequalities. The next theorem, giving explicit description of the parametric *AE* solution sets to any 2D linear system, follows from [8, Theorem 4.1] and Theorem 4.1.

Theorem 5.9 *A parametric AE solution set (4.1) to a 2-dimensional linear system (1.1)–(1.3) is described by the following inequalities*

$$|A(\dot{p})x - b(\dot{p})| \le \sum_{\mu=1}^{m} \delta_\mu |A_{\bullet\bullet\mu}x - b_{\bullet\mu}|\hat{p}_\mu, \tag{5.8}$$

$$|\Delta_{0,i} + \sum_{\mu=1,\mu\neq i}^{m} \Delta_{\mu,i}\dot{p}_\mu| \le \sum_{\mu=1,\mu\neq i}^{m} \delta_\mu |\Delta_{\mu,i}|\hat{p}_\mu, \qquad i \in M, \tag{5.9}$$

where $\delta_\mu := \{1 \text{ if } \mu \in \mathscr{E}, -1 \text{ if } \mu \in \mathscr{A}\}$*, M is the index set of the 2nd class existentially quantified parameters,* $\Delta_{\alpha,\beta}(x) := f_{\alpha,1}(x)f_{\beta,2}(x) - f_{\alpha,2}(x)f_{\beta,1}(x)$*, and* $f_{\lambda,1}(x)$*,* $f_{\lambda,2}(x)$ *are the components of the coefficient vector* $f_\lambda(x) := A_{\bullet\bullet\lambda}x - b_{\bullet\lambda}$ *of the parameter* p_λ *for* $\lambda \in \{\alpha, \beta\}$.

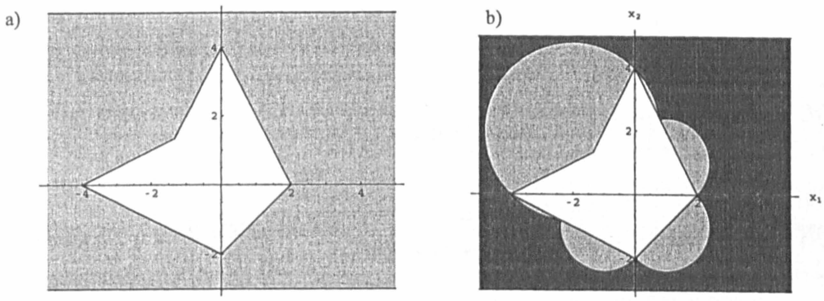

Fig. 5.3 The controllable non-parametric solution set a) and the parametric controllable solution set b) for the system from Example 5.3

For a system of two equations the above theorem implies: a) any parametric AE solution set is described by $2+m_1$ absolute-value inequalities, where m_1 is the number of 2nd class E-parameters; b) the maximal degree of the polynomial equations describing the boundary of a 2D parametric AE solution set is not greater than 2.

Example 5.3 Consider the parametric linear system $A(p)x = b(q)$, where

$$A(p) = \begin{pmatrix} p_1 & -p_2 \\ p_2 & p_1 \end{pmatrix}, \quad b(q) = \begin{pmatrix} 2q \\ 2q \end{pmatrix}, \quad p_1 \in [-2,2], p_2 \in [-1,2], q \in [1,2].$$

The parametric controllable solution set is described by the inequalities

$$\left|3+\frac{x_2}{2}\right| \le -1+2|x_1|+\frac{3|x_2|}{2}, \quad |-3x_1-3x_2| \le -|x_1+x_2|+2|x_1^2+x_2^2|$$
$$\left|3-\frac{x_1}{2}\right| \le -1+2|x_2|+\frac{3|x_1|}{2}, \quad \left|3x_1-3x_2-\frac{x_1^2}{2}-\frac{x_2^2}{2}\right| \le -|x_1-x_2|+3\frac{|x_1^2+x_2^2|}{2}.$$

The left two inequalities above are the so-called end-point inequalities which describe the non-parametric controllable solution set $\Sigma_{cont}(A([p]),b([q]))$. Since for $p=(2,2)^\top$ or $p=(-2,2)^\top$ both $A(p)$ and $A([p])$ are singular, both controllable solution sets are unbounded. The non-parametric one is presented in gray on Fig. 5.3 a) and both the parametric (in dark gray) and non-parametric controllable solution sets are presented on Fig. 5.3 b).

The webComputing service framework [7] is expanded by a program
http://cose.math.bas.bg/webMathematica/webComputing/ParametricAESSet.jsp
for generating explicit description of 2-dimensional parametric AE solution sets and for their graphical visualization.

6 Conclusion

The description of parametric AE solution sets by Fourier-Motzkin type parameter elimination is feasible, much faster and more compact than by quantifier elimination or other techniques. The description of a parametric AE solution set is more simple and usually involves less number of characterizing inequalities than the description of the corresponding united parametric solution set for the same system. Knowing the explicit description of an united parametric solution set, we can easily obtain the explicit description of any parametric *AE* solution set for the same system. Unfortunately, so far we know the explicit description of the united parametric solution set to only a few systems with fixed data dependencies. Therefore more research is necessary in this direction.

Many *AE* solution sets for a given parametric system are empty sets. The inequalities describing a parametric *AE* solution set present necessary and sufficient conditions for the solution set to be non-empty. If we do not know the so-called cross inequalities obtained by the elimination of 2nd class existentially quantified parameters, then the well-known end-point characterizing inequalities present a necessary condition for the parametric *AE* solution set to be non-empty. We proved various inclusion relations between different parametric *AE* solution sets corresponding to a non-parametric system. Knowing the description of a parametric *AE* solution set, we know the maximal degree of the polynomial equations describing the solution set boundary. We proved that all parametric tolerable solution sets are convex polyhedrons. We hope that the explicit description of parametric *AE* solution sets will facilitate exploring more properties and developing new numerical methods for the parametric *AE* solution sets.

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