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Harmonic proper almost complex structures on Walker 4-manifolds

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Harmonic proper almost complex structure on Walker 4-manifolds

Johann Davidov, Absar Ul Haq, Oleg Mushkarov

Abstract

Every Walker 4-manifold M, endowed with a canonical neutral metric, admits a specific almost complex structure called proper. In this note we find the conditions under which a proper almost complex structure is harmonic in the sense of C. Wood and as a section of the bundle $End(TM)$.

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1 Introduction

An almost complex structure on a Riemannian manifold (M, g), $dim\, M = 2n$, is called almost Hermitian if it is g-orthogonal. If a Riemannian manifold admits an almost Hermitian structure, it has many such structures. One way to see this is to consider the twistor bundle $\pi : \mathcal{Z} \to M$ whose fibre at a point p consists of all g-orthogonal complex structures $J_p : T_pM \to T_pM$ ($J_p^2 = -Id$) on the tangent space of M at p. The fibre is the compact Hermitian symmetric space $O(2n)/U(n)$ and its standard metric $-\frac{1}{2}Trace\, J_1 \circ J_2$ is a Kähler-Einstein. The Levi-Civita connection of (M, g) gives rise to a splitting of the tangent bundle of the twistor space $\mathcal{Z}$ into vertical and horizontal parts: $T\mathcal{Z} = \mathcal{V} \oplus \mathcal{H}$. The vertical space at a point J is the tangent space to the fibre through that point while the horizontal space is isomorphic to $T_{\pi(J)}M$ via the differential π_*. Then we have a natural Riemannian metric h on the twistor space defined as follows. On a vertical space, h is the metric of the fibre, on a horizontal one – the lift of the metric of M, and vertical and horizontal spaces are declared to be orthogonal. It follows from the Vilms theorem (see, for example, [?, Theorem 9.59]) that the projection map $\pi : (\mathcal{Z}, h) \to (M, g)$ is a Riemannian submersion with totally geodesic fibres (this can also be proved by a direct computation). Now suppose that (M, g) admits an almost Hermitian structure J, i.e. a section of $\pi : \mathcal{Z} \to M$. Take a section V with compact support K of the bundle $J^*\mathcal{V} \to M$, the pull-back under J of the vertical bundle $\mathcal{V} \to \mathcal{Z}$. There exists $\varepsilon > 0$ such that the exponential map exp_J is a diffeomorphism of the ε-ball in

$T_I\mathcal{Z}$ for every point I of the compact set $J(K)$. Set $J_t(p) = exp_{J(p)}[tV(p)]$ for $p \in M$ and $t \in (-\varepsilon, \varepsilon)$. Then J_t is a section of $\mathcal{Z}$, i.e. an almost Hermitian structure on (M, g) (such that $J_t = J$ on $M \setminus K$).

Thus it is natural to seek for "reasonable" criteria that distinguish some of the almost Hermitian structures among all of them. Motivated by the harmonic maps theory, C. Wood [10, 11] has suggested to consider as "optimal" those almost-Hermitian structures $J : (M, g) \to (\mathcal{Z}, h)$, which are critical points of the energy functional on sections of the twistor space $\mathcal{Z}$ under variations through sections of $\mathcal{Z}$. In general, these critical points are not harmonic maps, but, by analogy, in [10, 11] they are referred to as "harmonic almost-complex structures". The Euler-Lagrange equation for a harmonic almost-complex structure J is

$$[J, \nabla^*\nabla J] = 0, \tag{1.1}$$

where $\nabla^*\nabla$ is the rough Laplacian of (M, g) [10, 11].

Every almost complex structure on a smooth manifold M is a section of the bundle $T^*M \otimes TM$. If a Riemannian metric g on M is specified, we can endow this bundle with the metric induced by g and define the energy functional on its sections. The almost complex structures that are critical points of this functional under variations through sections of $T^*M \otimes TM$, i.e. harmonic sections, can also be considered as "distinguished". The Euler-Lagrange equation for such an almost complex structure is [5, 6]

$$\nabla^*\nabla J = 0. \tag{1.2}$$

This picture has an analog in the pseudo-Riemannian case. In this note we shall discuss the equations (1.1) and (1.2) for the so-called proper almost Hermitian structure (locally) defined on a Walker four manifold.

2 Walker manifolds

A Walker manifold is a triple (M, g, D) where M is an n-dimensional manifold, g an indefinite metric and D an r-dimensional parallel null distribution [13]. Of special interest are those manifolds admitting a field of null planes of maximum dimension $r = \frac{n}{2}$. Since the dimension of a null plane is $r \leq \frac{n}{2}$, the lowest possible case is that of $(++--)$-manifolds admitting a field of parallel null 2-planes.

Recall that, by a result of Walker [13], given a Walker metric g on a 4-manifold M, around each point of M there exist local coordinates (x, y, z, t) such that $D = \langle \frac{\partial}{\partial_x}, \frac{\partial}{\partial_y} \rangle$ and the matrix of g in these coordinates has the following form

$$g(x, y, z, t) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & a(x, y, z, t) & c(x, y, z, t) \\ 0 & 1 & c(x, y, z, t) & b(x, y, z, t) \end{pmatrix} \tag{2.1}$$

for smooth functions $a(x, y, z, t)$, $b(x, y, z, t)$, $c(x, y, z, t)$.

It is well known that the existence of a metric of signature $(++--)$ with structure group $SO_0(2,2)$ is equivalent to the existence of a pair of commuting almost complex structures [7], and, moreover, any such pseudo-Riemannian metric may be viewed as an indefinite almost Hermitian metric for a suitable almost complex structure. These almost complex structures are not uniquely determined. However, according to [3, Lemma 6], if M is a 4-manifold with a metric g with signature (+ + - -) and X, Y are orthogonal null vector fields linearly independent at every point of M, then the triple (g, X, Y) determines an orientation and a unique g- and orientation compatible almost complex structure J on M such that $JX = Y$. One such a structure associated with any four-dimensional Walker metric has been locally given in [8] and called the proper almost complex structure.

For a Walker metric an orthonormal frame can be specialized by using the canonical coordinates as follows

$$
\begin{aligned}
e_1 &= \frac{1-a}{2}\partial_x + \partial_z & e_2 &= -c\partial_x + \frac{1-b}{2}\partial_y + \partial_t \\
e_3 &= -\frac{1+a}{2}\partial_x + \partial_z & e_4 &= --c\partial_x - \frac{1+b}{2}\partial_y + \partial_t
\end{aligned}
\tag{2.2}
$$

Then the proper almost complex structure is defined by $Je_1 = e_2$, $Je_3 = e_4$ [8]. Thus we have

$$
\begin{aligned}
J\partial_x &= \partial_y, \quad J\partial_z = -c\partial_x + \frac{a-b}{2}\partial_y + \partial_t, \\
J\partial_y &= -\partial_x, \quad J\partial_t = \frac{a-b}{2}\partial_x + c\partial_y - \partial_z.
\end{aligned}
\tag{2.3}
$$

3 Harmonic proper almost complex structures

Let (M, g, J) be a Walker four manifold with a Walker metric g and proper almost complex structure J given in local coordinates by (2.1) and (2.3).

Theorem 3.1 *The proper almost complex structure J is harmonic if and only if*

$$a_{xy} + b_{xy} + c_{xx} + c_{yy} = 0, \quad a_{xx} - b_{yy} = 0. \tag{3.1}$$

Proof. Note that the Euler-Lagrange equation

$$[J, \nabla^*\nabla J] = 0$$

for a harmonic almost Hermitian structure can be written as

$$\nabla^*\nabla\Omega(X, Y) = \nabla^*\nabla\Omega(JX, JY), \quad X, Y \in TM,$$

where $\Omega(X, Y) = g(JX, Y)$ is the fundamental 2-form of (M, g, J). By the Weitzenböck formula, $\Delta\Omega = \nabla^*\nabla\Omega + S(\Omega)$ where

$$S(\Omega)(X, Y) = Trace\{Z \to (R(Z, Y)\Omega)(Z, X) - (R(Z, X)\Omega)(Z, Y)\}$$

(see, for example, [4]). We have

$$(R(Z,Y)\Omega)(Z,X) = -\Omega(R(Z,Y)Z,X) - \Omega(Z,R(Z,Y)X)$$
$$= g(R(Z,Y)Z,JX) + g(R(Z,X)Y,JZ).$$

Denote by ρ and ρ^* the Ricci and the $*$-Ricci tensors of the (pseudo) Hermitian structure (g,J) (recall that $\rho^*(X,Y) = trace\{Z \to R(JZ,X)JY\}$). Then

$$\nabla^*\nabla\Omega(X,Y) = \Delta\Omega(X,Y) + \rho(X,JY) - \rho(JX,Y) - 2\rho^*(X,JY)$$

Therefore J is harmonic if and only if

$$\Delta\Omega(X,Y) - \Delta\Omega(JX,JY) = 2[\rho^*(X,JY) + \rho^*(JX,Y)]. \tag{3.2}$$

Lemma 3.2 *The Laplacian of the 2-form Ω is given by*

$$\begin{aligned}\Delta\Omega = & - (a_{xy}+b_{xy})dx\wedge dz + (a_{xx}+b_{xx})dx\wedge dt \\ & + (a_{xy}+b_{xy})dy\wedge dt - (a_{yy}+b_{yy})dy\wedge dz \\ & + \frac{1}{2}(a(a_{xx}+b_{xx}) + b(a_{yy}+b_{yy}) + 2c(a_{xy}+b_{xy}))dz\wedge dt\end{aligned}$$

Proof. To compute the Laplacian $\Delta\Omega = d\delta\Omega + \delta d\Omega$, we note first that the dual frame $\{\alpha_1,...,\alpha_4\}$ of the orthonormal frame $\{e_1,...,e_4\}$ defined by (2.2) is

$$\alpha_1 = dx + \frac{1+a}{2}dz + cdt, \quad \alpha_2 = dy + \frac{1+b}{2}dt$$
$$\alpha_3 = -dx + \frac{1-a}{2}dz - cdt, \quad \alpha_4 = -dy + \frac{1-b}{2}dt.$$

Hence

$$dx = \frac{1-a}{2}\alpha_1 - \frac{1+a}{2}\alpha_3 - c(\alpha_2+\alpha_4), \quad dy = \frac{1-b}{2}\alpha_2 - \frac{1+b}{2}\alpha_4$$
$$dz = \alpha_1 + \alpha_3, \quad dt = \alpha_2 + \alpha_4.$$

We have $||\alpha_1||^2 = ||\alpha_2||^2 = 1$, $||\alpha_3||^2 = ||\alpha_4||^2 = -1$ and the Hodge operator $*$ of (M,g) acts on the 2- and 3-forms as follows

$$*(\alpha_1\wedge\alpha_2) = \alpha_3\wedge\alpha_4, \quad *(\alpha_1\wedge\alpha_3) = \alpha_2\wedge\alpha_4, \quad *(\alpha_1\wedge\alpha_4) = \alpha_3\wedge\alpha_2, \quad *^2 = id,$$
$$*(\alpha_1\wedge\alpha_2\wedge\alpha_3) = -\alpha_4, \quad *(\alpha_1\wedge\alpha_2\wedge\alpha_4) = \alpha_3,$$
$$*(\alpha_1\wedge\alpha_3\wedge\alpha_4) = \alpha_2, \quad *(\alpha_2\wedge\alpha_3\wedge\alpha_4) = -\alpha_1.$$

It follows that

$$\begin{aligned}*(dx\wedge dy) = & \, dx\wedge dy + c(dx\wedge dz) + b(dx\wedge dt) \\ & -a(dy\wedge dz) - c(dy\wedge dt) + (ab-c^2)(dz\wedge dt)\end{aligned}$$

$$*(dx\wedge dz) = dy\wedge dt + c(dz\wedge dt), \quad *(dx\wedge dt) = -dx\wedge dt - a(dz\wedge dt),$$
$$*(dy\wedge dz) = -dy\wedge dz + b(dz\wedge dt), \quad *(dy\wedge dt) = dx\wedge dz - c(dz\wedge dt),$$
$$*(dz\wedge dt) = dz\wedge dt.$$

and

$$*(dx \wedge dy \wedge dz) = dy + cdz + bdt, \quad *(dx \wedge dy \wedge dt) = -dx - adz - cdt$$
$$*(dx \wedge dz \wedge dt) = dt, \quad *(dy \wedge dz \wedge dt) = -dz.$$

The fundamental 2-form for the proper almost complex structure is

$$\Omega = dx \wedge dt - dy \wedge dz + \frac{1}{2}(a+b)dz \wedge dt.$$

Therefore, we have

$$*\Omega = -dx \wedge dt - dz \wedge dy - \frac{1}{2}(a+b)dz \wedge dt$$

Thus

$$d * \Omega = -\frac{1}{2}(a_x + b_x)dx \wedge dz \wedge dt - \frac{1}{2}(a_y + b_y)dy \wedge dz \wedge dt$$

Then

$$\delta\Omega = - * d * \Omega = -\frac{1}{2}(a_y + b_y)dz + \frac{1}{2}(a_x + b_x)dt,$$

which implies

$$\begin{aligned} d\delta\Omega = &-\tfrac{1}{2}(a_{xy} + b_{xy})dx \wedge dz - \tfrac{1}{2}(a_{yy} + b_{yy})dy \wedge dz \\ &+\tfrac{1}{2}(a_{xx} + b_{xx})dx \wedge dt + \tfrac{1}{2}(a_{xy} + b_{xy})dy \wedge dt \\ &+\tfrac{1}{2}(a_{yt} + b_{yt} + a_{xz} + b_{xz})dz \wedge dt. \end{aligned} \tag{3.3}$$

Next we compute $\delta d\Omega$. We have

$$d\Omega = \frac{1}{2}(a_x + b_x)dx \wedge dz \wedge dt + \frac{1}{2}(a_y + b_y)dy \wedge dz \wedge dt.$$

Hence

$$*d\Omega = \frac{1}{2}(a_x + b_x)dt - \frac{1}{2}(a_y + b_y)dz.$$

This gives

$$\begin{aligned} d * d\Omega = &\tfrac{1}{2}(a_{xx} + b_{xx})dx \wedge dt + \tfrac{1}{2}(a_{xy} + b_{xy})dy \wedge dt - \tfrac{1}{2}(a_{xy} + b_{xy})dx \wedge dz \\ &-\tfrac{1}{2}(a_{yy} + b_{yy})dy \wedge dz + \tfrac{1}{2}(a_{xz} + a_{yt} + b_{xz} + b_{yt})dz \wedge dt. \end{aligned}$$

Therefore

$$\begin{aligned} \delta d\Omega = - \star d \star d\Omega = &\tfrac{1}{2}(a_{xx} + b_{xx})dx \wedge dt - \tfrac{1}{2}(a_{xy} + b_{xy})dx \wedge dz \\ &+\tfrac{1}{2}(a_{xy} + b_{xy})dy \wedge dt - \tfrac{1}{2}(a_{yy} + b_{yy})dy \wedge dz \\ &+\tfrac{1}{2}(a(a_{xx} + b_{xx}) + 2c(a_{xy} + b_{xy}) + b(a_{yy} + b_{yy}) \\ &-a_{xz} - a_{yt} - b_{xz} - b_{yt})dz \wedge dt \end{aligned} \tag{3.4}$$

Now the result follows from (3.3) and (3.4). ■

Lemma 3.3 *The $*$-Ricci tensor of (M, g, J) is given by*

$$\rho^*(\partial_x, \partial_x) = \rho^*(\partial_y, \partial_y) = \rho^*(\partial_x, \partial_y) = \rho^*(\partial_y, \partial_x) = 0,$$

$$\rho^*(\partial_x, \partial_z) = \rho^*(\partial_t, \partial_y) = -\tfrac{1}{2}(b_{xx} - c_{xy}),$$

$$\rho^*(\partial_x, \partial_t) = -\rho^*(\partial_z, \partial_y) = -\tfrac{1}{2}(a_{xy} - c_{xx})$$

$$\rho^*(\partial_y, \partial_z) = -\rho^*(\partial_t, \partial_x) = -\tfrac{1}{2}(b_{xy} - c_{yy}),$$

$$\rho^*(\partial_y, \partial_t) = \rho^*(\partial_z, \partial_x) = -\tfrac{1}{2}(a_{yy} - c_{xy})$$

$$\rho^*(\partial_z, \partial_z) = \tfrac{1}{4}[a_x b_x + a_y(b_y - c_x) + b_y c_x + c_y(a_x - b_x) - c_x^2 - c_y^2$$
$$+2c(a_{xy} - c_{xx}) - (a - b)a_{yy} - 2a_{yt} - 2b_{xz} + (a - b)c_{xy} + 2c_{xt} + 2c_{yz}]$$

$$\rho^*(\partial_z, \partial_t) = -\tfrac{1}{4}(a - b)(a_{xy} - c_{xx}) - \tfrac{1}{2}c(a_{yy} - c_{xy})$$

$$\rho^*(\partial_t, \partial_z) = \tfrac{1}{4}(a - b)(b_{xy} - c_{yy}) - \tfrac{1}{2}c(b_{xx} - c_{xy})$$

$$\rho^*(\partial_t, \partial_t) = \tfrac{1}{4}[a_x b_x + a_y(b_y - c_x) + b_y c_x + c_y(a_x - b_x) - c_x^2 - c_y^2$$
$$2c(b_{xy} - c_{yy}) + (a - b)b_{xx} - 2a_{yt} - 2b_{xz} - (a - b)c_{xy} + 2c_{xt} + 2c_{yz}].$$

Proof. The traceless $*$-Ricci tensor $\rho_0^* = \rho^* - \frac{\tau^*}{4}g$ and the $*$-scalar curvature $\tau^* = Trace\,\rho^*$ have been computed in [1, 2]. The formulas there easily imply the lemma. ■

Proof of the theorem. Clearly, if equation (3.2) is satisfied for (X, Y), it also holds for (JX, JY). Moreover, the identity $\rho^*(X, Y) = \rho^*(JY, JX)$ implies that both sides of (3.2) are skew-symmetric and that (3.2) is automatically satisfied when $Y = JX$. It follows that (3.2) holds for every X, Y if and only if it holds for $X = \partial_x, Y = \partial_z$ and $X = \partial_x, Y = J\partial_z$. Thus the theorem follows from Lemmas 3.2 and 3.3.

Theorem 3.1 and [8, Theorems 2 and 3] imply the following.

Corollary 3.4 *(i) The proper almost complex structure is almost Kähler and harmonic if and only if*

$$a_x + b_x = 0, \quad a_y + b_y = 0, \quad c_{xx} + c_{yy} = 0, \quad a_{xx} - b_{yy} = 0..$$

(ii) The proper almost complex structure is integrable and harmonic if and only if

$$a_x - b_x = 2c_y, \quad a_y - b_y = -2c_x, \quad a_{xy} + b_{xy} = 0, \quad a_{xx} - b_{yy} = 0.$$

Theorem 3.1 and [2, Theorems 4, 5, 9 and 11] give

Corollary 3.5 *If the proper almost complex structure J is integrable, it is harmonic provided one of the following conditions holds:*
(1) The Walker metric g is self-dual;
(2) The proper Hermitian structure (g, J) is locally conformally Kähler;
(3) The structure (g, J) is $*$-Einstein;
(4) The metric g is Einstein.

Theorem 3.1 and [1, Theorems 2, 3 and 7] imply

Corollary 3.6 *If the almost Hermitian structure (g, J) is almost Kähler, the proper almost complex structure J is harmonic provided one of the following conditions holds:*
(1) The Walker metric g is self-dual;
(2) The structure (g, J) is $*$-Einstein;
(3) The metric g is Einstein.

Remark. The twistor space $\mathcal{Z}$ of an oriented four-dimensional Riemannian manifold (M, g) is a unit sphere bundle over M. A section σ of a sphere bundle of radius r over a Riemannian manifold is a harmonic section if and only if $\nabla^*\nabla\sigma = \frac{1}{r^2}||\sigma||^2\sigma$ where ∇ is the Levi-Civita connection of the base manifold (see, for example [12]). Thus in the four-dimensional case a compatible almost complex structure J on (M, g) is harmonic if and only if

$$\nabla^*\nabla J = ||J||^2 J. \tag{3.5}$$

It follows that this equation is equivalent to (1.1) in dimension four. This can directly be seen as follows. Clearly (3.5) implies (1.1). Suppose that J is a compatible almost complex structure on (M, g) satisfying condition (1.1). Let Ω be the fundamental 2-form of the almost Hermitian structure (g, J) normalized with unit length. Then (1.1) can be written as $\nabla^*\nabla\Omega(X, Y) = \nabla^*\nabla\Omega(JX, JY)$ for every $X, Y \in TM$ while (3.5) is equivalent to $\nabla^*\nabla\Omega = ||\sigma||^2\Omega$. If we endow M with the orientation determined by J, Ω is a section of the rank 3 bundle Λ_+T^*M of self-dual 2-forms. Take a local orthonormal frame of TM of the form $E_1, E_2 = JE_1, E_3, E_4 = JE_3$. Let J_2 and J_3 be the (local) almost complex structures for which $J_2E_1 = E_3$, $J_2E_4 = E_2$ and $J_3E_1 = E_4$, $J_3E_2 = E_3$. These structures are compatible with the metric g and we denote by Ω_2 and Ω_3 their normalized fundamental 2-forms. Then $\Omega_1 = \Omega, \Omega_2, \Omega_3$ is an othonormal frame of the bundle Λ_+T^*M. Thus, there are 1-forms α, β, γ such that

$$\nabla\Omega_1 = \alpha\Omega_2 + \beta\Omega_3, \quad \nabla\Omega_2 = -\alpha\Omega_1 + \gamma\Omega_3, \quad \nabla\Omega_3 = -\beta\Omega_1 - \gamma\Omega_2.$$

Note that $\nabla^*\nabla\Omega = -Trace\nabla^2\Omega$. For every $X \in TM$ we have

$$\begin{aligned}\nabla^2_{XX}\Omega = \nabla_X\nabla_X\Omega - \nabla_{\nabla_X X}\Omega = -[\alpha(X)^2 + \beta(X)^2]\Omega_1 \\ +[X(\alpha(X)) - \alpha(\nabla_X X) - \beta(X)\gamma(X)]\Omega_2 \\ +[X(\beta(X)) - \beta(\nabla_X X) - \alpha(X)\gamma(X)]\Omega_3.\end{aligned}$$

It follows that

$$\nabla^*\nabla\Omega = ||\Omega||^2\Omega - \kappa_2\Omega_2 - \kappa_3\Omega_3$$

where κ_2 and κ_3 are the traces of the quadratic forms $X(\alpha(X)) - \alpha(\nabla_X X) - \beta(X)\gamma(X)$ and $X(\beta(X)) - \beta(\nabla_X X) - \alpha(X)\gamma(X)$. Thus

$$\begin{aligned}&||\Omega||^2\Omega(X, Y) - \kappa_2\Omega_2(X, Y) - \kappa_3\Omega_3(X, Y) \\ &= ||\Omega||^2\Omega(JX, JY) - \kappa_2\Omega_2(JX, JY) - \kappa_3\Omega_3(JX, JY)\end{aligned}$$

for every $X, Y \in TM$. For $X = E_1$, $Y = E_3$ this identity gives $\kappa_2 = 0$; for $X = E_1$, $Y = E_4$ it implies $\kappa_3 = 0$. Thus $\nabla^*\nabla\Omega = ||\Omega||^2\Omega$.

In the pseudo-Riemannian case equation (1.1) is no longer equivalent to (3.5). Indeed, the proper almost complex structure on any Walker 4-manifold satisfies (3.5) since $||J||^2 = \sum_{i=1}^{4} ||Je_i||^2 = 0$ while it not always satisfies (1.1) (Theorem 3.1).

4 Proper almost complex structures as harmonic sections of $T^*M \otimes TM$

The Euler-Lagrange equation $\nabla^*\nabla J = 0$ is equivalent to $\nabla^*\nabla\Omega = 0$. By the Weitzenböck formula, the latter equation is equivalent to

$$\Delta\Omega(X,Y) = \rho(JX,Y) - \rho(X,JY) + 2\rho^*(X,JY), \quad X, Y \in TM. \tag{4.1}$$

Theorem 4.1 *The proper almost complex structure J on a Walker 4-manifold (M, g, J) is a harmonic section of the bundle $T^*M \otimes TM$ if and only if it is a harmonic almost complex structure.*

Proof. The Ricci tensor ρ has been computed in [8, Appendix B]. The formulas therein and Lemmas 3.2, 3.3 imply that equation (4.1) is satisfied if and only if

$$a_{xy} + b_{xy} + c_{xx} + c_{yy} = 0, \quad a_{xx} - b_{yy} = 0.$$

Thus the result follows from Theorem 3.1. ■

5 Examples

1. Let $\alpha(z,t)$, $\beta(z,t)$, $\gamma(z,t)$ be smooth functions depending only on the coordinates z and t. Set

$$(i)\ a = x^2 + y^2 + \alpha(z,t), \quad b = -x^2 - y^2 + x + \beta(z,t), \quad c = \gamma(z,t)$$
$$(ii)\ a = x^2 + y^2 + \alpha(z,t), \quad b = x^2 + y^2 + \beta(z,t), \quad c = xy + \gamma(z,t)$$
$$(iii)\ a = x^2 + y^2 + \alpha(z,t), \quad b = x^2 + y^2 + \beta(z,t), \quad c = \gamma(z,t)$$
$$(iv)\ a = x^2 + y^2 + \alpha(z,t), \quad b = -x^2 - y^2 + \beta(z,t), \quad c = \gamma(z,t)$$

Consider the Walker metric g whose canonical form is defined by means of the functions a, b, c. In all four cases the proper almost complex structure J is harmonic. In view of Corollaries 3.4 –3.6, let us note that the Walker metric is not self-dual or Einstein and the proper Hermitian structure (g, J) is not locally conformally Kähler or $*$-Einstein. In the cases (i) and (ii) J is not integrable and (g, J) is not almost Kähler. In the case (iii) the proper almost complex structure J is integrable and (g, J) is not almost Kähler. In the case (iv) it is almost Kähler while J is not integrable.

2. Clearly every Kähler complex structure is harmonic. Using [9], it is shown in the proof of [3, Theorem 7] that every complex 2-dimensional torus and every primary Kodaira surfaces admit Walker metrics which are Kähler-Einstein. Moreover, around every point of each of these surfaces there are local coordinates in which the metric has the form (2.1) with $a = b$, $c = 0$ and the complex structure is given by (2.3), so coincides with the corresponding proper complex structure.

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