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INTERVAL CRITERIA FOR FORCED OSCILLATION OF FUNCTIONAL DIFFERENTIAL EQUATIONS WITH γ-LAPLACIAN, DAMPING AND MIXED NONLINEARITIES

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Abstract. We consider a forced second order functional differential equation with γ-Laplacian, damping, and mixed nonlinearities in the form of

$$(r(t)\phi_\gamma(x'(t)))' + p(t)\phi_\gamma(x'(t)) + q_0(t)\phi_\beta(x(t)) + \int_a^b q(t,s)\phi_{\alpha(s)}(x(g(t,s)))d\zeta(s) = e(t),$$

where $\gamma, \beta \in [0,\infty)$, $-\infty < a < b \le \infty$, $\alpha \in C[a,b)$ is strictly increasing is such that $0 \le \alpha(a) < \mu < \alpha(b-)$ with $\beta > \gamma > \mu > 0$; $r,\ p,\ q_0,\ e \in C([t_0,\infty),\mathbb{R})$ with $r(t) > 0$ on $[t_0,\infty)$; $q \in C([0,\infty)\times[a,b))$; and $\zeta : [a,b) \to \mathbb{R}$ is nondecreasing. The function $g \in C([0,\infty)\times[a,b),\ [0,\infty))$ is such that $\lim_{t\to\infty} g(t,s) = \infty$, for $s \in [a,b)$. Interval oscillation criteria of the El-Sayed type and the Kong type are obtained. These criteria are further extended to equations with deviating arguments.

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Key words: Interval criteria, forced Oscillation, γ-Laplacian, nonlinear functional differential equations.

1. Introduction. We are concerned with the oscillatory behavior of forced second order functional differential equations with γ-Laplacian, damping and mixed nonlinearities in the form of

$$(1.1)\quad (r(t)\phi_\gamma(x'(t)))' + p(t)\phi_\gamma(x'(t)) + q_0(t)\phi_\beta(x(t)) + \int_a^b q(t,s)\phi_{\alpha(s)}(x(g(t,s)))d\zeta(s) = e(t),$$

where $\phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$, $\gamma, \beta \in [0,\infty), -\infty < a < b \leq \infty$, $\alpha \in C[a,b)$ is strictly increasing such that $0 \leq \alpha(a) < \mu < \alpha(b-)$ with $\beta > \gamma > \mu > 0$; r, p, q_0, $e \in C([t_0,\infty),\mathbb{R})$ with $r(t) > 0$ on $[t_0,\infty)$; $q \in C([0,\infty)\times[a,b))$; and $\zeta : [a,b) \to \mathbb{R}$ is nondecreasing. The function $g \in C([0,\infty)\times[a,b), [0,\infty))$ is such that $\lim_{t\to\infty} g(t,s) = \infty$, for $s \in [a,b)$. Our interest is to establish oscillation criteria for Eq. (1.1) without assuming that $p(t)$, $q_0(t)$, $q(t,s)$, and $e(t)$ are of definite sign. Here $\int_a^b f(s)\,d\zeta(s)$ denotes the Riemann-Stieltjes integral of the function f on $[a,b)$ with respect to ζ.

We note that as special cases, the integral term in the equation becomes a finite sum when $\zeta(s)$ is a step function and a Riemann integral when $\zeta(s) = s$.

As usual, a solution $x(t)$ of Eq. (1.1) is said to be oscillatory if it is defined on some ray $[T,\infty)$ with $T \geq 0$, and has an unbounded set of zeros. Eq. (1.1) is said to be oscillatory if every solution extendible throughout $[t_x,\infty)$ for some $t_x \geq 0$ is oscillatory.

In the last 50 years, there has been extensive work on oscillation and nonoscillation of various differential equations, see [1, 3, 4, 5, 6, 7, 8, 10, 19, 20, 21, 22, 31, 26] and the references cited therein. Special cases of the equation

$$(1.2)\qquad \left(r(t)\left(x'(t)\right)^\gamma\right)' + q_0(t)\,x^\gamma(t) + \sum_{j=1}^N q_j(t)\phi_{\alpha j}(x(t)) = e(t),$$

where $\phi_\alpha(u) := |u|^\alpha \operatorname{sgn} u$, γ is a quotient of odd positive integers and $\alpha_j > 0$, $j = 1,2,\ldots,N$, such that

$$\alpha_1 > \alpha_2 > \cdots > \alpha_m > \gamma > \alpha_{m+1} > \cdots > \alpha_n > 0.$$

has been studied by many authors. When $\gamma = N = 1$, $r(t) = 1$, $p(t) = q_0(t) = 0$, and $q_1(t) \geq 0$, Kartsatos [19, 20] initiated an approach for oscillation under the assmption that $e(t)$ is the second derivative of an oscillatory function. This method was further developed by different authors, See Keener [21], Kong and Wong [24], Kong and Zhang [25], Rankin [30], Skidmore and Leighton [32], Skidmore and Bowers [31], Teufel [39], and Wong [40].

Results were also obtained for oscillation of special cases of Eq. (1.2) without imposing the assumption that $e(t)$ is the second derivative of an oscillatory function. Most of them were for the case when $\gamma = 1$, $r(t) = 1$, and $p(t) = 0$. For instance, see Nasr [27] for $N = 1$ and $\alpha_1 > 1$, Sun and Wong [36] for $\alpha_j < 1$, and Sun and Wong [37] and Sun and Meng [35] for mixed nonlinearities. Among them, there were interval oscillation criteria which can be regarded as generalizations of the one by El-Sayed [9] for second order forced linear differential equations, and other interval oscillation criteria can be regarded as generalizations of the one by Kong [22] established initially for the second order homogeneous linear equations, see also [23]. Hassan, Erbe and Peterson [15] discussed the oscillation of an equation with p-Lapacian, more specifically, they established oscillation criteria of El-Sayed-type for the equation (1.2)

Hassan and Kong [16] considered the forced second order differential equations with γ-Laplacian and damping in the form of

$$\left(r(t)\phi_\gamma\left(x'(t)\right)\right)' + p(t)\phi_\gamma\left(x'(t)\right) + \sum_{j=0}^{N} q_j(t)\phi_{\alpha j}\left(x(t)\right) = e(t), \tag{1.3}$$

where $\alpha_j > 0$, $j = 0, 1, 2, \ldots, N$, such that

$$\alpha_j > \gamma,\ j = 1, 2, \ldots, m; \quad \text{and} \quad \alpha_j < \gamma,\ j = m+1, l+2, \ldots, N. \tag{1.4}$$

and $r,\ p,\ q_j,\ e \in C\left([0,\infty), \mathbb{R}\right)$ with $r(t) > 0$ on $[0,\infty)$. They established oscillation criteria of El-Sayed-type and Kong-type for Eq. (1.3). Sun and Kong [34] considered the equation

$$(r(t)x'(t))' + q_0(t)x(t) + \int_0^b q(t,s)\phi_{\alpha(s)}(x(t))d\zeta(s) = e(t).$$

Recently, Hassan and Kong [17] established interval oscillation criteria of both the El-Sayed-type and the Kong-type for the more general equation

$$(r(t)\phi_\gamma(x'(t)))' + q_0(t)\phi_\gamma(x(t)) + \int_0^b q(t,s)\phi_{\alpha(s)}(x(g(t,s)))d\zeta(s) = e(t).$$

Motivated by above, in this paper, we will establish interval oscillation criteria of both the El-Sayed-type and the Kong-type for the more general equation (1.1).

This paper is organized as follows: after this introduction, we state lemmas, in Section 2, we state oscillation criteria for (1.1) with $g(t,s) \equiv t$, in Section 3, we establish oscillation criteria for (1.1) with $g(t,s) \not\equiv t$.

2. Lemmas. We denote by $L_\zeta(a,b)$ the set of Riemann-Stieltjes integrables functions on $[a,b)$ with respect to ζ. Let $c\in(a,b)$ such that $\alpha(c)=\mu$. We further assume that

$$\alpha^{-1}\in L_\zeta(a,b)\quad \text{such that}\quad \int_a^c d\zeta(s)>0 \text{ and } \int_c^b d\zeta(s)>0.$$

To state our main results, we begin with the following lemmas which we will need in the proof of our main results. The following lemma generalizes [17, Lemma 2.1].

Lemma 2.1. *Let*

$$m:=\mu\left(\int_c^b d\zeta(s)\right)^{-1}\int_c^b \alpha^{-1}(s)\,d\zeta(s)$$

and

$$n:=\mu\left(\int_a^c d\zeta(s)\right)^{-1}\int_a^c \alpha^{-1}(s)\,d\zeta(s).$$

Then for any $\delta\in(m,n)$, there exists $\eta\in L_\zeta(a,b)$ such that $\eta(s)>0$ on $[a,b)$,

$$\int_a^b \alpha(s)\,\eta(s)\,d\zeta(s)=\mu \quad \text{and}\quad \int_a^b \eta(s)\,d\zeta(s)=\delta. \tag{2.1}$$

Proof. Let

$$\eta_1(s):=\begin{cases}0, & s\in(a,c)\\ \mu\alpha^{-1}(s)\left(\int_c^b d\zeta(s)\right)^{-1}, & s\in[c,b),\end{cases}$$

and

$$\eta_2(s):=\begin{cases}\mu\alpha^{-1}(s)\left(\int_a^c d\zeta(s)\right)^{-1}, & s\in(a,c)\\ 0, & s\in[c,b).\end{cases}$$

Clearly for $i=1,2$, $\eta_i\in L_\zeta(a,b)$ and

$$\int_a^b \alpha(s)\,\eta_i(s)\,d\zeta(s)=\mu.$$

Moreover,

$$\int_a^b \eta_1(s)\,d\zeta(s)=m \quad\text{and}\quad \int_a^b \eta_2(s)\,d\zeta(s)=n.$$

For $k \in [0,1]$ let

$$\eta(s,k) := (1-k)\,\eta_1(s) + k\eta_2(s), \quad s \in [a,b).$$

Then it is easy to see that

$$\int_a^b \alpha(s)\,\eta(s,k)\,d\zeta(s) = \mu.$$

Furthermore, since $\eta(s,0) = \eta_1(s)$ and $\eta(s,1) = \eta_2(s)$, we have

$$\int_a^b \eta(s,0)\,d\zeta(s) = m \quad \text{and} \quad \int_a^b \eta(s,1)\,d\zeta(s) = n.$$

By the continuous dependence of $\eta(s,k)$ on k there exists $k^* \in (0,1)$ such that $\eta(s) := \eta(s,k^*)$ satisfies

$$\int_a^b \eta(s)\,d\zeta(s) = \delta.$$

Note that $\eta(s) > 0$ for $s \in [a,b)$ and $\int_a^b \alpha(s)\,\eta(s)\,d\zeta(s) = \mu$ and the definitions of m and n gives $0 < m < 1 < n$. $\square$

The next Lemma is a generalized Arithmetic-Geometric mean inequality established in [34].

Lemma 2.2. *Let $u \in C[a,b)$ and $\eta \in L_\zeta(a,b)$ satisfying $u \geq 0$, $\eta > 0$ on $[a,b)$ and $\int_a^b \eta(s)\,d\zeta(s) = 1$. Then*

$$\int_a^b \eta(s)\,u(s)\,d\zeta(s) \geq \exp\left(\int_a^b \eta(s)\ln[u(s)]\,d\zeta(s)\right),$$

where we use the convention that $\ln 0 = -\infty$ and $e^{-\infty} = 0$.

3. Oscillation Criteria for (1.1) with $g(t,s) \equiv t$.

In this section, we establish oscillation criteria for equation (1.1) with $g(t,s) \equiv t$, namely,

$$(3.1)\quad \left(r(t)\phi_\gamma\left(x'(t)\right)\right)' + p(t)\phi_\gamma\left(x'(t)\right) + q_0(t)\,\phi_\beta(x(t)) + \int_a^b q(t,s)\,\phi_{\alpha(s)}(x(t))\;d\zeta(s) = e(t).$$

The first result provides an oscillation criterion of the El-Sayed-type.

Theorem 3.1. *Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ such that, for $i = 1, 2$*

$$q_0(t) \geq 0 \quad \text{for } t \in [a_i, b_i], \tag{3.2}$$

$$q\left(t, s\right) \geq 0, \quad \text{for } \left(t, s\right) \in [a_i, b_i] \times [a, b)\,, \tag{3.3}$$

and

$$\left(-1\right)^i e\left(t\right) \geq 0, \text{ for } t \in \left[a_i, b_i\right]. \tag{3.4}$$

Assume further that for $i = 1, 2$, there exist $u_i \in C^1\left[a_i, b_i\right]$ satisfying $u_i\left(a_i\right) = u_i\left(b_i\right) = 0$, $u_i\left(t\right) \not\equiv 0$ on $[a_i, b_i]$ and a continuous positive function $\rho(t)$ such that

$$\sup_{\delta \in (m,1]} \int_{a_i}^{b_i} \left[Q(t)|u_i(t)|^{\gamma+1} - \frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}}[(\gamma+1)|u_i'(t)| + |u_i(t)|\,|P(t)|]^{\gamma+1}\right] dt > 0, \tag{3.5}$$

where

$$P\left(t\right) := \frac{\rho'\left(t\right)}{\rho\left(t\right)} - \frac{p(t)}{r\left(t\right)}, \tag{3.6}$$

and

$$Q(t) := \hat{\delta}\rho(t)\left(q_0(t)\right)^{(\gamma-\mu)/(\beta-\mu)}\left(\hat{q}(t)\right)^{(\beta-\gamma)/(\beta-\mu)}, \tag{3.7}$$

with

$$\hat{\delta} := (\beta-\mu)(\beta-\gamma)^{(\gamma-\beta)/(\beta-\mu)}(\gamma-\mu)^{(\mu-\gamma)/(\beta-\mu)},$$

and

$$\hat{q}(t) := \left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} \exp\left(\int_a^b \eta\left(s\right)\ln\left[\frac{q\left(t,s\right)}{\eta\left(s\right)}\right] d\zeta\left(s\right)\right),$$

with $\eta\left(s\right)$ is defined as in Lemma 2.1 based on δ. Here we use the convention that $\ln 0 = -\infty$, $e^{-\infty} = 0$, and $0^{1-\delta} = 1$ and $\left(1-\delta\right)^{1-\delta} = 1$ for $\delta = 1$. Then Eq. (3.1) is oscillatory.

Proof. Assume Eq. (1.1) has an extendible solution $x(t)$ which is eventually positive or negative. Then, without loss of generality, assume $x\left(t\right) > 0$ for all $t \geq T \geq 0$, where T depends on the solution $x\left(t\right)$. When $x\left(t\right)$ is eventually negative, the proof follows the same way except that the interval $[a_2, b_2]$, instead of $[a_1, b_1]$, is used. Define

$$z(t) := \rho(t) \frac{r(t)\phi_\gamma(x'(t))}{\phi_\gamma(x(t))}, \ t \geq T. \tag{3.8}$$

Then

$$z'(t) = \rho(t)\left[\frac{(r(t)\phi_\gamma(x'(t)))'}{\phi_\gamma(x(t))} - \frac{r(t)\phi_\gamma(x'(t))(\phi_\gamma(x(t)))'}{(\phi_\gamma(x(t)))^2}\right] + \rho'(t)\frac{r(t)\phi_\gamma(x'(t))}{\phi_\gamma(x(t))}$$

$$= \rho(t)\left[\frac{(r(t)\phi_\gamma(x'(t)))'}{\phi_\gamma(x(t))} - \frac{r(t)\phi_\gamma(x'(t))}{\phi_\gamma(x(t))}\frac{\gamma x'(t)}{x(t)}\right] + \rho'(t)\frac{r(t)\phi_\gamma(x'(t))}{\phi_\gamma(x(t))}. \tag{3.9}$$

It follows from (1.1), (3.6) and (3.8) that for $t \geq T$,

$$z'(t) = -\rho(t) q_0(t) x^{\beta-\gamma}(t) - \rho(t)\int_a^b q(t,s)[x(t)]^{\alpha(s)-\gamma}\, d\zeta(s) + \rho(t)e(t)x^{-\gamma}(t)$$

$$+P(t)z(t) - \frac{\gamma|z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)r(t))^{\frac{1}{\gamma}}}. \tag{3.10}$$

From the assumption, there exists a nontrivial interval $[a_1, b_1] \subset [T, \infty)$ such that (3.3) and (3.4) hold with $i = 1$.

(I) We first consider the case where the supremum in (3.5) is assumed at $\delta = 1$. From (3.4) and (3.10), we have that for $t \in [a_1, b_1]$

(3.11)

$$z'(t) \leq -\rho(t) q_0(t) x^{\beta-\gamma}(t) - \rho(t) x^{\mu-\gamma}(t)\int_a^b q(t,s)[x(t)]^{\alpha(s)-\mu}\, d\zeta(s)$$
$$+P(t)z(t) - \frac{\gamma|z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)r(t))^{\frac{1}{\gamma}}}.$$

Let $\eta \in L_\zeta(a,b)$ be defined as in Lemma 2.1 with $\delta = 1$. Then η satisfies (2.1) with $\delta = 1$. This implies that

$$\int_a^b \eta(s)[\alpha(s) - \mu]\, d\zeta = 0.$$

Then, from Lemma 2.2, we get, for $t \in [a_1, b_1]$

$$\int_a^b q(t,s)[x(t)]^{\alpha(s)-\mu}\, d\zeta(s)$$
$$= \int_a^b \eta(s)\frac{q(t,s)}{\eta(s)}[x(t)]^{\alpha(s)-\mu}\, d\zeta(s)$$

$$\begin{aligned}
&\geq \exp\left(\int_a^b \eta(s)\ln\left(\frac{q(t,s)}{\eta(s)}[x(t)]^{\alpha(s)-\mu}\right)d\zeta(s)\right)\\
&= \exp\left(\int_a^b \eta(s)\ln\left[\frac{q(t,s)}{\eta(s)}\right]d\zeta(s)+\ln(x(t))\int_a^b \eta(s)[\alpha(s)-\mu]\,d\zeta(s)\right)\\
&= \exp\left(\int_a^b \eta(s)\ln\left[\frac{q(t,s)}{\eta(s)}\right]d\zeta(s)\right)=\hat{q}(t).
\end{aligned}$$

This together with (3.11) shows that

$$z'(t) \leq -\rho(t)\,q_0(t)\,x^{\beta-\gamma}(t)-\rho(t)\,\hat{q}(t)x^{\mu-\gamma}(t)+P(t)z(t)-\frac{\gamma\,|z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)\,r(t))^{\frac{1}{\gamma}}}. \tag{3.12}$$

Define

$$X := q_0^{1/(\beta-\gamma)}x \quad \text{and} \quad Y := \hat{q}q_0^{(\gamma-\mu)/(\beta-\gamma)}$$

and using the inequality in [11, Lemma 2.1]

$$X^{\beta-\gamma}+YX^{\mu-\gamma} \geq \hat{\delta}Y^{(\beta-\gamma)/(\beta-\mu)} \quad \text{for all } \beta>\gamma>\mu>0,$$

where

$$\hat{\delta} := (\beta-\mu)(\beta-\gamma)^{(\gamma-\beta)/(\beta-\mu)}(\gamma-\mu)^{(\mu-\gamma)/(\beta-\mu)},$$

we have

$$q_0x^{\beta-\gamma}+\hat{q}x^{\mu-\gamma} \geq \hat{\delta}\hat{q}^{(\beta-\gamma)/(\beta-\mu)}q_0^{(\gamma-\mu)/(\beta-\mu)}. \tag{3.13}$$

Substituting (3.13) into (3.12) and using the definition of Q, we obtain

$$z'(t) \leq -Q(t)+P(t)z(t)-\frac{\gamma\,|z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)\,r(t))^{\frac{1}{\gamma}}}, \quad \text{for } t\in[a_1,b_1], \tag{3.14}$$

where $Q(t)$ is defined by (3.7) with $\delta = 1$. Multiplying both sides of (3.14) by $|u_1(t)|^{\gamma+1}$, integrating from a_1 to b_1, and using integration by parts, we find that

$$\begin{aligned}
&\int_{a_1}^{b_1} Q(t)\,|u_1(t)|^{\gamma+1}\,dt\\
\leq\ &\int_{a_1}^{b_1}\Bigg\{(\gamma+1)\,\phi_\gamma(u_1(t))u_1'(t)\,z(t)+|u_1(t)|^{\gamma+1}P(t)z(t)\\
&\quad-\frac{\gamma\,|u_1(t)|^{\gamma+1}}{(\rho(t)\,r(t))^{\frac{1}{\gamma}}}|z(t)|^{\frac{\gamma+1}{\gamma}}\Bigg\}dt
\end{aligned}$$

$$\leq \int_{a_1}^{b_1} \Bigg\{ |u_1(t)|^{\gamma} \left[(\gamma+1) \left| u_1'(t) \right| + |u_1(t)| \, |P(t)| \right] \; |z(t)|$$

$$- \frac{\gamma \, |u_1(t)|^{\gamma+1}}{(\rho(t)\, r(t))^{\frac{1}{\gamma}}} |z(t)|^{\frac{\gamma+1}{\gamma}} \Bigg\} dt. \tag{3.15}$$

Let $\lambda := \dfrac{\gamma+1}{\gamma}$. Define A and B by

$$A^{\lambda} := \frac{\gamma \, |u_1(t)|^{\gamma+1}}{(\rho(t)\, r(t))^{\frac{1}{\gamma}}} |z(t)|^{\lambda},$$

and

$$B^{\lambda-1} := \frac{(\gamma \rho(t)\, r(t))^{\frac{1}{\gamma+1}}}{\gamma+1} \left[(\gamma+1) \left| u_1'(t) \right| + |u_1(t)| \, |P(t)| \right].$$

Using the inequality in [13] we have

$$\lambda A B^{\lambda-1} - A^{\lambda} \leq (\lambda-1) B^{\lambda}, \tag{3.16}$$

i.e.,

$$|u_1(t)|^{\gamma} \left[(\gamma+1) \left| u_1'(t) \right| + |u_1(t)| \, |P(t)| \right] \; |z(t)| - \frac{\gamma \, |u_1(t)|^{\gamma+1}}{(\rho(t)\, r(t))^{\frac{1}{\gamma}}} |z(t)|^{\lambda}$$

$$\leq \frac{\rho(t) r(t)}{(\gamma+1)^{\gamma+1}} \left[(\gamma+1) \left| u_1'(t) \right| + |u_1(t)| \, |P(t)| \right]^{\gamma+1},$$

which together with (3.15) implies that

$$\int_{a_1}^{b_1} Q(t) \, |u_1(t)|^{\gamma+1} \, dt \leq \int_{a_1}^{b_1} \frac{\rho(t) r(t)}{(\gamma+1)^{\gamma+1}} \left[(\gamma+1) \left| u_1'(t) \right| + |u_1(t)| \, |P(t)| \right]^{\gamma+1} dt.$$

This leads to a contradiction to (3.5).

(II) Now, we consider the case where the supremum in (3.5) is assumed at $\delta \in (m, 1)$. Then from (3.4), we see that, for $t \in [a_1, b_1]$,

$$\begin{aligned} z'(t) \;=\; & -\rho(t)\, q_0(t)\, x^{\beta-\gamma}(t) \\ & -\rho(t)\, x^{\mu-\gamma}(t) \left(\int_a^b q(t,s) \, [x(t)]^{\alpha(s)-\mu} \, d\zeta(s) - \rho(t) \, |e(t)| \, x^{-\mu}(t) \right) \\ & + P(t) z(t) - \frac{\gamma \, |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)\, r(t))^{\frac{1}{\gamma}}}. \end{aligned} \tag{3.17}$$

Let $\widetilde{\eta}(s) := \delta^{-1}\eta(s)$. Then, from (2.1), we have

$$\int_a^b \widetilde{\eta}(s)\, d\zeta(s) = 1 \quad \text{and} \quad \int_a^b \widetilde{\eta}(s)\left[\delta\alpha(s) - \mu\right] d\zeta = 0. \tag{3.18}$$

Hence, for $t \in [a_1, b_1]$

$$\begin{aligned} &\int_a^b q(t,s)\,[x(t)]^{\alpha(s)-\mu}\, d\zeta(s) + |e(t)|\, x^{-\mu}(t) \\ = &\int_a^b \widetilde{\eta}(s)\left(\delta\eta^{-1}(s)\, q(t,s)\,[x(t)]^{\alpha(s)-\mu} + |e(t)|\, x^{-\mu}(t)\right) d\zeta(s). \end{aligned} \tag{3.19}$$

Using the Arithmetic-geometric mean inequality, see [2, Page 17],

$$ch + dk \geq c^h d^k, \quad \text{where } c, d \geq 0,\ h, k > 0 \text{ and } h + k = 1,$$

with

$$c = \eta^{-1}(s)\, q(t,s)\,[x(t)]^{\alpha(s)-\mu},\ d = \frac{1}{1-\delta}\,|e(t)|\, x^{-\mu}(t),\ h = \delta \text{ and } k = 1-\delta,$$

we have that for $t \in [a_1, b_1]$ and $s \in [a, b)$

$$\begin{aligned} &\delta\eta^{-1}(s)\, q(t,s)\,[x(t)]^{\alpha(s)-\mu} + (1-\delta)\frac{|e(t)|}{1-\delta} x^{-\mu}(t) \\ &\qquad \geq \left[\frac{q(t,s)}{\eta(s)}\right]^{\delta} \left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} [x(t)]^{\delta\alpha(s)-\mu}. \end{aligned}$$

Substituting this into (3.19) and using Lemma 2.2 and (3.18), we see that,for $t \in [a_1, b_1]$,

$$\begin{aligned} &\int_a^b q(t,s)\,[x(t)]^{\alpha(s)-\mu}\, d\zeta(s) + |e(t)|\, x^{-\mu}(t) \\ \geq\ &\exp\left(\int_a^b \widetilde{\eta}(s) \ln\left(\left[\frac{q(t,s)}{\eta(s)}\right]^{\delta}\left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} [x(t)]^{\delta\alpha(s)-\mu}\right) d\zeta(s)\right) \\ =\ &\exp\left(\int_a^b \widetilde{\eta}(s)\left(\ln\left[\frac{q(t,s)}{\eta(s)}\right]^{\delta} + \ln\left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} + [\delta\alpha(s)-\mu]\ln x(t)\right) d\zeta(s)\right) \\ =\ &\left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta} \exp\left(\int_a^b \eta(s) \ln\frac{q(t,s)}{\eta(s)} d\zeta(s)\right) = \hat{q}(t). \end{aligned} \tag{3.20}$$

It follows from (3.17) and (3.20), that we get, for $t \in [a_1, b_1]$,

$$
\begin{aligned}
z'(t) &\leq -\rho(t)\, q_0(t)\, x^{\beta-\gamma}(t) - \rho(t)\, \hat{q}(t) x^{\mu-\gamma}(t) + P(t)z(t) - \frac{\gamma\, |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)\, r(t))^{\frac{1}{\gamma}}} \\
&\overset{(3.13)}{\leq} -Q(t) + P(t)z(t) - \frac{\gamma\, |z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)\, r(t))^{\frac{1}{\gamma}}}, \qquad (3.21)
\end{aligned}
$$

where Q is defined by (3.7) with $\delta \in (m, 1)$. The rest of the proof is similar to Part (I) and hence is omitted. □

Example 3.1. Consider the second order differential equation

$$
\begin{aligned}
\left((2+\cos 4t)\, (x'(t))^2\right)' - \sin t\, (x'(t))^2 + \cos t\, (x(t))^3 \qquad (3.22) \\
+ \int_0^1 \cos t\, \phi_{5s}(x(t)) ds = -e^t \cos 2t.
\end{aligned}
$$

Here we have

(i) $\alpha(s) = 5s,\ \xi(s) = s,\ \gamma = 2,\ \beta = 3,\ \mu = 1\ a = 0$ and $b = 1$;

(ii) $r(t) = 2 + \cos 4t,\ p(t) = -\sin t,\ q_0(t) = q(t,s) = \cot s,$ and $e(t) = -e^t \cos 2t$.

Note that

$$
m = \left(\int_{\frac{1}{5}}^1 ds \right)^{-1} \left(\int_{\frac{1}{5}}^1 \frac{1}{5s} ds \right) = \ln \sqrt[4]{5}.
$$

For any $\delta \in \left(\ln \sqrt[4]{5}, 1\right]$, we set

$$
\eta(s) := \frac{\delta}{5\delta - 1} s^{\frac{2-5s}{5\delta-1}},
$$

then (2.1) is satisfied. For any $T \in \mathbb{R}$, we choose $n \in \mathbb{N}$ so large that $2n\pi \geq T$ and let

$$
a_1 = 2n\pi,\ a_2 = b_1 = 2n\pi + \frac{\pi}{4},\ b_2 = 2n\pi + \frac{\pi}{2}.
$$

Let $\rho(t) = 2 + \cos 4t$, and for $i = 1, 2$ let $u_i(t) = \sin 4t$.Then

$$\int_0^{\frac{\pi}{4}} \left(\frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}} \left[(\gamma+1) \left| u_i'(t) \right| + \left| u_i(t) \right| \left| P(t) \right| \right]^{\gamma+1} \right) dt$$
$$= 4\int_0^{\frac{\pi}{4}} (2+\cos 4t)^2 \cos^3 4t dt = \frac{3}{2}\pi.$$

Therefore, it is easy to see that (3.5) is satisfied and hence Eq. (3.22) is oscillatory if

$$\sup_{\delta \in (\ln \sqrt[4]{5}, 1]} \int_0^{\frac{\pi}{4}} 2(2+\cos 4t)\sqrt{\cos t \ \hat{q}(t)} \sin^3 4t dt > \frac{3}{2}\pi,$$

where

$$\hat{q}(t) = \left[\frac{\left| e^t \cos 2t \right|}{1-\delta} \right]^{1-\delta} \exp\left(\int_a^b \eta(s) \ln \left[\frac{\cos t}{\eta(s)} \right] ds \right).$$

Following Philos [27], Kong [22], and Kong [23], we say that for any $a, b \in \mathbb{R}$ such that $a < b$, a function $H_i(t,s)$, $i = 1,2$, belongs to a function class $\mathcal{H}(a,b)$, denoted by $H_i \in \mathcal{H}(a,b)$, if $H_i \in C(\mathbb{D}, \mathbb{R})$, where $\mathbb{D} := \{(t,s) : b \geq t \geq s \geq a\}$, which satisfies

$$H_i(t,t) = 0, \quad H_i(b,s) > 0 \quad \text{and} \quad H_i(s,a) > 0 \quad \text{for } b > s > a, \tag{3.23}$$

and $H_i(t,s)$ has continuous partial derivatives $\partial H_i(t,s)/\partial t$ and $\partial H_i(t,s)/\partial s$ on $[a,b] \times [a,b]$ such that for $i = 1,2$,

$$\frac{\partial H_i(t,s)}{\partial t} + P(s) H_i(t,s) = (\gamma+1) h_{i1}(t,s) H^{\frac{\gamma}{\gamma+1}}(t,s) \tag{3.24}$$

and

$$\frac{\partial H_i(t,s)}{\partial s} + P(s) H_i(t,s) = (\gamma+1) h_{i2}(t,s) H^{\frac{\gamma}{\gamma+1}}(t,s), \tag{3.25}$$

where h_{i1}, $h_{i2} \in L_{loc}(\mathbb{D}, \mathbb{R})$. Next, we use the function class $\mathcal{H}(a,b)$ to establish an oscillation criterion for Eq. (1.1) of the Kong-type.

Theorem 3.2. *Suppose that for any $T \geq 0$ and for $i = 1,2$, there exist constants a_i and b_i with $T \leq a_i < b_i$ such that (3.3) and (3.4) hold. Assume further that for $i = 1,2$, there exist $c_i \in (a_i, b_i)$ and $H_i \in \mathcal{H}(a_i, b_i)$ and a continuous positive function $\rho(t)$ such that*

$$\sup_{\delta \in (m,1]} \left\{ \frac{1}{H_i(c_i, a_i)} \int_{a_i}^{c_i} \left[Q(s) H_i(s, a_i) - \rho(s) r(s) \left| h_{i1}(s, a_i) \right|^{\gamma+1} \right] ds \right.$$

$$+\frac{1}{H_i(b_i,c_i)}\int_{c_i}^{b_i}\left[Q(s)\,H_i(b_i,s)-\rho(s)\,r(s)\,|h_{i2}(b_i,s)|^{\gamma+1}\right]ds\Big\}>0, \tag{3.26}$$

where $P(t)$ and $Q(t)$ are defined by (3.6) *and* (3.7), *respectively. Then Eq.* (3.1) *is oscillatory.*

Proof. Assume Eq. (3.1) has an extendible solution $x(t)$ which is eventually positive or negative. Then, without loss of generality, assume $x(t)>0$ for all $t\geq T\geq 0$, where T depends on the solution $x(t)$. Define $z(t)$ by (3.8). From (3.14) and (3.21), we get that

$$z'(t)\leq -Q(t)+P(t)z(t)-\frac{\gamma\,|z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)\,r(t))^{\frac{1}{\gamma}}}. \tag{3.27}$$

Multiplying both sides of (3.27), with t replaced by s, by $H_1(b_1,s)$ and integrating with respect to s from c_1 to b_1, we find that

$$\begin{aligned}&\int_{c_1}^{b_1}Q(s)\,H_1(b_1,s)\,ds\\ \leq\ &-\int_{c_1}^{b_1}z'(s)\,H_1(b_1,s)\,ds+\int_{c_1}^{b_1}P(s)\,z(s)H_1(b_1,s)\,ds\\ &-\int_{c_1}^{b_1}\frac{\gamma\,|z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)\,r(t))^{\frac{1}{\gamma}}}H_1(b_1,s)\,ds.\end{aligned}$$

Using integration by parts and from (3.23) and (3.25), we obtain

$$\begin{aligned}&\int_{c_1}^{b_1}Q(s)\,H_1(b_1,s)\,ds\\ \leq\ & z(c_1)\,H_1(b_1,c_1)+\int_{c_1}^{b_1}\Big[(\gamma+1)\,h_{12}(b_1,s)\,H_1^{\frac{\gamma}{\gamma+1}}(b_1,s)\,z(s)\\ &-\frac{\gamma\,|z(s)|^{\frac{\gamma+1}{\gamma}}\,H_1(b_1,s)}{(\rho(s)\,r(s))^{\frac{1}{\gamma}}}\Big]ds\\ \leq\ & z(c_1)\,H_1(b_1,c_1)+\int_{c_1}^{b_1}\Big[(\gamma+1)\,|h_{12}(b_1,s)|H_1^{\frac{\gamma}{\gamma+1}}(b_1,s)\,|z(s)|\\ &-\frac{\gamma\,|z(s)|^{\frac{\gamma+1}{\gamma}}\,H_1(b_1,s)}{(\rho(s)\,r(s))^{\frac{1}{\gamma}}}\Big]ds.\end{aligned} \tag{3.28}$$

Let $\lambda = \dfrac{\gamma+1}{\gamma}$. Define A and B by

$$A^{\lambda} := \frac{\gamma \left|z(s)\right|^{\lambda} H_1(b_1,s)}{\left(\rho(s) r(s)\right)^{\frac{1}{\gamma}}} \quad \text{and} \quad B^{\lambda-1} := \left(\gamma\rho(s) r(s)\right)^{\frac{1}{\gamma+1}} \left|h_{12}(b_1,s)\right|.$$

Then, using the inequality (3.16), we get that

$$(\gamma+1)\left|h_{12}(b_1,s)\right| H_1^{\frac{\gamma}{\gamma+1}}(b_1,s)\left|z(s)\right| - \frac{\gamma\left|z(s)\right|^{\frac{\gamma+1}{\gamma}} H_1(b_1,s)}{\left(\rho(s) r(s)\right)^{\frac{1}{\gamma}}} \le \rho(s) r(s)\left|h_{12}(b_1,s)\right|^{\gamma+1}.$$

This together with (3.28) shows that

$$\frac{1}{H_1(b_1,c_1)} \int_{c_1}^{b_1} \left[Q(s) H_1(b_1,s) - \rho(s) r(s)\left|h_{12}(b_1,s)\right|^{\gamma+1}\right] ds \le z(c_1). \tag{3.29}$$

Similarly, multiplying both sides of (3.27), with t replaced by s, by $H_1(s,a_1)$ and integrating by parts from a_1 to c_1, we see that

$$\frac{1}{H_1(c_1,a_1)} \int_{a_1}^{c_1} \left[Q(s) H_1(s,a_1) - \rho(s) r(s)\left|h_{11}(s,a_1)\right|^{\gamma+1}\right] ds \le -z(c_1). \tag{3.30}$$

Combining (3.29) and (3.30) we get that

$$\frac{1}{H_1(c_1,a_1)} \int_{a_1}^{c_1} \left[Q(s) H_1(s,a_1) - \rho(s) r(s) h_{11}^{\gamma+1}(s,a_1)\right] ds$$
$$+\frac{1}{H_1(b_1,c_1)} \int_{c_1}^{b_1} \left[Q(s) H_1(b_1,s) - \rho(s) r(s) h_{12}^{\gamma+1}(b_1,s)\right] ds \le 0.$$

This contradicts (3.26) with $i=1$. This completes the proof. □

4. Oscillation Criteria for (1.1) with $g(t,s) \not\equiv t$. In this section we prove oscillation criteria for Eq. (1.1) with both cases of delay and advanced types. In the follwoing, we will use the notations:

$$g_*(t) = \inf_{s\in[a,b)} \{t, g(t,s)\} \text{ and } g^*(t) = \sup_{s\in[a,b)} \{t, g(t,s)\};$$

$$\psi_i(t,s) := \begin{cases} \delta_i(t,s), & g(t,s) < t, \\ \zeta_i(t,s), & g(t,s) > t; \end{cases}$$

with

$$\delta_i(t,s) := \frac{R(g(t,s), g(a_i,s))}{R(t, g(a_i,s))};$$

and

$$\zeta_i(t,s) := \frac{R(g(b_i,s), g(t,s))}{R(g(b_i,s), t)},$$

$$R(t,t_0) := \int_{t_0}^{t} \tilde{r}^{-\frac{1}{\gamma}}(u)\,du,\ \tilde{r}(t)$$
$$:= r(t)\left[\exp\int_0^t \frac{p(v)}{r(v)}dv\right] \text{ and } \hat{q}(t,s) := q(t,s)\left[\psi_1(t,s)\right]^{\alpha(s)}.$$

Theorem 4.1. *Suppose that for any $T \geq 0$ and for $i = 1, 2$, there exist constants $a_i,\ b_i \in [T,\infty)$ with $a_i < b_i$, such that*

$$q_0(t) \geq 0 \quad \textit{for } t \in [g_*(a_i), g^*(b_i)], \tag{4.1}$$

$$q(t,s) \geq 0 \quad \textit{for } (t,s) \in [g_*(a_i), g^*(b_i)] \times [a,b), \tag{4.2}$$

and

$$(-1)^i e(t) \geq 0, \textit{ for } t \in [g_*(a_i), g^*(b_i)]. \tag{4.3}$$

Assume further that for $i = 1, 2$, there exist $u_i \in C^1[a_i, b_i]$ satisfying $u_i(a_i) = u_i(b_i) = 0$, $u_i(t) \not\equiv 0$ on $[a_i, b_i]$ and a continuous positive function $\rho(t)$ such that

$$\sup_{\delta\in(m,1]}\int_{a_i}^{b_i}\left[\hat{Q}(t)|u_i(t)|^{\gamma+1} - \frac{\rho(t)r(t)}{(\gamma+1)^{\gamma+1}}[(\gamma+1)|u_1'(t)| + |u_1(t)|\,|P(t)|]^{\gamma+1}\right]dt>0,$$

where $P(t)$ is defined by (3.6) and

$$\hat{Q}(t) := \hat{\delta}\rho(t)\left(q_0(t)\right)^{(\gamma-\mu)/(\beta-\mu)}\left(\bar{q}(t)\right)^{(\beta-\gamma)/(\beta-\mu)}, \tag{4.4}$$

with

$$\hat{\delta} := (\beta-\mu)(\beta-\gamma)^{(\gamma-\beta)/(\beta-\mu)}(\gamma-\mu)^{(\mu-\gamma)/(\beta-\mu)},$$

and

$$\bar{q}(t) := \left[\frac{|e(t)|}{1-\delta}\right]^{1-\delta}\exp\left(\int_a^b \eta(s)\ln\left[\frac{\hat{q}(t,s)}{\eta(s)}\right]d\zeta(s)\right),$$

with $\eta(s)$ is defined as in Lemma 2.1 based on δ. Here we use the convention that $\ln 0 = -\infty$, $e^{-\infty} = 0$, and $0^{1-\delta} = 1$ and $(1-\delta)^{1-\delta} = 1$ for $\delta = 1$. Then Eq. (1.1) is oscillatory.

Proof. Assume Eq. (1.1) has an extendible solution $x(t)$ which is eventually positive or negative. Then, without loss of generality, we may assume $x(t)$, $x(g(t,s)) > 0$, for $t \in [T, \infty)$ and $s \in [a,b]$. Define $z(t)$ by (3.8). From (1.1) and (3.9), we have for $t \geq T$,

$$\begin{aligned} z'(t) &= -\rho(t)\, q_0(t)\, x^{\beta-\gamma}(t) \\ &\quad -\rho(t)\int_a^b q(t,s)\,\frac{[x(g(t,s))]^{\alpha(s)}}{[x(t)]^{\gamma}}\, d\zeta(s) + \rho(t)\, e(t) x^{-\gamma}(t) \\ &\quad +P(t)\, z(t) - \frac{\gamma\,|z(t)|^{\frac{\gamma+1}{\gamma}}}{(\rho(t)\, r(t))^{\frac{1}{\gamma}}}. \end{aligned} \tag{4.5}$$

From the assumption, there exist constants a_1 and b_1 with $a_1 < b_1$ and $[g_*(a_1), g^*(b_1)] \subset [t_0, \infty)$ such that (4.1), (4.2) and (4.3) hold with $i = 1$. From (1.1), we get, for $t \in [g_*(a_1), g^*(b_1)]$,

$$\begin{aligned} &\left(\tilde{r}(t)\phi_\gamma\left(x'(t)\right)\right)' \\ &= \left[\exp\int_0^t \frac{p(v)}{r(v)}dv\right]\left(r(t)\phi_\gamma\left(x'(t)\right)\right)' + \left[\exp\int_0^t \frac{p(v)}{r(v)}dv\right] p(t)\phi_\gamma\left(x'(t)\right) \\ &= \left[\exp\int_0^t \frac{p(v)}{r(v)}dv\right]\left[-q_0(t)\,\phi_\beta(x(t)) - \int_a^b q(t,s)\,\phi_{\alpha(s)}(x(g(t,s)))\, d\zeta(s) + e(t)\right] \\ &\leq 0. \end{aligned}$$

Then $\tilde{r}(t)\phi_\gamma\left(x'(t)\right)$ is nonincreasing on $[g_*(a_1), g^*(b_1)]$. Now we consider the following two cases:

Case (a): Delay type, i.e. $g(t,s) \leq t$, for $t \in [a,b]$ and $s \in [a,b]$. Since $\tilde{r}(t)\phi_\gamma\left(x'(t)\right)$ is nonincreasing on $[g_*(a_1), g^*(b_1)]$. Then

$$\begin{aligned} x(t) - x(g(t,s)) &= \int_{g(t,s)}^t \phi_\gamma^{-1}(\tilde{r}(u)\phi_\gamma(x'(u)))\tilde{r}^{-\frac{1}{\gamma}}(u)\, du \\ &\leq \phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')(g(t,s))\right]\int_{g(t,s)}^t \tilde{r}^{-\frac{1}{\gamma}}(u)\, du \\ &= \phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')(g(t,s))\right] R(t, g(t,s)), \end{aligned}$$

where ϕ_γ^{-1} is the inverse function of ϕ_γ, and so

$$\frac{x(t)}{x(g(t,s))} \le 1 + \frac{\phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')(g(t,s))\right]}{x(g(t,s))} R(t, g(t,s)). \tag{4.6}$$

We also see that for $t \in [a_1, g^*(b_1)]$

$$\begin{aligned} x(g(t,s)) &> x(g(t,s)) - x(g(a_1,s)) = \int_{g(a_1,s)}^{g(t,s)} \phi_\gamma^{-1}(\tilde{r}(u)\phi_\gamma(x'(u)))\tilde{r}^{-\frac{1}{\gamma}}(u)\,du \\ &\ge \phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')(g(t,s))\right] \int_{g(a_1,s)}^{g(t,s)} \tilde{r}^{-\frac{1}{\gamma}}(u)\,du \\ &= \phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')(g(t,s))\right] R(g(t,s), g(a_1,s)), \end{aligned}$$

which implies that for $t \in (a_1, g^*(b_1)]$

$$\frac{\phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')(g(t,s))\right])}{x(g(t,s))} < \frac{1}{R(g(t,s), g(a_1,s))}. \tag{4.7}$$

Therefore, the combination of (4.6) and (4.7) shows that for $t \in (a_1, g^*(b_1)]$

$$\frac{x(t)}{x(g(t,s))} < 1 + \frac{R(t, g(t,s))}{R(g(t,s), g(a_1,s))} = \frac{R(t, g(a_1,s))}{R(g(t,s), g(a_1,s))} = \frac{1}{\delta_1(t,s)}.$$

Hence

$$x(g(t,s)) > \delta_1(t,s)\, x(t), \quad \text{for } t \in [a_1, g^*(b_1)]. \tag{4.8}$$

Case (b): advanced type, i.e. $g(t,s) > t$, for $t \in [a,b]$ and $s \in [a,b]$. Since $\tilde{r}(t)\phi_\gamma\left(x'(t)\right)$ is nonincreasing on $[g_*(a_1), g^*(b_1)]$, we have, for $t \in [g_*(a_1), b_1]$

$$\begin{aligned} x(g(t,s)) - x(t) &= \int_t^{g(t,s)} \phi_\gamma^{-1}(\tilde{r}(u)\phi_\gamma(x'(u)))\tilde{r}^{-\frac{1}{\gamma}}(u)\,du \\ &\ge \phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')(g(t,s))\right] \int_t^{g(t,s)} \tilde{r}^{-\frac{1}{\gamma}}(u)\,du \\ &= \phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')(g(t,s))\right]\; R(g(t,s), t), \end{aligned}$$

and so

$$\frac{x(t)}{x(g(t,s))} \le 1 - \frac{\phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')(g(t,s))\right]}{x(g(t,s))} R(g(t,s), t). \tag{4.9}$$

Also, we see that, for $t \in [g_*(a_1), b_1]$

$$-x(g(t,s)) < x(g(b_1,s)) - x(g(t,s)) = \int_{g(t,s)}^{g(b_1,s)} \phi_\gamma^{-1}(\tilde{r}(u)\phi_\gamma(x'(u)))\tilde{r}^{-\frac{1}{\gamma}}(u)\,du$$

$$\begin{aligned} &\leq \phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')\left(g\left(t,s\right)\right)\right]\int_{g(t,s)}^{g(b_1,s)}\tilde{r}^{-\frac{1}{\gamma}}\left(u\right)du \\ &= \phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')\left(g\left(t,s\right)\right)\right]\ R\left(g\left(b_1,s\right),g\left(t,s\right)\right), \end{aligned}$$

which implies for $t\in[g_*\left(a_1\right),b_1)$, that

$$-\frac{\phi_\gamma^{-1}\left[\tilde{r}\phi_\gamma(x')\left(g\left(t,s\right)\right)\right]}{x\left(g\left(t,s\right)\right)}<\frac{1}{R\left(g\left(b_1,s\right),g\left(t,s\right)\right)}. \tag{4.10}$$

Thus, (4.9) and (4.10) imply, for $t\in[g_*\left(a_1\right),b_1)$

$$\frac{x\left(t\right)}{x\left(g\left(t,s\right)\right)}<1-\frac{R\left(g\left(t,s\right),t\right)}{R\left(g\left(b_1,s\right),g\left(t,s\right)\right)}=\frac{R\left(g\left(b_1,s\right),t\right)}{R\left(g\left(b_1,s\right),g\left(t,s\right)\right)}=\frac{1}{\zeta_1\left(t,s\right)}.$$

Hence

$$x\left(g\left(t,s\right)\right)>\zeta_1\left(t,s\right)x\left(t\right),\quad \text{for } t\in[g_*\left(a_1\right),b_1]. \tag{4.11}$$

From (4.8) and (4.11), we get

$$x\left(g\left(t,s\right)\right)\geq\psi_1\left(t,s\right)x\left(t\right),\quad \text{for } t\in[a_1,b_1] \text{ and } s\in[a,b).$$

Then (4.5) becomes, for two caes (a) and (b),

$$\begin{aligned} z'\left(t\right) \ \leq\ & -\rho\left(t\right)q_0\left(t\right)x^{\beta-\gamma}\left(t\right)-\rho\left(t\right)\int_a^b\hat{q}\left(t,s\right)\left[x\left(t\right)\right]^{\alpha(s)-\gamma}\ d\zeta\left(s\right)+\rho\left(t\right)e(t)x^{-\gamma}(t) \\ & +P\left(t\right)z\left(t\right)-\frac{\gamma\left|z\left(t\right)\right|^{\frac{\gamma+1}{\gamma}}}{\left(\rho\left(t\right)r(t)\right)^{\frac{1}{\gamma}}}, \end{aligned}$$

where $\hat{q}\left(t,s\right)=q\left(t,s\right)\left[\psi_1\left(t,s\right)\right]^{\alpha(s)}$. The rest of the proof is similar to that of Theorem 3.1 after (3.11) and hence is omitted. □

Theorem 4.2. *Suppose that for any $T\geq 0$ and for $i=1,2$, there exist constants a_i and b_i with $T\leq a_i<b_i$ such that* (4.1), (4.2) *and* (4.3) *hold. Assume further that for $i=1,2$, there exist $c_i\in(a_i,b_i)$ and $H_i\in\mathcal{H}(a_i,b_i)$ and a continuous positive function $\rho(t)$ such that*

$$\begin{aligned} \sup_{\delta\in(m,1]}\Bigg\{&\frac{1}{H_i\left(c_i,a_i\right)}\int_{a_i}^{c_i}\left[\hat{Q}\left(s\right)H_i\left(s,a_i\right)-\rho\left(s\right)r(s)\left|h_{i1}\left(s,a_i\right)\right|^{\gamma+1}\right]ds \\ &+\frac{1}{H_i\left(b_i,c_i\right)}\int_{c_i}^{b_i}\left[\hat{Q}\left(s\right)H_i\left(b_i,s\right)-\rho\left(s\right)r(s)\left|h_{i2}\left(b_i,s\right)\right|^{\gamma+1}\right]ds\Bigg\}>0, \end{aligned}$$

where $P(t)$ *and* $\hat{Q}(t)$ *are defined by* (3.6) *and* (4.4)*, respectively. Then Eq.* (3.1) *is oscillatory.*

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