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Weighted approximation by Meyer-König and Zeller operators

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Abstract

The weighted approximation errors of Meyer-König and Zeller operator is characterized for weights of the form $w(x) = x^{\gamma_0}(1-x)^{\gamma_1}$, where $\gamma_0 \in [-1,0], \gamma_1 \in \mathbb{R}$. Direct inequalities and strong converse inequalities of type A are proved in terms of the weighted K-functional.

Keywords: Meyer-König and Zeller operator, K-functional, Direct theorem, Strong converse theorem

AMS Subject Classification : 41A36, 41A25, 41A27, 41A17.

1 Introduction

The Meyer-Konig and Zeller (MKZ) operator is defined for functions $f \in C[0,1)$ by the formula

$$M_n(f,x) = \sum_{k=0}^{\infty} f\left(\frac{k}{n+k}\right) M_{n,k}(x) \quad \text{where} \quad M_{n,k}(x) = \binom{n+k}{k} x^k (1-x)^{n+1}. \tag{1.1}$$

Right after their appearance, the MKZ operators became a subject of serious investigations. The reason for this is the fact, that they allow approximating of functions unbounded at the point 1 (which is different, comparing with Bernstein polynomials). But the values of the function are taken at the points $\frac{k}{n+k}$, which creates additional difficulties working with these operators.

In this paper we investigate the weighted approximation of functions by the classical variant of MKZ operator in uniform norm $\|.\|_{[0,1)}$, i.e. we want to characterize the weighted error of approximation $\sup_{x\in[0,1)} |w(x)f(x)|$, where

$$w(x) = x^{\gamma_0}(1-x)^{\gamma_1}. \tag{1.2}$$

are the Jacobi weights.

In the unweighted case ($w(x) = 1$) the direct theorem is proved in [4], and the strong converse inequality of type A (in terminology of [3]) is proved in [6]. Regarding the weighted case, the first results are obtained by Becker and Nessel in [2], where they proved the direct theorems for some symmetrical weights $w(x) = \varphi^\alpha(x)$ where $\varphi(x) = x(1-x)^2$ is the weight function which is naturally connected with the second derivative of MKZ operators.

In [10] Totik established, that for $0 < \alpha \le 1$ and $\varphi(x) = x(1-x)^2$ the condition

$$\varphi^\alpha |\Delta_h^2(f,x)| \le K h^{2\alpha}$$

is equivalent to

$$M_n f - f = O\left(n^{-\alpha}\right).$$

In [9] the authors proved that for $0 \le \lambda \le 1$ and $0 < \alpha < 2$ the condition

$$|M_n f(x) - f(x)| = O\left(\left(\frac{\varphi^{(1-\lambda)/2}(x)}{\sqrt{n}}\right)^{\alpha}\right)$$

is equivalent to

$$\omega^2_{\varphi^{\lambda/2}}(f,t) = O(t^{\alpha}).$$

Here $\omega^2_{\varphi^{\lambda/2}}(f,t)$ are the modulus of Ditzian-Totik

$$\omega^2_{\varphi^{\lambda/2}}(f,t) = \sup_{0<h\le t} \sup_{x\pm h\varphi^{\lambda/2}(x)\in[0,1)} |\Delta^2_{h\varphi^{\lambda/2}(x)} f(x)|.$$

In [7] Holhoş proved the next direct inequality for weights $\gamma_0 = 0, \gamma_1 > 0$:

$$\|w(M_n f - f)\| \le 2\omega\left(f(1-e^{-t})e^{-\gamma_1 t}, \frac{1}{\sqrt{n}}\right) + \|wf\|\frac{\gamma_1 C(\gamma_1)}{\sqrt{n}}.$$

Before stating our main result, let us introduce some notations and definitions. The first derivative operator is denoted by $D = \frac{d}{dx}$. Thus, $Dg(x) = g'(x)$ and $D^2 g(x) = g''(x)$.

By $C[0,1)$ we denote the space of all continuous on $[0,1)$ functions. The functions from $C[0,1)$ are not expected to be continuous or bounded at 1. By $L_\infty[0,1)$ we denote the space of all Lebesgue measurable and essentially bounded in $[0,1)$ functions equipped with the uniform norm $\|\cdot\|_{[0,1)}$. For a weight function w we set

$$C(w)[0,1) = \{g \in C[0,1); \quad wg \in L_\infty[0,1)\},$$
$$W^2(w\varphi)[0,1) = \{g, Dg \in AC_{loc}(0,1) \qquad w\varphi D^2 g \in L_\infty[0,1)\},$$

where $AC_{loc}(0,1)$ consists of the functions which are absolutely continuous in $[a,b]$ for every $[a,b] \subset (0,1)$.

The weighted approximation error $\|w(f - M_n f)\|_{[0,1)}$ will be compared with the K-functional between the weighted spaces $C(w)[0,1)$ and $W^2(w\varphi)[0,1)$, which for every

$$f \in C(w)[0,1) + W^2(w\varphi)[0,1) = \{f_1 + f_2 : f_1 \in C(w)[0,1), f_2 \in W^2(w\varphi)[0,1)\}$$

and $t > 0$ is defined by

$$K_w(f,t)_{[0,1)} = \inf_{g\in W^2(w\varphi),\ f-g\in C(w)} \left\{\|w(f-g)\|_{[0,1)} + t\|w\varphi D^2 g\|_{[0,1)}\right\}. \tag{1.3}$$

Our main result is the following theorem, establishing a full equivalence between the K-functional $K_w\left(f, \frac{1}{n}\right)_{[0,1)}$ and the weighted error $\|w(M_n f - f)\|_{[0,1)}$.

Theorem 1.1. *For w defined by (1.2), where $\gamma_0 \in [-1,0], \gamma_1 \in \mathbb{R}$, there exist positive constants C_1, C_2 and L such that for every natural $n \ge L$ and for all*

$$f \in C(w)[0,1) + W^2(w\varphi)[0,1)$$

there holds

$$C_1\|w(M_n f - f)\|_{[0,1)} \le K_w\left(f, \frac{1}{n}\right)_{[0,1)} \le C_2\|w(M_n f - f)\|_{[0,1)}. \tag{1.4}$$

The proof is based on the method, used for the first time in [8]. Shortly, the idea is this: by making an appropriate transformation we go to Baskakov operators for which we have the needed estimations and go back by the inverse transformation.

2 A connection between Baskakov and MKZ operators

Following [8] we introduce a transformation T mapping functions defined on $[0,\infty)$ into functions defined on $[0,1)$. And we make the agreement that from now on we shall denote variables, functions and operators, defined in $[0,1)$ the usual way, and their analogs, defined in $[0,\infty)$, with tilde.

Now we give some notations and definitions.
The uniform norm on the interval $[0,\infty)$ we will denote $\|\cdot\|_{[0,\infty)}$ and we define the next function spaces.

$$C(\tilde{w})[0,\infty) = \{\tilde{g} \in C[0,\infty); \quad \tilde{w}\tilde{g} \in L_\infty[0,\infty)\},$$
$$W^2(\tilde{w}\tilde{\varphi})[0,\infty) = \left\{\tilde{g}, \tilde{D}\tilde{g} \in AC_{loc}(0,\infty) \quad \text{and} \quad \tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{g} \in L_\infty[0,\infty)\right\}.$$

The weighted error by Baskakov operators will be characterized by the next K-functional, defined for every function $\tilde{f} \in C(\tilde{w})[0,\infty) + W^2(\tilde{w}\tilde{\varphi})[0,\infty)$ and for every $t > 0$ by the formula

$$K_{\tilde{w}}(\tilde{f},t)_{[0,\infty)} = \inf\left\{\|\tilde{w}(\tilde{f}-\tilde{g})\|_{[0,\infty)} + t\left\|\tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{g}\right\|_{[0,\infty)}\right\}, \tag{2.1}$$

where the infimum is taken over functions $\tilde{g} \in W^2(\tilde{w}\tilde{\varphi})[0,\infty)$ such that $\tilde{f} - \tilde{g} \in C(\tilde{w})[0,\infty)$.

We start with the change of variable $\sigma : [0,1) \to [0,\infty)$ (used for the first time by V.Totik in [10]) given by

$$\tilde{x} = \sigma(x) = \frac{x}{1-x}. \tag{2.2}$$

Then the inverse change of variable $\sigma^{-1} : [0,\infty) \to [0,1)$ is

$$x = \sigma^{-1}(\tilde{x}) = \frac{\tilde{x}}{1+\tilde{x}}.$$

The transformation operator T, transforming a function $\tilde{f}$ defined on $[0,\infty)$ to a function f defined on $[0,1)$ is defined by

$$f(x) = T(\tilde{f})(x) = \lambda(x)(\tilde{f}\circ\sigma)(x), \quad \lambda(x) = 1-x. \tag{2.3}$$

Then the inverse operator T^{-1}, transforming a function f defined on $[0,1)$ to a function $\tilde{f}$ defined on $[0,\infty)$ is

$$\tilde{f}(\tilde{x}) = T^{-1}(f)(\tilde{x}) = \frac{1}{(\lambda\circ\sigma^{-1})(\tilde{x})}(f\circ\sigma^{-1})(\tilde{x}).$$

We want to estimate the weighted error by MKZ, so we define a new transformation operator S by

$$w(x) = S(\tilde{w})(x) = \frac{1}{\lambda(x)}(\tilde{w}\circ\sigma)(x). \tag{2.4}$$

and its inverse S^{-1} is

$$\tilde{w}(\tilde{x}) = S^{-1}(w)(\tilde{x}) = (\lambda\circ\sigma^{-1})(\tilde{x})(w\circ\sigma^{-1})(\tilde{x}).$$

Obviously we have:

$$\begin{aligned} wf &= S(\tilde{w})T(\tilde{f}) = (\tilde{w}\circ\sigma)(\tilde{f}\circ\sigma), \\ \tilde{w}\tilde{f} &= S^{-1}(w)T^{-1}(f) = (w\circ\sigma^{-1})(f\circ\sigma^{-1}). \end{aligned} \tag{2.5}$$

For the next lemmas, which are easily verified (see [8]), w is a weight in $[0,1)$ and $\tilde{w} = S^{-1}(w)$ is the according weight in $[0,\infty)$.

Lemma 2.1. *The operators T and its inverse T^{-1} are linear positive operators and the next equalities are true:*

$$T(\tilde{\varphi}\tilde{D}^2\tilde{f}) = \varphi D^2(T\tilde{f}),$$
$$T^{-1}(\varphi D^2 f) = \tilde{\varphi}\tilde{D}^2(T^{-1}f). \tag{2.6}$$

Lemma 2.2. *The operator $T : C(\tilde{w})[0,\infty) \to C(w)[0,1)$ is an one-to-one correspondence with*

$$\|wT(\tilde{f})\|_{[0,1)} = \|\tilde{w}\tilde{f}\|_{[0,\infty)}, \quad \|\tilde{w}T^{-1}(f)\|_{[0,\infty)} = \|wf\|_{[0,1)}.$$

Lemma 2.3. *The operator $T : W^2(\tilde{w}\tilde{\varphi})[0,\infty) \to W^2(w\varphi)[0,1)$ is an one-to-one correspondence with*

$$\|w\varphi D^2(T(\tilde{f}))\|_{[0,1)} = \|\tilde{w}\tilde{\varphi}\tilde{D}^2\tilde{f}\|_{[0,\infty)}, \quad \|\tilde{w}\tilde{\varphi}\tilde{D}^2(T^{-1}(f))\|_{[0,\infty)} = \|w\varphi D^2 f\|_{[0,1)}.$$

Lemma 2.4. *For every $f \in C(w)[0,1) + W^2(w\varphi)[0,1)$, $\tilde{f} = T^{-1}f$ and $t > 0$ we have*

$$K_w(f,t)_{[0,1)} = K_{\tilde{w}}(\tilde{f},t)_{[0,\infty)}.$$

The next lemma gives the connection between the MKZ operators M_n and the classical Baskakov operators [1].

Lemma 2.5. *For every f such that one of the series below is convergent and for every $n \in \mathbb{N}$ we have*

$$M_n(f)(x) = T(V_n(T^{-1}(f)))(x), \quad x \in [0,1). \tag{2.7}$$

Proof. From the definition of T we get

$$\begin{aligned}
T(V_n(T^{-1}(f)))(x) &= \lambda(x)(V_n(T^{-1}(f)) \circ \sigma^{-1})(x) \\
&= \frac{1}{1+\tilde{x}}(V_n(T^{-1}(f)(\tilde{x}) = \frac{1}{1+\tilde{x}}V_n(\tilde{f},\tilde{x}) \\
&= \frac{1}{1+\tilde{x}}\sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\tilde{x}^k}{(1+\tilde{x})^{n+k}}\tilde{f}\left(\frac{k}{n}\right) \\
&= \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{\tilde{x}^k}{(1+\tilde{x})^{n+k+1}}\frac{1}{(\lambda\circ\sigma^{-1})\left(\frac{k}{n}\right)}(f\circ\sigma^{-1})\left(\frac{k}{n}\right).
\end{aligned}$$

Since

$$\sigma^{-1}\left(\frac{k}{n}\right) = \frac{k/n}{1+k/n} = \frac{k}{n+k}$$

we have

$$(\lambda\circ\sigma^{-1})\left(\frac{k}{n}\right) = \lambda\left(\frac{k}{n+k}\right) = \frac{n}{n+k} \quad \text{and} \quad (f\circ\sigma^{-1})\left(\frac{k}{n}\right) = f\left(\frac{k}{n+k}\right).$$

Also

$$\frac{\tilde{x}^k}{(1+\tilde{x})^{n+k+1}} = \left(\frac{\tilde{x}}{1+\tilde{x}}\right)^k\frac{1}{(1+\tilde{x})^{n+1}} = x^k(1-x)^{n+1}.$$

Consequently

$$\begin{aligned}
T(V_n(T^{-1}(f)))(x) &= \sum_{k=0}^{\infty}\binom{n+k-1}{k}\frac{n+k}{k}x^k(1-x)^{n+1}f\left(\frac{k}{n+k}\right) \\
&= \sum_{k=0}^{\infty}\binom{n+k}{k}x^k(1-x)^{n+1}f\left(\frac{k}{n+k}\right) = M_n(f,x).
\end{aligned}$$

□

Lemma 2.6. *For every $f \in C(w)[0,1)$ and for every $n \in \mathbb{N}$ we have*

$$\left\|w(M_n f - f)\right\|_{[0,1)} = \left\|\tilde{w}(V_n \tilde{f} - \tilde{f})\right\|_{[0,\infty)}.$$

3 Proof of Theorem 1.1 and some other results for MKZ

First, we note that for a weight w, defined by (1.2), the according weight $\tilde{w}(\tilde{x})$ is

$$\tilde{w}(\tilde{x}) = S^{-1}(w)(\tilde{x}) = \tilde{x}^{\gamma_0}(1+\tilde{x})^{-1-\gamma_0-\gamma_1}. \tag{3.1}$$

Then the Theorem 1.1 follows from Lemma 2.6, Lemma 2.5 and Theorem 1 in [5].

From Lemma 2.6, Lemma 2.3 and Lemma 5 in [5] we obtain the next Jackson-type inequality.

Theorem 3.1. *For w, defined by* (1.2) *there exists a constant C such that for every natural $n \geq |1+\gamma_0+\gamma_1|$ we have*

$$\left\|w(M_n f - f)\right\|_{[0,1)} \leq \frac{C}{n} \left\|w\varphi D^2 f\right\|_{[0,1)}$$

for every function $f \in W^2(w\varphi)[0,1)$.

From the definition of T, Lemma 2.3, Lemma 2.5 and Lemma 7 in [5] we obtain the next Bernstein-type inequality.

Theorem 3.2. *For w, defined by* (1.2) *there exists a constant C such that for every natural $n \geq |1+\gamma_0+\gamma_1|$ we have*

$$\left\|w\varphi D^2 M_n f\right\|_{[0,1)} \leq Cn\|wf\|_{[0,1)}$$

for every function $f \in C(w)[0,1)$.

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