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SUBORDINATED MARKOV BRANCHING PROCESSES AND LÉVY PROCESSES

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Abstract. We consider the jump structure of the subordinated Lévy processes and subordinated Markov branching processes. Subordination provides a method of constructing a large subclass of Markov or Lévy processes

$$Y(t) = X(T(t)),$$

where $X(t)$ is a Markov or Lévy process and $T(t)$ is a continuous time subordinator independent of $X(t)$; that is a Lévy process with positive increments and $T(0) = 0$. Let $X(t)$ be a Lévy process. Then subordination preserves the independence and stationarity of the increments, but it changes their amplitudes and the total mass of the Lévy measure. Let $X(t)$ be a Markov branching process. Then subordination (owing to the independence of $X(t)$ and $T(t)$) preserves the Markov property, but it disturbs the branching property. The infinitesimal generator of the subordinated process $Y(t)$ involves the total progeny of reproduction. The intensity of the jump times depends on the subordinator's Bernstein function.

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1. Introduction. Branching processes and Lévy processes represent two exceptional classes of stochastic processes. Branching processes (Markovian or non-Markovian) describe the phenomena of multiplication (reproduction) of particles. The main assumption is the local independence of the evolution. Analytically this assumption leads to the additivity by the initial condition and is called the branching property. Lévy processes is a class of processes with stationary and independent increments starting from zero. The relation between Poisson process and Markov branching process, due to J. Lamperti, had been exchanged as a private communication between J. Lamperti, S. Watanabe and D. G. Kendal, and then published as a paragraph in the book of K. B. Athreya and P. E. Ney [1].

Nowadays, the random time change

$$Y(t) = X(T(t))$$

appears everywhere-in stochastic processes, in stochastic integrals and stochastic equations. It is a method to introduce some additional randomness, under the assumption of independence or dependence of the ground process $X(t)$ and random time process $T(t)$. Subordination in the sense of Bochner is a randomization of the time parameter, under the assumption of independence of X and T. On the other hand, the Lamperti's transform is based on the dependence. The random time $T(t)$ is defined via the additive functional of total progeny. It transforms the Markov branching process into Poisson process.

S. Bochner introduced the concept of subordination in 1955 [6], for Markov processes, Lévy process and corresponding semigroups. Subordination provides a method of constructing a large subclass of Markov or Lévy processes $Y(t) = X(T(t))$, where $X(t)$ is a Markov or Lévy process and $T(t)$ is a continuous time subordinator; that is a Lévy process with positive increments and $T(0) = 0$. There are two sources of randomness: the ground process $X(t)$ and time process $T(t)$, under the assumption that X and T are independent.

The properties of the subordinated semigroups have an independent interest in **functional analysis and potential theory**: see A. Carasso and T. Kato [8], also, Ch. Berg and G. Forst [2].

In **financial analysis**, subordinators represent the business times. They are studied in the books of: W. Schoutens [20], R. Cont and P. Tankov [9].

Our main interest is the trajectory properties of the subordinated process $Y(t)$. The problem is the discontinuities of $Y(t)$ which may be caused by the discontinuities of either $X(t)$ or $T(t)$.

The first part of this article concerning the Lévy process is a survey of known results, and the results on branching processes show how does the random observation time, under the assumption of independence of X and T disturbs the branching property.

Let $X(t)$ be a Lévy process. Then subordination preserves the independence and stationarity of the increments, but it changes their amplitudes and the total mass of the Lévy measure. The evolution of the Lévy process is governed by the Poisson random measure, which determines both the rate at which transitions occur and the associated transitions probabilities.

Section 2 reviews the definition and properties of the subordinated semigroups, based on the independent increments properties. In Section 3 we give the basic examples. Section 4 presents some Poisson random measures of the processes in order to describe the discontinuity of Y.

Suppose that the subordinator T is without drift. If $X(t)$ has continuous paths, then it is obvious that the only discontinuity of subordinated process can be caused by the jumps of the subordinator. If $X(t)$ is a Lévy process with unbounded Lévy measure, then the jump times of $Y(t)$ are the same as the jump times of $T(t)$. The independence of $X(t)$ and $T(t)$ ensures that a.s. no jump time of $X(t)$ lies in the closed range of $T(t)$. Consequently, the discontinuity of $X(t)$ can not influence the discontinuity of $Y(t)$. If $X(t)$ is a compound Poisson process, then the jump times of $Y(t)$ forms a subsequence of the jump times of $T(t)$. In this case, any randomness of $T(t)$ before it passes the level η_1 given by the first jump time of $X(t)$ is not reflected by $Y(t) = X(T(t)$.

The second part of the article is devoted to the branching processes. Let $X(t)$, $t = 0, 1, 2, \ldots$ be a Galton-Watson branching process and $T(t)$ be an integer-valued subordinator independent of the ground process $X(t)$, then $Y(t)$ represents a randomly indexed Galton-Watson branching process. This process was introduced by T. Epps [10] in 1996 for modeling of daily stock price as an alternative of the geometric Brownian motion. Assuming that $T(t)$ is a Poisson process, Epps obtained the asymptotic behavior of the moments. Under the assumption of independence, the random time had been generalized and defined by a general renewal process, see K. Mitov, I. Mitov and N. Yanev [15, 16, 17]. The authors study conditional limiting distributions. The asymptotic behavior of the moments and the probability for non-extinction is investigated in the critical and sub-critical branching processes.

The Sevastyanov's model of the age dependent reproduction is defined by the life-span u of the parent particle and the offspring number: $\eta(u)$, $u > 0$. In general, the integer-valued measure η depends on the age "u" of the parent-particle at the splitting time [21]. In [14], we suppose that the life-span is represented by a subordinator. Sevastyanov's model with motion of the particles is described by the family of renewal-type integral equations, where the motion of the particles and reproduction are subordinated by the same subordinator. In the model of spatial Markov branching processes, the subordination can be defined by the local time of motion process at some regular point, see [4].

In Section 5 we consider the jump transition kernel of the subordinated Markov branching processes (MBP) and Kolmogorov's backward equation. Obviously, for the MBP $X(t)$ the inter-arrival times are independent exponentially distributed, but with different rate depending on the state of the process. If $X(t) = k$ then the next inter-arrival time is a minimum of k independent exponentially distributed random variables. The amplitudes of the increments are all independent identically distributed (iid). The inter-arrival times for $Y(t) = X(T(t))$ are exponentially distributed with parameter $\psi_T(ka)$, where ψ_T is the Bernstein function of subordinator and a is the intensity of branching at the MBP $X(t)$. The amplitudes of the increments for $Y(t)$ are defined by averaging the transition probability of the ground process $X(t)$ by the Lévy measure Π_T of subordinator, namely:

$$\mathbb{K}(k, i-k) = \int_0^\infty P_{ki}(u)\Pi_T(du).$$

In Section 6 we suppose that $X(n)$, $n = 1, 2, \dots$ is a Galton-Watson branching process and $T(t)$ an independent subordinator represented by the Poisson process or by the integer-valued compound Poisson process. Obviously, $Y(t) = X(T(t)$ is an integer-valued Markov process starting from $Y(0) = 1$, but it is not a Lévy process, i.e. it is not a compound Poisson process and it is not a Markov branching process. The inter-arrival times are independent exponentially distributed, but the increments of $Y(t)$ are not independent.

2. Subordinated Lévy processes. Lévy process is an additive process with stationary increments $X = (X(t),\ t \geq 0)$ on a probability space $(\Omega_X, \mathcal{B}_X, P_X)$, which is a right continuous process having left limits and $X(0) = 0$. If $(\mathcal{F}_t, t \geq 0)$ denotes the natural filtration generated by X, then the increment $(X(t+u) - X(t))$ is independent of $\mathcal{F}_t$ and has the same law as $X(u)$ for every u, $t \geq 0$. A general Lévy process is a mixture of a continuous Brownian motion with drift and a pure jump process. The distributions of the increments of X are invariant under time and state space shifts. The distribution of X is determined by its transition probability measure $p_X(t, dx)$ and thus by its characteristic function. Let

$$E(\exp(i\lambda X(t))) = \exp(t f_X(\lambda)), \quad \lambda \in R, \quad t \geq 0,$$

where $f_X(\lambda)$ is the characteristic exponent, given by the Lévy-Khinchine formula

$$f_X(\lambda) = id_X\lambda - \frac{1}{2}\sigma^2\lambda^2 + \int_R (e^{i\lambda x} - 1 - i\lambda x \mathbf{1}_{\{|x|<1\}})\Pi_X(dx).$$

Obviously the drift d_X and variance σ^2 are constant, $i = \sqrt{-1}$ and

$$\int_{0+}^{\infty} (1 \ \wedge x^2)\Pi_X(dx) < \infty.$$

We say, that the Lévy process is defined by its triplet:

$$\{d_X, \sigma^2, \Pi_X(dx)\}.$$

The Lévy measure is the infinitesimal generator of the convolution semigroup generated by the transition probability:

$$\Pi_X(dx) = \lim_{t\to 0} \frac{p_X(t, dx)}{t}. \tag{1}$$

See for details [2, page 172]. The Lévy measure $\Pi_X(dx)$ represents the mean of the numbers of increments with altitude x. It is well known that any Lévy process is of unbounded variation if

$$\sigma^2 > 0$$

or

$$\int_{-1}^{+1} |x|\Pi_X(dx) = \infty.$$

For example, Meixner process is a pure jump process with unbounded variation, see [9]. In the particular choice of parameters, the Meixner $(2, 0, t)$ process can be constructed as a Brawnian motion subordinated by a series of independent and identically distributed Gamma processes with convenient normalization, see [5], [20, page 62].

Let $T = (T(t),\ t \geq 0)$ be a subordinator, i.e, the Lévy process with non decreasing sample paths on the probability space $(\Omega_T, \mathcal{B}_T, P_T)$ with natural filtration $\mathcal{G}_t$. Equivalently, this means that the Gaussian coefficient σ^2 is equal to zero and the Lévy measure Π_T does not charge the interval $]-\infty, 0]$ and fulfills

$$\int_{0+}^{\infty} (1 \ \wedge x)\Pi_T(dx) < \infty.$$

We denote the transition probability of T by $p_T(t, dx)$, $x \geq 0$, $t \geq 0$, and its Laplace transform by

$$E(\exp(-\lambda T(t))) = \exp(-t\psi_T(\lambda)), \quad \lambda > 0, \quad t > 0.$$

where $\psi_T : [0, +\infty[\to [0, +\infty[$ denotes the so-called Bernstein function verifying the following relation

$$\psi_T(\lambda) = d_T\lambda + \int_{0^+}^{\infty}(1 - e^{-\lambda x})\Pi_T(dx),$$

where d_T is a constant drift. All subordinators are of bounded variation, have positive drifts and jump measures concentrated on the interval $(0, \infty)$. The triplet of the subordinator is

$$\{d_T, 0, \Pi_T(dx)\}.$$

Natural examples of subordinators are: Gamma process, one-side stable, the quadratic variation of any Lévy process and so on. The compound Poisson process includes many explicitly known integer-valued Lévy processes.

Definition 2.1. *Let X and T be independent Lévy processes and suppose T is a subordinator. The subordinated process $Y = (Y(t), t \geq 0)$ is defined by*

$$Y(t) = X(T(t))$$

on the probability space

$$(\Omega_Y, \mathcal{B}_Y, P_Y),$$

where Ω_Y is the cartesian product $\Omega_X \times \Omega_T$. The Borelien σ-algebra and probability are defined by the tensor product:

$$\mathcal{B}_Y = \mathcal{B}_X \otimes \mathcal{B}_T, \quad P_Y = P_X \otimes P_T.$$

The transition probability is given by

$$p_Y(t, dy) = \int_0^{\infty} p_X(u, dy)p_T(t, du).$$

The main analytical properties of the subordinated Lévy processes are summarized in the following theorem.

Theorem 2.1. *The subordinated process Y has stationary independent increments and characteristic exponent given by*

$$f_Y(\lambda) = -\psi_T(-f_X(\lambda), \quad \lambda \in R, \quad t \geq 0.$$

The Lévy measure of the subordinated process is

$$\Pi_Y(dy) = d_T \Pi_X(dy) + \int_0^\infty p_X(u, dy)\Pi_T(du).$$

The drift of the subordinated process is:

$$d_Y = d_X d_T + \int_0^\infty \left(\int_{|x|<1} x p_X(u, dx) \right) \Pi_T(du).$$

The continuous part of the subordinated process has the coefficient $\sigma_Y^2 = d_T \sigma^2$. *The triplet of the subordinated process is*

$$\{d_Y, d_T\sigma^2, \Pi_Y(dx)\}.$$

Proof. The proof is based on the following conditional expectations:

$$E[\exp(i\lambda Y(t))] = E[E[\exp(i\lambda X(t)|T(t)]] =$$

$$E[\exp(T(t) f_X(\lambda))] = \exp(-t\psi_T(-f_X(\lambda))).$$

See for details: [2, page 172], [9, page 197], [19]. □

Definition 2.2. *The process* L *representing the first passage time of the subordinator* T *is defined by the following:*

$$L(u) = \inf\{t \geq 0 : T(t) > u, u \geq 0\}. \tag{2}$$

The process $L = (L(u), u \geq 0)$ takes values in the interval $(t \geq 0)$ and it is the right-continuous inverse of the process $T = (T(t), t \geq 0)$ taking values in the interval $(u \geq 0)$. The passage time above any fixed level is a.s. realized by a jump when the subordinator has no drift, see [3, page 77]. More precisely, the function $u \longrightarrow L(u)$ is non decreasing and hence $L(u-)$ exists. Since the event

$$\{T(t) > u\} = \bigcup_{\epsilon>0} \{T(t) > u + \epsilon\}$$

we have that the function $u \longrightarrow L(u)$ is right continuous and

$$L(u) = \lim_{\epsilon \downarrow 0} L(u + \epsilon).$$

The process $(L(u), u \geq 0)$ is right continuous (always) as $(T(t), t \geq 0)$ is right continuous, [18, page 190]. The process $(L(u), u \geq 0)$ is of continuous path if $(T(t),\ t \geq 0)$ has strictly increasing path, i.e. if the total mass of the Lévy measure is unbounded. For example, if $T(t)$ is a Gamma process then $L(u)$ is a continuous process. The probability distributions of T and L are related by the following

$$tP(L(u) \in dt)du = uP(T(t) \in du)dt,$$

see, V. M. Zolotarev [23].

Several filtrations are considered on the space Ω_Y. The natural filtration is denoted by

$$\mathcal{H}_t = \sigma(Y_\tau, \tau \leq t).$$

The Markov property of the subordinated process had been described by N. Bouleau, see [7], with the following filtration. Let us consider the family of events

$$\mathcal{A}_t = A_1 \otimes \{A_2 \cap (T(t) \geq u)\}, t \geq 0,$$

with

$$A_1 \in \mathcal{F}_u, A_2 \in \mathcal{G}_t, u \in [0, \infty],$$

where $\mathcal{F}_u$ and $\mathcal{G}_t$ are the natural filtrations of the processes $(X(u), u \geq 0)$ and $(T(t), t \geq 0)$, respectively. Then we construct the σ-algebra of this events. Denote it by $\mathcal{H}_t^* = \sigma(\mathcal{A}_t)$. The natural filtration $\mathcal{H}_t \subset \mathcal{H}_t^*$. The subordinated process $Y(t) = X(T(t))$ is strongly Markov process with respect to the filtration $\mathcal{H}_t^*$, see [7, page 67].

In order to study the stoping times, we consider the following filtration, define by

$$\mathcal{F}_{u+} = \bigcap_{\theta > u} \mathcal{F}_\theta,$$

and in the same way $\mathcal{G}_{t+}$, $\mathcal{H}_{t+}$, $\mathcal{H}_{t+}^*$.

The process $(L(u), u \geq 0)$ defined by (2) as the first passage time of the subordinator $T(t)$ across the level u is a $(\mathcal{G}_{t+})$ – stopping time and the following identity holds true a.s. for all $t > 0$ and $u > 0$:

$$\{T(t) < u\} = \{L(u) > t\}. \tag{3}$$

3. Examples. Compound Poisson process is a Lévy process with bounded total mass of the Lévy measure. Consequently, if one of X or T is a compound Poisson process, then the subordinated process is a compound Poisson process also.

Example 3.1 (Brownian motion subordinated by Poisson process, or by Gamma process, or by ε-stable process)**.** Brownian motion subordinated by Poisson process with intensity b is a compound Poisson process with the same intensity parameter b. The jumps time of Y are the same as the jumps time of T and the Lévy measure $\Pi_Y(dx) = bp_X(1, dx)$.

Brownian motion subordinated by Gamma process is the so called Variance-Gamma process, see [9, Chapter 4] and [20, page 57].

Brownian motion subordinated by ε-stable process has the characteristic exponent $f_Y(\lambda) = -(\lambda^2)^\varepsilon$. In the particular case: $\varepsilon = \frac{1}{2}$, the subordinated Brownian motion is a Cauchy process.

Example 3.2. (Poisson process with intensity a subordinated by Gamma (t, β) process)**.** We have the probability $P\{X(s) = k\}$ given by

$$p_X(s, \{k\}) = \frac{(as)^k}{k!} e^{-as}, k = 0, 1, 2 \dots$$

and Laplace transform $Ee^{-\lambda X(s)} = e^{-as(1-e^{-\lambda})}$. Probability density of the Gamma (t, β) subordinator is

$$p_T(t, dx) = \frac{1}{\Gamma(t)} \frac{dx}{\beta} \left(\frac{x}{\beta}\right)^{t-1} e^{-x/\beta},$$

where the well known function $\Gamma(t) = \int_0^\infty x^{t-1} e^{-x} dx$, $t\Gamma(t) = \Gamma(t+1)$ and $\Gamma(n+1) = n!$ for n integer. The Bernstein function $\psi_T(\lambda) = \log(1 + \beta\lambda)$ and the Lévy measure

$$\Pi_T(dx) = x^{-1} e^{-x/\beta} dx, \quad x > 0.$$

It is easy to calculate $\Pi_T(dx)$ by (1) as the following limit, namely

$$\Pi_T(dx) = \lim_{t \downarrow 0} \frac{1}{t\Gamma(t)} \frac{dx}{\beta} \left(\frac{x}{\beta}\right)^{t-1} e^{-x/\beta}.$$

The subordinated process $(Y(t), t \geq 0)$ is exactly Negative binomial process, with Laplace transform given by

$$Ee^{-\lambda Y(t)} = \left(\frac{p}{1 - qe^{-\lambda}} \right)^t,$$

where $p = 1 - q = \dfrac{1}{1 + \beta a}$. We have the transition probability:

$$P(Y(t) = k) = p^t q^k \frac{\Gamma(k + t)}{k!\Gamma(t)}.$$

The Lévy measure of the subordinated process Y is supported by the integers as follows

$$\Pi_Y(\{k\}) = \int_0^\infty \frac{e^{-xa}(xa)^k}{k!} \frac{e^{-x/\beta}}{x} dx = \frac{1}{k} q^k, \quad k = 1, 2, \ldots.$$

This is the well known logarithmic probability distribution. The Bernstein function of the subordinated process is

$$\psi_Y(\lambda) = \log \left(1 + a\beta(1 - e^{-\lambda}) \right).$$

As we are looking for the transmission of discontinuity, the most significant example for us the following Neyman process.

Example 3.3. (Poisson process subordinated by Poisson process)**.** Let X and T be two Poisson processes with intensity a and b respectively. Then the subordinated process Y is a compound Poisson process with intensity equal to the total mass of the Lévy measure Π_Y defined by:

$$\psi_Y(\infty) = b(1 - e^{-a}).$$

We see that the intensity of Y is less than the intensity of T. The Lévy measure is a zero truncated Poisson probability measure:

$$\Pi_Y(\{k\}) = \int_0^\infty p_X(s, \{k\}) b\delta(s - 1) ds = b \frac{a^k}{k!} e^{-a}, \quad k = 1, 2, \ldots.$$

Obviously, $\delta(s - 1)$ signifies the delta function: $\delta(s - 1) = 1$, iff $s = 1$. The transition probability of the Neyman process is:

$$p_Y(t, \{k\}) = \frac{a^k e^{-bt}}{k!} \sum_{j=0}^\infty \frac{j^k (bte^{-a})^j}{j!}, \quad k = 0, 1, 2, \ldots.$$

This probability distribution had been introduced by Neyman in 1939, see [11, page 368].

Example 3.4 (Gamma (t, β) process subordinated by Poisson process with intensity b)**.** Gamma process X subordinated by Poisson process is a compound Poisson process, with transition probability distribution characterized by the Bessel function. The jumps times of the subordinated process are the same as the jumps times of the subordinator. Namely, the Bernstein function

$$\psi_Y(\lambda) = b\left(1 - \frac{1}{1+\beta\lambda}\right) = \frac{b\beta\lambda}{1+\beta\lambda},$$

the Lévy measure is equal to

$$\Pi_Y(dx) = b\frac{e^{-x/\beta}}{\beta}dx$$

and has the total mass b. The transition probability:

$$p_Y(t, dx) = e^{-\left(bt+\frac{x}{\beta}\right)}\frac{1}{x}\sum_{k=0}^{\infty}\left(\frac{xbt}{\beta}\right)^k\frac{1}{k!\Gamma(k)}dx.$$

Bessel function of the first kind with parameter α denoted by $J_\alpha(z)$ is a solution of the equation:

$$x^2y'' + xy' + (x^2 - \alpha^2)y = 0$$

and has the following series expression

$$J_\alpha(z) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\alpha+1)}(\frac{z}{2})^{2k+\alpha},$$

absolutely convergent in the whole complex plane. In our example $\alpha = -1$, e.i. is a negative integer, the first term of the series is vanished and $J_{-n}(x) = (-1)^n J_n(x)$, $z = 2i\left\{\sqrt{\frac{xbt}{\beta}}\right\}$ and the transition probability is represented by:

$$p_Y(t, dx) = e^{-\left(bt+\frac{x}{\beta}\right)}\left(\frac{i}{x}\right)\left(\sqrt{\frac{xbt}{\beta}}\right)J_{-1}\left(2i\left(\sqrt{\frac{xbt}{\beta}}\right)\right).$$

Example 3.5 (Poisson process with intensity a subordinated by ε-stable process)**.** Bernstein function of the stable subordinator is given by:

$$\psi_T(\lambda) = \lambda^\varepsilon, 0 < \varepsilon < 1,$$

and the Lévy measure

$$\Pi_T(dx) = \frac{\varepsilon}{\Gamma(1-\varepsilon)} x^{-(1+\varepsilon)} dx.$$

Then the Laplace transform of the subordinated process is

$$Ee^{-\lambda Y(t)} = \exp\left(-ta^\varepsilon(1-e^{-\lambda})^\varepsilon\right).$$

The Lévy measure of the subordinated process is supported by the integers as follows

$$\Pi_Y(\{k\}) = \int_0^\infty \frac{e^{-ua}(ua)^k}{k!} \frac{du}{u^{\varepsilon+1}} \frac{\varepsilon}{\Gamma(1-\varepsilon)} = \frac{\varepsilon a^k \Gamma(k-\varepsilon)}{\Gamma(1-\varepsilon)k!}, \quad k = 1, 2, \ldots .$$

The process Y is a particular case of the so called discrete stable process.

The properties of the discrete stable infinitely divisible distribution had been developed by F. W. Stetel and K. Van Harn [22] and N. L. Johnson, A. W. Kemp and S. Kotz [11].

Example 3.6 (ε-stable process subordinated by a Poisson process)**.** Obviously, stable process subordinated by Poisson process with intensity b is a compound Poisson with Bernstein function

$$\psi_Y(\lambda) = b(1 - \exp(-\lambda^\varepsilon))$$

and total mass of the Lévy measure equal to b.

Example 3.7 (ε-stable process subordinated by a Gamma process)**.** The one sided stable process X subordinated by Gamma process with parameter $\beta = 1$ is called the Mittag-Leffler process. The Bernstein function of the subordinated process is given by

$$\psi_Y(\lambda) = \log(1 + \lambda^\varepsilon).$$

The Mittag-Leffler function is defined by:

$$E_\varepsilon(z) = \sum_{n=0}^\infty \frac{z^n}{\Gamma(1+\varepsilon n)}.$$

It is well known that

$$1 - E_\varepsilon(-x^\varepsilon)$$

is a probability distribution function on $x \geq 0$ for $\varepsilon \in (0, 1]$, $E_\varepsilon(0) = 1$ and

$$P(Y(1) \leq x) = 1 - E_\varepsilon(-x^\varepsilon).$$

The transition probability $p_Y(t, dx)$ can be expressed by the following series

$$P(Y(t) \leq x) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(t+k) x^{\varepsilon(t+k)}}{k! \Gamma(t) \Gamma(1 + \varepsilon(t+k))}.$$

It is easy to verify that the Laplace transform of $Y(t)$ is equal to the following

$$\frac{1}{\lambda^{\varepsilon t}} \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(t+k)}{\lambda^{\varepsilon k} k! \Gamma(t)}.$$

Obviously, the last sum represents the Taylor expansion of the following

$$\frac{1}{\lambda^{\varepsilon t}} \left[\sum_{k=0}^{\infty} \left[\frac{-1}{\lambda^\varepsilon} \right]^k \right]^t = \left[\frac{1}{1 + \lambda^\varepsilon} \right]^t.$$

see K. Sato [19, page 234 and page 439].

4. Poisson random measure. Poisson random measure describes the jump structure embedded in any Lévy process.

Definition 4.1. (Poisson random measure)**.** *Suppose that $(E, \mathcal{E}, \nu)$ is un arbitrary σ-finite measure space. Let $\mathcal{N} : \mathcal{E} \to (0, 1, 2, \dots)$ in such a way that the family $(\mathcal{N}(A) : A \in \mathcal{E})$ are random variables defined on the probability space $(\Omega, \mathcal{B}, P)$. Then $\mathcal{N}$ is called a Poisson random measure on $(E, \mathcal{E}, \nu)$ with intensity ν if*

- *for mutually disjoint $A_1, \dots, A_n$ in $\mathcal{E}$, the variables $\mathcal{N}(A_1), \dots, \mathcal{N}(A_n)$ are independent,*
- *for each $A \in \mathcal{E}$, $\mathcal{N}(A)$ is Poisson distributed with parameter $\nu(A)$,*
- *P-almost surely $\mathcal{N}$ is a measure.*

It is well known that if $\mathcal{N}$ is a Poisson random measure on $(E, \mathcal{E}, \nu)$ then the support of $\mathcal{N}$ is P-almost surely countable. If in addition, the intensity measure ν is a finite measure, then the support is P-almost surely finite. Let $E = \mathcal{B}_b(R_*^2)$ be the class of all bounded Borel subsets of

$$R_*^2 = R^2 \setminus \{(t, 0) : t \in R\}$$

away from the t-axis. Let $\Pi(dx)$ be the Lévy measure of any Lévy process. Define the product measure

$$d\nu(t, x) = dt \otimes \Pi(dx)$$

on $\mathcal{B}(\mathbb{R}_*^2)$. For $A \in \mathcal{B}_b(\mathbb{R}_*^2)$, let $\mathcal{N}(A)$ be a random variable on $(E, \mathcal{E})$ defined by

$$\mathcal{N}(A) = \text{Card}\{(t, x) \in A : X(t) - X(t_-) = x\}.$$

Then $\mathcal{N}(A)$ is Poisson distributed with the intensity measure $\nu(A)$. The system of random variables

$$\mathcal{N}(A) - \nu(A) : A \in \mathcal{B}_b(R_*^2)$$

forms an independent random measure with zero mean called compensated Poisson random measure denoted by

$$\mathcal{N}^0(A) = \mathcal{N}(A) - \nu(A).$$

Then the mean

$$E(\mathcal{N}^0(A)\mathcal{N}^0(B)) = \nu(A \cap B) \quad \text{for any} \quad A, B \in \mathcal{B}_b(R_*^2).$$

Note that unlike $\mathcal{N}$ the compensated measure $\mathcal{N}^0$ is neither integer valued nor positive, it is a signed measure. We have the following martingales and martingales measures, respectively for $X(u)$, $T(t)$ and $Y(t)$:

$$X^0(u) = X(u) - uEX(1), \quad \mathcal{N}_X^0(A) = \mathcal{N}_X(A) - \nu_X(A),$$

where $\nu_X(du, dx) = du \otimes \Pi_X(dx)$, and in the same way, for $T(t)$ and $Y(t)$.

Theorem 4.1. *The Lévy-Ito decomposition for the processes $X(u), T(t)$*

and $Y(t)$ *is as follows*

$$X(u) = d_X u + \sigma B(u) + \int_0^t \int_R x 1_{\{|x|<1\}} \mathcal{N}_X^0(du, dx) + \int_0^t \int_R x 1_{|x|\geq 1} \mathcal{N}_X(du, dx),$$

$$T(t) = d_T t + \int_0^t \int_{0^+}^{\infty} x \mathcal{N}_T(du, dx),$$

$$Y(t) = d_Y t + d_T \sigma B(t) + \int_0^t \int_R x 1_{\{|x|<1\}} \mathcal{N}_Y^0(du, dx) + \int_0^t \int_R x 1_{|x|\geq 1} \mathcal{N}_Y(du, dx),$$

where $B = (B(t), t \geq 0)$ *is a* 1*-dimensional Wiener process, independent of the system of Poisson random measure*

$$\{\mathcal{N}_X(E) : E \in \mathcal{B}_b(R_*^2)\}.$$

See [9], [12, page 236].

Now, consider the Poisson random measure of the processes $T = (T(t), t \geq 0)$, $X = (X(u), u \geq 0)$ and $Y = (Y(t), t \geq 0)$ described by the families of delta functions:

$$\mathcal{N}_T(A) = \sum_{i:(\theta_i,\gamma_i)\in A} \delta(t - \theta_i, u - \gamma_i), \quad A \in \mathcal{B}_b(R_*^2(t, u)), \quad t > 0. \tag{4}$$

$$\mathcal{N}_X(E) = \sum_{i:(\eta_i,\xi_i)\in E} \delta(u - \eta_i, x - \xi_i), \quad E \in \mathcal{B}_b(R_*^2(u, x)), \quad u > 0. \tag{5}$$

$$\mathcal{N}_Y(F) = \sum_{i:(z_i,\zeta_i)\in F} \delta(t - z_i, x - \zeta_i), \quad F \in \mathcal{B}_b(R_*^2(t, x)), \quad t > 0. \tag{6}$$

Theorem 4.2. *Suppose that* $X = (X(u), u \geq 0)$ *is a Lévy process but not a compound Poisson process. Let* $T(t)$ *be a pure jump subordinator without drift with Poisson random measure given by* (4)*. Then the subordinated Lévy process* $Y(t)$ *has jumps at the random points* $z_i = \theta_i$, $i = 1, 2, \ldots$ *with altitudes* $\zeta_i = X(\gamma_i)$ *and the Poisson random measure of* $Y(t)$ *is given by* (6)*.*

P r o o f. If $X(t)$ has continuous paths, then it is obvious that the only discontinuity of subordinated process can be caused by the jumps times of the subordinator. Otherwise, the both $X(u)$ and $T(t)$ have countable number of jumps. Since $T(t)$ is purely discontinuous, its range has zero Lebesgue measure. The independence of $X(u)$ and $T(t)$ ensures that a.s. no jump time of $X(u)$ lies in the closed range of $T(t)$. This way, the discontinuity of $X(u)$ can not influence discontinuity of $Y(t)$. If $X(u)$ has an unbounded Lévy measure then $X(u)$ is strictly increasing or decreasing. Consequently, the jumps altitude is equal to $\zeta_i = X(\gamma_i)$. Namely, by the strong Markov property and right continuity of the paths, we have:

$$\zeta_i = Y(z_i) - Y(z_i-) = X(T(\theta_i)) - X(T(\theta_i-)) =$$

$$X(T(\theta_i-) + \gamma_i) - X(T(\theta_i-)) = X(\gamma_i). \qquad \square$$

Theorem 4.3. *Suppose that $X = (X(u), u \geq 0)$ is a compound Poisson process without drift taking the form*

$$X(u) = \sum_{\eta_i \leq u} \xi_i 1_{[\eta_i, \infty)},$$

defined by its Poisson random measure (5)*. Let $T(t)$ be a pure jump subordinator without drift with Poisson random measure given by* (4)*. Then the subordinated Lévy process $(Y(t), t \geq 0)$ is a compound Poisson process defined by its Poisson random measure* (6)*. The random variables representing the inter-arrival times*

$$(z_1, z_{i+1} - z_i, i = 1, 2, \dots)$$

are iid with exponential distribution and z_1 is a first passage time of $T(t)$ the random level η_1, i.e. $z_1 = L(\eta_1)$. The sequence $(z_i,\ i = 1, 2, \dots)$ is a subsequence of the sequence $(\theta_i,\ i = 1, 2, \dots)$. The random variables $(\zeta_i,\ i = 1, 2, \dots)$ are iid and the amplitude $\zeta_1 = X(T(z_1))$.

Proof. Let the total mass of the Lévy measure $\Pi_X(-\infty, \infty) = m$. Obviously, in this case, the total mass of the Lévy measure $\Pi_Y(-\infty, \infty) = \psi_T(m)$. When $(X(u), u \geq 0)$ is a compound Poisson process, then

$$P(X(u) = 0) = e^{-um} > 0.$$

Since X and T are independent, there is a positive probability that $(Y(t) = 0, t > 0)$, namely:

$$P(Y(t) = 0) = E[P(X(T(t)) = 0|T(t))] = E[e^{-mT(t)}] = e^{-t\psi_T(m)}.$$

Consequently, any randomness of $T(t)$ before it passes the level η_1 given by the first jump time of $X(u)$ is not reflected by $Y(t) = X(T(t)$. The jump amplitude of $Y(t)$ is determined by its Lévy measure:

$$\Pi_Y(dy) = \int_0^\infty p_X(u, dy)\Pi_T(du). \qquad \square$$

See our significant example of Neyman process in the previous section.

5. Subordinated Markov branching processes. Subordinated Markov branching processes $(Y(t), t \geq 0)$ and the ground MBP $(X(u), u \geq 0)$ are described by the general pure jump Markov processes, [12, page 193] and [13, page 249]. They are right-continuous, piece-wise constant. All pure jump Markov processes with parameters $\{c(x), K(x,y)\}$ are defined by the sequence of the jump times $\tau_1, \tau_2, \ldots$ and the jumps amplitude $\Upsilon(x)$. The jump $\Upsilon(x)$ is independent of the past and has distribution that depends only on x. If the process is in the state x then it stays there for an exponential length of time τ_x with mean $\dfrac{1}{c(x)}$ after which it jumps from x to a new state $x + \Upsilon(x)$, where

$$P(\Upsilon(x) \leq y) = K(x, y).$$

The jump transition kernel is defined by:

$$c(x)P_x(X(\tau_x) \leq y) = c(x)P(x + \Upsilon(x) \leq y) = c(x)K(x, y - x).$$

The rate function $c(x)$ is always non-negative. If $c(x)$ is constant, $c(x) = c$, the process is called pseudo Poisson process, see [12, page 191]. For the integer-valued pure jump Markov processes the kernels may be specified by

$$K(k, j) = P(\Upsilon(k) = j).$$

The transition probabilities verify the following Kolmogorov backward equation:

$$\frac{d}{dt}P_{kj}(t) = c(k)\left\{\sum_{i=0}^{\infty} P(k + \Upsilon(k) = i)P_{ij}(t) - P_{kj}(t)\right\}. \tag{7}$$

Let $X = (X(t), t \geq 0)$, $X(0) = 1$, be a Markov branching process. Any Markov branching process is an integer-valued Markov chain determined by the random variable τ exponentially distributed representing the life-time of particles and the integer-valued random variable ξ representing the number of the off-springs particles with probability law

$$p(B) = P(\xi \in B), \quad p_k = P(\xi = k)$$

and probability generating function

$$h(s) = Es^{\xi}, |s| \leq 1.$$

Let

$$P(\tau > t) = \exp(-at).$$

Then the jump transition kernel of the MBP $(X(t), t \geq 0)$ is given by

$$c_X(k) = ka, K_X(k, B) = p(B + 1).$$

Obviously, the inter-arrival times are independent exponentially distributed, but with different rate depending on the state of the process. If $X(t) = k$, then the next inter-arrival time is a minimum of k independent exponentially distributed random variables. In a growing population the rate at which jumps occur increases with time. The amplitudes of the increments are all independent identically distributed represented by the random variable $\xi - 1$, i.e.

$$K_X(k, i) = p_{i+1}$$

does not depend on k. The jumps arrive, when one particle dies and gives birth of ξ off-springs, i.e. the increment is $\xi - 1$. The infinitesimal generator Δ_X is defined by the probability p_{i-k+1}:

$$P(k + \Upsilon(k) = i) = P(\xi = i + 1 - k).$$

The transition probabilities

$$P_{kj}(t) = P(X(t) = j | X(0) = k)$$

satisfy the Kolmogorov backward equation:

$$\frac{d}{dt} P_{kj}(t) = ka \left(\sum_{i=k-1}^{\infty} p_{i-k+1} P_{ij}(t) - P_{kj}(t) \right). \tag{8}$$

The branching property is manifested by the convolution relation

$$P_{kj}(t) = P_{1j}^{*k}(t),$$

see [1, page 106], [21, page 27].

Theorem 5.1 (Lamperti)**.** *Let $X(t)$ be a supercritical MBP having infinitesimal generating function $f_X(s) = a(h(s) - s)$, with $0 < a < \infty$, $h(0) = h'(0) = 0$, and $h'(1) < \infty$. Define the additive functional*

$$S(t) = \int_0^t X(u) du,$$

and its inverse

$$T(t) = S^{-1}(t) = \inf(u : S(u) = t).$$

Then the time scale process $Y(t) = X(T(t))$ is a compound Poisson process with Bernstein function given by:

$$a\left(\frac{h(\lambda)}{\lambda} - 1\right).$$

See [1, page 125].

In the following we suppose that the random time change process $T = (T(t), t \geq 0)$ is an independent subordinator **without drift** defined by its Bernstein function

$$\psi_T(\lambda) = \int_{0^+}^{\infty} (1 - e^{-\lambda x}) \Pi_T(dx). \tag{9}$$

Then (owing to the independence of X and T) the subordination in the sense of Bochner preserves the Markov properties but it disturbs the branching properties. Taking into consideration of Theorem 4.3 describing the jump-times of the subordinated Lévy processes, we must realize that the subordinated MBP even in the case of Poisson subordinator is not a pseudo Poisson process, see [12, page 191]. The jump times z_n of $Y(t)$ are a subsequence of the jump times θ_n of the subordinator $T(t)$. But the inter-arrival times of Y are not iid, because the ground process is not a Lévy process. The jump times τ_n of the ground process $X(t)$ influence the trajectories of $(Y(t), t \geq 0)$ by the first passage time of T the successive random levels τ_n. For this reason, we need the following result:

Theorem 5.2. *Let $T(t)$ be a subordintor with Bernstein function $\psi_T(\lambda)$ given by* (9)*. Consider the random variable τ exponentially distributed with parameter a. Then the first passage time the random level τ is a random variable θ exponentially distributed with parameter $\psi_T(a)$.*

P r o o f. Subordinator without drift does not hit any fixed level, it just overshoot. Following the definition (2) of the first passage time and its properties (3) we can calculate the probability of the event $(\theta > t)$.

$$P(\theta > t) = P(L(\tau) > t) = \int_{s=0}^{\infty} P(L(s) > t) P(\tau \in ds).$$

Integration by parts transforms the last integral into the Laplace transform of T.

$$\int_{s=0}^{\infty} P(T(t) < s) a e^{-as} ds = -\int_{s=0}^{\infty} \int_{x=0}^{s} p_T(t, dx) d(e^{-as}) = \int_{s=0}^{\infty} e^{-as} p_T(t, ds).$$

Obviously,

$$\lim_{s\downarrow 0} e^{-as}\int_{x=0}^{s} p_T(t,dx) = 0. \qquad \square$$

Denote the main characteristics of $Y(t)$ by:

$$K_Y(k,i) = \mathbb{K}(k,i),$$

$$c_Y(k) = \mathbb{C}(k),$$

and the transition probability of $Y(t)$

$$\mathbb{P}_{ki}(t) = P(Y(t) = i | Y(0) = k).$$

Theorem 5.3. *Let $X(t)$ be a MBP with transition probabilities $P_{kj}(t)$ satisfying the equation* (8). *Let $T(t)$ be an independent subordinator with a Bernstein function $\psi_T(\lambda)$ given by* (9). *Then the subordinated Markov process $Y(t) = X(T(t))$ has a jump transition kernel $\mathbb{C}(k)\mathbb{K}(k,j)$ given by:*

$$\mathbb{C}(k) = \psi_T(ka), \mathbb{K}(k, i-k) = \int_0^\infty P_{ki}(u)\Pi_T(du). \tag{10}$$

The transition probabilities $\mathbb{P}_{kj}(t)$ satisfy the following Kolmogorov backward equation:

$$\frac{d}{dt}\mathbb{P}_{kj}(t) = \psi_T(ka)\left\{\sum_{i=0}^{\infty}\int_0^\infty P_{ki}(u)\Pi_T(du)\mathbb{P}_{ij}(t) - \mathbb{P}_{kj}(t)\right\}. \tag{11}$$

Proof. Let $T(t) = u \in [\tau_m, \tau_{m+1})$, then u is a stoping time. The length of the interval $[u, \tau_{m+1})$ is exponentially distributed with the same parameter as the length of the interval $[\tau_m, \tau_{m+1})$. Let $X(u) = k$ then the length of the interval $[\tau_m, \tau_{m+1})$ is exponentially distributed with parameter ka. Following the previous theorem the process $Y(t)$ will stay at the point k a random time exponentially distributed with parameter $\psi_T(ka)$. Suppose, the first passage time of T across the random level τ_{m+1} is realize by the jump $\gamma_l\delta(t-\theta_l)$. Then the random point θ_l belongs to the sequence $\{z_1, z_2, \dots\}$, representing the jump times of the subordinated process $Y(t)$. Denote the values of the subordinator at the successive points by: $T(\theta_{l-1}) = u_{l-1}$ and $T(\theta_l) = u_l$. The subordinator is right continuous, consequently:

$$T(\theta_l -) = T(\theta_{l-1}), T(\theta_l) = u_l = u_{l-1} + \gamma_l.$$

Let $X(u_l) = i$, knowing that $X(u_{l-1}) = k$. The jumps amplitude is given by

$$\Upsilon(k) = X(u_l) - X(u_{l-1}) = i - k$$

and is determined by the evolution of the MBP $X(t)$ during the time interval with length γ_l, starting from the point k. Owing to the stationarity of the subordinator's increments, we have:

$$\mathbb{K}(k, i-k) = \int_0^\infty P_{ki}(u)\Pi_T(du). \qquad \square$$

Remark 5.1 (The semigroup approach)**.** Denote by

$$G_t f(x) = E_x f(X(t)), \mathbb{G}_t f(x) = \mathbb{E}_x f(Y(t))$$

for any measurable function $f(x)$. The infinitesimal generator Δ_X of the pure jump Markov processes is given by the following

$$\Delta_X f(x) = c(x)E\{f(x + \Upsilon(x)) - f(x)\}.$$

The infinitesimal generator Δ_Y of the subordinated Markov processes is given by the following

$$\Delta_Y f(x) = \int_0^\infty (G_u f(x) - f(x))\Pi_T(du).$$

The integral $\int_0^\infty$ is defined by the strong limit of $\int_u^v$ as $u \downarrow 0$ and $v \to \infty$. See [19, page 212, Theorem 32.1].

6. Randomly indexed Galton-Watson branching processes.

We consider a Galton-Watson process, [1, 21], defined by the sequence

$$(X(n), n = 1, 2, \dots), X(0) = 1,$$

as a ground process and integer valued subordinator defined by the Poisson or compound Poisson process $(T(t), t \geq 0)$, $T(0) = 0$. For example: $(T(t), t \geq 0)$ is a Neyman process, Pascal process or discrete stable subordinator.

Obviously, $(Y(t) = X(T(t)), t \geq 0)$ is an integer-valued Markov process starting from $Y(0) = 1$. We shall prove that, it is not a Lévy process, i.e. it is not a compound Poisson process and it is not a MBP. It is a pseudo Poisson process.

Denote by ξ the integer-valued random variable representing the number of the off-springs particles with probability law

$$p(B) = P(\xi \in B), \quad p_k = P(\xi = k)$$

and probability generating function

$$h(s) = Es^{\xi}, \quad |s| \leq 1.$$

Then the Galton-Watson process

$$X(n) = \sum_{i=1}^{X(n-1)} \xi_i,$$

where ξ_i are independent identically distributed (iid) random variable with the same law as ξ. The probability generating function of $X(n)$, denoted by

$$Es^{X(n)} = H_n(s)$$

is the n-fold composition of $h(s)$, namely:

$$H_1(s) = h(s), H_n = H_{n-1}(h(s)).$$

Suppose the total mass of the Lévy measure to the compound Poisson process is equal to m, then the subordinator can be represented by

$$T(t) = \sum_{\theta_i \leq t} \gamma_i 1_{[\theta_i, \infty)},$$

where the inter-arrival times $\theta_{i+1} - \theta_i$ are iid random variables exponentially distributed with parameter m and the increments are iid **integer-valued** random variables $\gamma_i = \gamma$, $i = 1, 2, \ldots$ with probability law

$$P(\gamma \in B) = \frac{\Pi_T(B)}{m}.$$

Particular case, the homogeneous Poisson process: the increments

$$\gamma_i = 1, \quad i = 1, 2, \ldots.$$

Randomly indexed Galton-Watson process realize the exceptional case of the subordinated processes when the range of subordinator coincides with the set of jump-times of the ground process, if T is a Poisson process. We shall consider separately the case: subordinator is a Poisson process or compound Poisson process.

Theorem 6.1. *Let $X(n)$ be a Galton-Watson process and $T(t)$ an independent subordinator represented by the Poisson process, then the subordinated process $Y(t)$ is a pseudo Poisson process with the following jump transition kernel:*

$$c_Y(k) = m, K_Y(k, B) = p^{*k}(B+1).$$

The probability generating function of the increments

$$Y(\theta_{n+1}) - Y(\theta_{n+1}-) = X(n+1) - X(n)$$

is given by

$$H_n\left(\frac{h(s)}{s}\right).$$

P r o o f. Obviously, in the case of Poisson subordinator, we have the following situation: If $t \in [\theta_n, \theta_{n+1})$ then $Y(t) = X(n)$. Let $Y(t) = k$, then

$$\Upsilon(k) = \sum_{i=1}^{k}(\xi_i - 1). \qquad \square$$

Theorem 6.2. *Let $X(n)$ be a Galton-Watson process and $T(t)$ an independent subordinator represented by the integer-valued compound Poisson process, then the subordinated process $Y(t)$ is a pseudo Poisson process with the following jump transition kernel:*

$$c_Y(k) = m, K_Y(k, B) = P^{*k}(X(\gamma) \in (B+1)).$$

P r o o f. In the case of compound Poisson subordinator, the range of subordinator is included in the set of jump-times of X. The increments of Y can be calculate as follows: If $t \in [\theta_n, \theta_{n+1})$ and $Y(t) = k$, then

$$\Upsilon(k) = \sum_{i=1}^{k}(X_i(\gamma) - 1),$$

where X_i **are independent copies of the Galton-Watson process** X starting with one particle and integer valued random variables $\gamma_i = \gamma$, $i = 1, 2, \dots$ are iid with probability

$$P(\gamma = j) = \frac{\Pi_T(\{j\})}{m}.$$

Consequently, the probability generating function of $\Upsilon(k)$, k-fixed, is

$$\left(\frac{\sum_{j=1}^{\infty} H_j(s)\Pi_T(\{j\})}{ms}\right)^k .$$

Denote by $F(t,s)$ the probability generating function of $Y(t)$, i.e.

$$F(t,s) = Es^{Y(t)}.$$

The process $Y(t)$ is right continuous, i.e. $Y(\theta_{n+1}-) = Y(\theta_n)$. The probability generating function of the increments

$$\Upsilon(k) = Y(\theta_{n+1}) - Y(\theta_n), \quad t \in [\theta_n, \theta_{n+1}), \quad Y(t) = k,$$

is given by

$$F\left(t, \frac{\sum_{j=1}^{\infty} H_j(s)\Pi_T(\{j\})}{ms}\right). \qquad \square$$

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