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SOME MULTIPLIER SEQUENCE SPACES OVER n-NORMED SPACES DEFINED BY A MUSIELAK–ORLICZ FUNCTION

Kuldip Raj, Sunil K. Sharma

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Abstract. In the present paper we introduce some multiplier sequence spaces over n-normed spaces defined by a Musielak–Orlicz function $\mathcal{M} = (M_k)$. We also study some topological properties and some inclusion relations between these spaces.

1. Introduction and preliminaries. The notion of difference sequence spaces was introduced by Kızmaz [9], who studied the difference sequence spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$. The notion was further generalized by Et and Colak [3] by introducing the spaces $l_\infty(\Delta^n)$, $c(\Delta^n)$ and $c_o(\Delta^n)$. Let w be the space of all real or complex sequences $x = (x_k)$. Let m, n be non-negative integers, then for $Z = l_\infty$, c and c_0, we have sequence spaces,

$$Z(\Delta^n_m) = \{x = (x_k) \in w\colon (\Delta^n_m x_k) \in Z\}$$

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where $\Delta_m^n x = (\Delta_m^n x_k) = (\Delta_m^{n-1} x_k - \Delta_m^{n-1} x_{k+m})$ and $\Delta_m^0 x_k = x_k$ for all $k \in \mathbb{N}$, which is equivalent to the following binomial representation

$$\Delta_m^n x_k = \sum_{v=0}^{n} (-1)^v \binom{n}{v} x_{k+mv}.$$

Taking $m = n = 1$, we get the spaces $l_\infty(\Delta)$, $c(\Delta)$ and $c_0(\Delta)$ introduced and studied by Kızmaz [9].

Let X be a linear metric space. A function $p\colon X \to \mathbb{R}$ is called paranorm, if

(1) $p(x) \geq 0$, for all $x \in X$;

(2) $p(-x) = p(x)$, for all $x \in X$;

(3) $p(x + y) \leq p(x) + p(y)$, for all $x, y \in X$;

(4) if (σ_n) is a sequence of scalars with $\sigma_n \to \sigma$ as $n \to \infty$ and (x_n) is a sequence of vectors with $p(x_n - x) \to 0$ as $n \to \infty$, then $p(\sigma_n x_n - \sigma x) \to 0$ as $n \to \infty$.

A paranorm p for which $p(x) = 0$ implies $x = 0$ is called total paranorm and the pair (X, p) is called a total paranormed space. It is well known that the metric of any linear metric space is given by some total paranorm (see [25], Theorem 10.4.2, P-183).

An Orlicz function M is a function, which is continuous, non-decreasing and convex with $M(0) = 0$, $M(x) > 0$ for $x > 0$ and $M(x) \longrightarrow \infty$ as $x \longrightarrow \infty$.

Lindenstrauss and Tzafriri [10] used the idea of Orlicz function to define the following sequence space. Let w be the space of all real or complex sequences $x = (x_k)$, then

$$\ell_M = \Big\{ x \in w\colon \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) < \infty \Big\}$$

which is called as an Orlicz sequence space. The space ℓ_M is a Banach space with the norm

$$\|x\| = \inf \Big\{ \rho > 0\colon \sum_{k=1}^{\infty} M\Big(\frac{|x_k|}{\rho}\Big) \leq 1 \Big\}.$$

It is shown in [10] that every Orlicz sequence space ℓ_M contains a subspace isomorphic to $\ell_p (p \geq 1)$. The Δ_2-condition is equivalent to $M(Lx) \leq kLM(x)$ for all values of $x \geq 0$, and for $L > 1$. An Orlicz function M can always be represented in the following integral form

$$M(x) = \int_0^x \eta(t)dt$$

where η is known as the kernel of M, is right differentiable for $t \geq 0, \eta(0) = 0, \eta(t) > 0$, η is non-decreasing and $\eta(t) \to \infty$ as $t \to \infty$.

A sequence $\mathcal{M} = (M_k)$ of Orlicz function is called a Musielak-Orlicz function see [13, 19]. A sequence $\mathcal{N} = (N_k)$ defined by

$$N_k(v) = \sup\{|v|u - M_k(u) \colon u \geq 0\},\ k = 1, 2, \ldots$$

is called the complementary function of a Musielak–Orlicz function $\mathcal{M}$. For a given Musielak–Orlicz function $\mathcal{M}$, the Musielak–Orlicz sequence space $t_{\mathcal{M}}$ and its subspace $h_{\mathcal{M}}$ are defined as follows

$$t_{\mathcal{M}} = \Big\{x \in w \colon I_{\mathcal{M}}(cx) < \infty \text{ for some } c > 0\Big\},$$

$$h_{\mathcal{M}} = \Big\{x \in w \colon I_{\mathcal{M}}(cx) < \infty \text{ for all } c > 0\Big\},$$

where $I_{\mathcal{M}}$ is a convex modular defined by

$$I_{\mathcal{M}}(x) = \sum_{k=1}^{\infty} M_k(x_k), x = (x_k) \in t_{\mathcal{M}}.$$

We consider $t_{\mathcal{M}}$ equipped with the Luxemburg norm

$$\|x\| = \inf\Big\{k > 0 \colon I_{\mathcal{M}}\Big(\frac{x}{k}\Big) \leq 1\Big\}$$

or equipped with the Orlicz norm

$$\|x\|^0 = \inf\Big\{\frac{1}{k}\Big(1 + I_{\mathcal{M}}(kx)\Big) \colon k > 0\Big\}.$$

A Musielak–Orlicz function $\mathcal{M} = (M_k)$ satisfies Δ_2-condition if each Orlicz function M_k satisfies Δ_2-condition.

A sequence of positive integers $\theta = (k_r)$ is called lacunary if $k_0 = 0$, $0 < k_r < k_{r+1}$ and $h_r = k_r - k_{r-1} \to \infty$ as $r \to \infty$. The intervals determined by θ

will be denoted by $I_r = (k_{r-1}, k_r)$ and $q_r = \dfrac{k_r}{k_{r-1}}$. The space of lacunary strongly convergent sequences N_θ was defined by Freedman et al.[4] as:

$$N_\theta = \Big\{ x \in w\colon \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} |x_k - l| = 0, \ \text{ for some } \ l \Big\}.$$

Strongly almost convergent sequence was introduced and studied by Maddox [11] and Freedman [4]. Parashar and Choudhary [20] have introduced and examined some properties of four sequence spaces defined by using an Orlicz function M, which generalized the well-known Orlicz sequence spaces $[C,1,p]$, $[C,1,p]_0$ and $[C,1,p]_\infty$. It may be noted here that the space of strongly summable sequences were discussed by Maddox [12]. Subsequently, difference sequence spaces have been discussed by several authors see [1, 2, 14, 15, 16, 18, 21, 22, 23, 24].

The concept of 2-normed spaces was initially developed by Gähler [5] in the mid of 1960's, while that of n-normed spaces one can see in Misiak [17]. Since then, many others have studied this concept and obtained various results, see Gunawan [6, 7] and Gunawan and Mashadi [8]. Let $n \in \mathbb{N}$ and X be a linear space over the field $\mathbb{K}$, where $\mathbb{K}$ is field of real or complex numbers of dimension d, where $d \geq n \geq 2$. A real valued function $\|\cdot, \ldots, \cdot\|$ on X^n satisfying the following four conditions:

(1) $\|x_1, x_2, \ldots, x_n\| = 0$ if and only if $x_1, x_2, \ldots, x_n$ are linearly dependent in X;

(2) $\|x_1, x_2, \ldots, x_n\|$ is invariant under permutation;

(3) $\|\alpha x_1, x_2, \ldots, x_n\| = |\alpha| \ \|x_1, x_2, \ldots, x_n\|$ for any $\alpha \in \mathbb{K}$, and

(4) $\|x + x', x_2, \ldots, x_n\| \leq \|x, x_2, \ldots, x_n\| + \|x', x_2, \ldots, x_n\|$

is called an n-norm on X and the pair $(X, \|\cdot, \ldots, \cdot\|)$ is called a n-normed space over the field $\mathbb{K}$.

For example, we may take $X = \mathbb{R}^n$ being equipped with the n-norm $\|x_1, x_2, \ldots, x_n\|_E =$ the volume of the n-dimensional parallelopiped spanned by the vectors $x_1, x_2, \ldots, x_n$ which may be given explicitly by the formula

$$\|x_1, x_2, \ldots, x_n\|_E = |\det(x_{ij})|,$$

where $x_i = (x_{i1}, x_{i2}, \ldots, x_{in}) \in \mathbb{R}^n$ for each $i = 1, 2, \ldots, n$ and $\|\cdot\|_E$ denotes the Euclidean norm. Let $(X, \|\cdot, \ldots, \cdot\|)$ be an n-normed space of dimension $d \geq n \geq 2$

and $\{a_1, a_2, \ldots, a_n\}$ be linearly independent set in X. Then the following function $\|\cdot, \ldots, \cdot\|_\infty$ on X^{n-1} defined by

$$\|x_1, x_2, \ldots, x_{n-1}\|_\infty = \max\{\|x_1, x_2, \ldots, x_{n-1}, a_i\| \colon i = 1, 2, \ldots, n\}$$

defines an $(n-1)$-norm on X with respect to $\{a_1, a_2, \ldots, a_n\}$. A sequence (x_k) in a n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to converge to some $L \in X$ if

$$\lim_{k\to\infty} \|x_k - L, z_1, \ldots, z_{n-1}\| = 0 \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

A sequence (x_k) in a n-normed space $(X, \|\cdot, \ldots, \cdot\|)$ is said to be Cauchy if

$$\lim_{k,\, p\to\infty} \|x_k - x_p, z_1, \ldots, z_{n-1}\| = 0 \text{ for every } z_1, \ldots, z_{n-1} \in X.$$

If every Cauchy sequence in X converges to some $L \in X$, then X is said to be complete with respect to the n-norm. Any complete n-normed space is said to be n-Banach space.

Let $\mathcal{M} = (M_k)$ be a Musielak–Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of positive reals such that $u_k \neq 0$ for all k, then we define the following sequence spaces in the present paper:

$$w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$$

$$= \Big\{x = (x_k) \in w \colon \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0, \quad \rho > 0,\ s \geq 0\Big\},$$

$$w^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$$

$$= \Big\{x = (x_k) \in w \colon \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0, \quad \text{for some } L,\ \ \rho > 0,\ s \geq 0\Big\}$$

and

$$w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$$

$$= \Big\{x = (x_k) \in w\colon \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} < \infty, \quad \rho > 0,\ s \geq 0\Big\}.$$

If we take $\mathcal{M}(x) = x$, we get

$$w_0^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) = \Big\{x = (x_k) \in w\colon \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} \Big[\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0, \quad \rho > 0,\ s \geq 0\Big\},$$

$$w^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) = \Big\{x = (x_k) \in w\colon \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} \Big[\| \frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0, \quad \text{for some } L,\ \ \rho > 0,\ s \geq 0\Big\}$$

and

$$w_\infty^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) = \Big\{x = (x_k) \in w\colon \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} \Big[\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} < \infty, \quad \rho > 0,\ s \geq 0\Big\}.$$

If we take $p = (p_k) = 1$ for all $k \in \mathbb{N}$, we have

$$w_0^\theta(\mathcal{M}, \Delta_l^m, u, s, \|\cdot, \ldots, \cdot\|) = \Big\{x = (x_k) \in w\colon \lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big] = 0, \quad \rho > 0,\ s \geq 0\Big\},$$

$$w^\theta(\mathcal{M}, \Delta_l^m, u, s, \|\cdot, \ldots, \cdot\|)$$

$$= \Big\{ x = (x_k) \in w \colon \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\| \frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big] = 0, \quad \text{for some } L, \ \ \rho > 0, \ s \geq 0 \Big\}$$

and

$$w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, s, \|\cdot, \ldots, \cdot\|) = \Big\{ x = (x_k) \in w \colon \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big] < \infty, \quad \rho > 0, \ s \geq 0 \Big\}.$$

If we take $\mathcal{M}(x) = x, s = 0$, $u = e = (1, 1, 1, \ldots, 1)$ then these spaces reduces to

$$w_0^\theta(\Delta_l^m, p, \|\cdot, \ldots, \cdot\|) = \Big\{ x = (x_k) \in w \colon \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[\| \frac{\Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} = 0, \ \ \rho > 0 \Big\},$$

$$w^\theta(\Delta_l^m, p, \|\cdot, \ldots, \cdot\|) = \Big\{ x = (x_k) \in w \colon \lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} \Big[\| \frac{\Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} = 0, \quad \text{for some } L, \ \ \rho > 0 \Big\}$$

and

$$w_\infty^\theta(\Delta_l^m, p, \|\cdot, \ldots, \cdot\|) = \Big\{ x = (x_k) \in w \colon \sup_r \frac{1}{h_r} \sum_{k \in I_r} \Big[\| \frac{\Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} < \infty, \ \rho > 0 \Big\}.$$

The following inequality will be used throughout the paper. If $0 \leq p_k \leq \sup p_k = H$, $K = \max(1, 2^{H-1})$ then

$$|a_k + b_k|^{p_k} \leq K\{|a_k|^{p_k} + |b_k|^{p_k}\} \tag{1.1}$$

for all k and $a_k, b_k \in \mathbb{C}$. Also $|a|^{p_k} \leq \max(1, |a|^H)$ for all $a \in \mathbb{C}$.

In this paper we study some topological properties and prove some inclusion relations between these spaces.

2. Main results.

Theorem 2.1. *Let $\mathcal{M} = (M_k)$ be a Musielak–Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be any sequence of strictly positive real numbers then the classes of sequences $w_0^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$, $w^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ and $w_{\infty}^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ are linear spaces over the field of complex number $\mathbb{C}$.*

Proof. Let $x = (x_k), y = (y_k) \in w_0^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ and $\alpha, \beta \in \mathbb{C}$. In order to prove the result we need to find some ρ_3 such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m(\alpha x_k + \beta y_k)}{\rho_3}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0.$$

Since $x = (x_k), y = (y_k) \in w_0^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$, there exist positive numbers $\rho_1, \rho_2 > 0$ such that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0$$

and

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m y_k}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0.$$

Define $\rho_3 = \max(2|\alpha|\rho_1, 2|\beta|\rho_2)$. Since M_k is non-decreasing, convex function and so by using inequality (1.1), we have

$$\begin{aligned}
&\frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m(\alpha x_k + \beta y_k)}{\rho_3}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} \\
&\le \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{\alpha u_k \Delta_l^m x_k}{\rho_3}, z_1, z_2, \ldots, z_{n-1}\| + \|\frac{\beta u_k \Delta_l^m y_k}{\rho_3}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} \\
&\le K \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} \\
&+ K \frac{1}{h_r} \sum_{k\in I_r} \frac{1}{2^{p_k}} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m y_k}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} \\
&\le K \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}
\end{aligned}$$

$$+ \ K\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k\Big[\|\frac{u_k\Delta_l^m y_k}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}$$

$\to 0$ as $r \to \infty$.

Thus we have $\alpha x + \beta y \in w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. Hence $w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ is a linear space. Similarly we can prove that $w^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ and $w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ are linear spaces.

Theorem 2.2. *Let $\mathcal{M} = (M_k)$ be a Musielak–Orlicz function, $p = (p_k)$ be a bounded sequence of positive real numbers and $u = (u_k)$ be a sequence of strictly positive real numbers. Then $w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ is a topological linear space paranormed by*

$$g(x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1\Big\},$$

where $H = \max(1, \sup_k p_k) < \infty$.

Proof. Clearly $g(x) \ge 0$ for $x = (x_k) \in w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. Since $M_k(0) = 0$ we get $g(0) = 0$. Again if $g(x) = 0$ then

$$\inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1\Big\} = 0.$$

This implies that for a given $\epsilon > 0$ there exist some $\rho_\epsilon (0 < \rho_\epsilon < \epsilon)$ such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho_\epsilon}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1.$$

Thus

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\epsilon}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \le \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho_\epsilon}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}}.$$

Suppose $(x_k) \ne 0$ for each $k \in \mathbb{N}$. This implies that $\Delta_l^m(x_k) \ne 0$ for each $k \in \mathbb{N}$. Let $\epsilon \to 0$ then

$$\|\frac{u_k\Delta_l^m x_k}{\epsilon}, z_1, z_2, \ldots, z_{n-1}\| \to \infty.$$

It follows that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\epsilon}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \to \infty.$$

Which is a contradiction. Therefore $\Delta_l^m(x_k) = 0$ for each k and thus $(x_k) = 0$ for each $k \in \mathbb{N}$. Let $\rho_1 > 0$ and $\rho_2 > 0$ be such that

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1$$

and

$$\Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m y_k}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1.$$

Let $\rho = \rho_1 + \rho_2$, then by using Minkowski's inequality, we have

$$\begin{aligned}
&\Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m (x_k+y_k)}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\le \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m x_k + u_k\Delta_l^m y_k}{\rho_1+\rho_2}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\le \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big(\frac{\rho_1}{\rho_1+\rho_2}\Big[\|\frac{u_k\Delta_l^m x_k}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\|\Big] \\
&+ \frac{\rho_2}{\rho_1+\rho_2}\Big[\|\frac{u_k\Delta_l^m y_k}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\|\Big]\Big)^{p_k}\Big)^{\frac{1}{H}} \\
&\le \Big(\frac{\rho_1}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&+ \Big(\frac{\rho_2}{\rho_1+\rho_2}\Big)\Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m y_k}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \\
&\le 1.
\end{aligned}$$

Since ρ, ρ_1 and ρ_2 are non-negative, so we have

$$\begin{aligned}
&g(x+y) \\
&= \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m (x_k+y_k)}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \le 1\Big\}
\end{aligned}$$

$$\leq \inf\Big\{(\rho_1)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}$$

$$+ \inf\Big\{(\rho_2)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m y_k}{\rho_2}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

Therefore $g(x+y) \leq g(x) + g(y)$. Finally we prove that the scalar multiplication is continuous. Let λ be any complex number. By definition

$$g(\lambda x) = \inf\Big\{\rho^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m \lambda x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

Thus

$$g(\lambda x) = \inf\Big\{(|\lambda| t)^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m x_k}{t}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\},$$

where $\frac{1}{t} = \frac{\rho}{|\lambda|}$. Since $|\lambda|^{p_r} \leq \max(1, |\lambda|^{\sup p_r})$, we have

$$g(\lambda x) \leq \max(1, |\lambda|^{\sup p_r}) \inf\Big\{t^{\frac{p_r}{H}} : \Big(\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m x_k}{t}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}\Big)^{\frac{1}{H}} \leq 1\Big\}.$$

So the fact that scalar multiplication is continuous follows from the above inequality. This completes the proof of the theorem. □

Theorem 2.3. *Let $\mathcal{M} = (M_k)$ be a Musielak–Orlicz function. If $\sup_k [M_k(x)]^{p_k} < \infty$ for all fixed $x > 0$, then*

$$w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subseteq w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|).$$

Proof. Let $x = (x_k) \in w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$, then there exists positive number ρ_1 such that

$$\lim_{r\to\infty} \frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho_1}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0.$$

Define $\rho = 2\rho_1$. Since M_k is non-decreasing, convex and so by using inequality (1.1), we have

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k}$$

$$\begin{aligned}
&= \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\| \frac{u_k \Delta_l^m x_k + L - L}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} \\
&\leq K \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} \frac{1}{2^{p_k}} M_k \Big[\| \frac{u_k \Delta_l^m x_k - L}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} \\
&+ K \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} \frac{1}{2^{p_k}} M_k \Big[\| \frac{L}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} \\
&\leq K \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\| \frac{u_k \Delta_l^m x_k - L}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} \\
&+ K \sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\| \frac{L}{\rho_1}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} \\
&< \infty.
\end{aligned}$$

Hence $x = (x_k) \in w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. □

Theorem 2.4. *Let* $0 < \inf p_k = h \leq p_k \leq \sup p_k = H < \infty$ *and* $\mathcal{M} = (M_k), \mathcal{M}' = (M_k')$ *be Musielak-Orlicz functions satisfying* Δ_2*-condition, then we have*

(i) $w_0^\theta(\mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(\mathcal{M} \circ \mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$;

(ii) $w^\theta(\mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w^\theta(\mathcal{M} \circ \mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$;

(iii) $w_\infty^\theta(\mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathcal{M} \circ \mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$.

Proof. Let $x = (x_k) \in w_0^\theta(\mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ then we have

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k' \Big[\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} = 0.$$

Let $\epsilon > 0$ and choose δ with $0 < \delta < 1$ such that $M_k(t) < \epsilon$ for $0 \leq t \leq \delta$. Let

$(y_k)^{p_k} = M'_k\Big[\|\dfrac{u_k\Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k}$ for all $k \in \mathbb{N}$. We can write

$$\frac{1}{h_r}\sum_{k\in I_r} k^{-s} M_k[y_k]^{p_k} = \frac{1}{h_r}\sum_{k\in I_r, y_k\le\delta} k^{-s} M_k[y_k]^{p_k} + \frac{1}{h_r}\sum_{k\in I_r, y_k\ge\delta} k^{-s} M_k[y_k]^{p_k}.$$

So we have

$$\begin{aligned}\frac{1}{h_r}\sum_{k\in I_r, y_k\le\delta} k^{-s} M_k[y_k]^{p_k} &\le [M_k(1)]^H \frac{1}{h_r}\sum_{k\in I_r, y_k\le\delta} k^{-s} M_k[y_k]^{p_k}\\ &\le [M_k(2)]^H \frac{1}{h_r}\sum_{k\in I_r, y_k\le\delta} k^{-s} M_k[y_k]^{p_k} \qquad (2.1)\end{aligned}$$

For $y_k > \delta, y_k < \dfrac{y_k}{\delta} < 1 + \dfrac{y_k}{\delta}$. Since $M_k's$ are non-decreasing and convex, it follows that

$$M_k(y_k) < M_k\big(1 + \frac{y_k}{\delta}\big) < \frac{1}{2}M_k(2) + \frac{1}{2}M_k\big(\frac{2y_k}{\delta}\big).$$

Since $\mathcal{M} = (M_k)$ satisfies Δ_2-condition, we can write

$$M_k(y_k) < \frac{1}{2}T\frac{y_k}{\delta}M_k(2) + \frac{1}{2}T\frac{y_k}{\delta}M_k(2) = T\frac{y_k}{\delta}M_k(2).$$

Hence,

$$\frac{1}{h_r}\sum_{k\in I_r, y_k\ge\delta} k^{-s} M_k[y_k]^{p_k} \le \max\Big(1, (T\frac{M_k(2)}{\delta})^H\Big)\frac{1}{h_r}\sum_{k\in I_r, y_k\le\delta} k^{-s}[y_k]^{p_k} \qquad (2.2)$$

From equation (2.1) and (2.2), we have $x = (x_k) \in w_0^\theta(\mathcal{M}\circ\mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. This completes the proof of (i). Similarly we can prove that

$$w^\theta(\mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w^\theta(\mathcal{M}\circ\mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$$

and

$$w_\infty^\theta(\mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathcal{M}\circ\mathcal{M}', \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|). \qquad \square$$

Theorem 2.5. *Let* $0 < h = \inf p_k = p_k < \sup p_k = H < \infty$. *Then for a Musielak–Orlicz function* $\mathcal{M} = (M_k)$ *which satisfies* Δ_2*-condition, we have*

(i) $w_0^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$;

(*ii*) $w^{\theta}(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$;

(*iii*) $w_{\infty}^{\theta}(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_{\infty}^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$.

Proof. It is easy to prove so we omit the details. □

Theorem 2.6. *Let $\mathcal{M} = (M_k)$ be a Musielak–Orlicz function and $0 < h = \inf p_k$. Then $w_{\infty}^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_0^{\theta}(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ if and only if*

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k(t)^{p_k} = \infty \tag{2.3}$$

for some $t > 0$.

Proof. Let $w_{\infty}^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_0^{\theta}(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. Suppose that (2.3) does not hold. Therefore there are subinterval $I_{r(j)}$ of the set of interval I_r and a number $t_0 > 0$, where

$$t_0 = \|\frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\| \text{ for all } k,$$

such that

$$\frac{1}{h_{r(j)}} = \sum_{k \in I_{r(j)}} k^{-s} M_k(t_0)^{p_k} \le K < \infty, m = 1, 2, 3, \ldots \tag{2.4}$$

let us define $x = (x_k)$ as follows:

$$\Delta_l^m x_k = \begin{cases} \rho t_0, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases}.$$

Thus, by (2.4), $x \in w_{\infty}^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. But $x \notin w_{\infty}^{0}(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. Hence (2.3) must hold.

Conversely, suppose that (2.3) holds and that $x \in w_{\infty}^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. Then for each r,

$$\frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} \le K < \infty. \tag{2.5}$$

Suppose that $x \notin w_0^{\theta}(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. Then for some number $\epsilon > 0$, there is a number k_0 such that for a subinterval $I_{r(j)}$, of the set of interval I_r,

$$\|\frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\| > \epsilon \text{ for } k \ge k_0.$$

From properties of sequence of Orlicz function, we obtain

$$M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} \geq M_k(\epsilon)^{p_k}$$

which contradicts (2.3), by using (2.5). Hence we get

$$w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_0^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|).$$

This completes the proof. □

Theorem 2.7. *Let $\mathcal{M} = (M_k)$ be a Musielak–Orlicz function. Then the following statements are equivalent:*

(i) $w_\infty^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$;

(ii) $w_0^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$;

(iii) $\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k(t)^{p_k} < \infty$ *for all* $t > 0$.

Proof. $(i) \Rightarrow (ii)$. Let (i) holds. To verify (ii), it is enough to prove

$$w_0^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subset w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|).$$

Let $x = (x_k) \in w_0^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. Then for $\epsilon > 0$ there exists $r \geq 0$, such that

$$\frac{1}{h_r} \sum_{k \in I_r} k^{-s}\Big[\|\frac{u_k\Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} < \epsilon.$$

Hence there exists $K > 0$ such that

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s}\Big[\|\frac{u_k\Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} < K.$$

So we get $x = (x_k) \in w_\infty^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$.
$(ii) \Rightarrow (iii)$. Let (ii) holds. Suppose (iii) does not hold. Then for some $t > 0$

$$\sup_r \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k(t)^{p_k} = \infty$$

and therefore we can find a subinterval $I_{r(j)}$, of the set of interval I_r such that

$$\frac{1}{h_{r(j)}}\sum_{k\in I_{r(j)}} k^{-s}M_k(\frac{1}{j})^{p_k} > j,\ j=1,2,3,\dots \tag{2.6}$$

Let us define $x=(x_k)$ as follows:

$$\Delta_l^m x_k = \begin{cases} \frac{\rho}{j}, & k\in I_{r(j)} \\ 0, & k\notin I_{r(j)} \end{cases}.$$

Then $x=(x_k)\in w_0^\theta(\Delta_l^m,u,p,s,\|\cdot,\dots,\cdot\|)$. But by (2.6), $x\notin w_\infty^\theta(\mathcal{M},\Delta_l^m,u,p,s,\|\cdot,\dots,\cdot\|)$, which contradicts (ii). Hence (iii) must holds.
$(iii)\Rightarrow(i)$. Let (iii) holds and suppose $x=(x_k)\in w_\infty^\theta(\Delta_l^m,u,p,s,\|\cdot,\dots,\cdot\|)$. Suppose that $x=(x_k)\notin w_\infty^\theta(\mathcal{M},\Delta_l^m,u,p,s,\|\cdot,\dots,\cdot\|)$, then

$$\sup_r\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k\Big[\|\frac{u_k\Delta_l^m x_k}{\rho},z_1,z_2,\dots,z_{n-1}\|\Big]^{p_k}=\infty. \tag{2.7}$$

Let $t=\|\frac{u_k\Delta_l^m x_k}{\rho},z_1,z_2,\dots,z_{n-1}\|$ for each k, then by (2.7)

$$\sup_r\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k(t)^{p_k}=\infty$$

which contradicts (iii). Hence (i) must holds. □

Theorem 2.8. *Let $\mathcal{M}=(M_k)$ be a Musielak-Orlicz function. Then the following statements are equivalent:*

(i) $w_0^\theta(\mathcal{M},\Delta_l^m,u,p,s,\|\cdot,\dots,\cdot\|)\subset w_0^\theta(\Delta_l^m,u,p,s,\|\cdot,\dots,\cdot\|)$;

(ii) $w_0^\theta(\mathcal{M},\Delta_l^m,u,p,s,\|\cdot,\dots,\cdot\|)\subset w_\infty^\theta(\Delta_l^m,u,p,s,\|\cdot,\dots,\cdot\|)$;

(iii) $\inf_r\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k(t)^{p_k}>0$ *for all* $t>0$.

Proof. $(i)\Rightarrow(ii)$. It is obvious.
$(ii)\Rightarrow(iii)$. Let (ii) holds. Suppose that (iii) does not hold. Then

$$\inf_r\frac{1}{h_r}\sum_{k\in I_r} k^{-s}M_k(t)^{p_k}=0 \text{ for some } t>0,$$

and we can find a subinterval $I_{r(j)}$, of the set of interval I_r such that

$$\frac{1}{h_{r(j)}} \sum_{k \in I_{r(j)}} k^{-s} M_k(j)^{p_k} < \frac{1}{j},\ j = 1, 2, 3, \ldots \tag{2.8}$$

Let us define $x = (x_k)$ as follows:

$$\Delta_l^m x_k = \begin{cases} \rho j, & k \in I_{r(j)} \\ 0, & k \notin I_{r(j)} \end{cases}.$$

Thus by (2.8), $x = (x_k) \in w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$ but $x = (x_k) \notin w_\infty^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. Which contradicts (ii). Hence (iii) must holds.

$(iii) \Rightarrow (i)$. Let (iii) holds. Suppose that $x = (x_k) \in w_0^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. Then

$$\frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k \Big[\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} \to 0 \text{ as } r \to \infty. \tag{2.9}$$

Again suppose that $x = (x_k) \notin w_0^\theta(\Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. for some number $\epsilon > 0$ and a subinterval $I_{r(j)}$, of the set of interval I_r. we have

$$\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \geq \epsilon \text{ for all } k.$$

Then from properties of the Orlicz function, we can write

$$M_k \Big[\| \frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1} \| \Big]^{p_k} \geq M_k(\epsilon)^{p_k}.$$

consequently, by (2.9), we have

$$\lim_{r \to \infty} \frac{1}{h_r} \sum_{k \in I_r} k^{-s} M_k(\epsilon)^{p_k} = 0$$

which contradicts (iii). Hence (i) must holds. □

Theorem 2.9. *Let $0 \leq p_k \leq q_k$ for all k and let $(\frac{q_k}{p_k})$ be bounded. Then*

$$w^\theta(\mathcal{M}, \Delta_l^m, u, q, s, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|).$$

P r o o f. Let $x = (x_k) \in w^\theta(\mathcal{M}, \Delta_l^m, u, q, s, \|\cdot, \ldots, \cdot\|)$, write

$$t_k = M_k\Big[\|\frac{u_k \Delta_l^m x_k}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{q_k}$$

and $\mu_k = \dfrac{p_k}{q_k}$ for all $k \in \mathbb{N}$. Then $0 < \mu_k \leq 1$ for all $k \in \mathbb{N}$. Take $0 < \mu \leq \mu_k$ for $k \in \mathbb{N}$. Define sequences (u_k) and (v_k) as follows:
For $t_k \geq 1$, let $u_k = t_k$ and $v_k = 0$ and for $t_k < 1$, let $u_k = 0$ and $v_k = t_k$. Then clearly for all $k \in \mathbb{N}$, we have

$$t_k = u_k + v_k, t_k^{\mu_k} = u_k^{\mu_k} + v_k^{\mu_k}.$$

Now it follows that $u_k^{\mu_k} \leq u_k \leq t_k$ and $v_k^{\mu_k} \leq v_k^{\mu}$. Therefore,

$$\begin{aligned} \frac{1}{h_r}\sum_{k\in I_r} t_k^{\mu_k} &= \frac{1}{h_r}\sum_{k\in I_r}(u_k^{\mu_k} + v_k^{\mu_k}) \\ &\leq \frac{1}{h_r}\sum_{k\in I_r} t_k + \frac{1}{h_r}\sum_{k\in I_r} v_k^{\mu}. \end{aligned}$$

Now for each k,

$$\begin{aligned} \frac{1}{h_r}\sum_{k\in I_r} v_k^{\mu} &= \sum_{k\in I_r}\Big(\frac{1}{h_r}v_k\Big)^{\mu}\Big(\frac{1}{h_r}\Big)^{1-\mu} \\ &\leq \Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{h_r}v_k\Big)^{\mu}\Big]^{\frac{1}{\mu}}\Big)^{\mu}\Big(\sum_{k\in I_r}\Big[\Big(\frac{1}{h_r}\Big)^{1-\mu}\Big]^{\frac{1}{1-\mu}}\Big)^{1-\mu} \\ &= \Big(\frac{1}{h_r}\sum_{k\in I_r} v_k\Big)^{\mu} \end{aligned}$$

and so

$$\frac{1}{h_r}\sum_{k\in I_r} v_k^{\mu} \leq \frac{1}{h_r}\sum_{k\in I_r} t_k + \Big(\frac{1}{h_r}\sum_{k\in I_r} v_k\Big)^{\mu}.$$

Hence $x = (x_k) \in w^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$. This completes the proof of the theorem. □

Theorem 2.10. (*i*) *If* $0 < \inf p_k \leq p_k \leq 1$ *for all* $k \in \mathbb{N}$, *then*

$$w^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathcal{M}, \Delta_l^m, u, s, \|\cdot, \ldots, \cdot\|).$$

(ii) If $1 \leq p_k \leq \sup p_k = H < \infty$, *for all* $k \in \mathbb{N}$, *then*

$$w^{\theta}(\mathcal{M}, \Delta_l^m, u, s, \|\cdot, \ldots, \cdot\|) \subseteq w^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|).$$

Proof. (i) Let $x = (x_k) \in w^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$, then

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0.$$

Since $0 < \inf p_k \leq p_k \leq 1$. This implies that

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big] \leq \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k},$$

therefore,

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big] = 0.$$

Therefore

$$w^{\theta}(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) \subseteq w^{\theta}(\mathcal{M}, \Delta_l^m, u, s, \|\cdot, \ldots, \cdot\|).$$

(ii) Let $p_k \geq 1$ for each k and $\sup p_k < \infty$. Let $x = (x_k) \in w^{\theta}(\mathcal{M}, \Delta_l^m, u, s, \|\cdot, \ldots, \cdot\|)$, then for each $\rho > 0$, we have

$$\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} = 0 < 1.$$

Since $1 \leq p_k \leq \sup p_k < \infty$, we have

$$\begin{aligned}
\lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big]^{p_k} & \\
\leq \lim_{r\to\infty} \frac{1}{h_r} \sum_{k\in I_r} k^{-s} M_k \Big[\|\frac{u_k \Delta_l^m x_k - L}{\rho}, z_1, z_2, \ldots, z_{n-1}\|\Big] & \\
= 0 & \\
< 1. &
\end{aligned}$$

Therefore $x = (x_k) \in w^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|)$, for each $\rho > 0$. Hence

$$w^\theta(\mathcal{M}, \Delta_l^m, u, s, \|\cdot, \ldots, \cdot\|) \subseteq w^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|).$$

This completes the proof of the theorem. □

Theorem 2.11. *If* $0 < \inf p_k \leq p_k \leq \sup p_k = H < \infty$, *for all* $k \in \mathbb{N}$, *then*

$$w^\theta(\mathcal{M}, \Delta_l^m, u, p, s, \|\cdot, \ldots, \cdot\|) = w^\theta(\mathcal{M}, \Delta_l^m, u, s, \|\cdot, \ldots, \cdot\|).$$

Proof. It is easy to prove so we omit the details.

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Kuldip Raj
e-mail: kuldeepraj68@rediffmail.com
Sunil K. Sharma
e-mail: sunilksharma42@yahoo.co.in
School of Mathematics
Shri Mata Vaishno Devi University
Katra-182320, J&K, India.

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