Serdica
Mathematical Journal
Сердика
Математическо списание

Serdica Math. J. **40** (2014), 41–54

Serdica
Mathematical Journal

Bulgarian Academy of Sciences
Institute of Mathematics and Informatics

ON THE SET-THEORETIC COMPLETE INTERSECTION PROPERTY FOR THE EDGE IDEALS OF WHISKER GRAPHS

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Communicated by V. Drensky

Abstract. We show that the edge ideals of some whisker graphs are set-theoretic complete intersections.

1. Introduction. Given a Noetherian commutative ring with identity R, the *arithmetical rank* (ara) of a proper ideal I of R is defined as the smallest integer s for which there exist s elements $a_1, \dots, a_s$ of R such that the ideal $(a_1, \dots, a_s)$ has the same radical as I. In this case we will say that $a_1, \dots, a_s$ generate I *up to radical.* In general $\mathrm{ht}(I) \leq \mathrm{ara}(I)$. If equality holds, I is called a *set-theoretic complete intersection.* We consider the case where R is a polynomial ring over a field K and I is the so-called *edge ideal* of a graph whose vertices are the indeterminates. Its set of generators is formed by the products of the pairs of indeterminates that form the edges of the graph. Thus I is generated by squarefree

2010 *Mathematics Subject Classification:* 13A15, 13F55, 14M10, 05C05, 05C38.
Key words: Set-theoretic complete intersection ideals, arithmetical rank, edge ideals, whiskers, sunlet graphs, cactus graphs.

monomials of degree 2, and is therefore a radical ideal. The arithmetical rank of edge ideals has recently been studied by several authors (see e.g. Kummini [8]) and explicitly determined for some special types of graphs. In many cases it has been proven that ara(I) coincides with the projective dimension of the quotient ring R/I, which, in general, according to a well-known result by Lyubeznik [9], provides a lower bound. This equality has been established for lexsegment edge ideals by Ene, Olteanu, Terai [5], for the edge ideals of acyclic graphs (the so-called *forests*) by Kimura and Terai [7] (extending a result by Barile [1]), for the graphs formed by one or two cycles connected through a path (*cyclic* and *bicyclic* graphs) by Barile, Kiani, Mohammadi and Yassemi [2], and for the graphs consisting of paths and cycles with a common vertex by Kiani and Mohammadi [6]. In all these cases, the arithmetical rank is independent of the field K.

As a consequence of the Auslander-Buchsbaum formula (see the proof of Corollary 5.1 for further details on this point), whenever an ideal of R generated by squarefree monomials is a set-theoretic complete intersection, it is a Cohen-Macaulay ideal. Dochtermann and Engström [4] proved that this latter property is fulfilled by the edge ideals of the graphs in which every vertex belongs to exactly one terminal edge (equivalently: every vertex of degree greater than one is adjacent to exactly one vertex of degree one). These graphs are those obtained by adding a *whisker* to each vertex of a given graph, i.e., by attaching a terminal edge to all its vertices. In the present paper we determine a large class of *whisker graphs* (which can have any number of cycles) that are set-theoretic complete intersections. This class includes all whisker graphs constructed on cyclic and byciclic graphs. It also includes all trees that give rise to Cohen-Macaulay edge ideals, and have been characterized by Villarreal [11]. The results presented in this paper are independent of the field K.

2. Preliminaries. A useful technique that provides an upper bound for the arithmetical rank of ideals is the following result due to Schmitt and Vogel.

Lemma 2.1 ([10], Lemma p. 249). *Let R be a commutative ring with identity and P be a finite subset of elements of R. Let $P_0, \dots, P_r$ be subsets of P such that*

(i) $\bigcup_{i=0}^{r} P_i = P$;

(ii) P_0 *has exactly one element;*

(iii) if p and p' are different elements of P_i $(0 < i < r)$, there is an integer i', with $0 \leq i' < i$, and an element in $P_{i'}$ which divides pp'.

We set $q_i = \sum_{p \in P_i} p^{e(p)}$*, where* $e(p) \geq 1$ *are arbitrary integers. We will write* (P) *for the ideal of* R *generated by the elements of* P*. Then*

$$\sqrt{(P)} = \sqrt{(q_0, \ldots, q_r)}.$$

In the following we will consider squarefree monomial ideals arising from graphs, the so-called *edge ideals*.

Definition 2.2. *Let* G *be a graph with vertex set* $V(G) = \{x_1, \ldots, x_n\}$*, with* $n \in \mathbb{N}$*,* $n \geq 1$*, and whose edge set is* $E(G)$*. Suppose that* $x_1, \ldots, x_n$ *are indeterminates over the field* K*. The edge ideal of* G *in the polynomial ring* $R = K[x_1, \ldots, x_n]$ *is the squarefree monomial ideal*

$$I(G) = \left(\left\{x_i x_j \;\middle|\; \{x_i, x_j\} \in E(G)\right\}\right).$$

For the sake of simplicity, we will use the same notation $x_i x_j$ *for the monomial and for the corresponding edge.*

Definition 2.3. *Let* G *be a graph and* x *a vertex of* G*. Adding a whisker to the vertex* x *of* G *means adding a new vertex* y *and the edge connecting* x *and* y *to* G*.*

For each vertex x_i of a graph G, we consider a new vertex y_i and add the whisker $x_i y_i$ to G. Let G' be the graph obtained in this way. We will call it the *whisker graph on* G.

Dochtermann and Engström [4] have shown the following result:

Theorem 2.4 ([4], Theorem 4.4)**.** *Let* G' *be the graph obtained by adding a whisker to all vertices of a graph on* n *vertices. Then the ideal* $I(G')$ *is Cohen-Macaulay and* $\operatorname{ht}(I(G')) = n$*.*

Proof. The Cohen-Macaulay property was proven in Theorem 4.4 [4]. For the second part of the claim it suffices to observe that $I(G')$ is pure (see Bruns-Herzog [3], Cor. 5.1.5) and that the ideal generated by the vertices of G is a minimal prime ideal of $I(G')$. □

3. The arithmetical rank of the edge ideals of whisker graphs on paths and cycles. In this section, we show that the edge ideals of the whisker graphs on line graphs and cycle graphs are set-theoretic complete intersections.

Let $n \in \mathbb{N}$, $n \geq 2$, and let L_n be the line graph (path) of length $n-1$, with vertex set $V(L_n) = \{x_1, \ldots, x_n\}$ and edge set $E(L_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n\}$.

For each vertex x_i consider a new vertex y_i and the whisker x_iy_i. We will adopt this notation throughout the paper. Call L'_n the graph obtained in this way.

Lemma 3.1. *With respect to the above notations,*

$$\mathrm{ara}(I(L'_n)) = \mathrm{ht}(I(L'_n)) = |V(L_n)| = n,$$

thus $I(L'_n)$ is a set-theoretic complete intersection.

Proof. If $n = 2$, set

$$\begin{aligned} q_0 &= x_1x_2 \\ q_1 &= x_1y_1 + x_2y_2. \end{aligned}$$

For each $n \geq 3$, set

$$\begin{aligned} q_0 &= x_1x_2 \\ q_1 &= x_1y_1 + x_2x_3 \\ &\vdots \\ q_{n-2} &= x_{n-2}y_{n-2} + x_{n-1}x_n \\ q_{n-1} &= x_{n-1}y_{n-1} + x_ny_n. \end{aligned}$$

Applying Lemma 2.1, we show that $I(L'_n) = \sqrt{(q_0, \ldots, q_{n-1})}$, which implies the claim. For $i = 0, \ldots, n-1$, we take P_i to be the set of the monomials of q_i. The assumptions of Lemma 2.1 are obviously fulfilled if $n = 2$. So let $n \geq 3$. Then *(i)* and *(ii)* hold true and, moreover, if $i \in \{1, \ldots, n-2\}$, the product of the two monomials in P_i is $x_iy_i \cdot x_{i+1}x_{i+2}$, which is a multiple of $x_ix_{i+1} \in P_{i-1}$, and the product of the two monomials in P_{n-1} is $x_{n-1}y_{n-1} \cdot x_ny_n$, which is a multiple of $x_{n-1}x_n \in P_{n-2}$. □

Definition 3.2. *Let $n \in \mathbb{N}$, $n \geq 3$. An n-sunlet graph (or n-sun graph) is a graph G with $2n$ vertices, obtained by adding a whisker to each vertex of a cycle graph C_n of length n.*

Given a cycle C_n with vertex set $V(C_n) = \{x_1, \ldots, x_n\}$ and edge set $E(C_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1\}$, we consider the n-sunlet graph S_n on C_n, obtained by adding to each vertex x_i of C_n a whisker, whose terminal vertex is y_i, for all $i = 1, \ldots, n$. Thus, S_n has vertex set $V(S_n) = \{x_1, \ldots, x_n, y_1, \ldots, y_n\}$ and edge set $E(S_n) = \{x_1x_2, x_2x_3, \ldots, x_{n-1}x_n, x_nx_1, x_1y_1, x_2y_2, \ldots, x_ny_n\}$.

Lemma 3.3. *For each $n \in \mathbb{N}$, $n \geq 3$, the edge ideal of the n-sunlet graph S_n is a set-theoretic complete intersection, namely*

$$\mathrm{ara}(I(S_n)) = \mathrm{ht}(I(S_n)) = |V(C_n)| = n.$$

P r o o f. We distinguish the following cases.
If $n = 3$, consider the following sums of monomials

$$\begin{aligned} q_0 &= x_1x_2 \\ q_1 &= x_1x_3 + x_2x_3 \\ q_2 &= x_1y_1 + x_2y_2 + x_3y_3. \end{aligned}$$

If $n = 4$, set

$$\begin{aligned} q_0 &= x_1x_2 \\ q_1 &= x_1x_4 + x_2x_3 \\ q_2 &= x_1y_1 + x_2y_2 + x_3x_4 \\ q_3 &= x_3y_3 + x_4y_4. \end{aligned}$$

Finally, for $n = 5$, set

$$\begin{aligned} q_0 &= x_1x_2 \\ q_1 &= x_1x_5 + x_2x_3 \\ q_2 &= x_1y_1 + x_4x_5 \\ q_3 &= x_2y_2 + x_3x_4 + x_3y_3x_5y_5 \\ q_4 &= x_3y_3 + x_4y_4 + x_5y_5. \end{aligned}$$

Now suppose that $n \geq 6$. In this case set

$$\begin{aligned} q_0 &= x_1x_2 \\ q_1 &= x_1x_n + x_2x_3 \\ q_2 &= x_2y_2 + x_3x_4 \\ &\vdots \\ q_{n-4} &= x_{n-4}y_{n-4} + x_{n-3}x_{n-2} \\ q_{n-3} &= x_1y_1 + x_{n-1}x_n \\ q_{n-2} &= x_{n-3}y_{n-3} + x_{n-2}x_{n-1} + x_{n-2}y_{n-2}x_ny_n \\ q_{n-1} &= x_{n-2}y_{n-2} + x_{n-1}y_{n-1} + x_ny_n. \end{aligned}$$

Then, in any case, we have $I(S_n) = \sqrt{(q_0, \ldots, q_{n-1})}$ by Lemma 2.1. We show that its assumptions are fulfilled by the sets $P_0, \ldots, P_{n-1}$, where, for all $i = 0, \ldots, n-1$, P_i is the set of monomials appearing in q_i. It is straightforward to verify that conditions (i) and (ii) are satisfied. Evidently condition (iii) is true if $n \in \{3, 4, 5\}$. We prove it for $n \geq 6$. The product of the monomials in P_1 is $x_1x_n \cdot x_2x_3$, which is a multiple of $x_1x_2 \in P_0$. For $i = 2, \ldots, n-4$, the product of the monomials of P_i is $x_iy_i \cdot x_{i+1}x_{i+2}$, which is a multiple of $x_ix_{i+1} \in P_{i-1}$. The product of the monomials of P_{n-3} is $x_1y_1 \cdot x_{n-1}x_n$, a multiple of $x_1x_n \in P_1$. In P_{n-2}, we can form three products: $x_{n-3}y_{n-3} \cdot x_{n-2}x_{n-1}$ and $x_{n-3}y_{n-3} \cdot x_{n-2}y_{n-2}x_ny_n$, which are multiples of $x_{n-3}x_{n-2} \in P_{n-4}$, and $x_{n-2}x_{n-1} \cdot x_{n-2}y_{n-2}x_ny_n$, which is a multiple of $x_{n-1}x_n \in P_{n-3}$. As for P_{n-1}, we have $x_{n-2}y_{n-2} \cdot x_{n-1}y_{n-1}$, which is a multiple of $x_{n-2}x_{n-1} \in P_{n-2}$, $x_{n-2}y_{n-2} \cdot x_ny_n$, which is an element of P_{n-2}, and $x_{n-1}y_{n-1} \cdot x_ny_n$ which is a multiple of $x_{n-1}x_n \in P_{n-3}$. This completes the proof. $\square$

4. The arithmetical rank of a large class of whisker graphs.

Consider the following family of graphs. For some integer $r \geq 0$, let $S_0, \dots, S_r$ be pairwise disjoint finite sets of paths and cycles (*blocks*) fulfilling the following conditions:

(*a*) $|S_0| = 1$;

(*b*) for all $i = 2, \dots, r$, and all $H \in S_i$,
$$V(H) \cap \bigcup_{\substack{K \in S_j \\ j \in \{0,\dots,i-2\}}} V(K) = \varnothing;$$

(*c*) for all $i = 1, \dots, r$, and all $H \in S_i$, there is $v \in V(H)$ such that
$$V(H) \cap \bigcup_{\substack{K \in S_j,\ K \neq H \\ j \in \{0,\dots,i\}}} V(K) = V(H) \cap \bigcup_{K \in S_{i-1}} V(K) = \{v\}.$$
In other words, every $H \in S_i$ has exactly one vertex in common with the union of the blocks belonging to $\bigcup_{j=0}^{i} S_j$, and this vertex belongs to some block $K \in S_{i-1}$, and to none of the blocks $L \in S_j$, with $j \leq i-2$.

(*d*) Two paths belonging to S can only intersect in their terminal vertices, and a path belonging to S can intersect a cycle belonging to S only in one of its terminal vertices.

Whenever $H \in S_i$, we will say that H has *rank* i.

Note that, as a consequence of condition (*c*), if H and H' are different blocks of rank i having one vertex in common, then this vertex belongs to some block of rank $i-1$, and is their unique common vertex. Moreover, if H is a block of rank i, then the block K of rank $i-1$ with which H has a vertex v in common is unique: if there were another block K' of rank $i-1$ containing v, then v would belong to some block of rank $i-2$, which would contradict condition (*b*).

Let $S_0 = \{G_0\}$ and consider the graph $G = \bigcup_{K \in S} K$.

An easy induction on the rank yields the following

Lemma 4.1. *We have*

$$|V(G)| = |V(G_0)| + \sum_{\substack{H \in S \\ H \neq G_0}} (|V(H)| - 1)\,.$$

Consider a graph G as above and let G' the graph obtained by adding a whisker to each vertex of G. As usual, call x_k the vertices of G and y_k the terminal vertices connected to x_k.

Theorem 4.2. *With respect to the notations introduced above,*

$$\mathrm{ara}(I(G')) = \mathrm{ht}(I(G')) = |V(G)|,$$

so that $I(G')$ is a set-theoretic complete intersection.

Proof. Let S and S_i be the sets defined above. Fix an element $G_0 \in S$. Let r be the maximum rank of the elements of S. If $r = 0$, the claim follows from Lemma 3.1 if G_0 is a path, and from Lemma 3.3 if G_0 is a cycle. So assume that $r > 0$. Suppose that $V(G_0) = \left\{x_1^0, \dots, x_{n_0}^0\right\}$, and call y_k^0 the terminal vertex of the whisker attached to x_k^0. Suppose that $x_{a_1}^0, \dots, x_{a_s}^0$ are the vertices that G_0 has in common with the elements of S_1. For all $j = 1, \dots, s$, let $G_{(1,j)} \in S_1$ be one of the blocks that has $x_{a_j}^0$ among its vertices (in Figure 1, $j = 1$). Let $x_{a_j}^1$ be a vertex of $G_{(1,j)}$ that is adjacent to $x_{a_j}^0$ (the one following $x_{a_j}^0$ in the clockwise order, if $G_{(1,j)}$ is a cycle). Let G'_0 be the subgraph of G' induced on the vertex set

$$V(G_0) \cup \left\{y_k^0 \;\middle|\; k \notin \{a_1, \dots, a_s\}\right\} \cup \left\{x_{a_1}^1, \dots, x_{a_s}^1\right\}.$$

Then G'_0 is a whisker graph on G_0. More precisely, the terminal vertex of the whisker attached to x_k^0 is x_k^1 if $k \in \{a_1, \dots, a_s\}$, and is y_k^0 otherwise. Hence, for all $j \in \{1, \dots, s\}$, the edge $x_{a_j}^0 x_{a_j}^1$ of $G_{(1,j)}$ is a whisker of G'_0.

Now let $i > 0$. Let $G_{(i,1)}, \dots, G_{(i,\beta)}$ be all graphs of S_i that have a certain vertex x^{i-1} in common with a given element G_{i-1} of S_{i-1} (see Figure 2). Fix an index $j \in \{1, \dots, \beta - 1\}$, and set $G_i = G_{(i,j)}$ (in Figure 2, $j = 1$). Let $V(G_i) = \left\{x_1^i, \dots, x_{n_i}^i\right\}$, and call y_k^i the terminal vertex of the whisker attached to x_k^i. We may assume that $x_1^i = x^{i-1}$. Let $x_{b_1}^i, \dots, x_{b_t}^i$ be the vertices of G_i that G_i has in common with some elements $G_{(i+1,1)}, \dots, G_{(i+1,t)}$ of S_{i+1}. This set of vertices may be empty (which is certainly the case if $i = r$). Note that these vertices are all different from x_1^i because, by definition of S_{i+1}, $G_{(i+1,j)}$ has no vertex in common with G_{i-1}. For all $j = 1, \dots, t$, let $x_{b_j}^{i+1}$ be a vertex of $G_{(i+1,j)}$

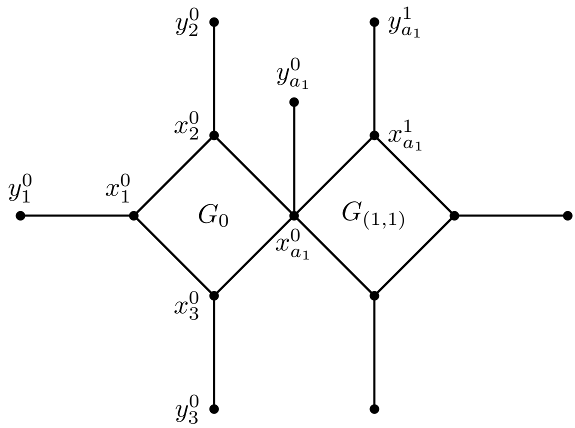

Fig. 1

adjacent to $x_{b_j}^i$ (the one following $x_{b_j}^i$ in the clockwise order, if $G_{(i+1,j)}$ is a cycle). Moreover, let z_j be a vertex of $G_{(i,j+1)}$ adjacent to x_1^i. Let G_i' be the subgraph of G' induced on the vertex set

$$V(G_i) \cup \{y_k^i \mid k \notin \{1, b_1, \ldots, b_t\}\} \cup \left\{x_{b_1}^{i+1}, \ldots, x_{b_t}^{i+1}\right\} \cup \{z_j\}.$$

Thus G_i' is a whisker graph on G_i. More precisely, the terminal vertex of the whisker attached to x_k^i is z_j if $k = 1$, is x_k^{i+1} if $k \in \{b_1, \ldots, b_t\}$, and is y_k^i otherwise. Hence, the edge $x_1^i z_j$ of $G_{(i,j+1)}$, and for all $j \in \{1, \ldots, t\}$, the edge $x_{b_j}^i x_{b_j}^{i+1}$ of $G_{(i+1,j)}$ are whiskers of G_i'.

Finally, set $\overline{G_i} = G_{(i,\beta)}$. Let $V\left(\overline{G_i}\right) = \left\{\overline{x}_1^i, \ldots, \overline{x}_{m_i}^i\right\}$, and call $\overline{y}_k^i$ the terminal vertex of the whisker attached to $\overline{x}_k^i$. We may assume that $\overline{x}_1^i = x^{i-1}$. Let $\overline{x}_{c_1}^i, \ldots, \overline{x}_{c_u}^i$ be the vertices of $\overline{G}_i$ that $\overline{G}_i$ has in common with some elements $\overline{G}_{(i+1,1)}, \ldots, \overline{G}_{(i+1,u)}$ of S_{i+1}. For all $j = 1, \ldots, u$, let $\overline{x}_{c_j}^{i+1}$ be a vertex of $\overline{G}_{(i+1,j)}$ adjacent to $\overline{x}_{c_j}^i$ (the one following $\overline{x}_{c_j}^i$ in the clockwise order, if $\overline{G}_{(i+1,j)}$ is a cycle).

Let $\overline{G}_i'$ be the subgraph of G' induced on the vertex set

$$V\left(\overline{G}_i\right) \cup \left\{\overline{y}_k^i \mid k \notin \{c_1, \ldots, c_u\}\right\} \cup \left\{\overline{x}_{c_1}^{i+1}, \ldots \overline{x}_{c_u}^{i+1}\right\}.$$

Thus $\overline{G}_i'$ is a whisker graph on $\overline{G}_i$. More precisely, the terminal vertex of the whisker attached to $\overline{x}_k^i$ is $\overline{x}_k^{i+1}$ if $k \in \{c_1, \ldots, c_u\}$, and is $\overline{y}_k^i$ otherwise. Hence, for all $j \in \{1, \ldots, u\}$, the edge $\overline{x}_{c_j}^i \overline{x}_{c_j}^{i+1}$ of $G_{(i+1,j)}$ is a whisker of $\overline{G}_i'$.

In Figure 2 the edges of the whisker graph $G_{(i,1)}'$ are dashed lines and the edges of the whisker graph $\overline{G}_i'$ are dotted lines. By means of the above construction, G' is subdivided in subgraphs that are whisker graphs and have pairwise no edge in common. Each of them is a whisker graph H' on an element

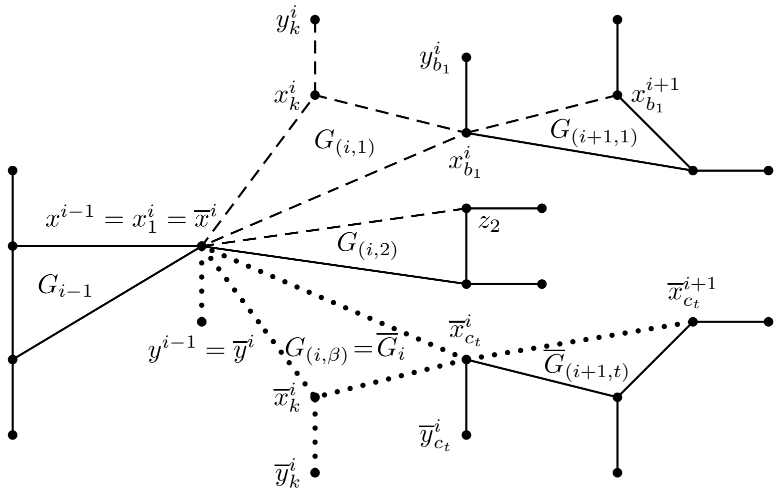

Fig. 2

H of S. Moreover, whenever $H \neq G_0$, exactly one of the edges of H is a whisker of K' for some other $K \in S$, which has a vertex in common with H and whose rank is equal to the rank of H, or to the rank of H minus one.

Now we construct a set of $|V(G)|$ polynomials that generate $I(G')$ up to radical. This set will be obtained by attaching a certain set of polynomials to each $H \in S$, and then taking the union of all this sets. First consider $H = G_0$. The set of polynomials attached to G_0 is Q_0, a set of $|V(G_0)|$ polynomials that generate $I(G'_0)$ up to radical, and are defined as in Lemma 3.1 if G_0 is a path, and as in Lemma 3.3 if G_0 is a cycle. Now let H be an element of S other than G_0. We will attach to H a set of $|V(H)| - 1$ polynomials. To this end, we will first apply Lemma 3.1 or Lemma 3.3 to construct a set of $|V(H)|$ polynomials that generate $I(H')$ up to radical, and then we will cancel one polynomial. Let us describe the procedure. Note that $H \in S_k$ for some $k \geq 1$. The elements of S_k that share a vertex with the same element G_{k-1} of S_{k-1} will be denoted, as above, $G_{(k,1)}, \ldots, G_{(k,j)}, \ldots, G_{(k,\beta)}$. Call Q_{k-1} the set of polynomials attached to G_{k-1}.

First suppose that $H = G_{(k,1)}$ ($k = i + 1$ in Figure 2). The edge $x^{k-1}_{b_1} x^k_{b_1}$ of $G_{(k,1)}$ is a whisker of $G'_{(k-1,1)}$. Arrange the vertices of $G_{(k,1)}$ in such a way that $x^{k-1}_{b_1}$, $x^k_{b_1}$ are the first two (those corresponding to x_1 and x_2 in the proofs of the aforementioned lemmas). Note that if $G_{(k,1)}$ is a path, $x^{k-1}_{b_1}$ is a terminal vertex, as is x_1 in the proof of Lemma 3.1, because $x_{b_1}^{k-1}$ is the vertex shared by $G_{(k,1)}$ and $G_{(k-1,1)}$. Then, applying the construction described in one of the lemmas, we obtain a set of $\left|V\left(G_{(k,1)}\right)\right|$ polynomials that generate $I\left(G'_{(k,1)}\right)$ up to radical, the first of which is $q_0 = x_{b_1}^{k-1}x_{b_1}^{k}$. We then omit this polynomial, and let $Q_{(k,1)}$ be the resulting set of polynomials. The quadratic monomials appearing in these polynomials are those corresponding to all edges of $G_{(k,1)}$ (with the only exception of the edge $x_{b_1}^{k-1}x_{b_1}^{k}$) and all whiskers of $G'_{(k,1)}$.

Now suppose that $H = G_{(k,j)}$ with $j \in \{2, \ldots, \beta\}$ ($k = i, j = 2$ in Figure 2). The edge $x_i^k z_j$ of $G_{(k,j)}$ is a whisker of $G'_{(k,j-1)}$. Arrange the vertices of $G_{(k,j)}$ in such a way that x_1^k, z_j are the first two. Then, as in the previous case, construct $\left|V\left(G_{(k,j)}\right)\right|$ polynomials that generate $I\left(G'_{(k,j)}\right)$ up to radical, the first of which is $q_0 = x_1^k z_j$. We then omit this polynomial, and let $Q_{(k,j)}$ be the resulting set of polynomials. The quadratic monomials appearing in these polynomials are those corresponding to all edges of $G_{(k,j)}$ (with the only exception of the edge $x_1^k z_j$) and all whiskers of $G'_{(k,j)}$.

Let Q be the union of the sets of polynomials defined above. Then, by Lemma 4.1, $|Q| = |V(G)|$. The claim follows if one can prove that $I(G') = \sqrt{(Q)}$. We show that this equality is a consequence of Lemma 2.1. Consider any arrangement of the sets of polynomials such that

(i) Q_0 is the first element,

(ii) for all indices k, j, Q_{k-1} precedes $Q_{(k,j)}$,

(iii) for all indices k, j, $Q_{(k,j-1)}$ precedes $Q_{(k,j)}$.

Let $T^0, \ldots, T^N$ be such an arrangement. For all i, call H_i the element of S associated with the set T^i in the construction described above. Moreover, for all r, let $G'_r = \bigcup_{i=0}^{r} H'_i$, so that $G' = G'_N$. We show, by (finite) induction on $r \geq 0$, that, for all r,

$$I(G'_r) = \sqrt{\left(\bigcup_{i=0}^{r} T^i\right)},$$

whence, in particular, $I(G') = \sqrt{(Q)}$, as claimed. For $r = 0$, the claim is true by the first step of the above construction, which, in view of condition (i), yields $I(G'_0) = \sqrt{(Q_0)} = \sqrt{(T^0)}$. So assume that $r \geq 1$ and that the claim is true for $r-1$. Let M be a set of minimal monomial generators of $I(G'_{r-1})$, and let $q_1, \ldots, q_s$ be the polynomials of T^r. Then, by induction

$$\sqrt{\left(\bigcup_{i=0}^{r} T^i\right)} = \sqrt{I(G'_{r-1}) + (T^r)} = \sqrt{(M) + (q_1, \ldots, q_s)}.$$

Now, with respect to the notation used in the above construction, H_r is either of the form $G_{(k,1)}$ or $G_{(k,j)}$, with $j \in \{2, \ldots, \beta\}$. In the first case, in view of condition (ii), we have that $Q_{k-1} = T^i$ for some $i < r$. Hence the monomial $x_{b_1}^{k-1} x_{b_1}^{k}$ (which corresponds to a whisker of G_{k-1}) belongs to M. Now, as shown in the proofs of Lemmas 3.1 and 3.3, for all $j = 1, \ldots, s$, the product of any two monomials of q_j is either divisible by a monomial appearing in q_h, for some $h < j$, or is divisible by $x_{b_1}^{k-1} x_{b_1}^{k}$. Recall that, according to the above construction, $\sqrt{\left(x_{b_1}^{k-1} x_{b_1}^{k}, q_1, \ldots, q_s\right)} = I(H'_r)$. By Lemma 2.1 it thus follows that $\sqrt{(M) + (q_1, \ldots, q_s)} = I(G'_{r-1}) + I(H'_r) = I(G'_r)$. The second case can be treated similarly, using condition (iii). □

Example 4.3. Let us give an application of the preceding result. Consider the following graph G:

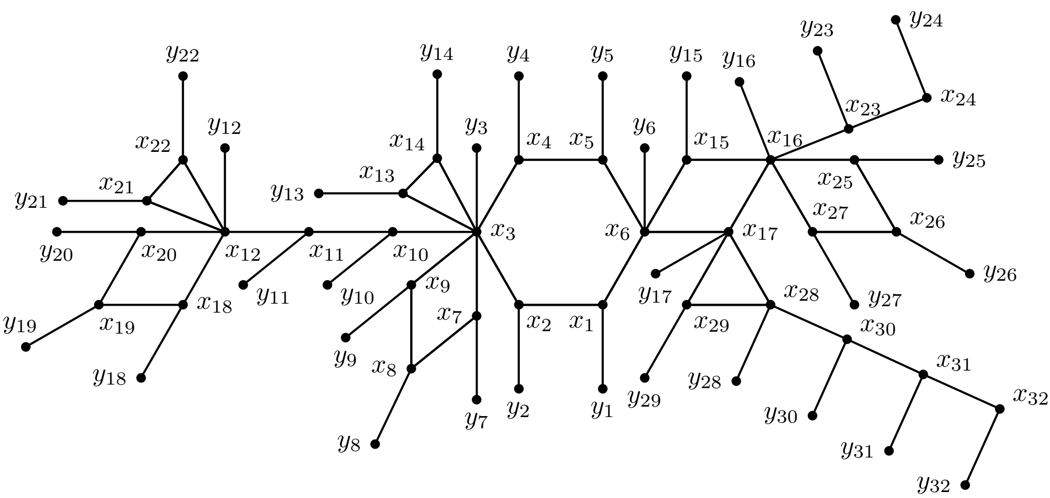

Fig. 3

The edge ideal of G is

$$I(G) = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_6, x_1x_6, x_3x_7, x_7x_8, x_8x_9, x_3x_9, x_3x_{10}, x_{10}x_{11}, x_{11}x_{12}, x_3x_{13}, x_{13}x_{14}, x_3x_{14}, x_6x_{15}, x_{15}x_{16}, x_{16}x_{17}, x_6x_{17}, x_{12}x_{18}, x_{18}x_{19}, x_{19}x_{20}, x_{12}x_{20}, x_{12}x_{21}, x_{21}x_{22}, x_{12}x_{22}, x_{16}x_{23}, x_{23}x_{24}, x_{16}x_{25}, x_{25}x_{26}, x_{26}x_{27}, x_{16}x_{27}, x_{17}x_{28}, x_{28}x_{29}, x_{17}x_{29}, x_{28}x_{30}, x_{30}x_{31}, x_{31}x_{32}, x_iy_i \mid i = 1, \dots, 32).$$

We define eleven sets of polynomials.

- The first set is:

$q_0 = x_1x_2$

$q_1 = x_1x_6 + x_2x_3$

$q_2 = x_2y_2 + x_3x_4$

$q_3 = \boxed{x_3x_7} + x_4x_5$

$q_4 = x_4y_4 + x_5x_6 + x_1y_1x_5y_5$

$q_5 = x_1y_1 + x_5y_5 + \boxed{x_6x_{15}}.$

- The second set is:

$q_6 = x_3x_9 + x_7x_8$

$q_7 = \boxed{x_3x_{10}} + x_7y_7 + x_8x_9$

$q_8 = x_8y_8 + x_9y_9.$

- The third set is:

$q_9 = \boxed{x_3x_{13}} + x_{10}x_{11}$

$q_{10} = x_{10}y_{10} + x_{11}x_{12}$

$q_{11} = x_{11}y_{11} + \boxed{x_{12}x_{18}}.$

- The fourth set is:

$q_{12} = x_3x_{14} + x_{13}x_{14}$

$q_{13} = x_3y_3 + x_{13}y_{13} + x_{14}y_{14}$

- The fifth set is:

$q_{14} = x_6x_{17} + x_{15}x_{16}$

$q_{15} = x_6y_6 + x_{15}y_{15} + x_{16}x_{17}$

$q_{16} = \boxed{x_{16}x_{23}} + \boxed{x_{17}x_{28}}.$

- The sixth set is:

$q_{17} = x_{12}x_{20} + x_{18}x_{19}$

$q_{18} = \boxed{x_{12}x_{21}} + x_{18}y_{18} + x_{19}x_{20}$

$q_{19} = x_{19}y_{19} + x_{20}y_{20}.$

- The seventh set is:

$q_{20} = x_{12}x_{22} + x_{21}x_{22}$

$q_{21} = x_{12}y_{12} + x_{21}y_{21} + x_{22}y_{22}.$

- The eighth set is:

$q_{22} = \boxed{x_{16}x_{25}} + x_{23}x_{24}$

$q_{23} = x_{23}y_{23} + x_{24}y_{24}.$

- The nineth set is:

$q_{24} = x_{16}x_{27} + x_{25}x_{26}$

$q_{25} = x_{16}y_{16} + x_{25}y_{25} + x_{26}x_{27}$

$q_{26} = x_{26}y_{26} + x_{27}y_{27}.$

- The tenth set is:

$q_{27} = x_{17}x_{29} + x_{28}x_{29}$

$q_{28} = x_{17}y_{17} + \boxed{x_{28}x_{30}} + x_{29}y_{29}.$

- The eleventh set is:

$q_{29} = x_{28}y_{28} + x_{30}x_{31}$

$q_{30} = x_{30}y_{30} + x_{31}x_{32}$

$q_{31} = x_{31}y_{31} + x_{32}y_{32}.$

We have that $I(G) = \sqrt{(q_0, \dots, q_{31})}$, whence $\mathrm{ara}(I(G)) = 32$.

5. Final remarks. The graphs G considered in Theorem 4.2 are sometimes referred to as *cactus graphs*. This class includes all bicyclic graphs. It also includes all trees, which can be characterized as the cactus graphs where all blocks are paths. The whisker graph on a tree is again a tree (and, conversely, if a whisker graph is a tree, it is obviously a whisker graph on a tree). In [11] Villarreal has shown that the edge ideal of a tree is Cohen-Macaulay if and only if it is a whisker graph. In view of the results presented by Kimura and Terai [7] we have the following characterization, which clarifies the role of whisker graphs in combinatorial commutative algebra.

Corollary 5.1. *Let G be a tree. The following conditions are equivalent.*
(a) *$I(G)$ is a set-theoretic complete intersection.*
(b) *$I(G)$ is Cohen-Macaulay.*
(c) *$I(G)$ is a pure.*
(d) *G is a whisker graph.*

Proof. According to the Auslander-Buchsbaum formula, we have

$$\mathrm{pd}(R/I(G)) = \operatorname{depth} R - \operatorname{depth}(R/I(G)) \geq \dim R - \dim(R/I(G)) = \mathrm{ht}(I(G)),$$

and, on the other hand, $\mathrm{ara}(I(G)) \geq \mathrm{pd}(R/I(G))$. Hence, whenever $\mathrm{ara}(I(G)) = \mathrm{ht}(I(G))$, one has that $\operatorname{depth}(R/I(G)) = \dim(R/I(G))$. This shows that $(a) \Rightarrow (b)$. The implication $(b) \Rightarrow (c)$ follows from [3], Cor. 5.1.5. According to [7], Theorem 1.1, we also have that $\mathrm{ara}(I(G)) = \mathrm{bight}(I(G))$, where the latter number (the so-called *big height*) denotes the maximum height of the minimal prime ideals of $I(G)$. This shows that $(c) \Rightarrow (a)$. Finally, the equivalence $(b) \Leftrightarrow (d)$ is Theorem 2.4 in [11]. □

In the case where G is any tree, Kimura and Terai give an explicit description of $\mathrm{ara}(I(G))$ polynomials generating $I(G)$ up to radical, which form a so-called *tree-like system*. For the whisker trees, a system of polynomials of the same type has been obtained in the present paper through a recursive construction.

Remark 5.2. All graphs G considered in this paper are supposed to be connected, but this assumption is by no means restrictive. In fact, in the general case, if $G_1, \ldots, G_s$ are the connected components of G, then the whisker graphs $G'_1, \ldots, G'_s$ are the connected components of the whisker graph G'. Since

$$\mathrm{ht}(I(G')) = \sum_{i=1}^{s} \mathrm{ht}(I(G'_i)) \leq \mathrm{ara}(I(G')) \leq \sum_{i=1}^{s} \mathrm{ara}(I(G'_i)),$$

if $I(G'_1), \ldots, I(G'_s)$ are set-theoretic complete intersections, then so is $I(G')$.

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Received June 19, 2013