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LIMIT OF THREE-POINT GREEN FUNCTIONS: THE DEGENERATE CASE

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Abstract. We investigate the limits of the ideals of holomorphic functions vanishing on three points in $\mathbb{C}^2$ when all three points tend to the origin, and what happens to the associated pluricomplex Green functions. This is a continuation of the work of Magnusson, Rashkovskii, Sigurdsson and Thomas, where those questions were settled in a generic case.

1. Introduction. Let Ω be a hyperconvex bounded domain in $\mathbb{C}^n$ containing the origin 0 and let $\mathcal{O}(\Omega)$ denote the space of holomorphic functions, respectively $PSH_-(\Omega)$ the space of nonpositive plurisubharmonic functions on Ω. For every subset S of Ω we let $\mathcal{I}(S)$ denote the ideal of all holomorphic functions vanishing on S. We consider ideals $\mathcal{I}$ such that their zero locus $V(\mathcal{I}) := \{z \in \Omega : f(z) = 0, \forall f \in \mathcal{I}\}$ is a finite set. Since the domain is pseudoconvex, there are

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finitely many global generators $\psi_j \in \mathcal{O}(\Omega)$ such that for any $f \in \mathcal{I}$, there exists $h_j \in \mathcal{O}(\Omega)$ such that $f = \sum_j h_j \psi_j$, see e.g. [2, Theorem 7.2.9, p. 190].

Definition 1.1 [6]**.** *Let $\mathcal{I}$ be an ideal of Ω, and ψ_j its generators. Then*

$$G_{\mathcal{I}}^{\Omega}(z) := \sup\left\{u(z) : u \in PSH_-(\Omega), u \le \max_j \log|\psi_j| + O(1)\right\}.$$

Note that the condition is meaningful only near $a \in V(\mathcal{I})$. In the special case when S is a finite set in Ω and $\mathcal{I} = \mathcal{I}(S)$, we write $G_{\mathcal{I}(S)} = G_S$. This case reduces to Pluricomplex Green functions with logarithmic singularities, already studied by many authors, e.g. Demailly [1], [8], Lelong [3], and Rashkovskii and Sigurdsson [6].

Following the lead of [4], we want to study the limit of G_{S_ε} when S_ε is a set of points tending to the origin, and relate that to the limit of the ideals $\mathcal{I}(S_\varepsilon)$ (in a sense to be specified below, see [4] for more details).

Definition 1.2. *Let $\mathcal{I}$ be an ideal such that its zero locus is a finite set. Then $\mathcal{I}$ is called a* complete intersection ideal *if it admits a set of n generators, where n is the dimension of the ambient space.*

The main result of [4], Theorem 1.11, states:

Theorem 1.3. *Let $\mathcal{I}_\varepsilon = \mathcal{I}(S_\varepsilon)$, where S_ε is a set of N points all tending to 0 and assume that $\lim_{\varepsilon\to 0} \mathcal{I}_\varepsilon = \mathcal{I}$. Then $(G_{\mathcal{I}_\varepsilon})$ converges to $G_{\mathcal{I}}$ locally uniformly on $\Omega \setminus \{0\}$ if and only if $\mathcal{I}$ is a complete intersection ideal.*

Furthermore, [4, Theorem 1.12, (i)] works out the limits of Green functions when $N = 3$ and the dimension is 2.

We need a notion of convergence of ideals, inspired by Hausdorff convergence. This is taken from [4].

Let Ω be a bounded pseudoconvex domain in $\mathbb{C}^n$. Let $E \subset \mathbb{C} \setminus \{0\}$ such that $\bar{E} \ni 0$ be the set of parameters along which we take limits. Convergence of holomorphic functions is always understood uniformly on compacta.

Definition 1.4. *If $(\mathcal{I}_\varepsilon)_{\varepsilon\in E}$ are ideals in $\mathcal{O}(\Omega)$, we define*

$$\liminf_{E\ni\varepsilon\to 0} \mathcal{I}_\varepsilon := \{f \in \mathcal{O}(\Omega) : \ \forall\varepsilon \in E, \exists f_\varepsilon \in \mathcal{I}_\varepsilon, \lim_{E\ni\varepsilon\to 0} f_\varepsilon = f\}.$$

Likewise $\limsup_{E\ni\varepsilon\to 0} \mathcal{I}_\varepsilon$ is the vector space generated by

$$\{f \in \mathcal{O}(\Omega) : \exists E' \subset E, 0 \in \overline{E'} \textit{ and } \forall\varepsilon \in E', f_\varepsilon \in \mathcal{I}_\varepsilon : \lim_{E'\ni\varepsilon\to 0} f_\varepsilon = f\}.$$

Note that a typical example of such an E' is a sequence tending to 0.

We say that $(\mathcal{I}_\varepsilon)_{\varepsilon\in E}$ converges *to* $\mathcal{I}$ *if* $\liminf_{E\ni\varepsilon\to 0}\mathcal{I}_\varepsilon = \limsup_{E\ni\varepsilon\to 0}\mathcal{I}_\varepsilon = \mathcal{I}$, *and write* $\lim_{E\ni\varepsilon\to 0}\mathcal{I}_\varepsilon = \mathcal{I}$.

Of course $\liminf_{E\ni\varepsilon\to 0}\mathcal{I}_\varepsilon \subset \limsup_{E\ni\varepsilon\to 0}\mathcal{I}_\varepsilon$, and they are both ideals.

Let $S_\varepsilon := \{a_1^\varepsilon, a_2^\varepsilon, a_3^\varepsilon\}$. For each pair of distinct indices, i, j, let $[a_i^\varepsilon - a_j^\varepsilon] = v_k^\varepsilon \in \mathbb{CP}^1$ where $\{i,j,k\} = \{1,2,3\}$. The cases which are studied in [4] are those where there exist $i \neq j$ such that $\lim_{E\ni\varepsilon\to 0} v_i^\varepsilon$ and $\lim_{E\ni\varepsilon\to 0} v_j^\varepsilon$ exist and are distinct. In those cases $\lim_{E\ni\varepsilon\to 0}\mathcal{I}(S_\varepsilon) = \mathfrak{M}_0^2$ (the square of the maximal ideal at zero, i.e. the set of functions vanishing at zero together with all their first derivatives), which is not a complete intersection ideal.

The main goal of this note is to investigate the asymptotic behavior of ideals and Green functions in the remaining (and most singular) case, when there exists $v \in \mathbb{C}^2$, with $\|v\| = 1$, such that

$$\lim_{E\ni\varepsilon\to 0} v_i^\varepsilon = [v] \text{ for } 1 \le i \le 3. \tag{1.1}$$

We use the notation $z\cdot\bar{w} := z_1\bar{w}_1 + z_2\bar{w}_2$ for $z, w \in \mathbb{C}^2$, and $\|z\|^2 := z\cdot\bar{z}$.

The notions we study do not depend on the order of the points in S_ε, nor does (1.1). We may number the three points so that for each ε,

$$||a_1^\varepsilon - a_2^\varepsilon|| \ge ||a_3^\varepsilon - a_2^\varepsilon|| \ge ||a_1^\varepsilon - a_3^\varepsilon||.$$

We perform a translation so that $a_1^\varepsilon = (0,0)$. Since the distance from a_1^ε to the origin tends to 0 by hypothesis, this does not change any of the limits we are studying, and we shall make this assumption henceforth.

Let θ be the (acute) angle between the complex lines directed by a_2^ε and a_3^ε, i.e. $\theta := \cos^{-1}\left(\dfrac{|a_2^\varepsilon\cdot\bar{a}_3^\varepsilon|}{\|a_2^\varepsilon\|\|a_3^\varepsilon\|}\right)$. Geometrically, this is the smallest possible angle between the real lines directed by $e^{it_1}a_2^\varepsilon$ and $e^{it_2}a_3^\varepsilon$ for any real numbers t_1, t_2.

Theorem 1.5. *With the above normalizations,* $\lim_{E\ni\varepsilon\to 0}\mathcal{I}(S_\varepsilon) = \mathfrak{M}_0^2$ *if and only if* $\lim_{E\ni\varepsilon\to 0}\dfrac{\|a_2^\varepsilon\|}{\theta} = 0$, *or equivalently*

$$\lim_{E\ni\varepsilon\to 0}\frac{\|a_2^\varepsilon\|}{|\det(\frac{a_2^\varepsilon}{\|a_2^\varepsilon\|}, \frac{a_3^\varepsilon}{\|a_3^\varepsilon\|})|} = 0, \tag{1.2}$$

where the determinant is taken with respect to an orthonormal basis.

Theorem 1.6. (1) *Suppose that* $\lim_{E\ni\varepsilon\to 0} v_j^\varepsilon = [e_1] := [1:0]$, *for* $1 \le j \le 3$, *and that* $\lim_{E\ni\varepsilon\to 0} M(\varepsilon) = m \in \mathbb{C}$, *where*

$$M(\varepsilon) := \frac{\det\left(\frac{a_2^\varepsilon}{\|a_2^\varepsilon\|}, \frac{a_3^\varepsilon}{\|a_3^\varepsilon\|}\right)\left(e_1 \cdot \frac{\bar{a}_2^\varepsilon}{\|a_2^\varepsilon\|}\right)^3}{\left(\frac{a_3^\varepsilon \cdot \bar{a}_2^\varepsilon}{\|a_2^\varepsilon\|} - \|a_2^\varepsilon\|\right)\frac{(a_3^\varepsilon \cdot \bar{a}_2^\varepsilon)}{\|a_3^\varepsilon\|\|a_2^\varepsilon\|}}, \tag{1.3}$$

then

$$\lim_{E\ni\varepsilon\longrightarrow 0} \mathcal{I}_\varepsilon = \left\langle z_2 - m z_1^2, z_1^3 \right\rangle. \tag{1.4}$$

(2) *Conversely, if* (1.4) *holds for some* $m \in \mathbb{C}$, *then* $\lim_{E\ni\varepsilon\to 0} v_j^\varepsilon = [e_1] := [1:0]$, *for* $1 \le j \le 3$ *and* (1.3) *holds.*

Of course the same result holds with any other unit vector instead of e_1. Note that, unlike the situation of Theorem 1.5, when the limit ideal is known, the limits of the directions v_j^ε are determined. Notice also that when (1.2) is verified, then $|M(\varepsilon)| \to \infty$. This will be made clear after equation (2.7).

Using some of the results of [4] and [7], we now draw consequences about the limits of the Green functions. The case not covered in [4, Theorem 1.12, (i)] is when $\lim_{E\ni\varepsilon\to 0} v_j^\varepsilon = [e_1] := [1:0]$, for $1 \le j \le 3$, so we make that hypothesis.

Theorem 1.7. (1) *If condition* (1.2) *is satisfied, then*

$$\lim_{E\ni\varepsilon\to 0} G_{\mathcal{I}_\varepsilon} = \max(2\log|z_1|, \frac{3}{2}\log|z_2|) + O(1).$$

(2) *If* $\lim_{E\ni\varepsilon\to 0} M(\varepsilon) = m \in \mathbb{C}$, *then*

$$\lim_{E\ni\varepsilon\to 0} G_{\mathcal{I}_\varepsilon} = \max(3\log|z_1|, \log|z_2 - m z_1^2|) + O(1).$$

Notice that although in case (1) the limit ideal does not depend on the common value of $\lim_{E\ni\varepsilon\to 0} v_j^\varepsilon$, $1 \le j \le 3$, the limit of the Green functions does.

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2. Proofs of Theorems 1.5 and 1.6.

2.1. Preliminary facts. Denote the Taylor expansion and Taylor polynomial of a holomorphic function f by

$$f(z) = f(z_1, z_2) = \sum_{j,k=0}^{\infty} a_{jk} z_1^j z_2^k; \quad P_m(f)(z) := \sum_{\substack{j,k \\ j+k \leqslant m}} a_{jk} z_1^j z_2^k.$$

It follows from the Cauchy formula on the distinguished boundary of $\mathbb{D}^2$ that

Lemma 2.1. *Let $m \in \mathbb{N}^*$, U a bidisk centered at $(0,0)$, relatively compact in $\mathbb{D}^2$. There exists $C = C(m, U)$ such that for any $f \in \mathcal{O}(\mathbb{D}^2)$ with $\sup\limits_{\mathbb{D}^2} ||f|| \leqslant 1$, there exist holomorphic functions $r_{j,k} \in \mathcal{O}(\mathbb{D}^2)$ satisfying : for $j + k = m + 1$, $\sup\limits_U |r_{j,k}| \leqslant C, 0 \leqslant j \leqslant m + 1$ and for $z = (z_1, z_2) \in U$, then*

$$f(z) = P_m(f)(z) + R_{m+1}(z) = P_m(f)(z) + \sum_{j=0}^{m+1} r_{j,m+1-j}(z) z_1^j z_2^{m+1-j}.$$

We also recall that given $x, y \in \mathbb{C}^2$, we always have $|\det(x,y)|^2 + |x \cdot \bar{y}|^2 = \|x\|^2 \|y\|^2$, so with our notations

$$\left| \det\left(\frac{a_2^\varepsilon}{\|a_2^\varepsilon\|}, \frac{a_3^\varepsilon}{\|a_3^\varepsilon\|}\right) \right| = \sin \theta,$$

and this proves the equivalence of (1.2) with $\lim\limits_{\varepsilon \to 0} \dfrac{\|a_2^\varepsilon\|}{\theta} = 0$.

2.2. Proof of the sufficiency in Theorem 1.5. Under the hypothesis (1.2), we will prove that $\limsup\limits_{\varepsilon} \mathcal{I}_\varepsilon \subset \mathfrak{M}_0^2 \subset \liminf\limits_{\varepsilon} \mathcal{I}_\varepsilon$.

Suppose that $f \in \limsup\limits_{\varepsilon} \mathcal{I}_\varepsilon$, then $f = \sum\limits_{i=1}^{N} f_i$, where for each i there exists $E_i \subset E$ such that $0 \in \overline{E}_i$ and a family of holomorphic functions $\{f_i^\varepsilon, \varepsilon \in E_i\}$, with $f_i^\varepsilon \in \mathcal{I}_\varepsilon$, $\varepsilon \in E_i$, converging to f_i uniformly on a fixed neighborhood U of the origin. It will be enough to show that each $f_i \in \mathfrak{M}_0^2$. We do this, dropping the index i from the notation henceforth and write $f_i = f$, $E_i = E'$.

Observe that all the Taylor coefficients of f^ε will have to converge. Since $f^\varepsilon(a_1^\varepsilon) = 0$, $a_{0,0}^\varepsilon = 0$ for any ε. Applying Lemma 2.1 for $m = 1$, if $U \Subset U' \Subset \Omega$,

$$f^\varepsilon(z_1, z_2) = a_{1,0}^\varepsilon z_1 + a_{0,1}^\varepsilon z_2 + R_2(z_1, z_2)$$

with $|R_2(z_1,z_2)| \le C\|z\|^2$, where C only depends on U, U' and $\sup_{U'}|f^\varepsilon|$.

Applying this to $z = a_i^\varepsilon$, dividing by $\|a_i^\varepsilon\|$ and writing $\nabla f^\varepsilon(0) := (a_{1,0}^\varepsilon, a_{0,1}^\varepsilon)$, we find

$$\frac{a_i^\varepsilon}{\|a_i^\varepsilon\|} \cdot \nabla f^\varepsilon(0) = O(\|a_i^\varepsilon\|), \quad i = 2, 3.$$

Write M for the 2×2 matrix with rows given by the coordinates of $\dfrac{a_2^\varepsilon}{\|a_2^\varepsilon\|}$ and $\dfrac{a_3^\varepsilon}{\|a_3^\varepsilon\|}$. Then $\|M\| = O(1)$ and $\|M^{-1}\| = O\left(|\det(\dfrac{a_2^\varepsilon}{\|a_2^\varepsilon\|}, \dfrac{a_3^\varepsilon}{\|a_3^\varepsilon\|})|^{-1}\right)$. Since by our choice of numbering, $\|a_3^\varepsilon\| \le \|a_2^\varepsilon\|$, we have

$$\nabla f^\varepsilon(0) = O(\|M^{-1}\|\|a_2^\varepsilon\|),$$

so that if condition (1.2) is met, then $\lim_{E'\ni\varepsilon\to 0} \nabla f^\varepsilon(0) = 0$, thus $f \in \mathfrak{M}_0^2$. We have proved that condition (1.2) implies that $\limsup_\varepsilon \mathcal{I}_\varepsilon \subset \mathfrak{M}_0^2$.

To prove the inclusion $\mathfrak{M}_0^2 \subset \liminf_\varepsilon \mathcal{I}_\varepsilon$, it will be easier to take suitable coordinates. We choose a new (varying) basis $\mathfrak{B}_\varepsilon$, with $e_1^\varepsilon := \dfrac{a_2^\varepsilon}{\|a_2^\varepsilon\|}$, and $e_2^\varepsilon = \tilde{e}_1^\varepsilon$, where for any $z = (z_1, z_2)$ we write $\tilde{z} := (-\bar{z}_2, \bar{z}_1)$. Let

$$z_1^\varepsilon = z \cdot \bar{e}_1^\varepsilon, z_2^\varepsilon = z \cdot \bar{e}_2^\varepsilon \tag{2.1}$$

be the coordinates of a point z in this new basis. Note that $x \cdot \bar{\tilde{y}} = -\det(x, y)$.

In $\mathfrak{B}_\varepsilon$,

$$a_2^\varepsilon = (\varepsilon', 0), \quad a_3^\varepsilon = (\rho, \delta\rho). \tag{2.2}$$

Then

$$\frac{a_3^\varepsilon \cdot \bar{a}_2^\varepsilon}{\|a_2^\varepsilon\|\|a_3^\varepsilon\|} = \frac{\rho}{|\rho|(1+|\delta|^2)^{1/2}},$$

so $\theta = \tan^{-1}|\delta|$ and (1.2) is now equivalent to $\lim_{E\ni\varepsilon\to 0} \dfrac{\varepsilon'}{\delta} = 0$. Note that by our numbering of the points,

$$|\rho|(1+|\delta|^2)^{1/2} \le |\varepsilon'|. \tag{2.3}$$

With those new coordinates, the following polynomials are in $\mathcal{I}_\varepsilon$:

$$\begin{aligned}
Q_1^\varepsilon(z) &= (z_1^\varepsilon)^2 - \varepsilon' z_1^\varepsilon - \frac{\rho - \varepsilon'}{\delta} z_2^\varepsilon = (z_1^\varepsilon)^2 + o(1);\\
Q_2^\varepsilon(z) &= z_2^\varepsilon\big(z_1^\varepsilon - \rho\big) = z_1^\varepsilon z_2^\varepsilon + o(1);\\
Q_3^\varepsilon(z) &= z_2^\varepsilon\big(z_2^\varepsilon - \delta\rho\big) = (z_2^\varepsilon)^2 + o(1).
\end{aligned}$$

Let $\alpha_{ij} := e_j^\varepsilon \cdot \bar{e}_i$, for $1 \leqslant i, j \leqslant 2$, so that

$$z_1 = \alpha_{11} z_1^\varepsilon + \alpha_{12} z_2^\varepsilon, \tag{2.4}$$

$$z_2 = \alpha_{21} z_1^\varepsilon + \alpha_{22} z_2^\varepsilon, \tag{2.5}$$

or in compact notation $z = Az^{(\varepsilon)}$. If we let

$$\begin{aligned} f_1^\varepsilon(z) &= \alpha_{11}^2 Q_1^\varepsilon(z) + 2\alpha_{11}\alpha_{12} Q_2^\varepsilon(z) + \alpha_{12}^2 Q_3^\varepsilon(z), \\ f_2^\varepsilon(z) &= \big(\alpha_{11}\alpha_{22} + \alpha_{12}\alpha_{21}\big) Q_2^\varepsilon(z) + \alpha_{11}\alpha_{21} Q_1^\varepsilon(z) + \alpha_{12}\alpha_{22} Q_3^\varepsilon(z), \\ f_3^\varepsilon(z) &= \alpha_{21}^2 Q_1^\varepsilon(z) + 2\alpha_{21}\alpha_{22} Q_2^\varepsilon(z) + \alpha_{22}^2 Q_3^\varepsilon(z), \end{aligned}$$

then

$$\begin{aligned} z_1^2 &= \lim_{\varepsilon \longrightarrow 0} f_1^\varepsilon(z) \in \liminf_{\varepsilon \longrightarrow 0} \mathcal{I}_\varepsilon; \\ z_1 z_2 &= \lim_{\varepsilon \longrightarrow 0} f_2^\varepsilon(z) \in \liminf_{\varepsilon \longrightarrow 0} \mathcal{I}_\varepsilon; \\ z_2^2 &= \lim_{\varepsilon \longrightarrow 0} f_3^\varepsilon(z) \in \liminf_{\varepsilon \longrightarrow 0} \mathcal{I}_\varepsilon, \end{aligned}$$

which proves that $\mathfrak{M}_0^2 \subset \liminf_{\varepsilon} \mathcal{I}_\varepsilon$.

Thus (1.2) is sufficient for the claimed convergence.

2.3. Proof of Theorem 1.6, direct part. We use the notations from (2.4) above. Recall that all the coefficients α_{ij} depend on ε. The hypothesis $\lim_{E \ni \varepsilon \to 0} v_j^\varepsilon = [e_1]$ implies $\lim_{E \ni \varepsilon \to 0} \alpha_{12} = \lim_{E \ni \varepsilon \to 0} \alpha_{21} = 0$, and $\lim_{E \ni \varepsilon \to 0} |\alpha_{11}| = \lim_{E \ni \varepsilon \to 0} |\alpha_{22}| = 1$, but $\arg \alpha_{11} = -\arg \alpha_{22}$ doesn't have to converge.

According to (2.2),

$$\rho = \frac{a_3^\varepsilon \cdot \bar{a}_2^\varepsilon}{\|a_2^\varepsilon\|}, \quad \delta = \frac{1}{\rho} \frac{\det(a_2^\varepsilon, a_3^\varepsilon)}{\|a_2^\varepsilon\|} = \frac{\det(a_2^\varepsilon, a_3^\varepsilon)}{a_3^\varepsilon \cdot \bar{a}_2^\varepsilon}. \tag{2.6}$$

From (1.3) we find

$$M(\varepsilon) = \frac{\delta \bar{\alpha}_{11}^3}{\rho - \varepsilon'}, \text{ thus } m = \lim_{\varepsilon} \frac{\delta \alpha_{22}}{(\rho - \varepsilon')\alpha_{11}^2}, \tag{2.7}$$

and we see that $M(\varepsilon)$ doesn't admit a finite limit when (1.2) holds, i.e. $\lim_{E \ni \varepsilon \to 0} \frac{\varepsilon'}{\delta} = 0$.

We now prove that $\langle z_2 - m z_1^2, z_1^3 \rangle \subset \liminf_{E \ni \varepsilon \to 0} \mathcal{I}_\varepsilon$. Let

$$g_{1,\varepsilon}(z) := (z \cdot \bar{e}_1^\varepsilon)((z - a_2^\varepsilon) \cdot \bar{e}_1^\varepsilon)((z - a_3^\varepsilon) \cdot \bar{e}_1^\varepsilon);$$

then $g_{1,\varepsilon} \in \mathcal{I}_\varepsilon$ for any ε and $z_1^3 = \lim_{E \ni \varepsilon \to 0} \alpha_{11}^3 g_{1,\varepsilon}(z)$, so $z_1^3 \in \liminf_{E \ni \varepsilon \to 0} \mathcal{I}_\varepsilon$.

On the other hand, $\mathcal{I}_\varepsilon$ contains

$$\frac{-\alpha_{22}\delta}{\rho - \varepsilon'} Q_1^\varepsilon(z) = \alpha_{22} z_2^\varepsilon - \frac{\alpha_{22}\delta}{\rho - \varepsilon'}(z_1^\varepsilon)^2 + o(1)$$
$$= z_2 - \frac{\alpha_{22}\delta}{(\rho - \varepsilon')\alpha_{11}^2} z_1^2 + o(1) = z_2 - m z_1^2 + o(1),$$

so $z_2 - m z_1^2 \in \liminf_{E \ni \varepsilon \to 0} \mathcal{I}_\varepsilon$.

To see that $\limsup_{E \ni \varepsilon \to 0} \mathcal{I}_\varepsilon \subset \langle z_2 - m z_1^2, z_1^3 \rangle$, we decompose any function $f \in \limsup_{E \ni \varepsilon \to 0} \mathcal{I}_\varepsilon$ in the same way as at the beginning of the proof of Theorem 1.5, and so we may work with an $f = \lim_{E' \ni \varepsilon \to 0} f^\varepsilon$, $f^\varepsilon \in \mathcal{I}_\varepsilon$.

Given any function f expressed in the (z_1, z_2)-coordinates, we denote $\hat{f}(z) = f(Az)$, so that $\hat{f}(z^{(\varepsilon)}) = f(z)$. Therefore $f^\varepsilon \in \mathcal{I}_\varepsilon$ means that $\widehat{f^\varepsilon}(0,0) = \widehat{f^\varepsilon}(\varepsilon', 0) = \widehat{f^\varepsilon}(\rho, \delta\rho) = 0$.

We write

$$\widehat{f^\varepsilon}(z_1^\varepsilon, z_2^\varepsilon) = \sum_{j,k} \hat{a}_{ij}^\varepsilon (z_1^\varepsilon)^j (z_2^\varepsilon)^k$$

(both the function and the coordinates depend on ε).

Applying Lemma 2.1 for $m = 2$, taking $a_{0,0}^\varepsilon = 0$ into account,

$$\widehat{f^\varepsilon}(z_1, z_2) = \hat{a}_{1,0}^\varepsilon z_1 + \hat{a}_{0,1}^\varepsilon z_2 + \hat{a}_{2,0}^\varepsilon z_1^2 + \hat{a}_{0,2}^\varepsilon z_2^2 + \hat{a}_{1,1}^\varepsilon z_1 z_2 + R_3(z_1, z_2)$$

with $|R_3(z_1, z_2)| \le C\|z\|^3$, uniformly in $\varepsilon \in E'$ by the convergence hypothesis.

Since $\widehat{f^\varepsilon}(\varepsilon', 0) = 0$ we have $\hat{a}_{1,0}^\varepsilon \varepsilon' + \hat{a}_{2,0}^\varepsilon \varepsilon'^2 + R_3(\varepsilon', 0) = 0$. Thus

$$\hat{a}_{1,0}^\varepsilon = -\hat{a}_{2,0}^\varepsilon \varepsilon' - \frac{R_3(\varepsilon', 0)}{\varepsilon'} \tag{2.8}$$

for any $\varepsilon \in E'$.

Thus $\dfrac{\partial \hat{f}}{\partial z_1}(0,0) = \lim_{E' \ni \varepsilon \to 0} \hat{a}_{1,0}^\varepsilon \bar{\alpha}_{11} = 0$.

Furthermore, from $\hat{f}^\varepsilon(\rho, \delta\rho) = 0$, and (2.8) we deduce

$$\begin{aligned} &\left[-\hat{a}_{2,0}^\varepsilon \varepsilon' - \frac{R_3(\varepsilon', 0)}{\varepsilon'} \right] \rho + \hat{a}_{0,1}^\varepsilon \delta\rho + \hat{a}_{2,0}^\varepsilon \rho^2 + \hat{a}_{0,2}^\varepsilon \delta^2 \rho^2 \\ &+ \hat{a}_{1,1}^\varepsilon \delta\rho^2 + R_3(\rho, \delta\rho) = 0, \end{aligned} \tag{2.9}$$

and dividing by $\rho(\rho-\varepsilon')$,

$$\hat{a}^{\varepsilon}_{2,0}+\hat{a}^{\varepsilon}_{0,1}\frac{\delta}{\rho-\varepsilon'}+\hat{a}^{\varepsilon}_{0,2}\delta^2\frac{\rho}{\rho-\varepsilon'}+\hat{a}^{\varepsilon}_{1,1}\frac{\delta}{\rho-\varepsilon'}\rho+\frac{R_3(\rho,\delta\rho)}{\rho(\rho-\varepsilon')}-\frac{R_3(\varepsilon',0)}{\varepsilon(\rho-\varepsilon')}=0.$$

We observe that $|\rho-\varepsilon'|\asymp\varepsilon'$. From (2.3), we see that $|\rho-\varepsilon'|\le 2\varepsilon'$. Conversely, by our choice of numbering, $\|a_2\|^2\ge\|a_2-a_3\|^2\ge\|a_3\|^2$, which is equivalent to $\|a_2\|^2\ge 2\operatorname{Re}(a_2\cdot\bar{a}_3)\ge\|a_3\|^2$. Therefore

$$|\rho-\varepsilon'|\ge\operatorname{Re}(\rho-\varepsilon')=\|a^{\varepsilon}_2\|-\frac{\operatorname{Re}(a^{\varepsilon}_2\cdot\bar{a}^{\varepsilon}_3)}{\|a^{\varepsilon}_2\|}\ge\frac{1}{2}\|a^{\varepsilon}_2\|=\frac{\varepsilon'}{2}, \tag{2.10}$$

q.e.d.

Since $R_3(\rho,\delta\rho)=O(\rho^3)$, we have $\lim\limits_{E\ni\varepsilon\to 0}\dfrac{R_3(\rho,\delta\rho)}{\rho(\rho-\varepsilon')}=0$,
and $\hat{a}^{\varepsilon}_{2,0}=-\hat{a}^{\varepsilon}_{0,1}\dfrac{\delta}{\rho-\varepsilon'}+o(1)$.

Now $f(z)=\lim\limits_{E'\ni\varepsilon\to 0}\widehat{f^{\varepsilon}}(z^{(\varepsilon)})$ and their Taylor coefficients converge as well, so since

$$\widehat{f^{\varepsilon}}(z^{(\varepsilon)})=\hat{a}^{\varepsilon}_{0,1}\left(\bar{\alpha}_{22}z_2-\frac{\delta}{\rho-\varepsilon'}\bar{\alpha}^2_{11}z_1^2\right)\\ +\hat{a}^{\varepsilon}_{0,2}\bar{\alpha}^2_{22}z_2^2+\hat{a}^{\varepsilon}_{1,1}\bar{\alpha}_{11}\bar{\alpha}_{22}z_1z_2+R_3(z^{(\varepsilon)}_1,z^{(\varepsilon)}_2)+o(1),$$

and $m=\lim\limits_{E\ni\varepsilon\to 0}\dfrac{\delta\bar{\alpha}^2_{11}}{(\rho-\varepsilon')\bar{\alpha}_{22}}$, the Taylor coefficients of f satisfy $a_{1,0}=0$, $a_{2,0}+ma_{0,1}=0$, i.e. $f\in\langle z_2-mz_1^2,z_1^3\rangle$, q.e.d.

2.4. Proof of the necessity in Theorem 1.5. Suppose that (1.2) doesn't hold, so there exists $E'\subset E$ with $0\in\overline{E'}\setminus E$ such that

$$\inf_{\varepsilon\in E'}\frac{\|a^{\varepsilon}_2\|}{|\det(\frac{a^{\varepsilon}_2}{\|a^{\varepsilon}_2\|},\frac{a^{\varepsilon}_3}{\|a^{\varepsilon}_3\|})|}>0. \tag{2.11}$$

This implies that $\lim\limits_{E'\ni\varepsilon\to 0}\det(\dfrac{a^{\varepsilon}_2}{\|a^{\varepsilon}_2\|},\dfrac{a^{\varepsilon}_3}{\|a^{\varepsilon}_3\|})=0$. Passing to a subsequence (which we omit in the notation), we may assume that $\lim\limits_{E'\ni\varepsilon\to 0}[a^{\varepsilon}_2]$ exists in $\mathbb{CP}^1$. We take an orthonormal basis (e_1,e_2) such that $\lim\limits_{E'\ni\varepsilon\to 0}[a^{\varepsilon}_2]=[e_1]$. Then $\lim\limits_{E'\ni\varepsilon\to 0}v_j=[e_1]$

for $j = 2, 3$. To study $[v_1] = [a_2^\varepsilon - a_3^\varepsilon]$, we recall that the coordinates of $a_3^\varepsilon - a_2^\varepsilon$ are $(\rho - \varepsilon', \delta\rho)$ in $\mathcal{B}_\varepsilon$ (2.2). Then by (2.10) and (2.6)

$$\left|\frac{\delta\rho}{\rho - \varepsilon'}\right| \preceq |\delta| = \frac{|\det(a_2^\varepsilon, a_3^\varepsilon)|}{\|a_2^\varepsilon\|^2\|a_3^\varepsilon\|^2 - |\det(a_2^\varepsilon, a_3^\varepsilon)|^2} \asymp \left|\det(\frac{a_2^\varepsilon}{\|a_2^\varepsilon\|}, \frac{a_3^\varepsilon}{\|a_3^\varepsilon\|})\right|,$$

so $\lim\limits_{E'\ni\varepsilon\to 0} v_1 = [e_1]$ as well: the first hypothesis of Theorem 1.6 is satisfied. To check the second, observe that

$$|M(\varepsilon)| \asymp \left|\frac{\det(\frac{a_2^\varepsilon}{\|a_2^\varepsilon\|}, \frac{a_3^\varepsilon}{\|a_3^\varepsilon\|})}{\frac{a_3^\varepsilon\cdot\bar{a}_2^\varepsilon}{\|a_2^\varepsilon\|} - \|a_2^\varepsilon\|}\right| \leq 2\left|\frac{\det(\frac{a_2^\varepsilon}{\|a_2^\varepsilon\|}, \frac{a_3^\varepsilon}{\|a_3^\varepsilon\|})}{\|a_2^\varepsilon\|}\right|$$

by (2.10). So by (2.11) $M(\varepsilon)$ remains bounded along a subsequence. Passing to a further subsequence E'', we assume that it converges to $m \in \mathbb{C}$. Applying the direct part of Theorem 1.6, we have $\lim\limits_{E''\ni\varepsilon\to 0} \mathcal{I}_\varepsilon = \langle z_2 - mz_1^2, z_1^3\rangle \neq \mathfrak{M}_0^2$. So the whole limit can't be $\mathfrak{M}_0^2$ either.

2.5. Proof of Theorem 1.6, converse. The reasoning is analogous to the one immediately preceding. By the sufficiency in Theorem 1.5, (1.2) does not hold. The proof above shows that there is a subsequence along which all v_j converge, and to the same limit v.

Along this subsequence, $M(\varepsilon)$ remains bounded, so there is a further subsequence along which it converges to some $m \in \mathbb{C}$, so the direct part of Theorem 1.6 shows that $\lim\limits_{E\ni\varepsilon\to 0} \mathcal{I}_\varepsilon = \langle \hat{z}_2 - m\hat{z}_1^2, \hat{z}_1^3\rangle$, where $\hat{z}_1, \hat{z}_2$ are the coordinates in an orthonormal basis $\hat{e}_1, \hat{e}_2$ so that $[\hat{e}_1] = v$. But if $v \neq [e_1]$, then $\langle \hat{z}_2 - m\hat{z}_1^2, \hat{z}_1^3\rangle$ is not the limit given by the hypothesis, a contradiction. In the same way, if the quantity in (1.3) didn't converge to m, it would converge to $m' \neq m$ along a subsequence, the ideals would converge to $\langle z_2 - m'z_1^2, z_1^3\rangle$, which contradicts the hypothesis.

3. Proof of Theorem 1.7.

In case (2), the limit ideal is a complete intersection ideal and we may apply Theorem 1.3.

In case (1), we refer to [7, Example 5.3]. Since the proof given there refers to an earlier version of the present paper with some technical imperfections, we write a slightly more careful proof here.

Given any $p \in \mathbb{N}^*$, define $\mathcal{I}_{(p)} := \lim\limits_{\varepsilon} \mathcal{I}_\varepsilon^p$, if it exists. Note that $\mathcal{I}_{(p)} \supset (\lim\limits_{\varepsilon} \mathcal{I}_\varepsilon)^p$.

If $\mathcal{I}_{(p)}$ is well defined for all p, [7, Theorem 1.1] shows that the limit G of the Green functions $G_{\mathcal{I}_\varepsilon}$ exists and is obtained as the regularized upper envelope of the scaled functions $\frac{1}{p}G_{\mathcal{I}_{(p)}}$, for $p \in \mathbb{N}^*$. Furthermore, its Monge-Ampère mass at the origin is equal to N. If furthermore the Monge-Ampère mass at the origin of $\frac{1}{p}G_{\mathcal{I}_{(p)}}$ is equal to N for some finite p, then by [7, Theorem 4.10] $G = \frac{1}{p}G_{\mathcal{I}_{(p)}}$.

In our case, $\mathcal{I}_{(2)} \supset (\mathfrak{M}_0^2)^2 = \mathfrak{M}_0^4 \ni z_1^4$ in particular. On the other hand,

$$\begin{aligned}(\mathcal{I}_\varepsilon)^2 \ni -\frac{\delta\alpha_{22}^3}{\rho-\varepsilon'}&\left(Q_1^\varepsilon(z^{(\varepsilon)})Q_3^\varepsilon(z^{(\varepsilon)}) - (Q_2^\varepsilon(z^{(\varepsilon)}))^2\right)\\ &= \alpha_{22}^3\left((z_2^\varepsilon)^3 - \frac{\delta\rho\varepsilon'}{\rho-\varepsilon'}z_1^\varepsilon z_2^\varepsilon + \delta\rho(\frac{\rho}{\rho-\varepsilon'}-1)(z_2^\varepsilon)^2\right.\\ &\qquad\left.+\frac{\delta^2\rho}{\rho-\varepsilon'}(z_1^\varepsilon)^2 z_2^\varepsilon + \frac{\delta(\varepsilon'-2\rho)}{\rho-\varepsilon'}z_1^\varepsilon(z_2^\varepsilon)^2\right)\\ &= \alpha_{22}^3(z_2^\varepsilon)^3 + o(1) = z_2^3 + o(1),\end{aligned}$$

so $z_2^3 \in \mathcal{I}_{(2)}$.

This implies that

$$\frac{1}{2}G_{\mathcal{I}_{(2)}}(z) \geq \frac{1}{2}\max(\log|z_1^4|, \log|z_2^3|) + O(1) = \max\left(2\log|z_1|, \frac{3}{2}\log|z_2|\right) + O(1).$$

Since the Monge-Ampère mass of this lower bound is 3, $G(z)$ itself is of this form.

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