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EMPIRICAL BAYES TWO-SIDED TEST FOR THE PARAMETER OF LINEAR EXPONENTIAL DISTRIBUTION FOR RANDOM INDEX*

Huang Juan

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Abstract. In the case of random index, the empirical Bayes two-side test rule for the parameter of linear exponential distribution is constructed. The asymptotically optimal property for the proposed EB test is obtained under suitable conditions. It is shown that the convergence rates of the proposed EB test rules can arbitrarily close to $O(n^{-\frac{1}{2}})$.

1. Introduction. Estimation and test of empirical Bayes (EB) have been investigated in many papers, in the particular for the exponential and scale exponential family, the readers are referred to literature (see [2, 4, 5, 6, 8]). Recently, Empirical Bayes test for the parameter of linear exponential distribution have been discussed [9, 10]. Usually, EB approaches are concerned with non-random index of size of the historical samples. In fact, we may meet the problem

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that size of the historical samples is a random index which takes positive integer valued random variable. Study on limit theory with random index has been acquired some results [11]. Up to now, Empirical Bayes test problem for the parameter of distribution family for random index hasn't been studied. Differing from the previous many study, in the case of random index, we will renew to construct empirical Bayes test rule for the parameter of linear exponential distribution is constructed.

Let X have a conditional density function for given $\theta^{[1]}$

$$f(x|\theta) = (\mu x + \theta)\exp\left\{-\theta x - \frac{1}{2}\mu x^2\right\}, \tag{1.1}$$

where μ is a known constant and θ is an unknown parameter. Denote sample space $\Omega = \{x \mid x > 0\}$ and parameter space $\Theta = \left\{\theta > 0 | \int_\Omega f(x \mid \theta)dx = 1\right\}$.

In this paper, we discuss the following two-sided test problem

$$H_0 : \theta_1 \le \theta \le \theta_2 \Leftrightarrow H_1 : \theta < \theta_1 \quad \text{or} \quad \theta > \theta_2, \tag{1.2}$$

where θ_1 and θ_2 are given positive constant, taking $\theta_0 = \frac{\theta_1 + \theta_2}{2}$ and $\gamma_0 = \frac{\theta_2 - \theta_1}{2}$, then two-sided test problem of (1.2) is equivalent with

$$H_0^* : |\theta - \theta_0| \le \gamma_0 \Leftrightarrow H_1^* : |\theta - \theta_0| > \gamma_0. \tag{1.3}$$

For hypothesis test problem (1.3), we have loss function

$$L_i(\theta, d_i) = (1-i)a[(\theta-\theta_0)^2 - \gamma_0^2]I_{[|\theta-\theta_0|>\gamma_0]} + ia[\gamma_0^2 - (\theta-\theta_0)^2]I_{[|\theta-\theta_0|\le\gamma_0]}, \quad i = 0, 1$$

where $a > 0$, $d = \{d_0, d_1\}$ is action space, d_0 and d_1 imply acceptance and rejection of H_0^*.

Assume that the prior distribution $G(\theta)$ of θ is unknown, we obtain randomized decision function

$$\delta(x) = P(\text{accept} \quad H_0^* \mid X = x). \tag{1.4}$$

Then, the Bayes risk function of $\delta(x)$ is shown by

(1.5)

$$\begin{aligned} R(\delta(x), G(\theta)) &= \int_\Theta \int_\Omega [L_0(\theta, d_0) f(x|\theta)\delta(x) + L_1(\theta, d_1) f(x|\theta)(1 - \delta(x))]dxdG(\theta) \\ &= a\int_\Omega \beta(x)\delta(x)dx + C_G, \end{aligned}$$

where

$$C_G = \int_\Theta L_1(\theta, d_1)dG(\theta), \beta(x) = \int_\Theta [(\theta-\theta_0)^2 - \gamma_0^2] f(x|\theta)dG(\theta). \tag{1.6}$$

The marginal density function of X is given by

$$f_G(x) = \int_\Theta f(x|\theta)dG(\theta) = \int_\Theta (\mu x + \theta)\exp\left(-\theta x - \frac{1}{2}\mu x^2\right) dG(\theta).$$

Let

$$p_G(x) = \int_\Theta \exp\left(-\theta x - \frac{1}{2}\mu x^2\right) dG(\theta).$$

Hence, $p_G^{(1)}(x) = -\int_\Theta (\mu x+\theta)\exp\left(-\theta x - \frac{1}{2}\mu x^2\right) dG(\theta) = -f_G(x)$, where $p_G^{(1)}(x)$ is derivative of $p_G(x)$, then

$$\int_x^\infty f_G(x)dx = p_G(x). \tag{1.7}$$

By (1.6) and simple calculation, we have

$$\beta(x) = f_G^{(2)}(x) + Q(x)f_G^{(1)}(x) - \mu Q(x)p_G(x) + S(x)f_G(x), \tag{1.8}$$

where $Q(x) = 2\mu x + 2\theta_0, S(x) = \mu^2 x^2 + 2\mu\theta_0 x + 3\mu + \theta_0^2 - \gamma_0^2$, and $f_G^{(1)}(x)$ and $f_G^{(2)}(x)$ are first and second order derivative of $f_G(x)$.

Using (1.5), Bayes test function is obtained as follows

$$\delta_G(x) = \begin{cases} 1, & \beta(x) \le 0, \\ 0, & \beta(x) > 0. \end{cases} \tag{1.9}$$

Further, we can get minimum Bayes risk

$$R(G) = \inf_\delta R(\delta, G) = R(\delta_G, G) = a\int_\Omega \beta(x)\delta_G(x)dx + C_G. \tag{1.10}$$

When the prior distribution of $G(\theta)$ is known and $\delta(x) = \delta_G(x)$, $R(G)$ is achieved. However, where $G(\theta)$ is unknown, so $\delta_G(x)$ can not be made use of, we need to introduce EB method.

The rest of this paper is structured as follows. Section 2 presents an EB test. In section 3, we obtain asymptotic optimality and the optimal rate of convergence of the EB test.

2. Construction of EB test function for random index. Under the following condition, we need to construct EB test function. We make the following assumptions: let $(X_1, \theta_1), \ldots, (X_{\tau_n}, \theta_{\tau_n})$ and (X, θ) be independent random vectors, the θ_i $(i = 1, \ldots, \tau_n)$ and θ are independently identically distributed (i.i.d.) and have the common prior distribution $G(\theta)$. Let $X_1, X_2, \ldots, X_{\tau_n}, X$ be mutually independent random variable sequence with the common marginal density function $f_G(x)$, where $X_1, X_2, \ldots, X_{\tau_n}$ are historical samples and X is present sample and τ_n is a discrete random index which takes positive integer values with known distribution. Assume $f_G(x) \in C_{s,\alpha}$, $x \in R^1$, where $C_{s,\alpha} = \{g(x)|g(x)$ is probability density function and has continuous s-th order derivative $g^{(s)}(x)$ with $|g^{(s)}(x)| \leq \alpha$, $s \geq 3$, α and s are natural numbers$\}$. First construct estimator of $\beta(x)$.

Let $K_r(x)$ $(r = 0, 1, \ldots, s-1)$ be a Borel measurable real function vanishing off $(0, 1)$ such that

$$(A1) \qquad \frac{1}{t!}\int_0^1 v^t K_r(v)dv = \begin{cases} (-1)^t, & t = r, \\ 0, & t \neq r, t = 0, 1, \ldots, s-1. \end{cases}$$

Denote $f_G^{(0)}(x) = f_G(x)$, $f_G^{(r)}(x)$ is the r order derivative of $f_G(x)$ $(r = 0, 1, \ldots, s-1)$. Similar to Prakasa [7], kernel estimation of $f_G^{(r)}(x)$ is defined by

$$(2.1) \qquad f_{\tau_n}^{(r)}(x) = \frac{1}{\tau_n h_n^{(1+r)}} \sum_{j=1}^{\tau_n} K_r\left(\frac{x - X_j}{h_n}\right),$$

where h_n is smoothing bandwidth and $\lim_{n\to\infty} h_n = 0$.

As $p_G(x) = \int_x^\infty f_G(x)dx = E\{I_{(X_i > x)}\}$, hence, $p_G(x)$ is defined as follows

$$(2.2) \qquad p_G(x) = \frac{1}{\tau_n}\sum_{i=1}^{\tau_n} I_{(X_i > x)}.$$

Thus, estimator of $\beta(x)$ is obtained by

$$(2.3) \qquad \beta_{\tau_n}(x) = f_{\tau_n}^{(2)}(x) + Q(x)f_{\tau_n}^{(1)}(x) - \mu Q(x)p_{\tau_n}(x) + S(x)f_{\tau_n}(x).$$

Hence, EB test function is defined by

$$(2.4) \qquad \delta_{\tau_n}(x) = \begin{cases} 1, & \beta_{\tau_n}(x) \leq 0, \\ 0, & \beta_{\tau_n}(x) > 0. \end{cases}$$

Let E denote the mathematical expectation with respect to the joint distribution of $X_1, X_2, \ldots, X_{\tau_n}$, we get overall Bayes risk of $\delta_{\tau_n}(x)$

$$R(\delta_{\tau_n}(x), G) = a \int_\Omega \beta(x) E[\delta_{\tau_n}(x)]dx + C_G. \tag{2.5}$$

If $\lim_{n\to\infty} R(\delta_{\tau_n}, G) = R(\delta_G, G)$, $\{\delta_{\tau_n}(x)\}$ is asymptotic optimality of EB test function; if $R(\delta_{\tau_n}, G) - R(\delta_G, G) = O(n^{-q})$, $q > 0$, $O(n^{-q})$ is asymptotic optimality convergence rates of EB test function of $\{\delta_{\tau_n}(x)\}$. Before proving the theorems, we give a series of lemmas.

Let $c, c_1, c_2, c_3, \ldots, c_8$ be different constants in different cases even in the same expression.

Lemma 2.1 (Lu, [3]). *Let $\{X_i, i \geq 1\}$ be independent identical distribution random samples, with $EX_i = 0$ and $E|X_i|^r < \infty$, $r \geq 2$, then*

$$E|\sum_{i=1}^{n} X_i|^r \leq cn^{\frac{r}{2}} E|X_i|^r.$$

Lemma 2.2. *$f_{\tau_n}^{(r)}(x)$ is defined by* (2.1)*. Let $X_1, X_2, \ldots, X_{\tau_n}$ be independent identically distributed random samples. Assume* $(A1)$ *holds, $\forall x \in \Omega$,*

(1) *If $f_G^{(r)}(x)$ is continuous function, $\lim_{n\to\infty} h_n = 0$ and $\lim_{n\to\infty} \frac{1}{h_n^{2r+2}} E\left(\frac{1}{\tau_n}\right) = 0$, we have*

$$\lim_{n\to\infty} E|f_{\tau_n}^{(r)}(x) - f_G^{(r)}(x)|^2 = 0.$$

(2) *If $f_G(x) \in C_{s,a}$, $h_n = n^{-\frac{1}{2+2s}}, E\left(\frac{1}{\tau_n}\right) = o(n^{-\gamma})$, where $\gamma = \frac{s-1}{s+1}$, $s \geq 2$, for $0 < \lambda \leq 1$, we get*

$$E|f_{\tau_n}^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} \leq c \cdot n^{-\frac{\lambda(s-r)}{1+s}}.$$

P r o o f. Proof of (1):

$$E|f_{\tau_n}^{(r)}(x) - f_G^{(r)}(x)|^2 \leq 2|Ef_{\tau_n}^{(r)}(x) - f_G^{(r)}(x)|^2 + 2Var(f_{\tau_n}^{(r)}(x)) := 2(I_1^2 + I_2), \tag{2.6}$$

where

$$\begin{aligned} Ef_{\tau_n}^{(r)}(x) &= E\left\{E\left[\frac{1}{mh_n^{1+r}}\sum_{i=1}^{m}K_r\left(\frac{x-X_i}{h_n}\right)|\tau_n=m\right]\right\} \\ &= E\left\{\frac{1}{h_n^{1+r}}E\left[K_r\left(\frac{x-X_i}{h_n}\right)|\tau_n=m\right]\right\} \\ &= h_n^{-(1+r)}E\left[K_r\left(\frac{x-X_1}{h_n}\right)\right] \\ &= h_n^{-(1+r)}\int_0^\infty K_r\left(\frac{x-y}{h_n}\right)f_G(y)dy \\ &= h_n^{-r}\int_0^1 K_r(u)f_G(x-h_nu)du. \end{aligned}$$

We obtain the following Taylor expansion of $f_G(x-h_nu)$ in x

$$f_G(x-h_nu)-f_G(x) = \frac{f'_G(x)}{1!}(-h_nu)+\frac{f''_G(x)}{2!}(-h_nu)^2+\cdots+\frac{f_G^{(s)}(x-\xi h_nu)}{s!}(-h_nu)^s,$$

where $0<\xi<1$.

Since $f_G(x)$ is continuous in x and $(A1)$, it is easy to see that

$$\begin{aligned} 0 &\le \lim_{n\to\infty}|Ef_{\tau_n}^{(r)}(x)-f_G^{(r)}(x)| = \lim_{n\to\infty}\left|\frac{1}{h_n^r}\int_0^1 K_r(u)f_G(x-b_nu)du - f_G^{(r)}(x)\right| \\ &\le \frac{1}{r!}\int_0^1 u^r|K_r(u)|\lim_{n\to\infty}\left|f_G^{(r)}(x-\xi h_nu)-f_G^{(r)}(x)\right|du = 0, \end{aligned}$$

we have

$$\lim_{n\to\infty} I_1^2 = \lim_{n\to\infty}\left|Ef_{\tau_n}^{(r)}(x)-f_G^{(r)}(x)\right|^2 = 0. \tag{2.7}$$

by Lemma 2.1, we get

$$
\begin{aligned}
I_2 &= \frac{1}{h_n^{2r+2}} D\left[\frac{1}{\tau_n}\sum_{i=1}^{\tau_n} K_r\left(\frac{x-X_i}{h_n}\right)\right] \\
&= \frac{1}{h_n^{2r+2}} E\left[\frac{1}{\tau_n}\sum_{i=1}^{\tau_n}\left(K_r\left(\frac{x-X_i}{h_n}\right) - EK_r\left(\frac{x-X_i}{h_n}\right)\right)\right]^2 \\
&= \frac{1}{h_n^{2r+2}} E\left\{\frac{1}{m^2}\left[E\left(\sum_{i=1}^{m} K_r\left(\frac{x-X_i}{h_n}\right) - EK_r\left(\frac{x-X_i}{h_n}\right)\right)^2 |\tau_n = m\right]\right\} \\
&\le c\frac{1}{h_n^{2r+2}} E\left\{\frac{1}{m}\left[E\left(K_r\left(\frac{x-X_1}{h_n}\right)\right)^2 |\tau_n = m\right]\right\} \\
&\le c\frac{1}{h_n^{2r+2}} E\left(\frac{1}{\tau_n}\right).
\end{aligned}
\tag{2.8}
$$

when $\frac{1}{h_n^{2r+2}} E\left(\frac{1}{\tau_n}\right) \to 0$, we have

$$
\lim_{n\to\infty} I_2 = 0. \tag{2.9}
$$

Substituting (2.7) and (2.9) into (2.6), the proof of (1) is completed.

Proof of (2): Similar to (2.6), we can show that

$$
\begin{aligned}
E|f_{\tau_n}^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} &\le 2[Ef_{\tau_n}^{(r)}(x) - f_G^{(r)}(x)]^{2\lambda} + 2[Varf_{\tau_n}^{(r)}(x)]^{\lambda} \\
&:= 2(J_1^{2\lambda} + J_2^{\lambda}).
\end{aligned}
\tag{2.10}
$$

We obtain the following Taylor expansion of $f_G(x-h_nu)$ in x

$$
f_G(x-h_nv) = f_G(x) + \frac{f_G'(x)}{1!}(-h_nv) + \frac{f_G''(x)}{2!}(-h_nv)^2 + \cdots + \frac{f_G^{(r)}(x-\xi h_nv)}{r!}(-h_nv)^r,
$$

where $0 < \xi < 1$.

Since $(A1)$ and $f_G(x) \in C_{s,\alpha}$, we have

$$
|Ef_{\tau_n}^{(r)}(x) - f_G^{(r)}(x)| \le \int_0^1 |K_r(v)| h_n^{s-r} v^s \left|\frac{f_G^{(s)}(x-\xi h_nv)}{s!}\right| dv \le c\cdot h_n^{s-r}.
$$

when $h_n = n^{-\frac{1}{2+2s}}$, we get

$$
J_1^{2\lambda} = |Ef_{\tau_n}^{(r)}(x) - f_G^{(r)}(x)|^{2\lambda} \le c\cdot n^{-\frac{\lambda(s-r)}{s+1}}. \tag{2.11}
$$

By (2.7), when $h_n = n^{-\frac{1}{2+2s}}$, $E\left(\frac{1}{\tau_n}\right) = o(n^{-\gamma})$, where $\gamma = \frac{s-1}{s+1}$, we have

$$J_2^\lambda \le c_1\left[(h_n^{2r+2})^{-1}\right]^\lambda \left[E\left(\frac{1}{\tau_n}\right)\right]^\lambda \le c\cdot n^{-\frac{\lambda(s-r)}{1+s}}. \tag{2.12}$$

Substituting (2.11) and (2.12) into (2.9), obviously, proof of (2) is completed. □

Lemma 2.3 (Van, [2]). *$R(\delta_G, G)$ and $R(\delta_{\tau_n}, G)$ are defined by* (1.9) *and* (2.4)*, then*

$$0 \le R(\delta_{\tau_n}, G) - R(\delta_G, G) \le a\int_\Omega |\beta(x)|P(|\beta_{\tau_n}(x) - \beta(x)| \ge |\beta(x)|)dx.$$

Lemma 2.4. *$P_G(x)$ and $p_{\tau_n}(x)$ are defined by* (1.7) *and* (2.2)*. Let $X_1, X_2, \ldots, X_{\tau_n}$ be independent identical random samples, then, for $0 < \lambda \le 1$, $E\left[\left(\frac{1}{\tau_n}\right)^\lambda\right] = O(n^{-\lambda})$, we have*

$$E|p_{\tau_n}(x) - p_G(x)|^{2\lambda} \le cn^{-\lambda}.$$

Proof. Since

$$Ep_{\tau_n}(x) = E\left\{\frac{I_{(X_1>x)}}{m(X_1)}\right\} = \int_x^\infty \frac{f_G(y)}{m(y)}dy = \int_x^\infty p(y)dy = p_G(x),$$

we get ϕ_{τ_n} is an unbiased estimator of $p_G(x)$.

Applying moment monotone inequality, we have

$$\left(E|p_{\tau_n}(x) - p_G(x)|^{2\lambda}\right)^{\frac{1}{2\lambda}} \le \left(E|p_{\tau_n}(x) - p_G(x)|^2\right)^{\frac{1}{2}}.$$

That is to say

$$E\left|p_{\tau_n}(x) - p_G(x)\right|^{2\lambda} \le \left(E|p_{\tau_n}(x) - p_G(x)|^2\right)^\lambda := J.$$

By Lemma 2.1, we can easily get

$$\begin{aligned} J &= E|p_{\tau_n}(x) - p_G(x)|^{2\lambda} \le E[E|p_{\tau_n}(x) - p_G(x)|^{2\lambda}|\tau_n = m] \\ &\le cE\left[\left(\frac{1}{\tau_n}\right)^\lambda\right] \le cn^{-\lambda}. \end{aligned}$$

The proof of Lemma 2.4 is completed. □

3. Asymptotic optimality and convergence rates.

Theorem 3.1. *$\delta_{\tau_n}(x)$ is defined by (2.4). Let $X_1, X_2, \ldots, X_{\tau_n}$ be independent identical random sample. Assume (A1) and the following regularity conditions hold.*

(1) $\lim_{n\to\infty} h_n = 0$ *and* $\lim_{n\to\infty} \frac{1}{h_n^6} E\left(\frac{1}{\tau_n}\right) = 0$,

(2) $\int_\Theta \theta^2 dG(\theta) < \infty$,

(3) $f_G^{(2)}(x)$ *is continuous function of x,*

we get

$$\lim_{n\to\infty} R(\delta_{\tau_n}, G) = R(\delta_G, G).$$

Proof. By Lemma 2.2, we have

$$0 \le R(\delta_{\tau_n}, G) - R(\delta_G, G) \le a \int_\Omega |\beta(x)| p(|\beta_{\tau_n}(x) - \beta(x)| \ge |\beta(x)|) dx.$$

Witting $\Psi_{\tau_n}(x) = |\beta(x)| p(|\beta_{\tau_n}(x) - \beta(x)| \ge \beta(x)|)$. Hence, $\Psi_{\tau_n}(x) \le |\beta(x)|$.

Again by (1.6) and Fubini theorem, we can get

$$\int_\Omega |\beta(x)| dx \le |\theta_0^2 - \gamma_0^2| + \int_\Theta \theta^2 dG(\theta) + 2|\theta_0| \int_\Theta \theta dG(\theta) < \infty.$$

Applying domain convergence theorem, then

$$0 \le \lim_{n\to\infty} R(\delta_{\tau_n}, G) - R(\delta_G, G) \le \int_\Omega [\lim_{n\to\infty} \Psi_{\tau_n}(x)] dx, \tag{3.1}$$

If Theorem 3.1 holds, we only need to prove $\lim_{n\to\infty} \Psi_{\tau_n}(x) = 0$ a.s.x,

By Markov's and Jensen's inequations, then

$$\begin{aligned}\Psi_{\tau_n}(x) \; &\le \left[E\left|f_{\tau_n}^{(2)}(x) - f_G^{(2)}(x)\right|^2\right]^{\frac{1}{2}} + |Q(x)| \left[E\left|f_{\tau_n}^{(1)}(x) - f_G^{(1)}(x)\right|^2\right]^{\frac{1}{2}} \\ &+\mu|Q(x)\left[E\left|p_{\tau_n}(x) - p_G(x)\right|^2\right]^{\frac{1}{2}} + |S(x)| \left[E\left|f_{\tau_n}(x) - f_G(x)\right|^2\right]^{\frac{1}{2}}.\end{aligned}$$

Again by Lemma 2.1 (1) and Lemma 2.4, for fixed $x \in \Omega$, when $r = 0, 1, 2$

and $\lambda = 1$, we get

$$
\begin{aligned}
0 &\le \lim_{n\to\infty} \Psi_{\tau_n}(x) \\
&\le \left[\lim_{n\to\infty} E|f^{(2)}_{\tau_n}(x) - f^{(2)}_G(x)|^2\right]^{\frac{1}{2}} \\
&\quad + |Q(x)| \left[\lim_{n\to\infty} E|f^{(1)}_{\tau_n}(x) - f^{(1)}_G(x)|^2\right]^{\frac{1}{2}} \\
&\quad + \mu|Q(x)| \left[\lim_{n\to\infty} E|p_n(x) - p_G(x)|^2\right]^{\frac{1}{2}} \\
&\quad + |S(x)| \left[\lim_{n\to\infty} E|f_{\tau_n}(x) - f_G(x)|^2\right]^{\frac{1}{2}} = 0.
\end{aligned} \tag{3.2}
$$

Substituting (3.2) into (3.1), proof of Theorem 3.1 is completed. □

Theorem 3.2. *$\delta_{\tau_n}(x)$ is defined by (2.4). Let $X_1, X_2, \ldots, X_n$ be independent identical random samples. Assume (A1) and the following regularity conditions hold.*

(4) *$f_G(x) \in C_{s.\alpha}$, where $s \ge 3$,*

(5) *$h_n = n^{-\frac{1}{2+2s}}, E\left(\frac{1}{\tau_n}\right) = o(n^{-\gamma})$, where $\gamma = \frac{s-1}{s+1}$,*

(6) *for $0 < \lambda \le 1$ and $m = 0, 1, 2$, $\int_\Omega x^{m\lambda}|\beta(x)|^{1-\lambda}dx < \infty$.*

we have

$$R(\delta_{\tau_n}, G) - R(\delta_G, G) = O\left(n^{-\frac{\lambda(s-4)}{2(s+2)}}\right).$$

P r o o f. By Lemma 2.2 and Markov's inequations, then

$$\begin{aligned} 0 \le R(\delta_{\tau_n}, G) - R(\delta_G, G) &\le c_1 \int_\Omega |\beta(x)|^{1-\lambda} E|f_{\tau_n}^{(2)}(x) - f_G^{(2)}(x)|^\lambda dx \\ &+ c_2 \int_\Omega |\beta(x)|^{1-\lambda} |Q(x)|^\lambda E|f_{\tau_n}^{(2)}(x) - f_G^{(2)}(x)|^\lambda dx \\ &+ c_3 \int_\Omega |\beta(x)|^{1-\lambda} \mu^\lambda |Q(x)|^\lambda E|p_{\tau_n}(x) - p_G(x)|^\lambda dx \\ &+ c_4 \int_\Omega |\beta(x)|^{1-\lambda} |S(x)|^\lambda E|f_{\tau_n}(x) - f_G(x)|^\lambda dx \\ &= A_n + B_n + C_n + D_n. \end{aligned} \tag{3.3}$$

By Lemma 2.2 (2) and condition (6), we get

$$A_n \le c_1 n^{-\frac{\lambda(s-2)}{2(s+1)}} \int_\Omega |\beta(x)|^{1-\lambda} dx \le c_5 n^{-\frac{\lambda(s-2)}{2(s+1)}}. \tag{3.4}$$

$$B_n \le c_2 n^{-\frac{\lambda(s-2)}{2(s+2)}} \int_\Omega |\beta(x)|^{1-\lambda} |Q(x)|^\lambda dx \le c_6 n^{-\frac{\lambda(s-2)}{2(s+1)}}. \tag{3.5}$$

$$D_n \le c_4 n^{-\frac{\lambda s}{2(s+1)}} \int_\Omega |\beta(x)|^{1-\lambda} |S(x)|^\lambda dx \le c_8 n^{-\frac{\lambda s}{2(s+1)}}. \tag{3.6}$$

Using Lemma 2.4 and condition (6), we can obtain

$$C_n \le c_3 n^{-\frac{\lambda}{2}} \mu^\lambda \int_\Omega |\beta(x)|^{1-\lambda} |Q(x)|^\lambda dx \le c_7 n^{-\frac{\lambda}{2}}. \tag{3.7}$$

Substituting (3.4)–(3.7) into (3.3), we have

$$R(\delta_{\tau_n}, G) - R(\delta_G, G) = O\left(n^{-\frac{\lambda(s-2)}{2(s+1)}}\right).$$

Proof of Theorem 3.2 is completed. □

Remark. When $\lambda \to 1$, $O\left(n^{-\frac{\lambda(s-2)}{2(s+1)}}\right)$ is arbitrarily close to $O\left(n^{-\frac{1}{2}}\right)$.

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School of Science
Guangdong Ocean University
Zhanjiang, China, 524088
e-mail: huangjuan401178@163.com

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