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AVAILABILITY OF A REPAIRABLE k-OUT-OF-n: G SYSTEM WITH REPAIR TIMES ARBITRARILY DISTRIBUTED

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Abstract. An k-out-of-n: G system is a system that consists of n components and works if and only if k components among the n work simultaneously. The system and each of its components can be in only one of two states: working or failed. When a component fails it is put under repair and the other components stay in the "working" state with adjusted rates of failure. After repair, a component works as new and its actual lifetime is the same as initially. If the failed component is repaired before another component fails, the $(n-1)$ components recover their initial lifetime. The lifetime and time of repair are independent. In this paper, we propose a technique to calculate the mean time of repair, the probability of various states of the system and its availability by using the theory of distribution.

1. Introduction. The k-out-of-n system structure is a very popular type of redundant fault-tolerant systems. It finds wide applications in both industrial and military systems. These systems include the multidisplay system

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in cockpits, the multiengined system in an airplane, the multipump system in a hydraulic control system. In a communications system with three transmitters, the average message load may be such that at least two transmitters must be operational at all times or critical messages may be lost. The transmission subsystem functions as a 2-out-of-3: G system. Systems with spares may also be represented by the k-out-of-n system model. In the case of an automobile with four tires, for example, usually one additional spare tire is carried on the vehicle. Thus, the vehicle can be driven as long as at least 4-out-of-5 tires are in good condition.

In this paper, we consider the repairable case where $k = n - 1$, i.e. a repairable $(n - 1)$-out-of-n: G system which works if and only if at least (k) components among the n work simultaneously. Few papers analyze this kind of systems. Gaver [3] and Jack [4] consider a 2-unit parallel system, Gherda & Boushaba [2] analyze a 2-out-of-3 system when the distributions of the time of failure and the time of repair are general, Gherda & Boushaba [1] analyze a $(n - 1)$-out-of-n G-system. In this work, we generalize the result of Gherda & Boushaba [1]: We suppose that all components and the system have either one of two states: a “working” state or a “failed” state. When a component fails it is put under repair and the other components stay in the “working” state with adjusted rates of failure. After repair, the component works as new and its actual lifetime is the same as initially. If the failed component is repaired before another component fails, the $(n-1)$ components recover their initial lifetime. The lifetime and time of repair are independent.

In this paper, we propose a technique to calculate the mean time of failure, the probability of various states of our system and its availability by using the theory of distribution. Finally, we give some numerical examples.

2. Notation.

- C_i: component of the system, $i = 1, 2, \dots, n$.
- E_s: the state of the system, $s = 0, 1, 2$.
- We say that the system is in state E_s at time t if there are exactly s failed components at time t.
- λ_i: rate of failure of component i when all components j, $(i \neq j)$ are working.
- λ'_i: rate of failure of component i when component j, $(i \neq j)$ is “failed”.

- X_i: time of repair of component i.
- $P_1(t;x)$: the density of the probability of event: “only one is fails at time t and it is under repair since a time x”.
- $P_2(t;x)$: the density of probability of event: “Two components j and k are “failed” and components i are in the “working” state at time t since time x; $i \in \{1,\dots,n\} \,/\, \{j;k\}$”.
- $P_0(t)$: probability of the event: “No component is “failed” at time t”.
- $\phi(s)$: The Laplace transform of the distribution of T.
- A: The availability of the system.
- G_i: the distribution function of X_i and $\overline{G_i} = 1 - G_i$.
- u_i: hazard rate function

$$1 - G_i(x) = \varrho^{-\int_0^x u_i(y)dy} \text{ and } g_i(x) = \frac{dG_i}{dx} = \varrho^{-\int_0^x u_i(y)dy} u_i(x).$$
$$P_0 = \lim_{t\to\infty} P_0\left(t\right), P_1\left(x\right) = \lim_{t\to\infty} P_1\left(t,x\right), P_2\left(x\right) = \lim_{t\to\infty} P_2\left(t,x\right).$$

- G_i^*, $\overline{G_i}^*$: the Laplace transform of G_i and $1 - G_i$.
- $(f)^*$: the inverse Laplace transform of f.
- $*$: convolution product.

3. Analysis.

Hypothesis.

- We consider a k-out-of-n: G repairable system. When a component fails it is put under repair and the other components stay in the “working” state with adjusted rates of failure. After repair, a component works as new and its actual lifetime is the same as initially.
- Elements of the system are identical.
- For $i \neq j$ we have $\lambda_i = \lambda_j$, $\lambda'_i = \lambda'_j$, $\mu_i\left(x\right) = \mu_j\left(x\right)$, $G_i(t) = G_j(t)$, $M_i(t) = M_j(t)$, $h_i(t) = h_j(t)$, λ and λ' given.

Theorem 1. *The availability of a system k-out-of-n at time t* $P_{k+1}(t)$ *is the solution of the following integro-differential system.*

$$
(1)\quad \begin{cases}
1) P_0(t+\Delta t) = d\Delta \int_0^t P_1(t,x)\,\mu(x)\left[1-(n-1)\lambda'\right]dx + \\
\qquad d\Delta \int_0^t P_0(t,x)\left[1-\mu(x)\,dx\right]\left[1-(n-1)\lambda'\right] + \circ(\Delta t) \\
2) s = 1,\dots,k-1 \\
P_s(t+\Delta t) = \Delta t \int_0^t P_{s+1}(t,x)\,\mu(x)\left[1-(n-s-1)\lambda'\right]dx + \\
\qquad \Delta t \int_0^t P_s(t)\left[1-(n-s)\lambda' - su(x)\,dx\right] + \\
\qquad \Delta t \int_0^t P_{s-1}(t,x)\left[1-(n-s+1)\lambda'\right]\left[1-(s-1)\mu(x)\right]dx + \circ(\Delta t) \\
3) s = k \\
P_k(t+\Delta t) = \Delta t \int_0^t P_k(t)\left[1-(n-k)\lambda' - ku(x)\,dx\right] + \\
\qquad \Delta t \int_0^t P_{k-1}(t,x)\left[1-(n-k+1)\lambda'\right]\left[1-(k-1)\mu(x)\right]dx + \circ(\Delta t) \\
4) s = k+1 \\
P_{k+1}(t+\Delta t) = \Delta t \int_0^t P_k(t)\left[(n-k)\lambda'\right]ku(x)\,dx + \circ(\Delta t) \\
5) \sum_0^k P_s(t) = 1 - P_{k+1}(t)
\end{cases}
$$

P r o o f. Let $N(t)$ and $X(t)$ be two random variables such that $N(t) = i$ if the system is in the state E_i and $X(t) = x$ if the component which failed at time t is under repair since date x:

1) $\boldsymbol{s = 0}$

The event $N(t+\Delta t) = 0$; it can be obtained in 2 different ways:

At time t, $N(t) = 0$ and during the interval of time $[t; t+\Delta t]$ there are no failures, or $N(t) = 1$ and during Δt the failed element is repaired and nothing is tobe down.

The probability of this event is

$$(2)\qquad \begin{cases} P_0\,(t+\Delta t) = \Delta t \displaystyle\int_0^t P_1\,(t,x)\,\mu\,(x)\left[1-(n-1)\,\lambda'\right]dx \\ +\Delta t \displaystyle\int_0^t P_0\,(t)\left[1-(n-1)\lambda' - u\,(x)\,dx\right] + \circ\,(\Delta t) \end{cases}$$

2) $\boldsymbol{1 < s < k}$

Consider the event $N(t+\Delta t) = s$; it can be obtained in 3 different ways:

– At time t, $N(t) = s$ and during the interval of time $[t; t+\Delta t]$ there are no failures, and no repair, the probability of this event is

$$(3)\qquad P_s\,(t)\left[1-(n-s)\,\lambda'\Delta t\right]\left[1-su(x)\Delta t\right] + \circ\,(\Delta t)\,.$$

Or at time t, $N(t) = s+1$ and during the interval of time $[t; t+\Delta t]$ there is a faulty item is returned to service and no failures occurred, the probability of this event is:

$$(4)\qquad \Delta t \int_0^t P_{s+1}\,(t,x)\,\mu\,(x)\left[1-(n-s-1)\,\lambda'\right]dx + \circ\,(\Delta t)$$

Or at time t, $N(t) = s+1$ and during the interval of time $[t; t+\Delta t]$ there is a failed component and no repairs or complete, the probability of this event is:

$$(5)\qquad \Delta t \int_0^t P_{s-1}\,(t,x)\left[1-(n-s+1)\,\lambda'\right]\left[1-(s-1)\,\mu\,(x)\right]dx + \circ\,(\Delta t)$$

So, the probability of this event is

$$(6)\qquad \begin{cases} P_s\,(t+\Delta t) = \Delta t \displaystyle\int_0^t P_{s+1}\,(t,x)\,\mu\,(x)\left[1-(n-s-1)\,\lambda'\right]dx + \\ \Delta t \displaystyle\int_0^t P_s\,(t)\left[1-(n-s)\lambda' - su\,(x)\,dx\right] + \\ \Delta t \displaystyle\int_0^t P_{s-1}\,(t,x)\left[1-(n-s+1)\,\lambda'\right]\left[1-(s-1)\,\mu\,(x)\right]dx + \circ\,(\Delta t) \end{cases}$$

3) $\boldsymbol{s = k}$

$$(7)\qquad \begin{cases} P_k\,(t+\Delta t) = \Delta t \displaystyle\int_0^t P_k\,(t)\left[1-(n-k)\lambda' - ku\,(x)\,dx\right] \\ \Delta t \displaystyle\int_0^t P_{k-1}\,(t,x)\left[1-(n-k+1)\,\lambda'\right]\left[1-(k-1)\,\mu\,(x)\right]dx + \circ\,(\Delta t) \end{cases}$$

4) $\boldsymbol{s = k + 1}$

$$P_{k+1}(t + \Delta t) = \Delta t \int_0^t P_k(t) \left[(n-k)\lambda' \right] ku(x)\, dx + \circ(\Delta t) \tag{8}$$

5)

$$\sum_{s=0}^{k} P_s(t + \Delta t) = 1 - P_{k+1}(t + \Delta t) \tag{9}$$

□

Corollary 1. *For $k = n - 1$ the solution for the system introducted in Proposition 1 is given by:*

$$P_0(t) = \exp - \left(\sum_{i=1}^{n} \lambda_i \overline{G_i^*} \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \lambda'_j \right) \right) t \tag{10}$$

and

$$P_1(x) = \begin{cases} \lambda_i P_0(t) \left[\exp - \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \lambda'_j \right) x \right] \overline{G_i}(x), & if \ t > x \\ 0, \quad otherwise \end{cases} \tag{11}$$

with conditions $P_0 = \lim_{t \to \infty} P_0(t) = 0$ *and* $P_1(x) = \lim_{t \to \infty} P_1(t, x) = 0$.

P r o o f. Let $N(t)$ and $X(t)$ be two random variables such that $N(t) = i$ if the system is in the state E_i and $X(t) = x$ if the component which failed at time t is under repair since date x.

Consider the event $N(t + dt) = n$; it can be obtained in $(n + 1)$ different ways:

– At time t, $N(t) = n$ and during the interval of time $[t; t + dt]$ there are no failures, the probability of this event is $P_0(t) \left[1 - \left(\sum_{1}^{n} \lambda_i \right) dt \right]$.

Or at time t, $N(t) = n - 1$ and during the interval time $[t, t + dt]$ the faulty component A_i is returned to service, the probability of this event is

So, the probability of this event is:

— Or at time t, $N(t) = n-1$ and during the interval of time $[t; t+dt]$; the failed component A_i is repaired, the probability of this event is

$$dt \int_0^t P_{1,i}(t,x)\, u_i(x)\, dx + 0(dt), \ i = 1, 2, \ldots, n.$$

Then the probability of this event is

$$\begin{aligned} &P_0(t+dt) = P_0(N(t+dt) = 0) \\ &= P_0(t)\left[1 - \left(\sum_{i=1}^{n} \lambda_i\right) dt\right] \\ &+ \sum_{i=1}^{n}\left(1 - \left[\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda'_j\right] dt\right) dt \int_0^t P_{1,i}(t,x)\, u_i(x)\, dx \end{aligned}$$

when $dt \to 0$ we obtain:

$$\frac{dP_0}{dt} = -\sum_{i=1}^{n}\left[P_0(t)\lambda_i + \int_0^t P_{1,i}(t,x)\, u_i(x)\, dx\right] \tag{12}$$

Consider the event $N(t+dt) = 1$; it can be obtained in $2n$ different ways:

– $N(t+dt) = 1$ and $X(t+dt) = x + dt$: If $N(t) = 1$, $X(t) = x \geq 0$ and the repair of the failed component is not finished at time $t + dt$: the probability of this event noted by $P_{1;i}(t+dt; x+dt)$ is given by:

$$\begin{aligned} &P_{1,i}(t+dt, x+dt) \\ &= \begin{cases} P_{1,i}(t,x)\left[1 - \left[\sum_{\substack{j=1 \\ j\neq i}}^{n}\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda'_j\right) + u_i(x)\right] dt\right] + \circ(dt) \ if \ t > x \\ 0, \quad \text{otherwise} \end{cases} \end{aligned}$$

when $dt \to 0$ we obtain

$$\begin{aligned} &\frac{\partial\theta(t-x)\, P_{1,i}(t,x)}{\partial t} + \frac{\partial\theta(t-x)\, P_{1,i}(t,x)}{\partial x} \\ &= -\left[\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda'_j\right) + u_i(x)\right] \theta(t-x)\, P_{1,i}(t,x) \end{aligned}$$

where θ is the indicator function.

– Or $N(t+dt) = n-1$ and $X(t+dt) = dt$: If $N(t) = n$ and a failure occurs during the interval of time $[t; t+dt]$: This case is given by the initial conditions:

$$P_{1,i}(t,0) = \lambda_i P_0(t) \qquad i = 1, n \quad and \qquad P_0(0) = 1 \qquad \square$$

3.1. Calculation of the state probabilities $P_0(t)$ and $P_1(t;x)$ of a system. To calculate $P_0(t)$ and $P_1(t;x)$ we have to use the following propositions.

Proposition 1 [5]. *$S * T$ admits a Laplace transform for $\zeta > b = \max(b_1, b_2)$ equal to the product of Laplace transform of S and T.*

Corollary 2 [5]. *If $T = (F(p))^*$ then $T^{(m)} = \left(F^{(m)}(p)\right)^*$. Indeed $T^{(m)} = T * \delta^{(m)}$ and $\delta^{(m)} = \left(p^{(m)}\right)^*$*

Proposition 3 [5]. *A holomorphic function $\Im(p)$ is a Laplace transformed of a distribution $T \in D'_+$ if and only if there exists a semi plan, which it is majorized in modulus with a polynomial in $|p|$.*

Let $P_0(s)$ and $P_{1;i}(s;x)$ be the Laplace transform of $P_0(t)$ and $P_1(t;x)$ respectively. Because the indicator function $\theta(t-x)$ is a regular distribution, we can use the derivative of (13) in the sense of distribution:

$$\frac{\partial}{\partial t}\left[\theta(t-x)P_1(t,x)\right] + \frac{\partial}{\partial x}\left[\theta(t-x)P_1(t,x)\right] = -\left[\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j\right) + u_i(x)\right]\theta(t-x)P_1(t,x)$$

by taking the Laplace transform of the two members of this last equality we obtain

$$sP_1(s,x) + \frac{d}{dx}P_1(s,x) = -\left[\left(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j\right) + u_i(x)\right]P_1(s,x)$$

so

$$P_1(s,x) = \lambda_i P_0(s) \exp - \left(s + \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j \right) \right) x \exp\left(- \int_0^x u_i(y) dy \right)$$

$$P_1(s,x) = \lambda_i P_0(s) \exp - \left(\left(s + \sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j \right) x \right) \overline{G_i}(x) \tag{13}$$

where $P_1(s,x)$ the Laplace transform of $\theta(t-x) P_1(t,x)$. We note that

$$|P_{1,i}(s,x)| \leq \lambda_i P_0(s) \frac{1}{\left(s + \sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j \right) x + 1} \left| \overline{G_i}(x) \right| \leq \lambda_i$$

by applying a Propositions 2, 3 and Corollary 2 to $P_{1,i}(s,x)$, which verifies this majorally, we obtain

$$\theta(t-x) P_1(t,x) = \left(\frac{d}{dt} \right)^2 f$$

where f is the inverse transform of $\frac{P_1(s,x)}{s^2}$. We note that $\frac{P_1(s,x)}{s^2} P_1(s;x)$ is the Laplace transform of the convolution product. By using (13):

$$\begin{aligned} \frac{P_1(s,x)}{s^2} &= \lambda_i \exp - \left(x \sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j \right) \overline{G_i}(x) P_0(s) \varrho^{-sx} \frac{1}{s^2} \\ &= \lambda_i \exp - \left(x \sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j \right) \overline{G_i}(x) \left(P_0(t) * \delta_\varepsilon(t-x) * (t) \right) \end{aligned}$$

where δ_ε is the Dirac function. The inverse Laplace transform of $P_{1;i}(s;x)$ is given by:

$$\left(\frac{d}{dt}\right)^2 \lambda_i \exp-\left(x\sum_{\substack{j=1\\j\neq i}}^{n}\lambda'_j\right)\overline{G_i}\left(x\right)P_0\left(t\right)*\delta_\varepsilon\left(t-x\right)*t$$

$$=\lambda_i \exp-\left(x\sum_{\substack{j=1\\j\neq i}}^{n}\lambda'_j\right)\overline{G_i}\left(x\right)P_0\left(t\right)$$

Now, we obtain

$$\frac{dP_0\left(t\right)}{dt}=-\left(\sum_{i=1}^{n}\lambda_i\right)P_0\left(t\right)+P_0\left(t\right)\sum_{i=1}^{n}\lambda_i G_i^*\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda'_j\right)$$

By taking the Laplace transform of the two members of this equality we obtain

$$P_0\left(s\right)=\left[s+\left(\sum_{i=1}^{n}\lambda_i\right)-\sum_{i=1}^{n}\lambda_i\left(G_i^*\sum_{\substack{j=1\\j\neq i}}^{n}\lambda'_j\right)\right]^{-1}$$

By using the inverse Laplace transformed, we obtain

$$P_0\left(t\right)=\exp-\left(\sum_{i=1}^{n}\lambda_i\overline{G_i^*}\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda'_j\right)\right)t \tag{14}$$

and

$$P_1\left(x\right)=\begin{cases}\lambda_i P_0\left(t\right)\left[\exp-\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda'_j\right)x\right]\overline{G_i}\left(x\right), & si\ t>x\\ 0, & \text{otherwise}\end{cases} \tag{15}$$

with conditions $P_0=\lim_{t\to\infty}P_0\left(t\right)=0$ and $P_1\left(x\right)=\lim_{t\to\infty}P_1\left(t,x\right)=0$.

3.2. Calculation of $\boldsymbol{P_2(t;x)}$. The event $N(t+dt)=n-2$ can be obtained in two ways:

$1-N(t+dt)=n-2, X(x+dt)=dt$; if $N(t)=n-1, X(t)=x\geq 0$, during the interval of time $[t;t+dt]$, one component among the working components fails and the repair of the failed component is not finished. This event has the probability:

$$P_2\left(t,x\right)=P_1\left(t,x\right)\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right)dt\left[1-u_i\left(x\right)dt\right]+0(dt)$$
$$=\begin{cases} P_1\left(t,x\right)\left[\left(\sum\limits_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right)dt\right], & \text{if } x<t\\ 0, & \text{otherwise}\end{cases}$$

$2-$ Or $N(t+dt)=n-2, X(x+dt)=dt$, if $N(t)=n-2$, any repair is used and no failure occurs during the interval of time $[t;t+dt]$:This event has the probability:

$$\begin{cases} P_2\left(t,x\right)\left[1-\left(\lambda_i'+\sum\limits_{\substack{j=1\\ j\neq i}}^{n}u_j(x)\right)dt\right], & \text{if } x<t\\ 0, & \text{otherwise}\end{cases}$$

so

$$P_2\left(t+dt,x+dt\right)=\begin{cases} P_1\left(t,x\right)\left[\left(\sum\limits_{\substack{j=1\\ j\neq i}}^{n}u_j(x)\right)dt\right] & \\ +P_2\left(t,x\right)\left[1-\left(\lambda_i'+\sum\limits_{\substack{j=1\\ j\neq i}}^{n}u_j(x)\right)dt\right], & \text{if } \ x<t\\ 0, & \text{otherwise}\end{cases}$$

when $dt \to 0$ we obtain

$$\frac{\partial P_2(t,x)}{\partial t} + \frac{\partial P_2(t,x)}{\partial x} = \begin{cases} \left(\sum_{\substack{j=1\\ j\neq i}}^{n} u_j(x)\right) P_1(t,x) - u_i(x) P_2(t,x), & \text{if } x < t \\ 0, & \text{otherwise} \end{cases}$$

by using the Laplace transform and the initial condition $P_{2,i}(t;t) = 0$, we obtain

$$(s,x)\left[s + u_i(x)\right] P_2(s,x) + \frac{\partial}{\partial x} P_{2,i}(s,x) - P_1(s,x) \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j\right) = 0 \tag{16}$$

one solution of this equation is given by

$$P_{2,i}(s,x) = \left[\exp \alpha(x)\right] \left[k + \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j\right) \int P_{1,i}(s,x) \exp(-\alpha(x))\, dx\right]$$

where $\alpha(x)$ is a primitive of $-[s + u_i(x)]$ and k a constant. Now by using the convolution product, the solution of (6) is given by

$$\begin{aligned} &P_{2,i}(s,x) = \left[\exp(-sx)\right] \left[\exp - \int u_i(x)\, dx\right] * \\ &\left[K + \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j\right) \lambda_i P_0(s) \int \exp - \left[s + \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j\right)\right] * \right. \\ &\left. x\overline{G_i}(x) \exp(sx) \exp\left(\int u_i(x)\, dx\right) dx\right] \end{aligned}$$

finally, we obtain:

$$\begin{aligned} P_{2;i}(s;x) = K\delta^*(t-x) \exp\left(-\int u_i(x)\, dx\right) + P_0(s)\, \delta^*(t-x)\, \lambda_i \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j\right) \\ \times \exp\left(-\int u_i(x)\, dx\right) \int \exp - \left(\sum_{\substack{j=1\\ j\neq i}}^{n} \lambda'_j\right) x dx \end{aligned}$$

by using the inverse Laplace transform and the initial condition $P_2(t;t) = 0$; $(K = 0)$, we obtain

$$P_2(t;x) = \lambda_i x \exp(-\lambda_i x)\overline{G_i}\,(x)\left(\sum_{i=1}^{n}\lambda_i\left[\overline{G^*}\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right)\right]\right)^{-1} *$$
$$\left(1-\exp\left(-t\sum_{i=1}^{n}\lambda_i\overline{G_I^*}\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right)\right)\right)$$

and

$$P_1\,(t) = \int_0^{\infty} P_1\,(t,x)\,dx = \frac{\exp\left(-t\sum\limits_{i=1}^{n}\lambda_i\overline{G_I^*}\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right)\right)-1}{\sum\limits_{i=1}^{n}\lambda_i\left[\overline{G_i^*}\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right)\right]}\times\lambda_i\left(\overline{G_i^*}\right)'\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right).$$

3.3. Characteristics of the system.

3.3.1. Mean time of failure.

Corollary 3. *Under the conditions in Proposition* 2 *the Mean time of failure is given by*

$$E(T) = \frac{\sum\limits_{i=1}^{n}\frac{\lambda_i}{\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right)}\left[\left(\overline{G_i}\right)^*\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right)\right]}{\left[\sum\limits_{i=1}^{n}\lambda_i\overline{G_i^*}\left(\sum_{\substack{j=1\\ j\neq i}}^{n}\lambda_j'\right)\right]^2} \tag{17}$$

P r o o f. Let Φ be the Laplace transform of $P_{1;i}(t;x)$: $\Phi\left(s\right)=\displaystyle\int_{0}^{\infty}P_{1,i}\left(s,x\right)dx$

$$\Phi\left(s\right)=\sum_{i=1}^{n}\int_{0}^{\infty}\frac{\lambda_i}{s+\sum_{i=1}^{n}\lambda_i\overline{G_i^{*}}\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)}\exp\left[-\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)x\right]\overline{G_i}\left(x\right)dx$$

$$=\frac{\lambda_i}{s+\sum_{i=1}^{n}\lambda_i\overline{G_i^{*}}\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)}\sum_{i=1}^{n}\int_{0}^{\infty}\exp\left[-\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)x\right]\overline{G_i}\left(x\right)dx$$

Then, we have

$$\Phi\left(s\right)=\frac{\sum\limits_{i=1}^{n}\dfrac{\lambda_i}{\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)}\left[\overline{G_i}^{*}\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)\right]}{\left[s+\sum\limits_{i=1}^{n}\lambda_i\overline{G_i^{*}}\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)\right]}$$

finally

$$E(T)=-\Phi'\left(0\right)=\frac{\sum\limits_{i=1}^{n}\dfrac{\lambda_i}{\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)}\left[\left(\overline{G_i}\right)^{*}\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)\right]}{\left[\sum\limits_{i=1}^{n}\lambda_i\overline{G_i^{*}}\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)\right]^{2}} \tag{18}$$

□

3.2.2. Availability of the system. Let $A(t;x)$ the availability of the system, then $A(t;x)=P_0(t)+\displaystyle\sum_{i=1}^{n}P_1,(t,x)$

Corollary 4. *Under the conditions in Proposition* 2 *the availability of the system is given by*

$$\left[1+\sum_{i=1}^{n}\lambda_i\exp\left[-\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)x\right]\overline{G_i}\left(x\right)\right]\times\exp-\left[\sum_{i=1}^{n}\lambda_i\overline{G_i^{*}}\left(\sum_{\substack{j=1\\j\neq i}}^{n}\lambda_j'\right)\right]t. \tag{19}$$

3.3.3. Case of n non identical components. In the stability case, we obtain the following equations for the system:

$$\begin{cases} P_0 \sum_{i=1}^{n} \lambda_i = \sum_{i=1}^{i=n} \mu_i(x) P_1\left(x\right) \\ \frac{\partial}{\partial x} P_1(x) = -\left[\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda_j'\right) + \mu_i(x)\right] P_1\left(x\right) \\ \frac{\partial}{\partial x} P_2(x) + \mu_i(x) P_i\left(x\right) = \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda_j'\right) P_1\left(x\right) \end{cases}$$

this system becomes

$$\begin{cases} P_1\left(0\right) = \lambda_i P_0 + \sum_{\substack{j=1 \\ j\neq i}}^{n} P_1\left(0\right)\left(1 - G_i^*\left(\lambda_j'\right)\right) \\ P_1\left(x\right) = P_1\left(0\right) \exp\left[-\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda_j'\right) x\right] \overline{G_i}\left(x\right) \\ P_2(x) = P_1\left(0\right)\left(1 - \exp\left[-\left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda_j'\right) x\right]\right) \overline{G_i}\left(x\right) \end{cases}$$

by using this hypothesis: $P_1\left(0\right) = \lambda_i P_0 + \sum_{\substack{j=1 \\ j\neq i}}^{n} \int_0^{\infty} P_j(x)\mu_j(x)dx$, (conditions at the limits), we obtain

$$\begin{cases} P_{1,i}(0) = \lambda_i P_0 + \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} P_{1,j}(0) \left[1 - G_j^* \left(\sum\limits_{\substack{k=1 \\ j\neq k}}^{n} \lambda'_k \right) \right] \\ \text{and} \\ P_0 + \sum\limits_{i=1}^{i=1} \left[\int_0^\infty P_1(x)\,dx + \int_0^\infty P_i(x)\,dx \right] = 1 \end{cases}$$

so

$$\begin{aligned} P_0 &= 1 - \sum_{i=1}^{i=n} \left[\int_0^\infty P_1(x)\,dx + \int_0^\infty P_i(x)\,dx \right] \\ &= 1 - \left[\sum_{i=1}^{i=n} P_1(0) \right] * \left[\int_0^\infty \exp\left[- \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda'_k \right) x \right] \overline{G_i}(x)\,dx \right. \\ &\left. + \int_0^\infty \overline{G_i}(x) \left[1 - \exp\left[- \left(\sum_{\substack{j=1 \\ j\neq i}}^{n} \lambda'_k \right) x \right] \right] dx \right] \\ &= 1 - \sum_{i=1}^{i=n} P_1(0)\, E(X_i) \end{aligned}$$

we can write the solution in the following form:

$$\begin{cases} P_{1,i}(0) = \lambda_i P_0 + \sum\limits_{\substack{j=1 \\ j\neq i}}^{n} P_{1,j}(0) \left[1 - G_j^* \left(\sum\limits_{\substack{j=1 \\ j\neq i}}^{n} \lambda'_k \right) \right] \\ P_0 = 1 - \sum\limits_{i=1}^{n} P_{1,i}(0)\, E(X_i) \end{cases}$$

finally we derive:

$$P_1(0) = \frac{1}{\sum_{i=1}^{n} \prod_{j\neq i} \left[2 - G_j^* \left(\sum_{\substack{k=1 \\ j\neq k}}^{n} \lambda'_k \right) \right] E(X_i)} * (B - C + D) \tag{20}$$

where

$$B = \prod_{j \neq i} [2 - G_j^* \left(\sum_{\substack{k=1 \\ j \neq k}}^{n} \lambda'_k \right)]$$

$$C = P_0 \prod_{j \neq i} \left[2 - G_j^* \left(\left[2 - G_j^* \left(\sum_{\substack{k=1 \\ j \neq k}}^{n} \lambda'_k \right) \right] \right) \right]$$

$$D = \left[2 \left(\sum_{\substack{j=1 \\ j \neq i}}^{n} \lambda_j \right) - 4\lambda_i - \sum_{j \neq i} (\lambda_j - \lambda_i) G_j^* \left(\lambda'_i + \lambda'_j \right) \right] P_0$$

3.3.4. Case of n identical components. In the stability case ($t \to \infty$), we obtain the following equations for the system:

$$\begin{cases} \lambda P_0 = u(x) P_1 \ (x) \\ \dfrac{\partial}{\partial x} P_1(x) = - \left[(n-1) \lambda' + u(x) \right] P_1 \left(x \right) \\ \dfrac{\partial}{\partial x} P_2(x) + u(x) P_2 \left(x \right) = \left((n-1) \lambda' \right) P_1 \left(x \right) \end{cases}$$

by using the initial conditions and conditions at the limits: $P_1(0) = 0$ and

$$P_1 \left(0 \right) = \lambda P_0 + (n-1) \int_0^{\infty} P_1(x) u(x) dx$$

so

$$\begin{cases} P_1 \left(0 \right) = \lambda P_0 + (n-1) P_1 \left(0 \right) \left[1 - G^* \left((n-1) \lambda' \right) \right] \\ P_0 + n \left[\int_0^{\infty} P_1 \left(x \right) dx + \int_0^{\infty} P_2 \left(x \right) dx \right] = 1 \end{cases}$$

by the same argument as in the non identical case, we derive

$$\text{(21)} \quad \begin{cases} P_1 \left(0 \right) = \dfrac{1}{n \left[\left[2 - G^* \left((n-1) \lambda' \right) \right] E(X) \right]^{n-1}} \left[2 - G^* \left((n-1) \lambda' \right) \right]^{n-1} \\ -P_0 \left[2 - G^* \left(\left[2 - G^* \left((n-1) \lambda' \right) \right] \right) \right]^{n-1} + [2n-6] \lambda P_0 \end{cases}$$

Remark 1. When $n = 3$, we obtain the result given in [3, p. 200].

3.4. Numerical example. Consider the case where $n = 2$ and the stability case ($t \to \infty$): When the components are non identical, the equations of the system are:

$$\begin{cases} (\lambda_1 + \lambda_2) P_0 = \sum_{i=1}^{2} u_i(x) P_{1,i}(x) \\ \frac{\partial}{\partial x} P_1(x) = -\left[\lambda'_{3-i} + u_i(x)\right] P_1(x) \\ P_1(x) = P_1(0)\left[\exp\left(\lambda'_{3-i}x\right)\overline{G_i}(x)\right] \\ P_2(x) + u(x)P_2(x) = P_1(0)\left(1 - \overline{G_i}(x)\exp\left(\lambda'_{3-i}x\right)\right) \end{cases}$$

By combining these equations, we obtain:

$$P_1(0) = \lambda_i P_0 + \int_0^\infty P_2(x)\, u_{3-i}(x) dx$$

$$P_2(0) = 0$$

$$A = P_0 \left[1 + \sum_{i=1}^{2} \frac{\frac{1}{\lambda_i} G_i^*\left(\lambda'_{3-i}\right)\left[\lambda_i + \lambda_{3-i}\overline{G_i^*}\left(\lambda'_{3-i}\right)\right]}{G_{3-i}^*(\lambda'_i) + G_i^*(\lambda'_i) - G_{3-i}^*(\lambda'_i)\, G_i^*\left(\lambda'_{3-i}\right)}\right]$$

where

$$P_0 = \frac{G_1^*(\lambda'_2) + G_2^*(\lambda'_1) - G_1^*(\lambda'_2)\, G_2^*(\lambda'_1)}{1 - \overline{G_1^*}(\lambda'_2)\,\overline{G_2^*}(\lambda'_1) + \sum_{i=1}^{2} E(X_i)\left[\lambda_i + \lambda_{3-i}\overline{G_i^*}\left(\lambda'_{3-i}\right)\right]}$$

In the identical case, we have

$$P_0 = \frac{2G^*(\lambda') - \left(G^*(\lambda')\right)^2}{1 - \left(\overline{G^*}(\lambda')\right)^2 + 2E(X)\left[\lambda + \lambda\overline{G^*}(\lambda')\right]} \tag{22}$$

$$A = P_0 \left[1 + 2\frac{\frac{1}{\lambda}\left[\lambda + \lambda\overline{G^*}(\lambda')\right]}{2 - G^*(\lambda')}\right] \tag{23}$$

$$= \frac{2\frac{1}{1+\lambda'} - \left(\frac{1}{1+\lambda'}\right)^2}{3 - \left(1 - \frac{1}{1+\lambda'}\right)^2 + 2\left(1 - \frac{1}{1+\lambda'}\right)}.3 = 3P_0 \tag{24}$$

In the following, we will give some numerical examples when we consider the cases:

– X is constant, $X = E(X)$, $\overline{G_1^*}\left(\lambda'\right) = 1 - \exp\left(-\lambda'\right)$

– X follows the exponential law: $\overline{G^*}\left(\lambda'\right) = 1 - \left(1 + \lambda'\right)^{-1}$

– X follows the law $\Gamma\left(\frac{1}{16}\right)$

we note $m = \frac{\lambda'}{\lambda}$

If $\lambda = 0 : 375$ and $\lambda = 0 : 750$; the values of $E(X)$ for different values of m and different laws of X are given in the following tables, respectively:

Table 1. Values of $E(X)$, $\lambda = 0 : 375$

m	$X = ct$	$X \to \exp(1)$	$X \to \Gamma\left(\frac{1}{16}\right)$
0.5	13.2	33.7	64
1	8.5	9.77	23
1.5	4.1	5	13
2	2.5	3	9

Table 2. Values of $E(X)$, $\lambda = 0 : 75$

m	$X = cte$	$X \to \exp(1)$	$X \to \Gamma\left(\frac{1}{16}\right)$
0.5	8.5	9.77	23
1	2.5	3.1	9
1.5	1.3	1.7	5.3
2	0.6	1.1	3.7

The values of the probability states and the availability , for various laws of X, are given in the following tables:

a) X is a constant

Table 3. Values of the probability states and the availability where X is a constant

m	λ	P_0	P_1	P_2	A	λ	P_0	P_1	P_2	A
0.5	0.375	0.52	0.217	0.02	0.96	0.75	0.31	0.28	0.06	0.88
1	0.375	0.48	0.216	0.04	0.91	0.75	0.24	0.26	0.11	0.77
1.5	0.375	0.43	0.215	0.07	0.86	0.75	0.18	0.24	0.16	0.67
2	0.375	0.38	0.214	0.09	0.82	0.75	0.13	0.22	0.21	0.58

b) X follows an exponential law with parameter 1

Table 4. Values of the probability states and the availability where X follows an exponential law with parameter 1

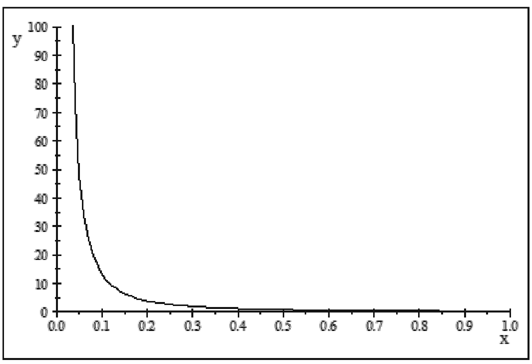
m	λ	P_0	P_1	P_2	A	λ	P_0	P_1	P_2	A
0.5	0.375	0.53	0.2	0.04	0.92	0.75	0.49	0.21	0.04	0.92
1	0.375	0.49	0.18	0.07	0.86	0.75	0.43	0.19	0.09	0.81
1.5	0.375	0.46	0.17	0.10	0.80	0.75	0.38	0.16	0.14	0.72
2	0.375	0.43	0.16	0.12	0.75	0.75	0.35	0.14	0.17	0.64

c) X follows the law $\rightarrow \Gamma\left(\frac{1}{16}\right)$

Table 5. Values of the probability states and the availability where X follows the law $\rightarrow \Gamma\left(\frac{1}{16}\right)$

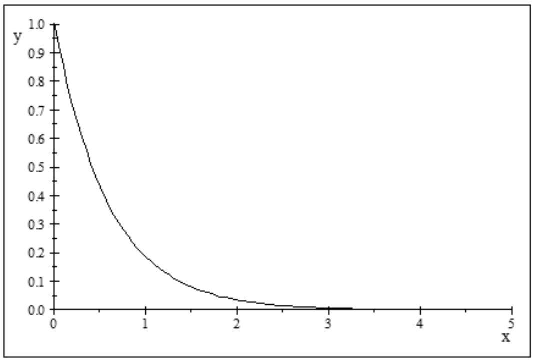
m	λ	P_0	P_1	P_2	A	λ	P_0	P_1	P_2	A
0.5	0.375	0.355	0.064	0.26	0.48	0.75	0.270	0.07	0.30	0.41
1	0.375	0.352	0.045	0.28	0.44	0.75	0.265	0.046	0.32	0.36
1.5	0.375	0.35	0.036	0.29	0.42	0.75	0.263	0.35	0.33	0.33
2	0.375	0.34	0.03	0.30	0.40	0.75	0.260	0.03	0.34	0.31

3.4.1. Graphical representation of the availability of system. X follows an exponential law with parameter 1



$y = E(\lambda')\ \lambda = 0.75\ n = 3\ k = 2$

Fig. 1. Plots Mean time of failure $E(X)$



$y = A(t), \lambda = 0.75,\ \lambda\prime = 0.75,\ n = 3, k = 2$

Fig. 2. Plots of availability $y = A(t)$

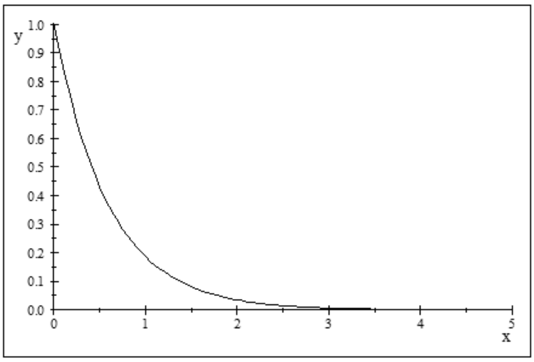


Fig. 3. Plots of availability $y = A(\lambda')$, where $\lambda = 0.75$

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