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ACADEMICIAN LJUBOMIR ILIEV AND THE CLASSICAL COMPLEX ANALYSIS

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In his memoir *Uber die Anzahl der Primzahlen unter einer gegebenen Grosse, Monatsber. der Konigl. Preuss. Akad. der Wiss. zu Berlin aus dem Jahr* 1859 (1860), 671–680, B. Riemann stated the famous hypothesis about the non-trivial zeros of the functions $\zeta(s)$, which is neither proved nor rejected till now, namely that all they are on the line $\mathrm{Re}\, s = 1/2$. It is equivalent to the hypothesis that the introduced by him entire function

$$\xi(z) = s(s-1)\pi^{-s/2}\Gamma(s/2)\zeta(s), \quad s = 1/2 + z$$

has only real zeros.

At the end of 19-th century but mostly in the first decades of 20-th one, the efforts of a number of mathematicians including first-class ones as Jensen, Pólya, Hardy and Titchmarsh turned to the problem of zero-distribution of entire functions defined as Fourier transforms of the kind

$$(1) \qquad \int_a^b F(t)\exp(izt)\,dt, \quad -\infty \le a < b \le \infty.$$

Indisputable motive for their investigations is Riemann's representation of the function ξ in this form with an even function F on the interval $(-\infty, \infty)$. This subjects attracts also L. Tchakalov and N. Obrechkov. The first Bulgarian publications in this field, due to them, are influenced by the results of Pólya about the zero-distribution of entire functions of the kind

$$\int_{-a}^{a} f(t)\exp(izt)\,dt, \quad 0 < a < \infty \tag{2}$$

and of their particular cases

$$\int_{0}^{a} f(t)\cos zt\,dt \tag{3}$$

and

$$\int_{0}^{a} f(t)\sin zt\,dt \tag{4}$$

This direction becomes also one of the fields of intensive studies of academician Iliev.

An essential role in Pólya's investigations plays an algebraic statement, most frequently called the theorem of Kakeya, saying that if $a_0 < a_1 < \cdots < a_n$, $n \in \mathbb{N}$, then the zeros of the polynomials $\sum_{k=0}^{n} a_k z^k$ are in the unit disk $D := \{z \in \mathbb{C} : |z| < 1\}$. Using it and applying the method of variation of the argument, Pólya obtains his famous result for reality and mutually interlacing of the zeros of entire functions (3) and (4) provided the function f is positive and increasing in the interval $(0, a)$.

Another approach to the problem of zero-distribution of entire functions of the kind (3) and (4) is due to academician Iliev. It is based on his result that if the zeros of the algebraic polynomial P of degree $n \in \mathbb{N}$ are in the region $\{z \in \mathbb{C} : |z| > 1\}$ and P^* is the polynomial defined by $P^*(z) = z^n\overline{P(1/\overline{z})}$, then the zeros of the polynomial $P(z) + \gamma z^k P^*(z)$, $|\gamma| = 1$, $k \in \mathbb{N}_0$, are on the unit circle. This assertion, as well as the successful use of an algebraic result of N. Obreshkov, leads to the most essential achievement of academician Iliev. It says that if the function f is positive and increasing in the interval $(0, a)$, $0 < a < \infty$, and the zeros of the algebraic polynomial p are in the strip $\{z \in \mathbb{C} : \lambda \le \operatorname{Re} z \le \mu\}$, then the zeros of the polynomial

$$\int_{0}^{a} f(t)\{p(z+t) + \gamma p(z-t)\}\,dt, \quad |\gamma| = 1, \tag{5}$$

are in the same strip too. The classical results of Pólya can be obtained by setting $p(z) = z^n$, $\gamma = \pm 1$, and letting n to go to infinity. Indeed, then the polynomials

$$P_n(f; z) = \int_{0}^{a} f(t)\left\{\left(1 + \frac{izt}{n}\right)^n + \gamma\left(1 - \frac{izt}{n}\right)^n\right\} dt, \quad n \in \mathbb{N},$$

have only real zeros and, moreover,

$$\lim_{n\to\infty} P_n(f;z) = \int_0^a f(t)\{\exp(izt) + \gamma\exp(-izt)\}\,dt$$

uniformly on each bounded subset of $\mathbb{C}$.

A brilliant realization of one of the most fruitful ideas of academician Iliev concerns the class $E(a)$ of entire functions (3) having only real zeros. If $A(a)$, $0 < a < \infty$, denotes the set of the real functions $x(t)$, $t \in \mathbb{R}$, such that $x(a) = 0$ and, moreover, $x'(it)$, $t \in \mathbb{R}$, is a restriction to the real axes of a function of the Laguerre–Pólya class, i.e. it is either a real polynomial with only real zeros or an uniform limit of such polynomials. A witty algorithm ensures "reproduction" of the class. Its first application is that if $x(t) \in A(a)$, $x(0) > 0$, and $\lambda > -1$, then the entire function

$$\int_0^a x^\lambda(t)\cos zt\,dt$$

has only real zeros. The particular case when $x(t) = 1 - t^{2q}$, $q \in \mathbb{N}$, leads to a result of Pólya saying that the entire function

$$\int_0^1 (1 - t^{2q})\cos zt\,dt$$

has only real zeros.

The next application is one of the most significant achievements of academician Iliev which states that if $\varphi(t)$, $t \in \mathbb{R}$, is a real, nonnegative, and even function, such that $\varphi'(it)$ is a restriction to the real axes of a function from the Laguerre–Pólya class, then the entire function

$$\int_0^\infty \exp(-\varphi(t))\cos zt\,dt$$

has only real zeros. The particular case when $\varphi(t) = a\cosh t$, $a > 0$, is the well-known result of Pólya for the reality of the zeros of the entire function

$$\int_0^\infty \exp(-a\cosh t)\cos zt\,dt.$$

Weierstrass gave the first example of a convergent power series which is non-continuable outside its circle of convergence, i.e. this circle is the domain of existence for the analytic function defined by its sum. This example became a starting point of a great number of investigations on the singular points of functions defined by convergent power series and their analytical non-continuability.

The contributions of academician Iliev in this field are obtained mainly under the influence of the works of such experts in the classical complex analysis as Hadamard, Ostrowski, Fabry and Szegö.

Due to Szegö is the result that if each of the coefficients of the power series

$$\sum_{n=0}^{\infty} a_n z^n \tag{6}$$

is equal to one of the finitely many distinct complex numbers $d_1, d_2, \dots, d_s, d_j \neq d_k$, $j \neq k$, then it is either analytically non-continuable outside the unit disk, or the Maclorain series of a rational function of the kind

$$\frac{P(z)}{1-z^m}, \quad m \in \mathbb{N},$$

where P is an algebraic polynomial, and this is possible if and only if the sequence of its coefficients is periodic after some subscript. Essential generalizations, extensions and various modifications of this result are obtained by academician Iliev. Typical one is the assertion for the series (6) with coefficients of the kind $a_n \gamma_n c_n$, $n \in \mathbb{N}_0$. If the members of the sequence $\{\gamma_n\}_0^\infty$ accept finitely many values and for some $\alpha \in \mathbb{R}$ the sequence $\{c_n n^\alpha\}$ has a finite number of limit points all different from zero, then the requirement for non-periodicity after each subscript of the sequence $\{\gamma_n\}$ is a sufficient condition for analytical non-continuability of the series (6).

It seems that academician Iliev was the first who obtained Szegö's type theorems for Dirichlet series of he kind

$$\sum_{n=0}^{\infty} \gamma_n c_n \exp(-\lambda_n s)$$

In the first decades of the past century, mainly after the works of P. Köbe, a new branch of Geometric Function Theory came into being. It is known now as Theory of Univalent Functions. Its main object is the class S of functions f which are holomorphic and univalent in the unit disk and are normalized by the conditions $f(0) = 0$, $f'(0) = 1$, i.e. the functions with Maclorain expansion of the kind

$$f(z) = z + a_2 z^2 + a_3 z^3 + \dots$$

in the unit disk and such that $f(z_1) \neq f(z_2)$ whenever $z_1 \neq z_2$. Different subclasses of S, e.g. defined by additional requirements for convexity of the image

$f(D)$ of the unit disk by means of a function $f \in S$, or by starlikeness of this image with respect to the zero point, are also studied. The central object of the efforts of a great number of investigators are the theorems of deformation and coefficients estimates. The famous Bieberbach's conjecture, that $|a_n| \leq n$, $n = 2, 3, \dots$, which remained open till its confirmation by Lui de Brange, was one of the stimuli for publishing a great number of papers in prestigious journals and other editions. In this field, except Köbe and Bieberbach, many mathematicians as Littlewood, Hayman, Lëwner, Szego, Golusin and others have remarkable contributions.

Academician Iliev didn't remain indifferent to this direction. The results, published in his papers devoted to the univalent functions, were created during a very short period. The main attention of their author was directed to the class S_k of the k-symmetric functions f_k from the class S, i.e. to those of them having Maclorain expansion of the kind

$$f_k(z) = z + a_1^{(k)} z^{k+1} + a_2^{(k)} z^{2k+1} + \dots . \tag{7}$$

One of the first results of academician Iliev is influenced by a theorem of Szegö about the divided difference of the functions in the class S. The successful use of a similar theorem of Goluzin for the class Σ of the functions f meromorphic and univalent in the region $\{\mathbb{C} \setminus \overline{D}\} \cup \{\infty\}$ and normalized by $f(\infty) = \infty$, $f'(\infty) = 1$, leads him to the inequalities

$$\frac{1 - r^2}{(1 + r^2)^{4/k}} \leq \left| \frac{f_k(z_1) - f_k(z_2)}{z_1 - z_2} \right| \leq \frac{1}{(1 - r^2)(1 - r^k)^{4/k}} \tag{8}$$

for each function $f_k \in S_k$ provided that $0 < |z_j| \leq r < 1$, $j = 1, 2$, $z_1 \neq z_2$. Its application leads to the result that the exact radius of univalence of the partial sums

$$\sigma_n^{(2)} = z + a_3^{(2)} z^3 + \dots + a_{2n+1}^{(2)} z^{2n+1}, \quad n = 1, 2, 3, \dots \tag{9}$$

of the function from the class S_2, i.e. the class of odd functions in S, is equal to $1/\sqrt{3}$.

Another application is that the partial sum

$$\sigma_n(z) = z + a_2 z^2 + \dots + a_n z^n$$

of a function from the class S is univalent in the circle $|z| < 1 - 4 \log n/n$ for each $n \geq 15$, which improves a result of V. Levin. Similar result for the partial sums (9) of the functions from the class S_2 is that they are univalent in the circle $|z| < \sqrt{1 - 3\log n/n}$ for $n \geq 12$.

The problem for the radius of univalence of the partial sums

$$\sigma_n^{(3)}(z) = z + a_1^{(3)} z^4 + \cdots + a_n^{(3)} z^{3n+1}$$

of the 3-symmetric functions is also treated by academician Iliev. As a result it is obtained that it is $\sqrt[3]{3}/2$ for $n \neq 2$. The proof is based on coefficients estimates for the functions from the class S_3 as well as on the left inequality in (9). For $n = 2$ its exactness is proved directly by the method of Löwner.

The inequality

$$\left| \frac{f_k(z_1) - f_k(z_2)}{z_1 - z_2} \right| \geq \frac{1 - r^2}{(1 + r^2)^{2/k}}, \quad |z_j| \leq r, o < r < 1, z_1 \neq z_2$$

is obtained under the additional assumption that the function f_k is convex. It is exact for $k = 1, 2$, i.e. for the functions from the class S as well as for the odd functions in this class. By its help the circle defined by the inequality $|z| < \{1 - (1 + 2/k)\log(n + 1)/(n + 1)\}^{1/k}$ is found where the partial sum

$$z + a_1^{(k)} z^{k+1} + \cdots + a_n^{(k)} z^{nk+1}$$

of a function $f_k \in S_k$ is univalent for $n > \exp(k\sqrt{2k})/(2 + k) - 1$.

An inequality for the divided difference of bounded functions in the class S_k is obtained, i.e. for the functions $f_k \in S_k$ such that $f_k(D)$ is a bounded domain.

The already mentioned contributions of academician Iliev and many others, e.g. for the inequality of Hamburger and Turan, for the problem of Pompeiu as well as for the numerical method based on the Newton iterations assigned him a merited position of one of the distinguished experts in actual areas of mathematical analysis where his effort has been directed during several decades in the past century.